Homework1

October 11, 2018

```
In [168]: import numpy as np
          import pandas as pd
```

0.1 Problem 1

0.2 Part 1, 2

0.3 Part 3

```
In [3]: # Define softmax and ReLu
        relu = np.vectorize(lambda z: np.fmax(0,z))
        softmax = lambda x: np.exp(x)/(np.exp(x).sum(axis=0, keepdims=True))
        def predict(y):
            return np.argmax(y,axis=0)
```

```
In [4]: def ff_nn_2_ReLu(x,W1,W2,W3,b1,b2,b3):
            a_1 = np.dot(W1,x) + b1
            h1 = relu(a_1)
            a_2 = np.dot(W2,h1) + b2
            h2 = relu(a_2)
            h2
            a_3 = np.dot(W3,h2) + b3
            y = softmax(a_3)
            return y
```

0.4 Part 4

```
In [133]: x = np.array([[1,0,0],[-1,-1,1]])
          W1 = np.array([[1,0],[-1,0],[0,.5]])
          W2 = np.array([[1,0,0],[-1,-1,0]])
          W3 = np.array([[1,1],[0,0],[-1,-1]])
          b1 = np.array([[0],[0],[1]])
          b2 = np.array([[1],[-1]])
          b3 = np.array([[1],[0],[0]])
          y = ff_nn_2_ReLu(x,W1,W2,W3,b1,b2,b3)
          y
```

```
Out[133]: array([[0.94649912, 0.84379473, 0.84379473],
                 [0.04712342, 0.1141952 , 0.1141952 ],
                 [0.00637746, 0.04201007, 0.04201007]])
```

For each x_i, the probability of x_i in class j is $(i, j)^{th}$ term of the matrix. For $1 \leq$ i $\leq$ n, 1$\leq$ j $\leq$ m, where n is number of samples (number of columns in x) and m is number of classes (assuming index starts at 1). Hence, for this input, all x_1, x_2, x_3 will be calssified in class 1 (assuming index starts at 0)

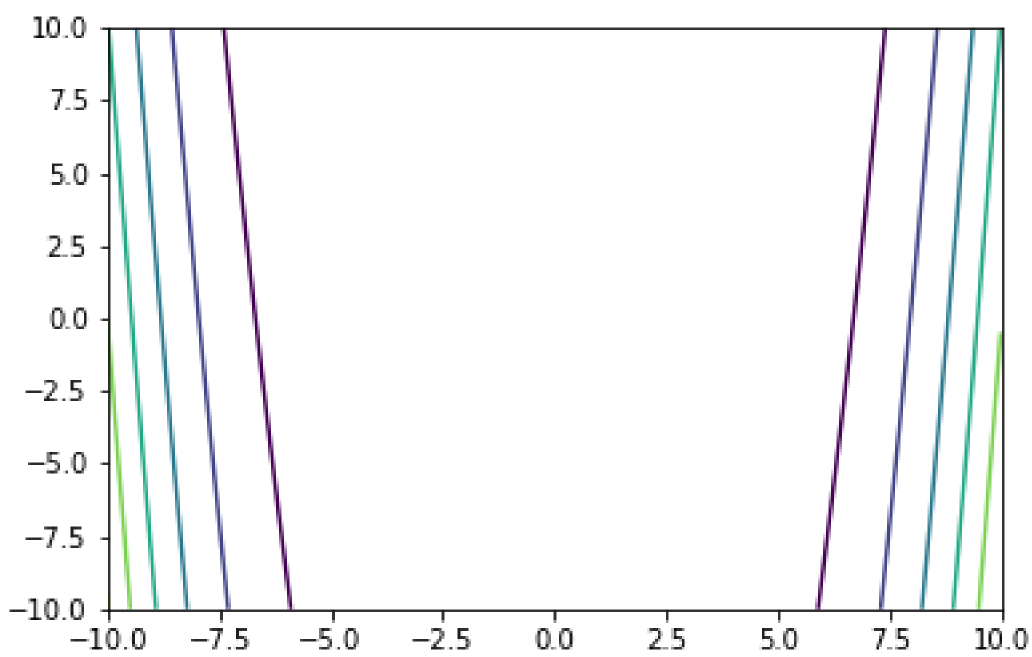
0.5 Problem 2

0.6 Part 1

$\frac{\delta f(x,y)}{\delta x} = -2(1-x) + 200(y-x^2)(-2x) = 2x - 2 - 400x(y-x^2) = 2x - 2 - 400(xy - x^3)$ $\frac{\delta f(x,y)}{\delta y} = 2 * 100(y-x^2) = 200(y-x^2)$

0.7 Part 2

In [19]:
```
import matplotlib.pyplot as plt
import numpy as np
# --- Paraboloid Function ---
delta = 0.025
x = np.arange(-10.0, 10.0, delta)
y = np.arange(-10.0, 10.0, delta)
X, Y = np.meshgrid(x, y)
Z = (1-X)**2 + 100*(Y-X**2)**2
fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
```



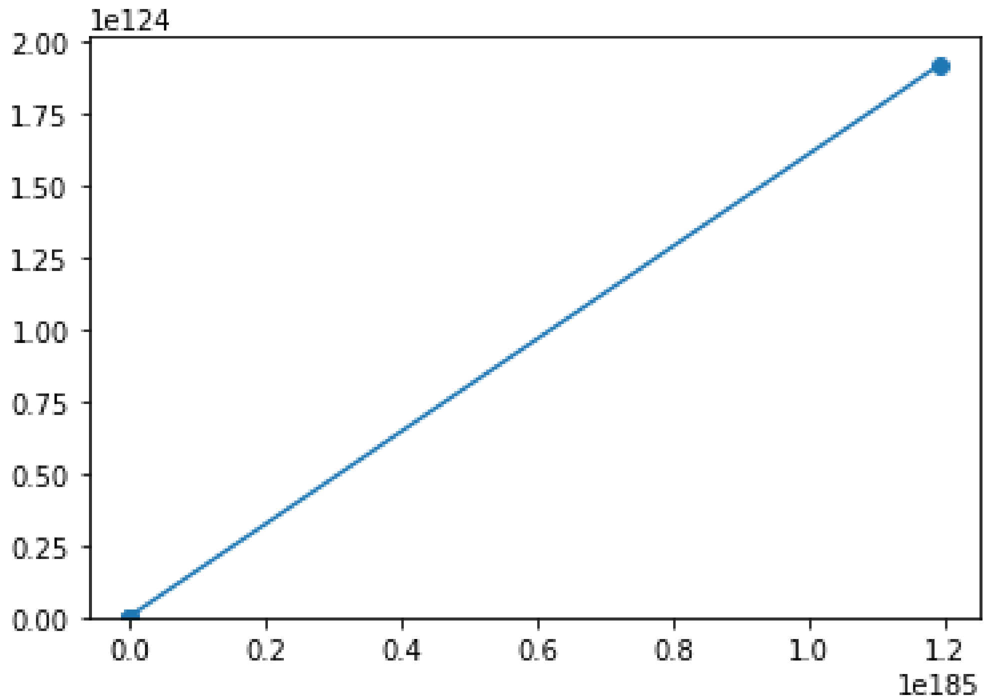
In [101]:
```
# --- Setting up gradient ---
def grad_f(vector):
```

```
        x, y = vector
        df_dx = 2*x - 2 - 400*(x*y-x**3)
        df_dy = 200*(y-x**2)
        return np.array([df_dx, df_dy])

    # --- Grad Descent ----
    def grad_descent(starting_point=None, iterations=10, learning_rate=6):
        if starting_point:
            point = starting_point
        else:
            point = np.random.uniform(-5,5,size=2)
            print('Initial point is',point)
        trajectory = [point]

        for i in range(iterations):
            grad = grad_f(point)
            point = point - learning_rate * grad
            trajectory.append(point)
        print('After iterations, the point is ',point)
        return np.array(trajectory)
```

In [102]:
```
# --- Visualize Trajectory ---
np.random.seed(10)
traj = grad_descent(iterations=50,learning_rate=10)

fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
x= traj[:,0]
y= traj[:,1]
plt.plot(x,y,'-o')
```

```
Initial point is [ 2.71320643 -4.79248051]
After iterations, the point is  [nan nan]


/Users/FrankWang/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:4: RuntimeWarning
  after removing the cwd from sys.path.
/Users/FrankWang/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:4: RuntimeWarning
  after removing the cwd from sys.path.
/Users/FrankWang/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:5: RuntimeWarning
  """
```

Out[102]:
```
[<matplotlib.lines.Line2D at 0x144754978>]
```

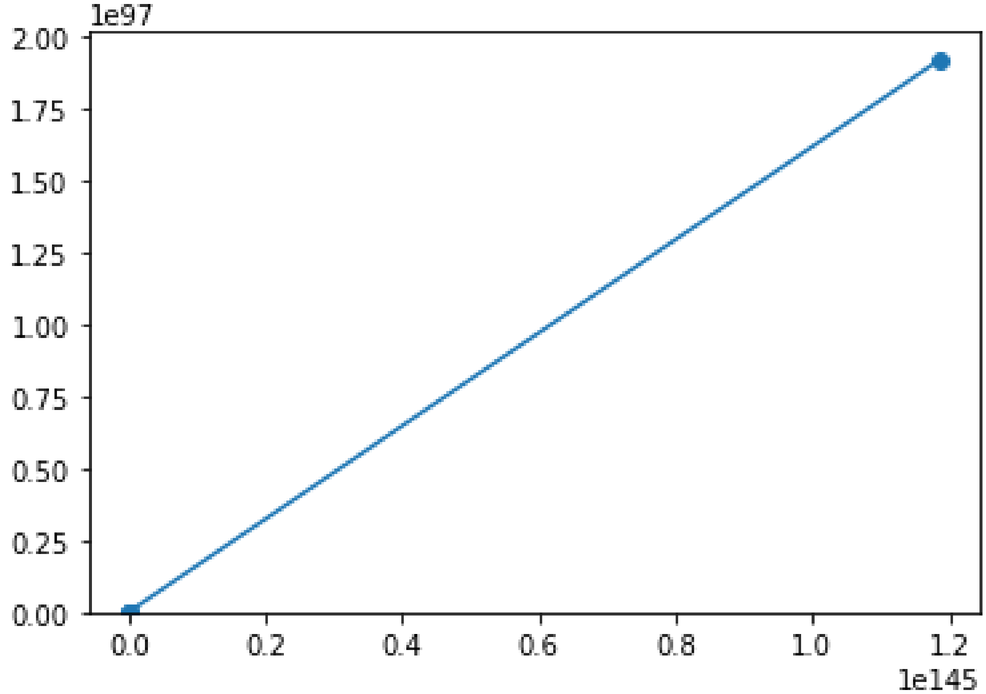


In [103]:
```
# --- Visualize Trajectory ---
np.random.seed(10)
traj = grad_descent(iterations=50,learning_rate=1)

fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
x= traj[:,0]
y= traj[:,1]
plt.plot(x,y,'-o')
```

```
Initial point is [ 2.71320643 -4.79248051]
After iterations, the point is  [nan nan]
```

```
/Users/FrankWang/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:4: RuntimeWarning
  after removing the cwd from sys.path.
/Users/FrankWang/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:4: RuntimeWarning
  after removing the cwd from sys.path.
```

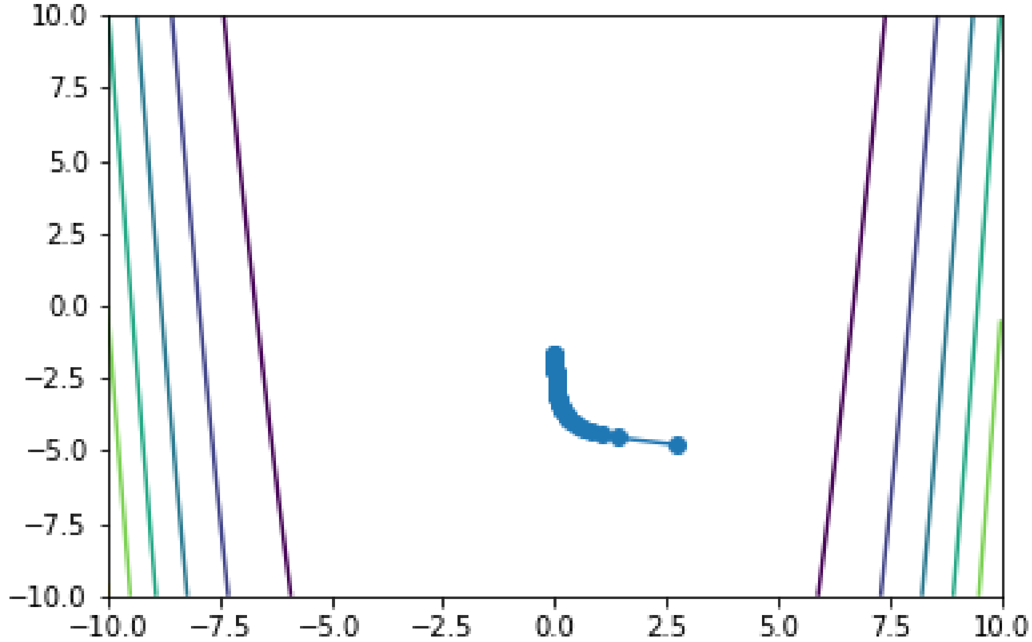
Out[103]:
```
[<matplotlib.lines.Line2D at 0x144816a90>]
```




```
In [104]: # --- Visualize Trajectory ---
          np.random.seed(10)
          traj = grad_descent(iterations=50,learning_rate=0.0001)

          fig, ax = plt.subplots()
          CS = ax.contour(X, Y, Z)
          x= traj[:,0]
          y= traj[:,1]
          plt.plot(x,y,'-o')
```

```
Initial point is [ 2.71320643 -4.79248051]
After iterations, the point is  [ 0.00513221 -1.65101579]
```

```
Out[104]: [<matplotlib.lines.Line2D at 0x1448fba20>]
```



This function is not convex and the minimum point is at (-1,-1). When the learning rate is large, the trajectories diverage. Large learning rates will skip the optimal point. On the other hand, when learning rate is small, it becomes more stable (this can be seen from the last trajectory). Here, small learning rate takes each step smaller and it will not skip the optimal point, but it requires more iterations to find the optimal point. Hence, choosing a right learning rate (tuning parameter) is important

0.8 Part 4

0.9 Gradient Descent with Momentum algorithm

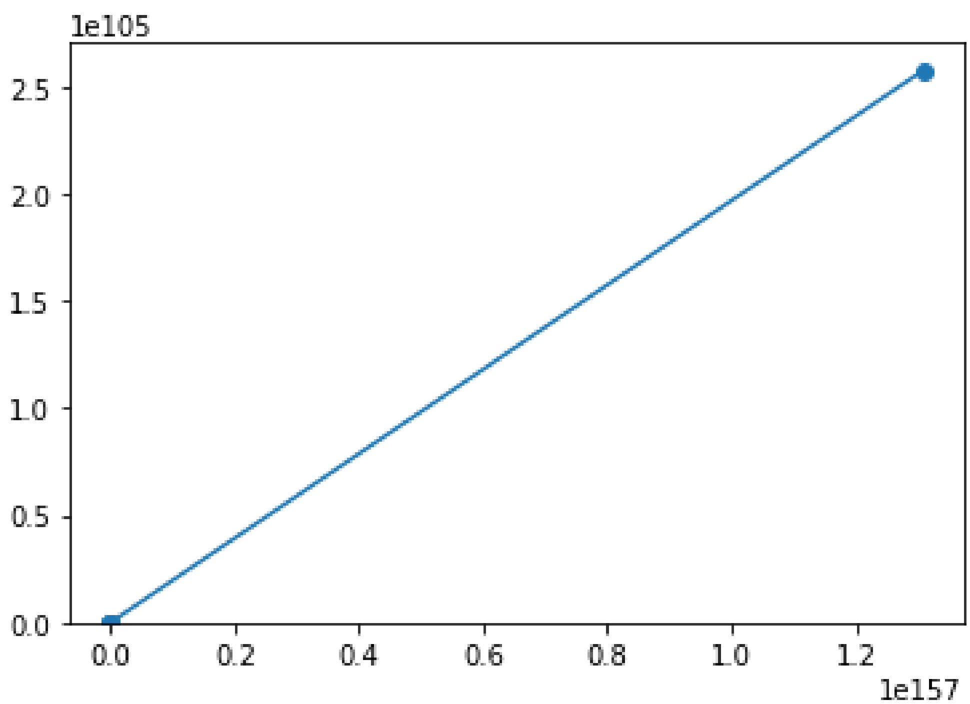
In [105]:
```
# --- Gradient with momentum ---
def grad_descent_with_momentum(starting_point=None, iterations=10, alpha=.9, epsilon=
    if starting_point:
        point = starting_point
    else:
        point = np.random.uniform(-5,5,size=2)
    trajectory = [point]
    v = np.zeros(point.size)
    print('Initial point is',point)

    for i in range(iterations):
        grad = grad_f(point)
        v = alpha*v + epsilon*grad
        point = point - v
        trajectory.append(point)
```

```
        print('After iterations, the point is ',point)
        return np.array(trajectory)
```

```
In [106]: # --- Visualizing trajectory --
          np.random.seed(10)
          traj = grad_descent_with_momentum(iterations=100, epsilon=2, alpha=.7)

          fig, ax = plt.subplots()
          CS = ax.contour(X, Y, Z)
          x= traj[:,0]
          y= traj[:,1]
          plt.plot(x,y,'-o')
```

```
Initial point is [ 2.71320643 -4.79248051]
After iterations, the point is  [nan nan]


/Users/FrankWang/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:4: RuntimeWarning
  after removing the cwd from sys.path.
/Users/FrankWang/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:5: RuntimeWarning
  """
/Users/FrankWang/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:4: RuntimeWarning
  after removing the cwd from sys.path.
/Users/FrankWang/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:5: RuntimeWarning
  """
```

```
Out[106]: [<matplotlib.lines.Line2D at 0x144816518>]
```

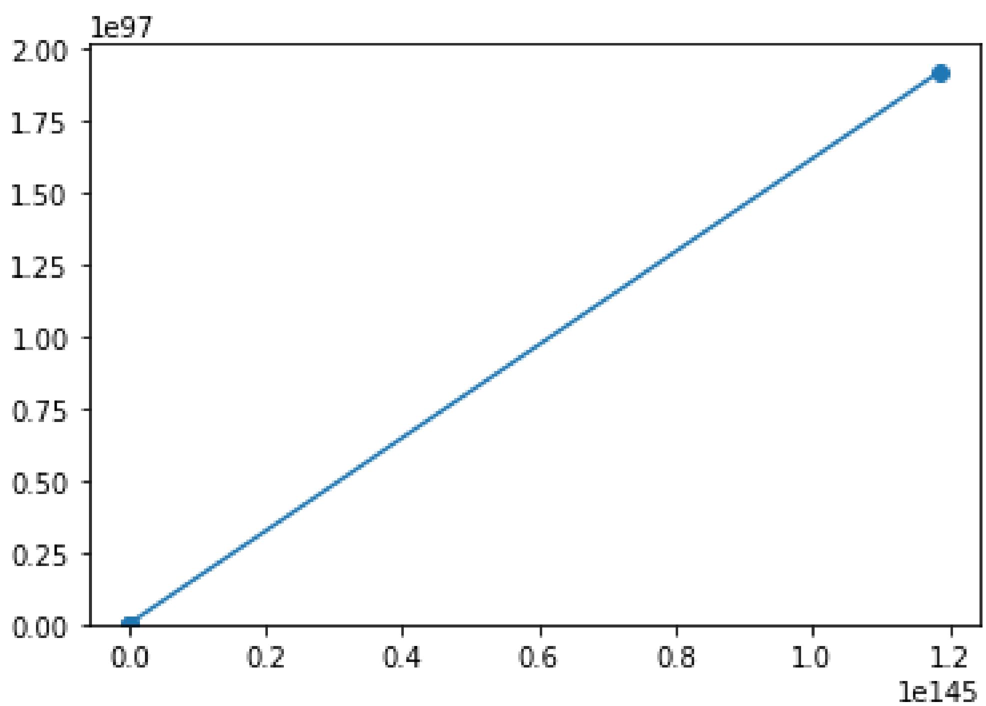


In [107]:
```
# --- Visualizing trajectory --
np.random.seed(10)
traj = grad_descent_with_momentum(iterations=50, epsilon=1, alpha=.5)

fig, ax = plt.subplots()
CS = ax.contour(X, Y, Z)
x= traj[:,0]
y= traj[:,1]
plt.plot(x,y,'-o')
```

```
Initial point is [ 2.71320643 -4.79248051]
After iterations, the point is  [nan nan]
```

```
/Users/FrankWang/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:4: RuntimeWarning
  after removing the cwd from sys.path.
/Users/FrankWang/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:4: RuntimeWarning
  after removing the cwd from sys.path.
```

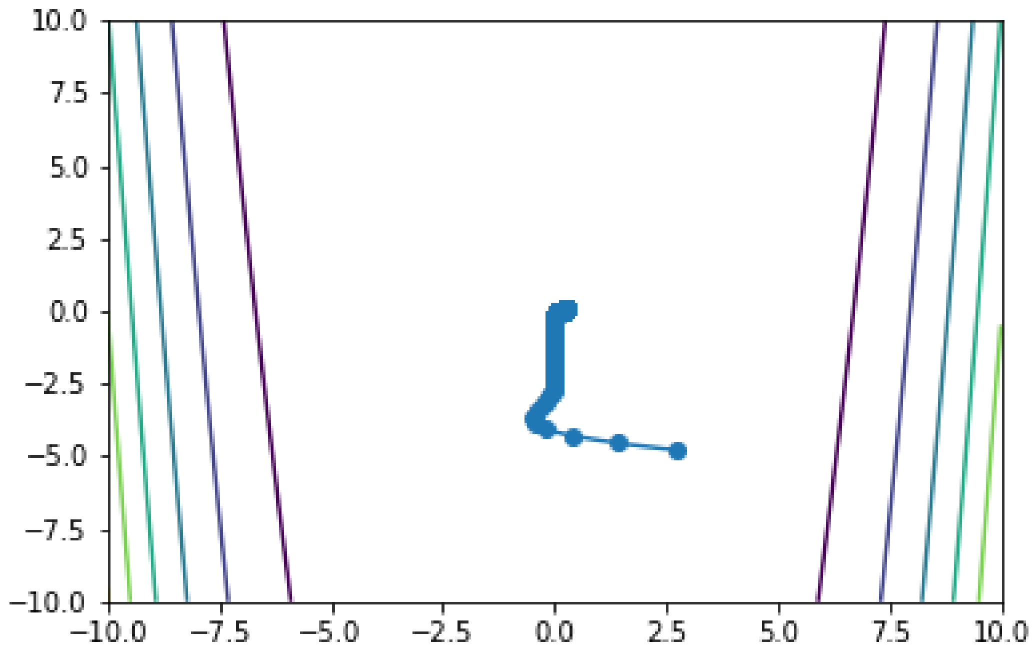
Out[107]: [<matplotlib.lines.Line2D at 0x144a820b8>]



```
In [118]: # --- Visualizing trajectory --
          np.random.seed(10)
          traj = grad_descent_with_momentum(iterations=1000, epsilon=0.0001, alpha=.5)

          fig, ax = plt.subplots()
          CS = ax.contour(X, Y, Z)
          x= traj[:,0]
          y= traj[:,1]
          plt.plot(x,y,'-o')
```

```
Initial point is [ 2.71320643 -4.79248051]
After iterations, the point is  [0.28519766 0.07826446]
```

```
Out[118]: [<matplotlib.lines.Line2D at 0x145215c18>]
```



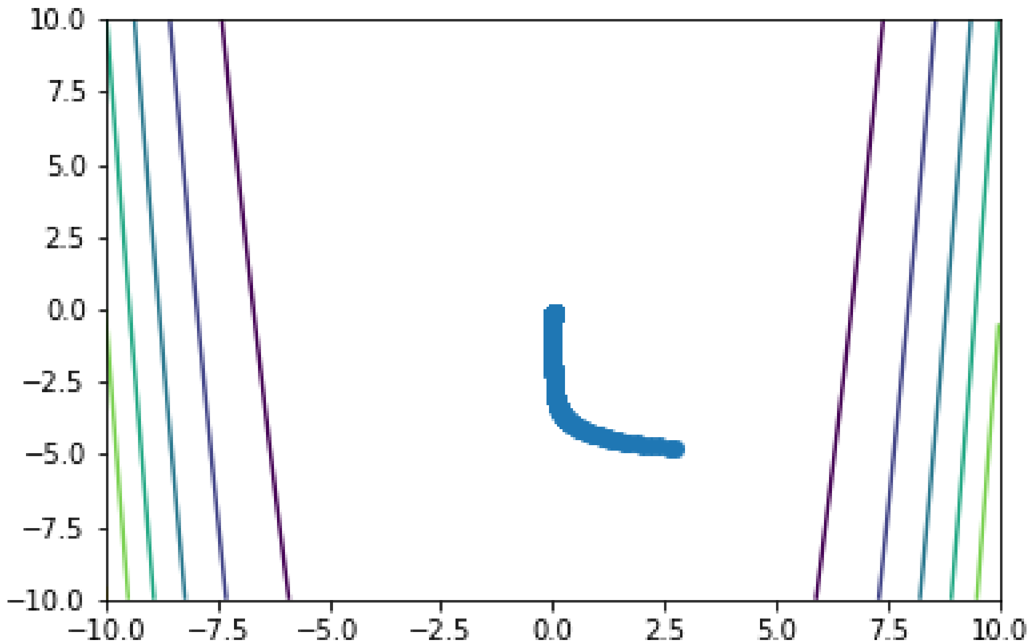
From all trajectories above, we can see epsilon is similar to learning rate in part c. Larger epsilon makes the algorithm skip the optimal point and diverage. On the other hand, small epsilon makes it more stable but takes more iterations to reach optimal point. For example, in the last plot, initial point is (2.71,-4.79) and it moves toward optimal point. After 1000 iterations, it ends at (0.285,0.0783). Next, I will try a smaller epsilon and see how it performs.

```
In [119]: # --- Visualizing trajectory --
          np.random.seed(10)
          traj = grad_descent_with_momentum(iterations=1000, epsilon=0.00001, alpha=.5)

          fig, ax = plt.subplots()
```

```
CS = ax.contour(X, Y, Z)
x= traj[:,0]
y= traj[:,1]
plt.plot(x,y,'-o')
```

```
Initial point is [ 2.71320643 -4.79248051]
After iterations, the point is  [ 0.01632369 -0.08119181]
```

```
Out[119]: [<matplotlib.lines.Line2D at 0x13f27f630>]
```



In the plot above, I changed epsilon 10 times smaller. Initial points were the same, but the second one moved slower than the first one.

1 Problem 3

1.1 Part 1

Let x_i be ith input for $x_1, x_2, \ldots, x_n$ and x_i is a 2D vector (2×1), W_1 be the first weight matrix

$$W_1 = \begin{bmatrix} w_{111} & w_{121} \\ w_{112} & w_{122} \\ w_{113} & w_{123} \end{bmatrix}$$

.

W_2 be the second weight matrix

$$W_2 = \begin{bmatrix} w_{211} & w_{221} & w_{231} \\ w_{212} & w_{222} & w_{232} \end{bmatrix}$$

.

W_3 be the third weight matrix

$$W_3 = \begin{bmatrix} w_{311} & w_{321} \\ w_{312} & w_{322} \\ w_{313} & w_{323} \end{bmatrix}$$

.

Then, with loss function

$$\boldsymbol{z_1} = w_1\boldsymbol{x} + \boldsymbol{b_1}, \boldsymbol{h_1} = f(\boldsymbol{z_1})$$
$$\boldsymbol{z_2} = w_2\boldsymbol{h_1} + \boldsymbol{b_2}, \boldsymbol{h_2} = f(\boldsymbol{z_2})$$
$$\boldsymbol{a} = w_3\boldsymbol{h_2} + \boldsymbol{b_3}, \boldsymbol{\hat{y}} = g(\boldsymbol{a})$$
$$L(\boldsymbol{y}, \boldsymbol{\hat{y}}) = y_1 log(\hat{y}_1) + y_2 log(\hat{y}_2) + y_3 log(\hat{y}_3)$$

where f and g are Relu and softmax function respectively. Take gradient with respect to each parameters, we have

$$\frac{\delta L}{\delta \boldsymbol{a}} = \frac{\delta L}{\delta \boldsymbol{\hat{y}}}\frac{\delta \boldsymbol{\hat{y}}}{\delta \boldsymbol{a}} = \boldsymbol{\hat{y}} - \boldsymbol{y}$$

$$\frac{\delta L}{\delta \boldsymbol{b_3}} = \frac{\delta L}{\delta \boldsymbol{a}}\frac{\delta \boldsymbol{a}}{\delta \boldsymbol{b_3}} = (\boldsymbol{\hat{y}} - \boldsymbol{y})$$

$$\frac{\delta L}{\delta \boldsymbol{w_3}} = \frac{\delta L}{\delta \boldsymbol{a}}\frac{\delta \boldsymbol{a}}{\delta \boldsymbol{w_3}} = (\boldsymbol{\hat{y}} - \boldsymbol{y})\boldsymbol{h_2}$$

$$\frac{\delta L}{\delta \boldsymbol{b_2}} = \frac{\delta L}{\delta \boldsymbol{a}}\frac{\delta \boldsymbol{a}}{\delta \boldsymbol{h_2}}\frac{\delta \boldsymbol{h_2}}{\delta z_2}\frac{\delta z_2}{\delta \boldsymbol{b_2}} = (\boldsymbol{\hat{y}} - \boldsymbol{y}) * \boldsymbol{w_3} * \mathbb{1}(z_2 > 0) * 1$$

$$\frac{\delta L}{\delta \boldsymbol{w_2}} = \frac{\delta L}{\delta \boldsymbol{a}}\frac{\delta \boldsymbol{a}}{\delta \boldsymbol{h_2}}\frac{\delta \boldsymbol{h_2}}{\delta z_2}\frac{\delta z_2}{\delta \boldsymbol{w_2}} = (\boldsymbol{\hat{y}} - \boldsymbol{y}) * \boldsymbol{w_3} * \mathbb{1}(z_2 > 0) * \boldsymbol{h_1}$$

$$\frac{\delta L}{\delta \boldsymbol{b_1}} = \frac{\delta L}{\delta \boldsymbol{a}}\frac{\delta \boldsymbol{a}}{\delta \boldsymbol{h_2}}\frac{\delta \boldsymbol{h_2}}{\delta z_2}\frac{\delta z_2}{\delta \boldsymbol{h_1}}\frac{\delta \boldsymbol{h_1}}{\delta z_1}\frac{\delta z_1}{\delta \boldsymbol{b_1}} = (\boldsymbol{\hat{y}} - \boldsymbol{y}) * \boldsymbol{w_3} * \mathbb{1}(z_2 > 0) * \boldsymbol{w_2} * \mathbb{1}(z_1 > 0) * 1$$

$$\frac{\delta L}{\delta \boldsymbol{w_1}} = \frac{\delta L}{\delta \boldsymbol{a}}\frac{\delta \boldsymbol{a}}{\delta \boldsymbol{h_2}}\frac{\delta \boldsymbol{h_2}}{\delta z_2}\frac{\delta z_2}{\delta \boldsymbol{h_1}}\frac{\delta \boldsymbol{h_1}}{\delta z_1}\frac{\delta z_1}{\delta \boldsymbol{w_1}} = (\boldsymbol{\hat{y}} - \boldsymbol{y}) * \boldsymbol{w_3} * \mathbb{1}(z_2 > 0) * \boldsymbol{w_2} * \mathbb{1}(z_1 > 0) * \boldsymbol{x}$$

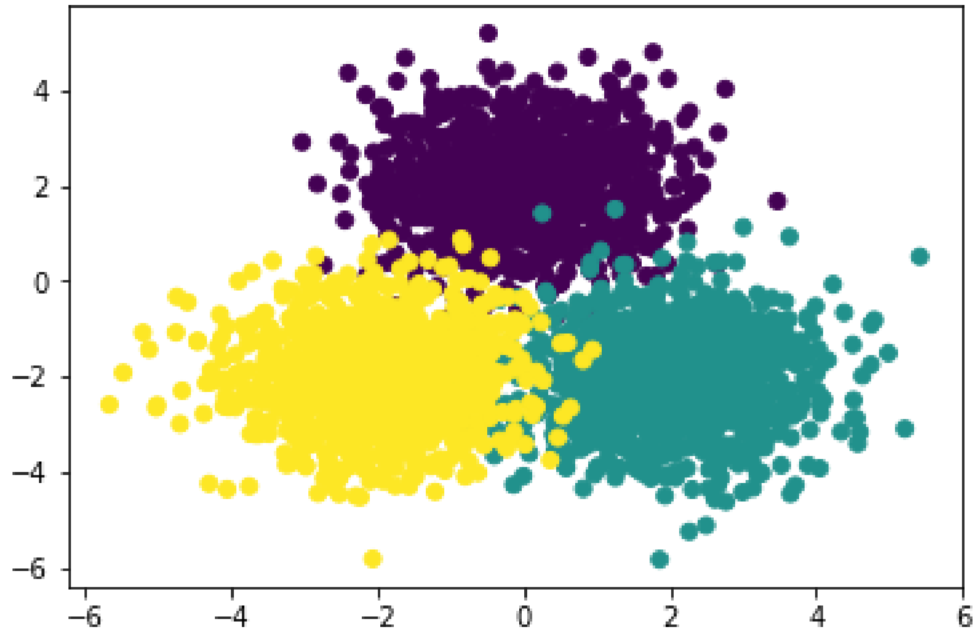
1.2 Part 2

```
In [377]: def grad_f(X,H1,H2,Y,Y_hat,parameters):
              # Unpacking parameters
              W1,b1,W2,b2,W3,b3 = parameters
              # Gradients - ReLU


              dW3 = np.dot((Y_hat - Y).T,H2)
              db3 = np.reshape((Y_hat - Y).sum(axis=0),(3,1))
              db2 = np.reshape((np.dot((Y_hat - Y),W3)* (H2 > 0)).sum(axis = 0),(2,1))
              dW2 = np.dot((np.dot((Y_hat - Y),W3)* (H2 > 0)).T,H1)
              db1 = np.reshape(np.sum(np.dot(np.dot((Y_hat - Y),W3)* (H2 > 0),W2)*(H1>0),axis =
              dW1 = np.dot((np.dot(np.dot((Y_hat - Y),W3)* (H2 > 0),W2)*(H1>0)).T,X)

              return [dW1,db1,dW2,db2,dW3,db3]
```

1.3 Part 3

```
In [378]: covm = [[1, 0], [0, 1]]
          x = np.concatenate((np.random.multivariate_normal(mean = [0, 2], cov=covm, size = 100
                              np.random.multivariate_normal(mean = [2,-2], cov=covm, size =1000)
                              np.random.multivariate_normal(mean = [-2, -2], cov=covm, size =100
          y = np.zeros(3000)
          y[1000:2000] = 1
          y[2000:3000] = 2
          plt.scatter(x[:,0], x[:,1], c=y)
```

```
Out[378]: <matplotlib.collections.PathCollection at 0x118d3fc18>
```



1.4 Part 4

```
In [379]: Y = pd.Series(y)
          Y = pd.get_dummies(Y).as_matrix()
          X = x
```

```
/Users/FrankWang/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:2: FutureWarning:
```

```
In [380]: W1 = np.array([[1,0],[-1,0],[0,.5]])
          W2 = np.array([[1,0,0],[-1,-1,0]])
          W3 = np.array([[1,1],[0,0],[-1,-1]])
```

```
b1 = np.array([[0],[0],[1]])
b2 = np.array([[1],[-1]])
b3 = np.array([[1],[0],[0]])
```

In [381]:
```
ReLu = np.vectorize(lambda z: np.fmax(0,z))
softmax = lambda z: np.exp(z)/(np.sum(np.exp(z),axis=1))[:,np.newaxis]
def predict(Y_hat):
    return np.argmax(Y_hat, axis=1)
```

In [382]:
```
def loss(y, y_hat):
    tot = y * np.log(y_hat)
    return -tot.sum()
def forward(X,parameters):
    # Unpacking parameters
    W1,b1,W2,b2,W3,b3 = parameters
    # Forward pass
    a1 = np.dot(x,W1.T) +b1.T
    H1 = ReLu(a1)
    a2 = np.dot(H1,W2.T) + b2.T
    H2 = ReLu(a2)
    a3 = np.dot(H2,W3.T) + b3.T
    Y_hat = softmax(a3)
    return H1,H2,Y_hat
```

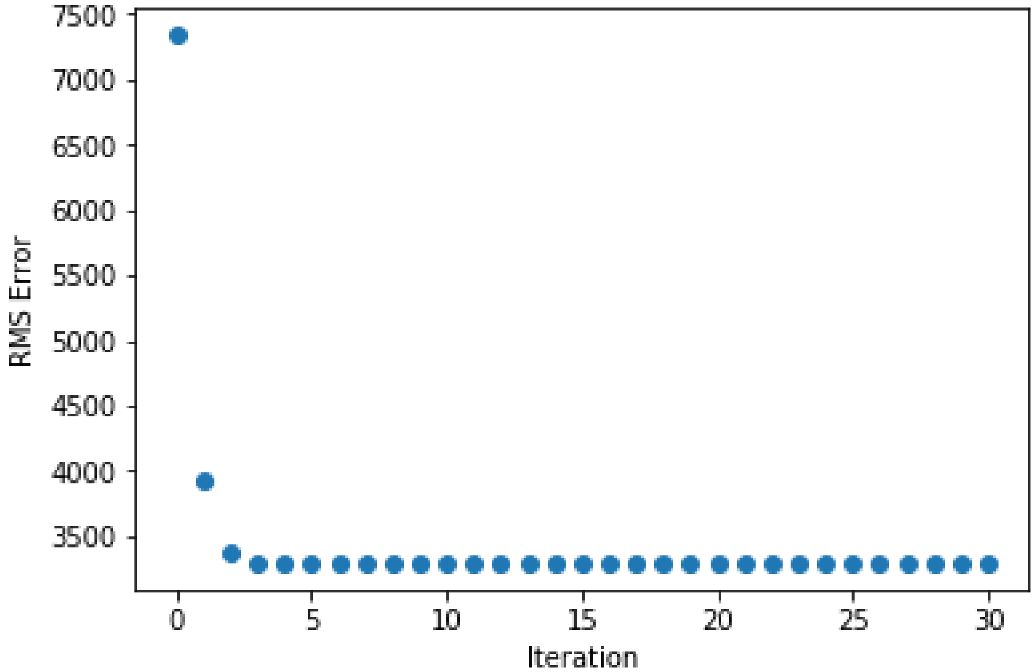
In [383]:
```
def grad_descent(x, y, parameters, iterations=10, learning_rate=1e-2):
    point = parameters
    trajectory = [point]
    losses = [loss(y, forward(X,point)[2])]

    for i in range(iterations):
        H1,H2,Y_hat = forward(x,point)
        grad = grad_f(x,H1,H2, y,Y_hat,point)
        point = np.subtract(point, [i*learning_rate for i in grad])
        trajectory.append(point)
        losses.append(loss(y,forward(X,point)[2]))
    return (np.array(trajectory), losses)
```

In [384]:
```
parameters = [W1,b1,W2,b2,W3,b3]
```

In [385]:
```
traj, losses = grad_descent(X,Y,parameters,iterations=30,learning_rate=1e-3)
```

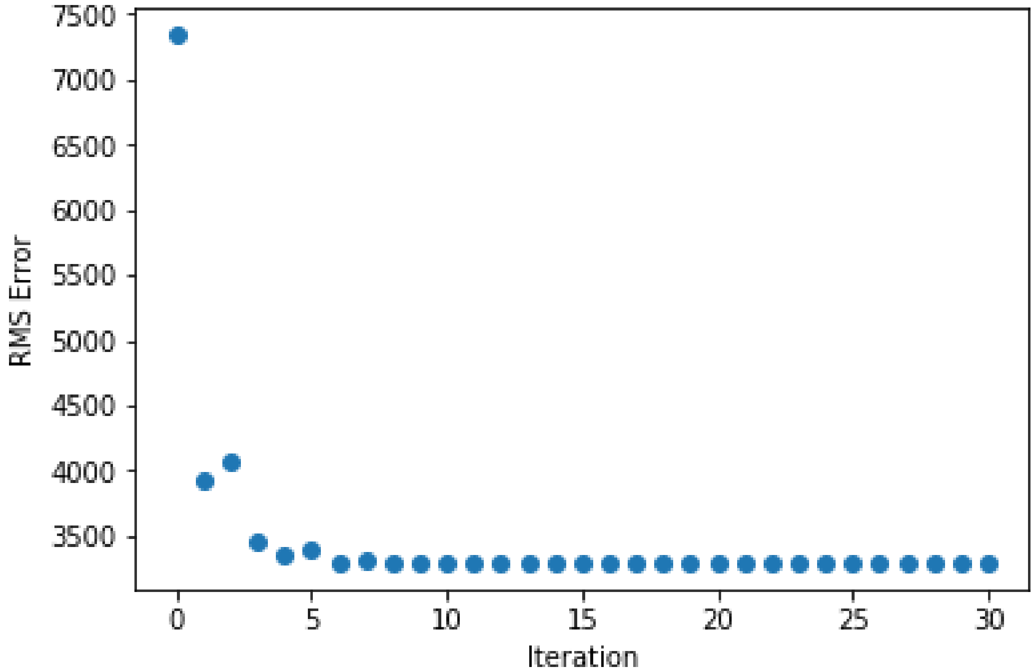
In [386]:
```
plt.plot(losses,'o')
plt.ylabel("RMS Error")
plt.xlabel("Iteration")
```

Out[386]:
```
Text(0.5,0,'Iteration')
```



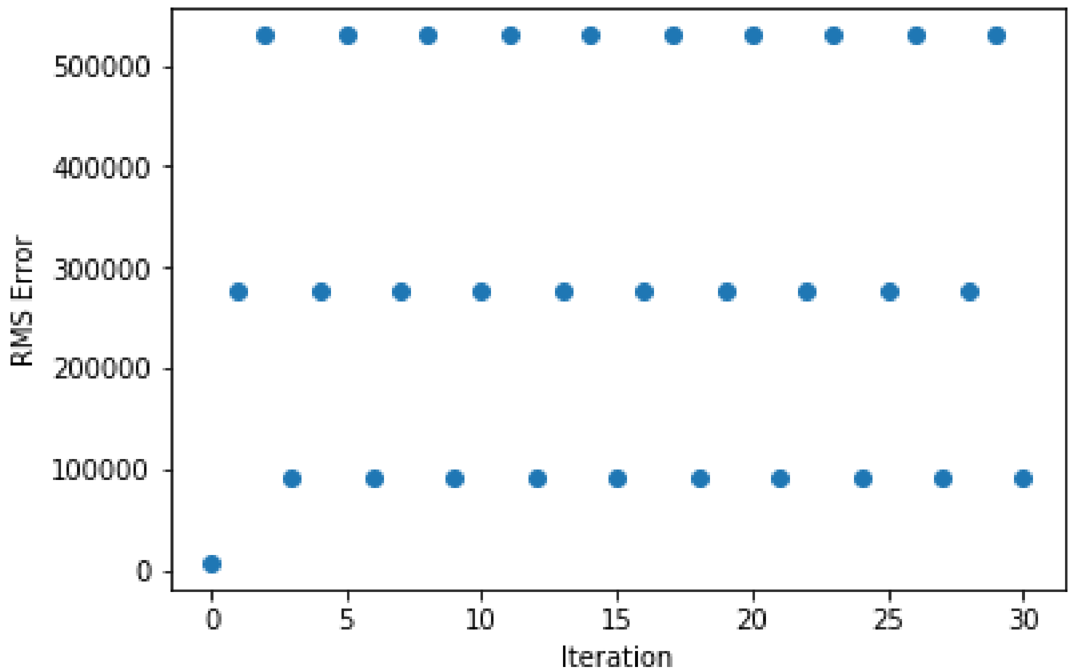
1.5 Part 5

```
In [387]: def grad_descent_with_momentum(x, y, parameters, iterations=10, alpha=.9, epsilon=10)
              point = parameters
              trajectory = [point]
              losses = [loss(y, forward(X,point)[2])]
              v = [np.zeros(np.shape(i)) for i in parameters]

              for i in range(iterations):
                  H1,H2,Y_hat = forward(x,point)
                  grad = grad_f(x,H1,H2, y,Y_hat,point)
                  #v = alpha*v + epsilon*grad
                  v = np.add([alpha*i for i in v], [i*epsilon for i in grad])
                  point = point - v
                  trajectory.append(point)
                  losses.append(loss(y,forward(X,point)[2]))
              return (np.array(trajectory), losses)

In [388]: traj, losses = grad_descent_with_momentum(X,Y,parameters,iterations=30,alpha=.5,epsi

In [389]: plt.plot(losses,'o')
          plt.ylabel("RMS Error")
          plt.xlabel("Iteration")

Out[389]: Text(0.5,0,'Iteration')
```



Both gradient descent and Momentum gradient descent have similar rate of convergence. But tuning parameters are important because smaller tuning parameter (learning rate in gradient descent and epsilon in Momentum gradient descent) will have a smaller step at each iteration and have a slower rate of convergence. However, if the tuning parameter is very large, it will skip the optimal point and become not stable. For example in the following plot, I change learning rate to 0.1 in gradient descent.

```
In [392]: traj, losses = grad_descent(X,Y,parameters,iterations=30,learning_rate=0.1)
          plt.plot(losses,'o')
          plt.ylabel("RMS Error")
          plt.xlabel("Iteration")

Out[392]: Text(0.5,0,'Iteration')
```



In the plot above, because learning rate is too large, it skipped the optimal point at each iteration and error is jumping around at each iteration. Hence, it is very important to choose a good tunning parameter