Introduction to 2^{K-p} Fractional Factorial Experiments

- A 2^K factorial experiment is an economical special case of a general factorial experiment
  - It minimizes the number of levels being investigated
  - Thus is it reduces the overall number of experimental conditions

- However, 2^K can still be a very large number of conditions even for moderate K

- In a 2^{K-p} fractional factorial experiment we also investigate K factors but in just a fraction of the conditions

- Rather than experimenting with all 2^K conditions, we specially select 2^{K-p} of them
  - When $p = 1$ we investigate K factors in half as many conditions
  - When $p = 2$ we investigate K factors in a quarter of the conditions

- The value p dictates the degree of *fractioning* and is typically chosen to
  - Minimize the number of experimental conditions m, given a fixed number of design factors K, or
  - Maximize the number of design factors K, given a fixed number of conditions m

- **Principle of effect sparsity:** in the presence of several factors, variation in the response is likely to be driven by a small number of main effects and low-order interactions.

  ~40% of ME's are significant
  ~10% of 2FI's . . .
  ~5% of 3FI's

- But consider the linear predictor from the full 2^K factorial experiment. There are:
  - K main effect terms
  - $\binom{K}{2}$ two-factor interaction terms
  - $\binom{K}{3}$ three-factor interaction terms
  - $\vdots$
  - $\binom{K}{K} = 1$ K-factor interaction term

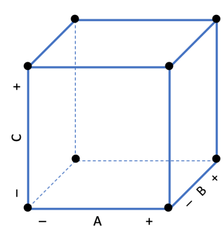
  This is a total of $\sum_{k=1}^{K} \binom{K}{k} = 2^K - 1$ estimated effects and just $K + \binom{K}{2}$ of these are main effects and two-factor interactions.

- In light of effect sparsity, it is as a waste of resources to estimate higher order interaction terms.
  - It would be a better use of resources to estimate the main effects and low-order interactions of a larger number of factors.

- So how do we choose *which* 2^{K-p} conditions to run?

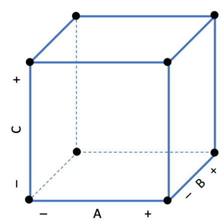
- Consider the following three examples as motivation:
  - **The $\mathbf{2^{3-1}}$ Example:** In this example we consider a one-half fraction of the 2^3 design which explores $K = 3$ factors (A,B,C) in $m = 4$ conditions rather than 8. The design matrix associated with a full 2^3 design and a visualization of the full 2^3 design are shown below. The question of primary interest is: *which* $m = 4$ conditions do we choose for the 2^{3-1} experiment?

Condition	Factor A	Factor B	Factor C
1	-1	-1	-1
2	$+1$	-1	-1
3	-1	$+1$	-1
4	$+1$	$+1$	-1
5	-1	-1	$+1$
6	$+1$	-1	$+1$
7	-1	$+1$	$+1$
8	$+1$	$+1$	$+1$



– **The 2^{4-1} Example:** In this example we consider a one-half fraction of the 2^4 design which explores $K = 4$ factors (A,B,C,D) in $m = 8$ conditions rather than 16. The design matrix associated with a full 2^4 design and a visualization of the full 2^4 design are shown below. Similar to the 2^{3-1} example, the question of primary interest is: *which* $m = 8$ conditions do we choose for the 2^{4-1} experiment?

Condition	Factor A	Factor B	Factor C	Factor D
1	−1	−1	−1	−1
2	+1	−1	−1	−1
3	−1	+1	−1	−1
4	+1	+1	−1	−1
5	−1	−1	+1	−1
6	+1	−1	+1	−1
7	−1	+1	+1	−1
8	+1	+1	+1	−1
9	−1	−1	−1	+1
10	+1	−1	−1	+1
11	−1	+1	−1	+1
12	+1	+1	−1	+1
13	−1	−1	+1	+1
14	+1	−1	+1	+1
15	−1	+1	+1	+1
16	+1	+1	+1	+1



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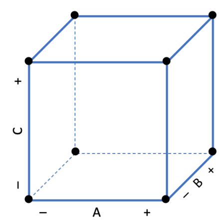
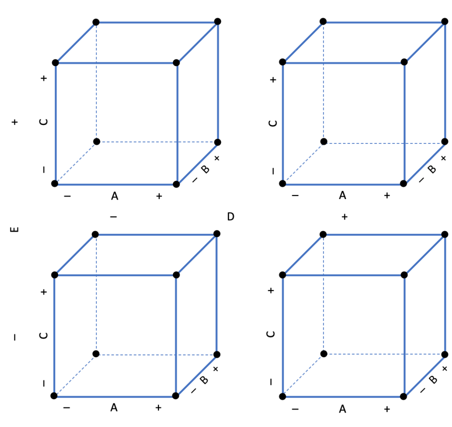


D+

which 4 to pick

– **The 2^{5-2} Example:** In this example we consider a one-quarter fraction of the 2^5 design which explores $K = 5$ factors (A,B,C,D,E) in $m = 8$ conditions rather than 32. The design matrix associated with a full 2^5 design and a visualization of the full 2^5 design are shown below. Similar to the previous two examples, the question of primary interest is: *which* $m = 8$ conditions do we choose for the 2^{5-2} experiment?

Condition	Factor A	Factor B	Factor C	Factor D	Factor E
1	-1	-1	-1	-1	-1
2	$+1$	-1	-1	-1	-1
3	-1	$+1$	-1	-1	-1
4	$+1$	$+1$	-1	-1	-1
5	-1	-1	$+1$	-1	-1
6	$+1$	-1	$+1$	-1	-1
7	-1	$+1$	$+1$	-1	-1
8	$+1$	$+1$	$+1$	-1	-1
9	-1	-1	-1	$+1$	-1
10	$+1$	-1	-1	$+1$	-1
11	-1	$+1$	-1	$+1$	-1
12	$+1$	$+1$	-1	$+1$	-1
13	-1	-1	$+1$	$+1$	-1
14	$+1$	-1	$+1$	$+1$	-1
15	-1	$+1$	$+1$	$+1$	-1
16	$+1$	$+1$	$+1$	$+1$	-1
17	-1	-1	-1	-1	$+1$
18	$+1$	-1	-1	-1	$+1$
19	-1	$+1$	-1	-1	$+1$
20	$+1$	$+1$	-1	-1	$+1$
21	-1	-1	$+1$	-1	$+1$
22	$+1$	-1	$+1$	-1	$+1$
23	-1	$+1$	$+1$	-1	$+1$
24	$+1$	$+1$	$+1$	-1	$+1$
25	-1	-1	-1	$+1$	$+1$
26	$+1$	-1	-1	$+1$	$+1$
27	-1	$+1$	-1	$+1$	$+1$
28	$+1$	$+1$	-1	$+1$	$+1$
29	-1	-1	$+1$	$+1$	$+1$
30	$+1$	-1	$+1$	$+1$	$+1$
31	-1	$+1$	$+1$	$+1$	$+1$
32	$+1$	$+1$	$+1$	$+1$	$+1$



which 8 to pick

Designing 2^{K-p} Fractional Factorial Experiments

which 2^{K-p} conditions to pick given 2^K conditions

Aliasing

- The first step in constructing a 2^{K-p} fractional factorial design is to write out the model matrix (when $n = 1$) for a *full* 2^{K-p} design

  → where we treat K-p as the total # of factors

- **2^{3-1} Example:** The model matrix (when $n = 1$) for a full 2^2 design with factors A and B is shown below:

Condition	I	A	B	AB	C
1	+1	−1	−1	+1	+1
2	+1	+1	−1	−1	−1
3	+1	−1	+1	−1	−1
4	+1	+1	+1	+1	+1

  – Rather than asking "which 4 conditions from a full 2^3 design do I run?", we now ask "in which of the four conditions in a full 2^2 design should I run factor C at its low versus high levels?"

- We use the ±1's in the AB interaction column to dictate, for a given condition, whether to run factor C at its low or high levels.

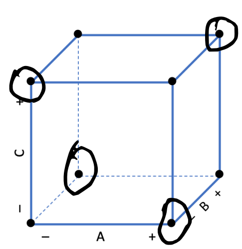
& orthogonality is kept

"Aliasing"

copy the conditions layout of AB to C

- What results is a prescription for experimenting with $K = 3$ factors in $2^{3-1} = 4$ conditions

- This is a 2^{3-1} fractional factorial design. We visualize it as follows:



- **Principal fraction:** The conditions selected by associating the levels of C with the ±1's in the AB column

- **Complementary fraction:** The conditions selected by associating the levels of C with $-AB$

- What we did there is called **aliasing**

associate the MAIN EFFECT of the new factor with the I.E. of two existing factors

- We call $C = AB$ the **design generator**

- When we do this, we **confound** the interaction effect with the main effect of the new factor

cannot separately estimate these effects

- In an ordinary 2^2 experiment with factors A and B, the AB column of the model matrix is used to estimate IE_{AB}
  - But due to the $C = AB$ aliasing, the AB column now jointly quantifies the main effect of C *and* the AB interaction effect

→ Previously, $\widehat{IE}_{AB} = \frac{\bar{Y}_{A^+ \cap B^+} + \bar{Y}_{A^- \cap B^-}}{2} - \frac{\bar{Y}_{A^+ \cap B^-} + \bar{Y}_{A^- \cap B^+}}{2}$

→ $\widehat{ME}_C = \bar{Y}_{C^+} - \bar{Y}_{C^-}$

Now, because of aliasing $\widehat{ME}_C = \widehat{IE}_{AB}$

So either of these 2 quantities estimate BOTH IE_{AB} AND ME_C

and we can't separate IE_{AB} and ME_C estimations (the price)

- This is the price we pay for using fewer conditions than what is prescribed by the full 2^K design.

turns out this problem not only impacts C & AB confounding

$B = AC$

$A = BC$

The Defining Relation

- In the 2^{3-1} example we aliased C with the AB interaction
  - We saw that this means the main effect of C and the AB interaction effect are confounded.
  - However, the aliasing (and hence confounding) doesn't stop there.
- Upon closer inspection we find that the main effects of A and B are now also aliased with interaction effects
- This becomes evident when we consider the **defining relation**: $I = ABC$ is the defining relation

Design generator: $C = AB$

$\Rightarrow C \times C = AB \times C \Rightarrow A(BC) = I \Rightarrow A = BC$

$" I$; element-wise product of the model matrix columns

$(1,1,1,1)^T$

- This may be used to uncover all aliases by multiplying it by any effect:

$A \times I = A(ABC)$; $B \times I = BABC = AB^2C$

$A = IBC$; $B = AC$

$= BC$

- Every main effect is aliased with a two factor interaction
- **Introducing aliasing anywhere causes confounding everywhere**
- **2^{4-1} Example:**
  - To construct this factorial design we consider the model matrix (when $n = 1$) associated with a full 2^3 design:

$3 = K - P$
$= 4 - 1$

Condition	I	A	B	C	AB	AC	BC	ABC
1	+1	−1	−1	−1	+1	+1	+1	−1
2	+1	+1	−1	−1	−1	−1	+1	+1
3	+1	−1	+1	−1	−1	+1	−1	+1
4	+1	+1	+1	−1	+1	−1	−1	−1
5	+1	−1	−1	+1	+1	−1	−1	+1
6	+1	+1	−1	+1	−1	+1	−1	−1
7	+1	−1	+1	+1	−1	−1	+1	−1
8	+1	+1	+1	+1	+1	+1	+1	+1

  - We need to choose one interaction column to alias a new factor D with
    - We could choose AB, AC, BC or ABC. Which one is the *right* choice?

let H_A $D = ABC$

then the principle of effect sparsity indicates IE_{ABC} is likely low

$\Rightarrow \widehat{IE}_{ABC} = \widehat{ME}_D$ is more about ME_D

* The complete aliasing structure is:

Defining Relation $\rightarrow$ $I = ABCD$

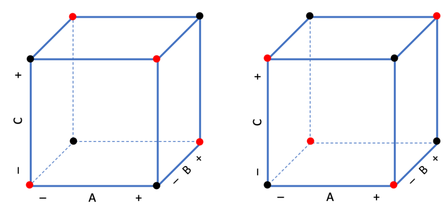
$A = BCD$ $\qquad$ $AB = CD$

$B = ACD$ $\qquad$ $BC = AD$

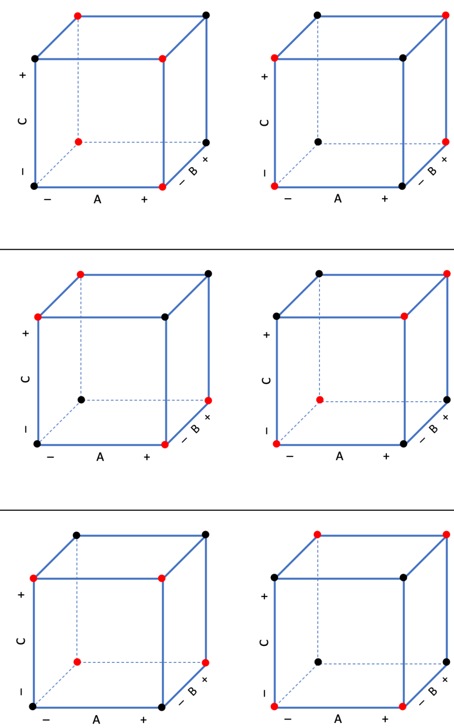
$C = ABD$ $\qquad$ $AC = BD$

$D = ABC$

– What would have happened if we had chosen $D = AB$ or $D = AC$ or $D = BC$ as design generators instead of $D = ABC$?

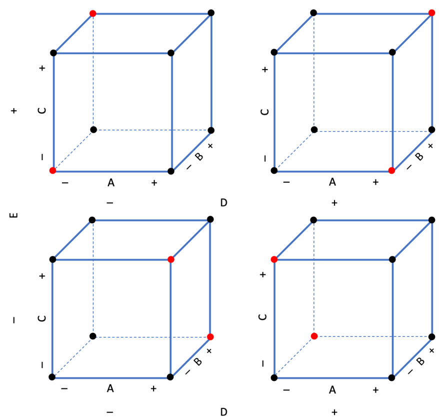


D^- $\qquad$ D^+



– Which one of these designs is best?

- **2^{5-2} Example:**
  - In addition to choosing an alias for factor D like we just did for the 2^{4-1} design, we also need to choose an alias for factor E
  - The 2^{5-2} fractional factorial design that results from these choices is visualized below:



  - In general, the number of design generators will always equal p
  - These design generators give rise to the following defining relation:
  - As usual, this may be used to determine the complete aliasing structure:

$$A = BCD = ABCE = DE$$
$$B = ACD = CE = ABDE$$
$$C = ABD = BE = ACDE$$
$$D = ABC = BCDE = AE$$
$$E = ABCDE = BC = AD$$
$$AB = CD = ACE = BDE$$
$$AC = BD = ABE = CDE$$

  - In general the number of effects aliased with a given effect is $2^p - 1$
  - Thus, in a 2^{K-p} fractional factorial design, every effect estimate actually jointly quantifies 2^p effects

- **SUMMARY:** To design a 2^{K-p} fractional factorial experiment, you must
  - Look at the model matrix (with $n = 1$) for a full 2^{K-p} design with $K - p$ factors
  - Choose p interaction columns to alias an additional p factors with
  - Use the ± 1's in these columns to dictate, for each condition, whether the p additional factors are run at their low or high levels

But how do we know *which* interactions to choose??

Resolution

- Due to the confounding that results from aliasing a new main effect with an existing interaction, it is important to think carefully about *which* interaction to choose as an alias
  - It is best to avoid aliasing a new factor with an interaction that is likely to be significant
  - High order interaction terms (that are unlikely to be significant) are good choices for aliases
- This notion is quantified by the **resolution** of the fractional factorial design.
  - A design is resolution R if main effects are aliased with interaction effects involving at least $R-1$ factors
- The easiest way to determine R is by looking at the defining relation
  - Each of the terms in the equivalence is referred to as a *word*
  - The length of the shortest word is the resolution of the design
  - The defining relations for the 2^{3-1}, 2^{4-1}, and 2^{5-2} designs are:

$$I = ABC$$

$$I = ABCD$$

$$I = ABCD = BCE = ADE$$

- General notation:

$$2_R^{K-p}$$

- In general, higher resolution designs are to be preferred over lower resolution designs
  - Resolution IV and V designs are to be preferred over a resolution III design

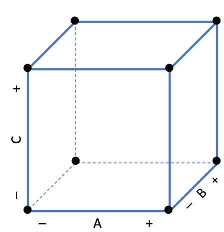
- The resolution of a fractional factorial experiment is determined by two things:
  1. The degree of fractioning desired (i.e., the size of p relative to K).
  2. The design generators chosen for aliasing.

- Given K and p, we should choose design generators that *maximize resolution*

- Let us return to the 2^{4-1} example.

Design Generator	Defining Relation
$D = ABC$	$I = ABCD$
$D = AB$	$I = ABD$
$D = AC$	$I = ACD$
$D = BC$	$I = BCD$

- Another way to justify the maximum resolution criterion is by the **projective property** of fractional factorial designs
  - A resolution R fractional factorial design can be projected into a full factorial design on *any subset* of $R-1$ factors
  - Let's visualize this with the 2^{3-1} design:



- This property can be exploited when analyzing the experimental data

- Maximizing R maximizes the size of the projected full factorial design

Minimum Aberration

- The maximum resolution criterion is one way to choose design generators

- But what if several choices lead to the same resolution? Then how do we choose?

- Consider a 2^{7-2}_{IV} design which is resolution IV and explores $K = 7$ factors in $m = 32$ conditions
  - Three design generator configurations that all give rise to a 2^{7-2}_{IV} design are shown below:

Design	Design Generators	Defining Relation
1	$F = ABC$, $G = ABD$	$I = ABCF = ABDG = CDFG$
2	$F = ABC$, $G = CDE$	$I = ABCF = CDEG = ABDEFG$
3	$F = ABCD$, $G = ABCE$	$I = ABCDF = ABCEG = DEFG$

  - How should we choose among these? Is one better than the others?
    * We can compare these designs on the basis of how many words of length 4 appear in the defining relation
    * Design 3 minimizes this number, and hence minimizes the number of main effects aliased with the lowest-order interactions

  - In general, for a given resolution R the **minimum aberration design** is one which minimizes the number of minimum-length words in the defining relation.

  - These designs are preferred since they minimize the number times main effects are aliased with the lowest order ($(R-1)$-factor) interactions

Optional Exercises:

- Calculations: 5, 10, 16
- Communication: 1(e)