$S^1 \hookrightarrow \mathbb{R}^3 \cup * \cong S^3$

Conway Knot is not Slice

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Last time $\begin{cases} \text{3-dim} \\ \text{classical} \end{cases}$ $\longrightarrow$ $\left.\begin{matrix} {}^{*}\text{4-dim} \\ \text{"quantum"} \end{matrix}\right\}$ Today

Goals for today

Major Goals

- Introduce the background and history of the Conway knot and the related Conway sliceness conjecture.
- Establish the relation between knot theory and 4-manifolds.
  - Kirby calculus.
- Introduce some more recent knot invariants:
  - Jones Polynomial
  - Khovanov's work
  - Lee's work
  - Rasmussen's work
- Sketch the proof of Conway knot not being slice.
  - We need to believe a lot of facts here!

The Conway knot

The Conway knot

- Picture is on the right!
- Corssing number: 11. unknotting # : 1.
- Has the same Alexander polynomial and Conway Polynomial as the unknot.

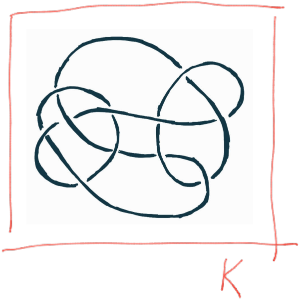
The Conway Sliceness Conjecture

Is the Conway knot **slice**?

Answer

Topologically yes, smoothly no.

Today



Slice knots

S^3 $\cup$ S^2 $\subseteq$ U

Recall unknot

A knot in S^3 is said to be trivial (or unknot) if it bounds an embedded disk in S^3.

"Sliceness" is the 4-dimensional analogy for the unknot.

Sliceness of a knot

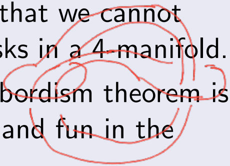
Most knots are NOT slice! not slice 4_1

A knot in S^3 is said to be slice (topologically) if it bounds a smoothly embedded (resp. locally flat) disk in B^4. 3_1

slice : "ribbon"

Why do we care about sliceness?

- The fact that not all knots are slice means that we cannot remove all self-intersections of immersed disks in a 4-manifold.
- This leads to the fact that the smooth h-cobordism theorem is false in dimension 4, hence all the wildness and fun in the world of 4-manifolds.



Why is the problem so hard?

Why so hard?

1 - The Conway knot is a positive mutant of a slice knot: the Kinoshita-Teresaka knot.
- A lot of obstructions of a knot being slice is preserved by positive mutation.

2 - Moreover, the alexander polynomial of the Conway knot is the same as the unknot.
- Hence the Conway knot has no known non-vanishing obstructions.

More on the Conway problem

History of the problem

- Fox first establish the idea of concordance and sliceness in 1962. $\rightsquigarrow$ embedding $\Sigma^4 \to X^4$.
- Conway discovered the Conway and Kinoshita-Teresaka knot in 1970, the first was later named after him. However, at the time, the two knots could not be distinguished in isotopy.
- The two knots were first distinguished in isotopy by Riley in 1971.
- Freedman proved that both knots are topological slice in 1984.
- The Kinoshita-Teresaka knot was proved to be slice in the 90s.
- Examples of non-slice mutants of slice knots were first found in 2001 by Kirk and Livinston.
- Conway knot was finally proved to be non-slice in 2018 by Piccirillo. Today.

Skeleton of proving the Conway knot is not slice

Step 1

We prove that two knots with the same **knot trace** have to be both slice or both non-slice. In this way we can replace the Conway knot with a hopefully easier knot to deal with.

1

Step 2

2

We use **dualizable links** techniques to build a knot K' that has the same knot trace as the Conway knot.

Step 3

Fortunately the knot K' has a non-vanishing obstruction of being slice: the Rasmussen s-invariant.

3

Handlebody decomposition

(Manifold / thickened)

S/I
Morse Theory

Handle decomposition: "thickened" version of CW decomposition

3-d

- A handle decomposition of a smooth n-manifold M is a union $\emptyset = M_{-1} \subset M_0 \subset \ldots \subset M_n = M$ where M_i is obtained from M_{i-1} by attaching i-handles.
- Any smooth manifold has a handle decomposition.

attaching sphere

core.

2

Handlebodies: "thickened" version of cells

- A k-handle is a "thickened version of k-cells", i.e. a manifold $D^k \times D^{n-k}$.
- $\partial D^k \times D^{n-k}$ is called the attaching region, and $\partial D^k \times 0$ is called the attaching sphere.

1-

0

2-h.

belt sphere.

1-handle.

2d

Examples

Here is a handle decomposition of a torus.

1-h

0-handle

≅

1-h

Examples of Kirby diagrams

Idea of Kirby diagrams

Next we will introduce **Kirby diagrams**: an effective way to represent smooth 4-manifolds using knot diagrams (and a little bit more data).

$\mathbb{Z}$.

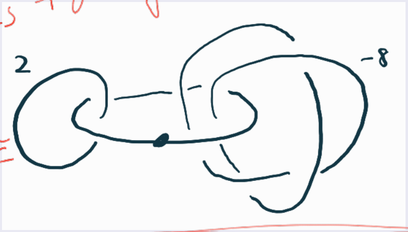
Examples

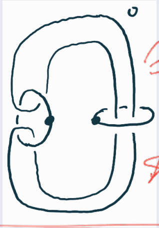
$\cup$

Here are some examples of Kirby diagrams, just to have a feeling.

knots/links + framings + •

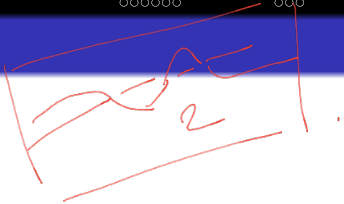
$S^4 \cong$





$\cong T^2 \times D^2$

Basic Kirby diagrams



(background)

0-handles

- The 0-handle is a 4-ball D^4, with boundary $S^3 \cong \mathbb{R}^3 \cup *$.
- Each connected smooth manifold yields a handle decomposition with a unique 0-handle.
- We draw attaching regions of other handles in the plane, representing $\mathbb{R}^3$. (Just like how we draw knots)

plane $\xleftarrow{\text{draw}} \mathbb{R}^3 \simeq S^3 \xrightarrow{\text{bounds}} D^4$.

More Kirby diagrams

core. attaching sphere

2-handles

- The two handle is a thickened disc D^2, and $\partial D^2 = S^1$. $\to S^3 \supset$ knot.
- Thus we can think of a 2-handle as a framed knot. The knot describes an embedding of S^1 in S^3, and the framing determines which way to thicken the disk bounded by the knot. (classifying D^2 handles are S^1).

Knot trace

The **knot trace** of a knot K is a 4-manifold $X(K)$ or $X_0(K)$ obtained by attaching a 0-framed 2-handle along the knot K to the 4-ball viewed as a 0-handle.

More Kirby diagrams

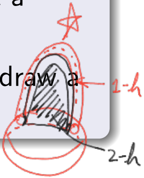
1-handles

- The 1-handle is a thickened interval D^1, and $\partial D^1 = S^0$.
- We draw 1-handles by two balls (magically connected in some higher dimension).
- Note that 2-handles can be attached on 1-handles.

1-handles: dotted circle notation (carved out notation)

- Alternatively, we can think of 1-handles as carving out a 0-framed two handle.
- This can be obtained by identifying the two balls and draw a dotted circle (as a carved out 2-handle).

$D^3 \cup 1\text{-}h \cup 2\text{-}h \cong D^3 \cong$



More Kirby Diagrams

3 and 4 handles: not in the picture

- We don't typically draw 3 and 4-handles.
- Thus a Kirby diagram is well-defined up to the attaching of 3 and 4-handles.
- Given a fixed Kirby diagram, all closed manifolds obtained by attaching only 3 and 4-handles are diffeomorphic to each other.

Kirby calculus - handle cancellation

Cancellation theorem

A $(k-1)$-handle and a k-handle can be cancelled if the attaching sphere of the latter intersects the belt sphere of the first transversally in a unique point (regardless of framings).

Cancellation of handles in Kirby diagrams

- Cancelling 2 handles and 3 handles: directly remove a 0-framed unknot.

  $^{0}\bigcirc \cup$ 3-handle $= \emptyset$

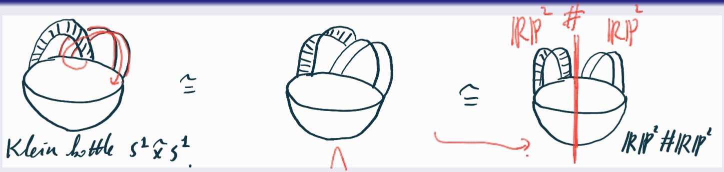
- Cancelling 1 handles and 2-handles: remove a 2-handle along with its meridian.

  = nothing

- What if there are other 2-handles on the 1-handle? (We need to be careful).

Kirby calculus - handle slide

Intuition: $\mathbb{RP}^2 \# \mathbb{RP}^2$ is the klein bottle



Sliding 2-handles

- For two handles of the same index, we can isotope the attaching sphere of the first handle on top of ~~one~~ the other without changing the diffeomorphism type.
- For sliding 2-handles in Kirby diagrams,
  - The new attaching sphere becomes the bandsum of the knots.
  - The framing is modified by $n_i, n_j \mapsto n_i + n_j \pm 2 \cdot lk(K_i, K_j)$.
- We can slide 2-handle over 1-handles since we treat 1-handles as hollowed out 2-handles.

Abandoning the Conway knot

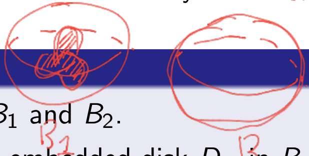
Theorem

A knot K is slice if and only if its knot trace $X(K)$ embeds smoothly in S^4.

Corollary 1

If two knots K and K' have diffeomorphic knot traces, then K is slice if and only if K' is slice.

With this corollary, it is safe for us to replace the Conway knot 4-ball. with an easier knot to deal with!



Proof of $\Rightarrow$

- S^3 decomposes S^4 into two 4-balls B_1 and B_2.
- If K sits in S^3, it bounds a smoothly embedded disk D_K in B_1 by definition of sliceness.
- $X(K) \cong B_2 \cup \overline{\nu(D_k)}$, which is smoothly embedded in S^4.

Proof cont.

Proof of $\Leftarrow$

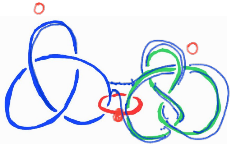
- Consider a piecewise linear embedding $F : S^2 \to X(K)$ such that its image consists of:
  - The cone over the knot K.
  - The core of the 2-handle.
- Consider the piecewise embedding $i \circ F : S^2 \to S^4$, where i is the given embedding into S^4. Note that $i \circ F$ is smooth away from the cone point $i(p)$.
- If we cut out a sufficiently small neighbourhood of the cone point, we have a smooth embedding: $S^2 \backslash \nu(F^{-1}(p)) \hookrightarrow S^4 \backslash \nu(i(p))$. Notice that:
  - $S^2 \backslash \nu(F^{-1}(p)) \cong D^2$ and $S^4 \backslash \nu(i(p)) \cong B^4$.
  - The image of the boundary of $S^2 \backslash \nu(F^{-1}(p))$ under F is the knot K we started with.

Dualizable links

Dualizable links

A dualizable link L is a three component link with components B(blue), G(green), and R(red) satisfying: $(R = \mathcal{U})$

- The sublink $B \cup R$ in S^3 is isotopic to $B \cup \mu_B$, where μ denotes the meridian.
- The sublink $G \cup R$ is isotopic to $G \cup \mu_G$.
- $lk(B, G) = 0$.



Relation to 4-manifolds

For a dualizable link L, we can associate a 4-manifold by considering B and G as 0-framed 2-handles and R as the 1-handle (in the dotted circle notation).

Dualizable links produce knots with the same trace

Theorem

For a dualizable link L and its associated 4-manifold X, we can find associated knots K and K' such that $X \cong X(K) \cong X(K')$.

proof of theorem

- Isotope L such that the knot R has no self-crossing. $R =$
- Slide the 2-handle G over B and cancel the 1-handle to get a 0-framed 2-handle represented by the knot K.
- Do the exact same procedure the other way around to get K'.
- Since handle slide and cancellation do not change the diffeomorphism type of the 4-manifold, we have $X \cong X(K) \cong X(K')$.

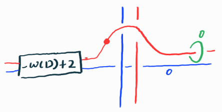
Existence of dualizable links

Existence theorem

For any knot K with unknotting number 1, there exists a dualizable link L such that its associated 4-manifold is diffeomorphic to $X(K)$.

Proof: constructing the trace

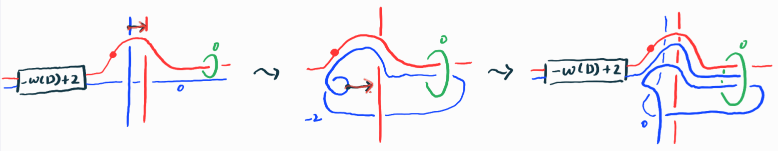
- Define $B := K$, and focus on the distinguished crossing (assume WLOG positive).
- Define R as a parallel of B away from this crossing. Note that R is the unknot.
- Define G to the the meridian of R.
- R and G are a cancelling pair so the associated manifold is exactly $X(B) = X(K)$.



Proof cont.

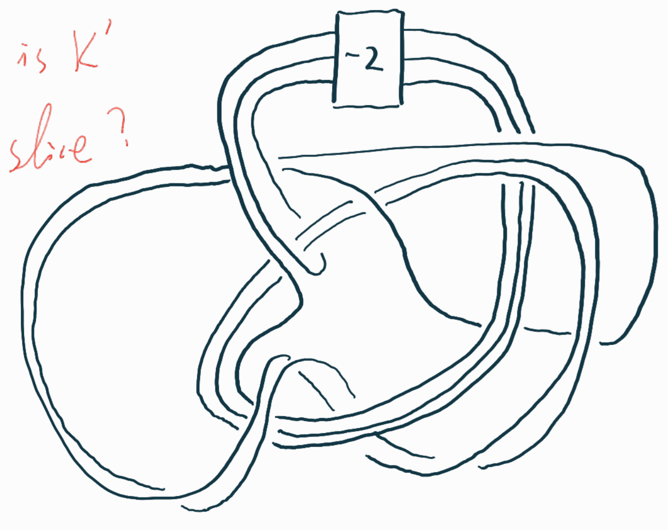
Proof cont.: sliding to a dualizable link

- We slide the handles via the indicated arrows and yield the second and third pictures.
- In the second picture, B acts as a meridian of R.
- In the third picture, R acts as a meridian of B. Check that this indeed defines a dualizable link.



Our knot K'

Why is K' not slice?



Jones Polynomial

Kauffman bracket

The Kauffman bracket $\langle - \rangle$ is a function from unoriented link diagrams to Laurent polynomials $\mathbb{Z}[q^{-1}, q]$ characterized by:

- $\langle \emptyset \rangle = 1$, $\langle D \sqcup \bigcirc \rangle = (q^{-1} + q)\langle D \rangle$.
- $\langle D \rangle = \langle D_0 \rangle - q \langle D_1 \rangle$, where D, D_0, D_1 corresponds to:

However, the Kauffman bracket is NOT a knot invariant.

Jones polynomial (Unreduced)

- The Jones polynomial is an oriented link invariant defined by

$$J(L) := (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle \in \mathbb{Z}[q, q^{-1}].$$

Bar-Natan.

Khovanov Homology

Heegaard - Floer Homology? Alexander

Overview

- The Khovanov Homology is a categorification of the Jones polynomial.
- Accordingly, The Kauffman Bracket becomes the Khovanov Bracket, which takes values in **chain complexes of graded vector spaces**.

Degree shift

The degree shift is the operator $\{l\}$ on **graded vector spaces** that shifts the dimension up by l.

Height shift

The height shift is the operator $[s]$ on **chain complexes** that shifts the place by s.

Khovanov Homology cont.

Khovanov bracket

The Khovanov bracket $[\![-]\!]$ is a function from unoriented link diagrams to chain complexes of graded vector spaces (graded in $\mathbb{Z}[q, q^{-1}]$) characterized by:

- $[\![\emptyset]\!] = 0 \to \mathbb{Z} \to 0$.
- $[\![D \sqcup \bigcirc]\!] = V \otimes [\![D]\!]$, where V denotes the vector space of dimension $q + q^{-1}$.
- $[\![D]\!] = \mathcal{F}(\to [\![D_0]\!] \to [\![D_1]\!]\{1\} \to 0)$, where the operator $\mathcal{F}$ "flattens" a double complex into a single complex by taking direct sums along the diagonals.

Khovanov homology

The Khovanov homology $Kh(L)$ is the homology of the complex of graded vector spaces $[\![L]\!][-n_-]\{n_+ - 2n_-\}$.

Lee's progress and Rasmussen's s-invariant

Rise to spectral sequence

- Lee modified the Khovanov homology to a spectral sequence whose E_2 page is exactly $Kh(L)$.
- The spectral sequence converges into a homology called Lee homoogy $KhL(L)$.

Theorem (Lee)

For any knot K, the total Lee homology $KhL(K) \cong \mathbb{Q} \oplus \mathbb{Q}$. Moreover, both generators are located in the grading $i = 0$.

Theorem/Definition (Rasmussen)

For any knot K, the generators of $KhL(K)$ locate in the gradings $(i, j) = (0, s(K) \pm 1)$. The integer $s(K)$ is called the Rasmussen's s-invariant. Moreove, if K is slice, $s(K) = 0$.

Calculation of Rasmussen's s-invariant

Original Calculation (2018)

- First calculate the Khovanov homology using the Skein relation. (Bar-Natan)
- Use spectral sequence techniques to see which generators of Khovanov homology survive to the E_∞ page.
- Deduce the Rasmussen's s-invariant accordingly.

Recent Developments (2020).

- To simplify the knots, we can use "Snappy" in Sage, with the method K.simplify('global').
- To calculate the s-invariant, we can use the Mathematica package "KnotTheory", with method "sInvariant".

Our knot K'

For the knot K' constructed before, $\boxed{s(K') = 2,}$ thus is not slice.

Finishing the proof

Putting everything together

- The Conway knot K is a knot of unknotting number 1, thus there exists a dualizable link L whose associated 4-manifold is exactly the knot trace of K.
- The other associated knot K' has the same knot trace as the Conway knot. Thus K' is slice if and only the Conway knot is.
- The knot K' is not slice since its has non-vanishing Rasmussen's s-invariant.
- Thus we conclude that the Conway knot is not slice.

Significance of this paper

Smooth PC: is every smooth $X \simeq_{Top} S^n$ diffeomorphic to S^n?

Importance of this paper

- The idea of dualizable links can be generalized into a notion called *RBG* link, and can be used to construct homeomorphic but not diffeomorphism knot traces.
- The notion of sliceness can be generalized to framed knots and to arbitrary closed 4-manifolds, and the Rasmussen's s-invariant turns out to be the most useful slice obstructions in S^4, $\#^n \mathbb{CP}^2$, and $\#^n \overline{\mathbb{CP}^2}$.
- With similar techniques, we can attempt to construct exotic 4-spheres (promising yet still unsuccessful).

Manifold X s.t. $X \simeq_{Top} S^4$ but $X \not\simeq_{Diff} S^4$.

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Reference (Recent Developments)

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