A NOTE ON HOLOMORPHIC QUADRATIC DIFFERENTIALS

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1. Overview

This is a lecture note for the 10^{th} talk of the seminar "Geometry of Teichmüller Spaces", and we will focus on the idea of holomorphic quadratic differentials.

Throughout this note, we will first give two equivalent definitions of holomorphic quadratic differentials, one global and the other local. An efficient way to study local presentations of holomorphic quadratic differentials is by a special collection of coordicate charts called "natural coordinates". We will also see that holomorphic quadratic differentials behaves like the "complex analytic counterpart" of measured foliations.

Then we will study two ways to construct quadratic differentials on Riemann surfaces, the first is by gluing polygons, and the second is by branched covers. The last part of the note will focus on the vector space of quadratic differentials on a fixed Riemann surface, where we will define a topology for the vector space, as well as a dimension count. In fact, this vector space can be seen as the cotangent space of the Teichmüller space.

2. Definitions and properties

In this section, we will discuss the basic definitions of holomorphic quadratic differentials, as well as standard methods of studying quadratic differentials, including the interchanging role of quadratic differentials and measured foliations, natural coordinates, and a revisit to Euler-Poincaré Formula.

We start with the global definition of holomorphic quadratic differentials.

Definition 1 (holomorphic quadratic differential)**.** A holomorphic quadratic differential on a Riemann surface X is a section of the bundle $S^2(T^*X) = (T^*X \otimes T^*X)/\mathbb{F}_2 \to X$ (The symmetric square of the holomorphic cotangent bundle over X).

There is an equivalent description of holomorphic quadratic differentials in terms of local coordinates: Given an atlas $\{z_\alpha : U_\alpha \to \mathbb{C}\}$, a holomorphic quadratic differential can be locally written as $\{\phi_\alpha(z_\alpha)dz_\alpha{}^2\}$ such that:

- Each $\phi_\alpha : U_\alpha \to \mathbb{C}$ is a holomorphic function with a finite set of zeros.
- (Change of coordinates) For each α, β, $\phi_\beta(z_\beta)(\frac{dz_\beta}{dz_\alpha})^2 = \phi_\alpha(z_\alpha)$. This is the same thing as saying $\phi_\beta(z)(\frac{d\psi}{dz})^2 = \phi_\alpha(z)$ where ψ denotes the change of coordinate biholomorphism.

Under this local description, it is clear that the order of a zero is well-defined. (since the derivative of a biholomorphism is never 0.)

The connection between the global and local definition of holomorphic quadratic differentials is that under both interpretations, a quadratic differential can be seen as a complex valued map of the holomorphic tangent bundle $TX \to \mathbb{C}$. In local coordinates, the map defined from the quadratic differential $\phi(z)dz^2$ is given by $q : (z_0, \alpha) \mapsto \phi(z_0)\alpha^2$.

After defining the holomorphic quadratic differentials, we will discuss some of its properties.

2.1. Quadratic differentials induce measured foliations.

Given a holomorphic quadratic differential q, we can define two associated measured foliations: a horizontal foliation, and a vertical foliation.

To define a horizontal (vertical) foliation, we take the union of

- the set of zeros of the quadratic differential q.

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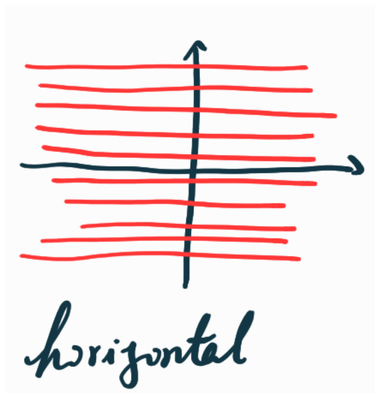
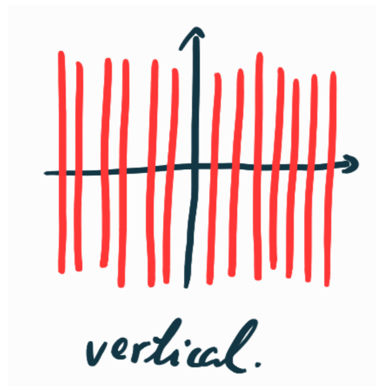
- the set of smooth path with tangent vectors evaluate to positive (negative) real numbers under q. (Use the map we just defined)

Then we define a transverse measure of the foliation. If $q = \phi(z)dz^2$ locally, the measure of the horizontal (vertical) foliation is given by $\mu(\gamma) = \int_\gamma |Im(\sqrt{\phi(z)}dz)|$. (resp. $\int_\gamma |Re(\sqrt{\phi(z)}dz)|$.)

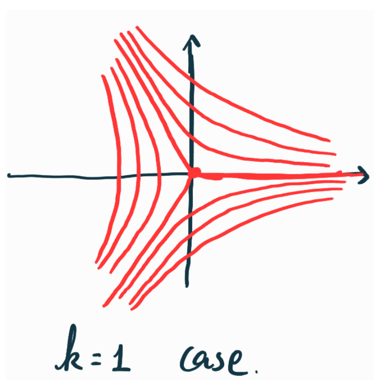
Before we see why this definition defines a measured foliation, we first do two examples.

1) When $q(z) = dz^2$ locally. In this case, the associated map $TX \to \mathbb{C}$ is given by $(z, \alpha) \mapsto \alpha^2$. Thus the horizontal foliation is the set of horizontal lines, and the vertical foliation is the set of vertical lines (which is why they have their names). In this example, the measures of the foliations are just the ones induced by $|dy|$ and $|dx|$.





2) When $q(z) = z^k dz^2$ locally. In this case, the associated map $TX \to \mathbb{C}$ is given by $(z, \alpha) \mapsto z^k\alpha^2$. We observe the set of points z_0 such that z_0^{k+2} is real and is positive. (These are just $k+2$ straight rays spanned by the $(k+2)^{th}$ roots of unity). This defines a singular point with $k+2$ prongs. Thus we know that the horizontal foliation is just a standard $(k+2)$-pronged measured foliation. The following plot is the model for the horizontal foliation when $k = 1$.



The vertical foliation is just an additional rotation of $\pi/(k+2)$.

Now we answer the question why the definition above defines a measured foliation. The general idea is that

- In the case of $q(z) = z^k dz^2$, we already know the above defines a measured foliation. (Recall that we don't need a transverse measure near a singular point).
- In the next subsection, we will show, by the mean of natural coordinates, that any holomorphic quadratic differential can be locally written as $z^k dz^2$.

2.2. **Natural coordinates.** We introduce natural coordinates by the following theorem/definition.

Theorem 1 (natural coordinates)**.** *Given any Riemann surface X and any holomorphic quadratic differential q defined on X. For every point in X, there is a coordinate such that $q(z) = z^k dz^2$ locally, where k is the order of zero at z. This coordinate is called the natural coordinate.*

Proof. We divide the proof into two cases: when q is not a zero, and when q is a zero of order k. Here we only provide proof for the first case.

Case 1: if $q(p) \neq 0$. We choose any coordinate chart $z : U \to \mathbb{C}$ centered at p. In the local sense, $z(p) = 0$ and $q(z) = \phi(z)dz^2$ such that $\phi(0) \neq 0$.

Since a holomorphic quadratic differential is required to have only finite number of zeros, we can fine a sub-chart U' small enough such that $\phi(z)$ is zero-free in this chart.

Then we define "natural coordinates" by the map $\zeta = \eta \circ z : U' \to \mathbb{C}$, where $\eta(z) = \int_0^z \sqrt{\phi(w)}dw$ along any path from 0 to z.

We then need to show that $q(\zeta) = d\zeta^2$, i.e. $\psi(\zeta) \equiv 1$.

By the fundamental theorem of calculus, $d\eta = \frac{d\zeta}{dz} = \sqrt{\phi(z)}$. Now we rewrite q in the new coordinate $q(\zeta) = \psi(\zeta)d\zeta^2$. By change of coordinates formula we know that $\psi(\zeta)(\frac{d\zeta}{dz})^2 = \phi(z)$. Thus it follows that $\psi \equiv 1$.

Case 2: if p is a zero of order k. We use the exact same technique as above on $\phi(z)/z^k$ instead of directly on $\phi(z)$. The details of the proof is omitted. □

2.3. **Revisit Euler-Poincaré Formula.** We recall that the Euler-Poincaré Formula states that $2\chi(s) = \Sigma(2 - p_s)$ where p_s is the number of prongs at a singular point s in a measured foliation. We also know that an order-k zero of a holomorphic quadratic differential has $(k + 2)$ prongs in the corresponding horizontal/vertical foliation. Thus by combining the previous two results, we have the following theorem:

Theorem 2. *For a closed Riemann surface with genus g, a holomorphic quadratic differential vanishes exactly at $4g - 4$ points, counting multiplicity.*

2.4. **Relation to singular Euclidean metrics.** It is worth to note that a holomorphic quadratic differential on a Riemann surface defines a singular Euclidean metric (a flat Riemannian metric outside the set of zeros for the differential) given by $|\phi(z)|(dx^2 + dy^2)$.

In this case, the area form is give by $|\phi(z)|dx \wedge dy$, and the length form is given by $|\phi(z)|^{\frac{1}{2}}\sqrt{dx^2 + dy^2}$.

2.5. **An Example.** At last we do an example and see explicitly what are the holomorphic quadratic differentials and their corresponding horizontal/vertical foliations on the torus.

Let $\Lambda \subset \mathbb{C}$ be a lattice, and $X = \mathbb{C}/\Lambda$ be the torus, with $\pi : \mathbb{C} \to X$ being the quotient map. Choose an open cover $\{U\}$ for X, and denote the corresponding atlas by $\{\pi|_{\tilde{U}} : \tilde{U} \to U\}$. In this sense, all transition maps are just translations in the complex plane.

For a quadratic differential $q = \phi(z)dz^2$ locally, we know that the derivative of the transition maps (translation) are 1, thus we know ϕ can be extended to an entire doubly periodic function. Liouville's Theorem ensures that ϕ is a constant function.

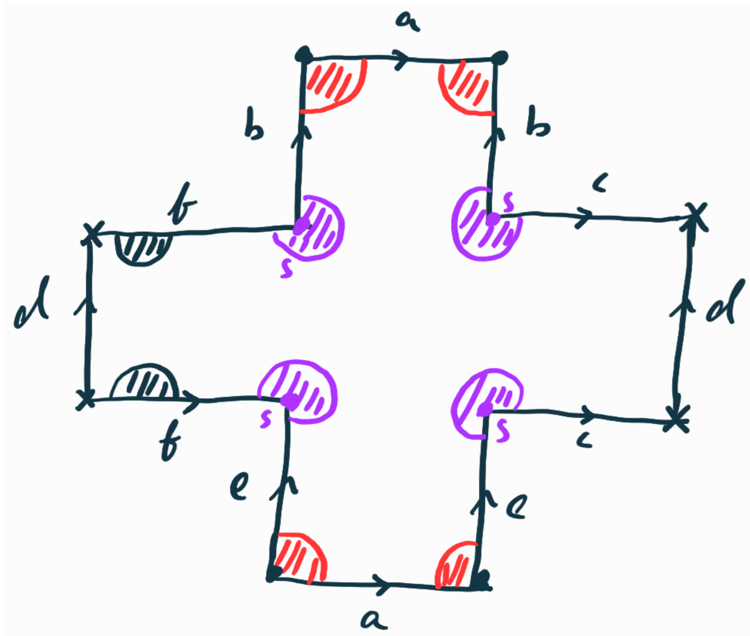
Thus we conclude that the set of holomorphic differentials on the torus X is bijective to the complex numbers $\mathbb{C}$.

Moreover, we note that for the quadratic differential corresponding to $z \in \mathbb{C}$, its associated horizontal foliation consists of straight lines meeting the x-axis with the angle of $-arg(z)/2$.

3. Constructions of Quadratic differentials

In this section, we will provide two different ways to construct quadratic differentials, the first by polygons, and the second by branched covers. Either construction is capable of building any holomorphic quadratic differential on any Riemann surface.

3.1. **Construction via polygons.** We first observe the example of the "Swiss Cross". General constructions have the same idea.



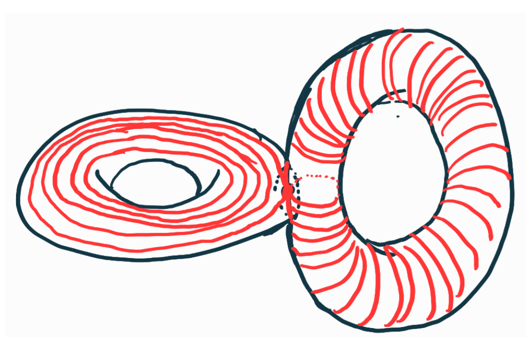
The Swiss cross has 4 types of coordinate charts:

(1) For points in the interior of the cross, we use the one chart viewed as a subset of the complex plane $\mathbb{C}$.
(2) For points in the interior of the edges, we take two (black) semi-disks and they together glue into a disk in the complex plane.
(3) For the "8 vertices" on the "outside", we take four (red) quarter-disks and they together glue into a disk in the complex plane.
(4) For the "4 vertices" on the "inside", we take four (purple) three-quarters-disks, apply a corresponding $z \mapsto z^{\frac{1}{3}}$ and glue into a disk in the complex plane. Note that the "4 points" glue into a single point, and we denote that single point s.

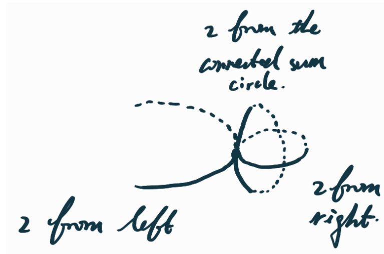
Note. The above described atlas glues into a genus 2 surface and defines a complex structure.

Indeed, every chart beyond the last one is viewed directly as subsets of $\mathbb{C}$, thus have biholomorphic transition maps. The only potential problem lies at the point s, but there is only one chart covering the point s so there is no transition functions to worry about.

Next we construct a quadratic differential by setting $q(z) = dz^2$ on the first three type to charts and $q(z) = 9z^4dz^2$ near s. Under this quadratic differential, the associated horizontal foliation is just horizontal lines on the Swiss cross. The following picture depicts the horizontal foliation on the glued genus 2 surface.



Note that s is a zero for the quadratic differential, thus is a singular point for the horizontal foliation. By Euler-Poincaré formula, s should have 6 prongs. Indeed, in the above example, there are 6 prongs: two comes from the left torus, two comes from the right, and the last two comes from the connected sum circle, as described by the following picture.

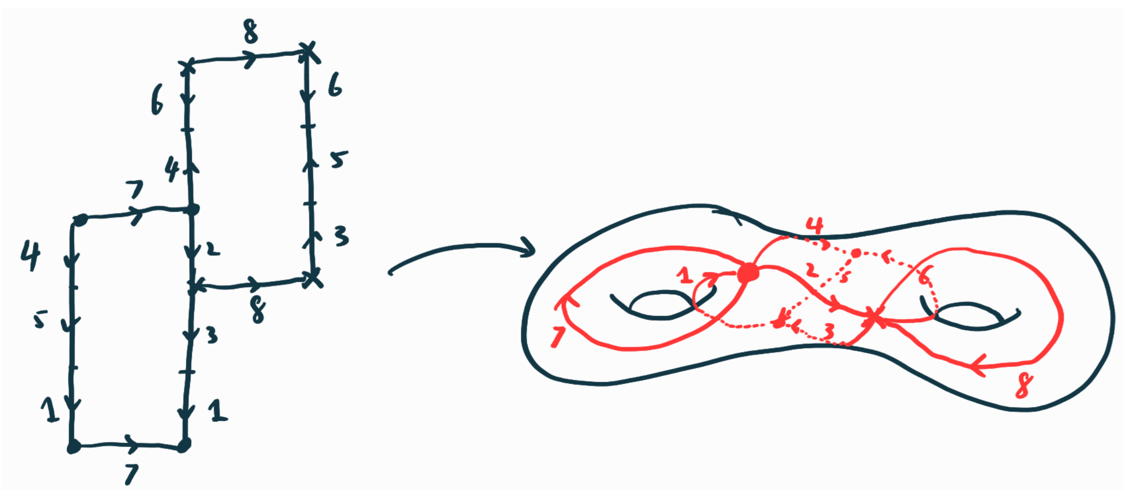


Now we finally check that the above local description indeed glue into a global quadratic differential. It is enough to check transitions between the "big chart" and the "special chart" around s. Indeed $(3w)^2 = (\frac{dz}{dw})^2 = 9w^2$.

We also note that the associated Riemannian metric just comes from the "big chart" as a subset of $\mathbb{C}$.

The above model construction can be generalized to other quadratic differentials on the Swiss cross. We just change the expression of q on the "big chart" to αdz^2 for some $\alpha \in \mathbb{C}$ and the expression for the chart around s is then automatically determined.

The construction can also be generalized to other Riemann surfaces. The following picture defines a second example for construction via polygons. In general, we use $q = dz^2$ for the large chart, and the local presentation for the corners have to be compatible accordingly.



In fact, it is worth to note that any quadratic differential on any Riemann surface can be constructed in this way, given by the following theorem.

Theorem 3. *Every holomorphic quadratic differential can be realized by some polygonal construction.*

Proof. (sketch)

- Natural coordinates tells us how to cut up the given Riemann surface into finitely many rectangles.
- Place these rectangles in Euclidean space and record side identifications.

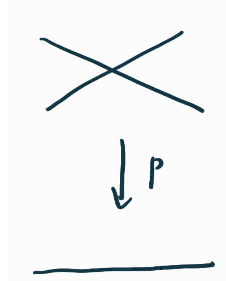
□

3.2. **Construction via branched covers.** We first recall the definition of a branched cover between Riemann surfaces.

Definition 2. A continuous map $p : X \to Y$ between two Riemann surfaces is called a branched cover if it can locally be written as the form $z \mapsto z^k$. In this sense,

- If $k = 1$, x is said to be unramified, or "good point".
- If $k > 1$, x is said to be a ramification point or "bad point". k is called the degree of ramification.

The following plot gives a local model for a branched cover, with the intersection point of the two lines behaving like a ramification point of degree 2.



We quickly note that given a branched cover $p : X \to Y$ an a complex structure on Y, we can pullback the complex structure on X. Our goal here is to see that we can also pullback a holomorphic quadratic differential q on Y back to X.

We start with a chart U in X. There is a corresponding chart $p(U)$ in Y. Assume that in these coordinates p is given by a holomorphic function $z \mapsto \psi(z)$. Further assume that $q = \phi(z)dz^2$ locally ion $p(U)$. In this case, we can pull back q to a quadratic differential $\tilde{q}$ on X locally given by $\phi(\psi(z))(d\psi(z))^2dz^2$.

Note that x is a ramification point of degree k iff ψ has a zero of order k at the origin, and q has a zero of order m at x iff ϕ has a zero of order m at the origin. In this setting, $\tilde{q}$ has a zero of order $km+2(k-1)$ at x. This matches the calculation from measured foliations that the horizontal foliation has $k(m+2)$ prongs.

Just like the polygonal construction, any holomorphic quadratic differential on any Riemann surfaces can be constructed by branched covers. The general construction is given as follows: we start from the torus, where we already know a complete classification of quadratic differentials on the torus (bijective to $\mathbb{C}$). Now any Riemann surface can be seen as a branched cover of the standard torus so that we can use the above construction to yield any quadratic differentials on any Riemann surface.

4. The vector space of quadratic differentials

Fix a Riemann surface X. It is easy to observe that the collection of a all quadratic differentials on X, denoted $QD(X)$, is a $\mathbb{C}$-vector space. In this section, we will first give a metric on the vector space $QD(X)$, then we will give a specific dimension count for $QD(X)$.

4.1. **A metric on $QD(X)$.**

For any quadratic differential, we define a norm for q on X by

$$||q|| = \int_X |q|$$

.

Note that with this definition, $|q|$ denotes the area form associated to the singular Riemannian metric we developed before. Then this norm induces a metric and defines a topology.

4.2. **A dimension count.**

Recall that we already know the dimension count for holomorphic quadratic differentials on the genus 1 surface (torus), which is of complex dimension 1 or real dimension 2. The following method only works on surfaces of genus larger or equal to 2. We use polygonal constructions for this dimension count.

We start with a polygon P such that

- If we identify pair of parallel sides, we get the Riemann surface S of genus g.
- Each vertex of the polygon maps to a point in S with total Euclidean angle 3π

One such example is given by the second example we have earlier (not the Swiss cross). In fact, such examples exist for every genus g.

Moreover, the set of these polygons is of the same dimension as $QD(S)$. We reason as follows:

- Each such polygon induces a holomorphic quadratic differential by dz^2 in the interior of the polygon.
- Note that the second condition above means that each vertex of the polygon is a simple zero of the surface S and there are no other zeros.
- The set of these polygons is of codimension zero of all polygons giving holomorphic quadratic differentials on S.

Thus we only need to count the dimension of the set of polygons of the above described type.

Step 1: We count sides of P. Bu Euler-Poincaré formula, there are $4g-4$ zeros for the quadratic differential, thus there are $4g-4$ (distinct) vertices of P. Thus by condition 2, we know that the interior angles of P sum up to $3\pi(4g-4) = (12g-12)\pi$. Consequently, we know that P has $12g-10$ sides.

Let's have a dimension count:

$$
\begin{aligned}
&+2(12g-10)(\text{ each side has 2 directions to choose})\\
&-(12g-10)(\text{ sides matches in pairs})\\
&-2\text{ (the last pair is determined by the others)}\\
&=12g-12
\end{aligned}
$$

Step 2: At last we note that as we change the polygon, we change the complex structure, which is what we don't want. Thus we need to subtract the dimension of all complex structures, which is the dimension for Teichmüller spaces, which is $(6g-6)$.

Thus we finally conclude that $dim(QD(X)) = 6g-6$.

References

[1] Benson Farb and Dan Margalit, *A Primer on Mapping Class Groups*, 2012.