Combined notes for exotic Smooth Structures on Closed 4-manifolds

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Contents

Chapter 1

Introduction

In this article, we will survey all known techniques of constructing exotic smooth structures on closed 4-manifolds.

In chapter 1,

In chapter 2,

In chapter 3,

In chapter 4,

In the appendices,

Chapter 2

Rational blowdown

In this chapter, we will discuss an important construction in the studies of exotic closed 4-manifolds: rational blow-downs. The main idea is, as the name suggests, "inverse" to the blow-up operation. Namely we wish to replace a neighbourhood of a configuration of spheres by a rational homology 4-ball.

To study rational blow-downs, as well as later chapters of this survey, one essential tool that we will use is the Seiberg-Witten invariants, the basics of which theory are discussed in the appendix, except for some specific applications (like the Laurent series form and its gluing results, which will be introduced throughout the chapters).

We will setup the procedure of rational blow-downs in the first section. In section 2, we will discuss how rational blow-downs affect gauge theories invariants. Note that in [6], the theory was initially partially setup in the sense of Donaldson invariants, but currently we prefer the more concise and elegant Seiberg-Witten invariants.

2.1 Setup of rational blow-downs

In this section we will setup the constructions of rational blow-downs.

We start by fixing a positive integer p and consider the manifold associated to the following plumbing diagram:

Plots/Cp-plumbing.png

Namely, we take one copy of the disk bundle over S^2 with Euler classes $-(p+2)$ and $(p-1)$ copies of the disk bundle over S^2 with Euler classes -2, and plumb them together via the above Dynkin diagram. We call this plumbed 4-manifold C_p. Note that the manifold C_p has the following Kirby diagram. (Notes on basic Kirby diagrams are discussed in the appendix.)

Plots/Cp-kirby.png

As we briefly said in the introduction, we wish to replace the plumbed manifold C_p by a rational homology 4-ball. To do so, we need the boundary of the plumbed 4-manifold to be homeomorphic to the boundary of a rational homology 4-ball, in particular, a lens space. This fact is proved by the following lemma.

Lemma 2.1. *The boundary of the plumbed 4-manifold ∂C_p is the lens space $L(p^2, p-1)$. Thus $\pi_i(\partial C_p) \cong \mathbb{Z}/p^2$.*

Proof. Recall that when the plumbing diagram is linear with coefficients $[a_1, a_2, ..., a_n]$, the resulting 4-manifold has its boundary homeomorphic to a lens space $L(p, q)$, where

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \frac{1}{a_n}}}$$

.

In this case, we pick $[a_1.a_2, ...a_n]$ to be $[p+2, 2, ..., 2]$ with $(p-2)$ many 2's, we have exactly

$$\frac{p^2}{p-1} = p + 2 - \cfrac{1}{2 - \cfrac{1}{\ddots - \frac{1}{2}}}$$

.

This proves the lemma.

$\square$

With the above lemma, we can now construct a rational 4-ball that has the same boundary with C_p. We consider the 4-manifold B_p defined by the following Kirby diagram.

Plots/Bp-kirby.png

We claim the following lemma.

Lemma 2.2. *The 4-manifold ∂C_p has the same boundary with the rational homology 4-ball B_p. Moreover, $B_p \cup_\partial \overline{C_p}$ is diffeomorphic to $\#(p-1)\mathbb{CP}^2$.*

Proof. We wish to construct a Kirby diagram for the manifold $B_p \cup_\partial \overline{C_p}$ and use handle moves to show diffeomorphism with $\#(p-1)\mathbb{CP}^2$.

To do so, we start by drawing out the Kirby diagram for the double of C_p, i.e. for $DC_p = C_p \cup_\partial \overline{C_p}$. This is shown in the following image.

Plots/DCp-kirby.png

Note that each -1-framed 2-handle for C_p on the left gives rise to a once linked 0-framed 2-handle for its double, as does the 0-framed 2-handle on the right. Details of constructing Kirby diagrams for the double of a manifold is discussed in the appendices.

Now we will try to modify this manifold into our target manifold $B_p \cup_\partial \overline{C_p}$. Recall from the Kirby diagrams for C_p and B_p that we can modify C_p to B_p by interchanging the dot and the 0-framing and blowing-down the $(p-1)$ 2-handles on the left.

Now we view the above Kirby diagram as a relative Kirby diagram for the pair (DC_p, C_p). If we do the above said surgery to the second space to modify it into the rational 4-ball B_p, the first space will become the space $B_p \cup_\partial \overline{C_p}$. This is because all relative handles are left unchanged, i.e. the $- \cup_\partial \overline{C_p}$ part is left unchanged. This process is illustrated by the diagram below.

Plots/surgered-DCp-kirby.png

Thus we have construced a Kirby diagram for the manifold $B_p \cup_\partial \overline{C_p}$. Next we will do handle moves to show diffeomorphism with $\#(p-1)\mathbb{CP}^2$.

We start by sliding the $(p-1)$ framed 2-handle over its meridians (the $(p-1)$ once linked 1-framed 2-handles on the left). This process is illustrated by the diagrams below. The result will contain a "complicated" component on the top and $(p-1)$ disjoint 1-framed 2-handles on the bottom.

Plots/surgered-DCp-is-CP2.png

However, we can now use the 0-framed meridian on the right to unlink the two large circles. Thus the top "complicated" component can in fact be seen as a cancelling 1-handle/2-handle pair and 2-handle/3-handle pair. Thus we conclude that the Kirby diagram on the right is in fact the manifold $\#(p-1)\mathbb{CP}^2$. This proves the lemma.

$\square$

With the above two lemmas, the phrase "rational blow-down" finally makes sense, namely to replace an embedded copy of C_p to a copy of B_p. Thus we give the following definition.

Definition 2.3 (Rational blow-down)**.** Consider a 4-manifold X such that the plumbing C_p embeds inside, namely $X = C_p \cup_{L(p^2,p-1)} X^0$ for some 4-manifold X^0. We define its rational blow-down to be the manifold $X_{(p)} := B_p \cup_{L(p^2,p-1)} X^0$.

Before we move on to its effects on gauge theoretic invariants, we quickly analyse its effects in the intersection form.

Note.

2.2 Effects on Seiberg-Witten invariants

In this section, we wish to understand how the Seiberg-Witten invariants are affected during the process of rational blow-downs. While we will mention only a few specific examples in this chapter, we will see in the next two chapters that rational blow-downs can be used to build logarithm transformations and consequently knot surgeries. Moreover, the change of Seiberg-Witten invariants under log transforms and knot surgeries are both based on the very theorem we will discuss in this section. It is safe to say that rational blow-downs are the building blocks of almost all gauge-theory based constructions of exotic 4-manifolds.

The main theorem goes as follows:

Theorem 2.4.

2.3 Basic Examples

Chapter 3

Logarithm Transformation

In this chapter, we will discuss another method of producing exotic closed 4-manifolds: logarithm transformations. The main idea is to cut out a (or more) homologically interesting thickened torus, and glue another thickened torus back on via a 3-torus diffeomorphism on the boundary. Such a surgery will modify the Seiberg-Witten invariants, which we will use to distinguish different diffeomorphism types, but will sometimes leave the homeomorphism type unchanged.

We will start in section 1 with the construction of logarithm transform on elliptic surfaces. In section 2, we will generalize the construction in general smooth 4-manifolds. In section 3, we shall see that a log transform can be seen as a rational blow-down after some relugar blow-ups, aand thus we can carefully calculate how log transforms modifies the Seiberg-Witten invariants. And at last in section 4, we will study the action of log transforms on Kirby diagrams. The basics of Kirby diagrams are also discussed in the appendices.

3.1 Log Transform on Elliptic surfaces

We start by looking at an elliptic fibration $E \xrightarrow{p} \mathbb{CP}^1$ with generic torus fibre. Choose a small disk $D \subset \mathbb{CP}^1$ such that its preimage $p^{-1}(D) =: E|_D$ contains only regular fibres. Parameterize $E|_D$ as $D^2 \times S^1 \times S^1$.

Then log transforms can be seen as "Dehn twists" on the last factor. To be specific, we fix a positive integer p, cut $E|_D$ open as $D^2 \times S^1 \times [0,1]$, twist one end by an angle of $2\pi/p$ and glue back. In other words, we choose a a diffeomorphism $\phi : \mathbb{T}^3 \to \mathbb{T}^3$ such that $\phi_* : H_1(\mathbb{T}^3) \to H_1(\mathbb{T}^3)$ has multiplicity p in the first factor. The process is illustrated by the following image.

`Plots/twist.png`

This surgery results in a new fibration $E_p|_D \to D$.

3.2 Generalized log transforms

More generally, we can

3.3 Log Transforms are Rational Blow-downs

3.3.1 Near-cusp embedded torus

3.3.2 The Laurent Series Form of Seiberg-Witten Invariants

3.3.3 Construction of Log Transforms via Rational Blow-downs

Finally we wish to study how (generalized) log transforms modify the series form of Seiberg-Witten series, whether on near-cusp embedded tori or not. To study this, we will see that the operation of a log transform can be seen as several blow-ups followed by a rational blow-down. Thus we can calculate the Seiberg-Witten invariants via the blow-up formula and the main theorem from last chapter.

To start with, we consider a 4-manifold X with a

Theorem 3.1. *The manifold obtained by is diffeomorphic to the manifold obtained by.*

Proof. □

3.4 The action on Kirby Diagrams

Chapter 4

Knot Surgery

In this chapter, we will discuss another method of producing exotic closed 4-manifolds: knot surgery. The main idea is to cut out a homologically interesting thickened torus, and replace it by a homologically non-distinguishable thickened S^3-knot complement.

To study knot surgeries, the two main tools that we will use are the Seiberg-Witten invariants agaian, and the Alexander polynomial. Section 1 will be focusing on Alexander polynomials, along with other definitions and properties we might need during the study of knot surgeries.

In section 2, we will rigorously define the knot surgery, and state the main theorem first proved by Fintushel and Stern in [5], the proof of which will be delayed till section 5. In section 3, we will see some examples and applications of the knot surgery. And at last in section 4, we will study the action of knot surgeries on Kirby diagrams.

4.1 Preparations

4.1.1 Alexander Polynomials

In this section, we will have a brief recall of Alexander polynomials.

Definition 4.1. For an oriented knot K, we define the Alexander polynomial Δ_K via the skein relation

$$\Delta_{K^+}(t) = \Delta_{K^-}(t) + (t^{1/2} + t^{-1/2}) \cdot \Delta_{K^0}(t)$$

, where the three knots K^+, K^-, K^0 only differ in one single crossing, as illustrated in the following diagram.

Plots/skein-relation.png

Moreover, we normalize the Laurent polynomial via the initial condition $K_{\bigcirc}(t) = 1$, where $\bigcirc$ means the unknot.

From this definition, it is easy to calculate the Alexander polynomial of a knot. Yet the above definition only concerns the knot diagrams, thus is not *a priori* well-defined. Why the above defined Laurent polynomial is invariant under Reidemeister moves is given by the following lemma.

Lemma 4.2. *The Alexander polynomial is well-defined.*

Proof. □

Finally, we will give some quick examples and calculations of Alexander polynomials.

Example.

Example.

Example.

4.2 The Knot Surgery

4.3 Examples

4.4 The action on Kirby Diagrams

In this section, we will study how a knot surgery will affect the Kirby diagram of a given 4-manifold. The construction and study in this section follows S. Akbulut's paper [7].

We will first consider a "universal example", namely how the Kirby diagram changes when the thickened torus $T^2 \times B^2$ is changed into a thickened knot complement $(S^3 \backslash K) \times S^1$. Then we can put the torus in the near-cusp scenario and study how Kirby diagrams change in a general case. Note that we will use the trefoil knot as an example in all the figures used in this section.

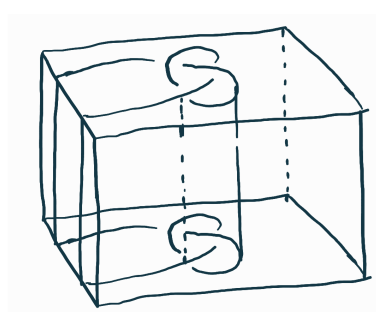
We start by raising the following question:

Question. What does the manifold $(S^3 \backslash K) \times S^1$ look like?

We start by observing that $(S^3 \backslash K) \times S^1$ can be seen as two copies of $(S^3 \backslash K) \times I$ glued along $S^3 \backslash K \coprod S^3 \backslash K$. Thus we first describe the manifold $(S^3 \backslash K) \times I$.

Up to attaching a 3-handle, the manifold $(S^3 \backslash K) \times I$ can be obtained by removing the tubular neighbourhood of a embedded arc (including the knot structure) from B^3 and thicken it by crossing with I. This process is illustrated by the following pictures.

Plots/S3-K-box-cut.png

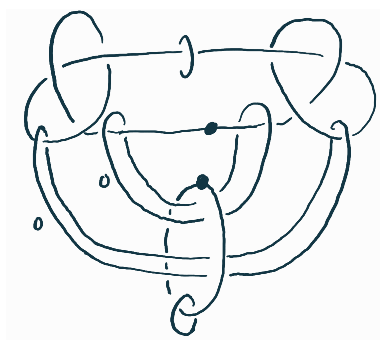


In the above figures, each "horizontal layer" is a copy of B^3, and the "removed handle" (the figure on the left) can be seen as a 3-handle. By gluing the 3-handle to the described manifold (on the right) along the boundary in each "layer", we obtain the thickened knot complement $(S^3 \backslash K) \times I$.

Equivalently, we can see that $(S^3 \backslash K) \times I$ can be obtained by removing the slice disc bounded by $K \# -K$ from B^4. This gives us a Kirby diagram for $(S^3 \backslash K) \times I$, following the dotted circle notation.

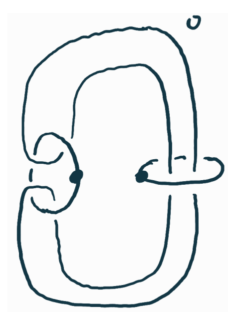
Plots/thickened-S3-K-I-kirby.png

From this, we build the manifold $(S^3\backslash K) \times S^1$. We need to glue a second copy of $(S^3\backslash K) \times I$ upside down to the first copy. Equivalently, we need to identify the two boundary components on the top and bottom of $(S^3\backslash K) \times I$. Consequently, we need to glue in a single 1-handle and two 2-handles (obtained from identifying the two 1-handles α and β on the boundary components). Thus we obtain the following Kirby diagram of $(S^3\backslash K) \times S^1$, again up to attaching 3-handles.



Question. How does knot surgery change the Kirby diagram in the universal case?

Now we start with a Kirby diagram of $T^2 \times B^2$ (as illustrated below), and we wish to modify it to the Kirby diagram for $(S^3\backslash K) \times S^1$ we just obtained.



Then we wiggle the Kirby diagram a little bit and introduce a cancelling 2-handle/3-handle pair. We slide the new 2-handle over the old one, and obtain the following Kirby diagram depicted on the right.

Plots/surgery-process-no-cusp.png

Now we compare the newly obtained Kirby diagram for $T^2 \times B^2$ and the Kirby diagram of $(S^3 \backslash K) \times S^1$ obtained before.

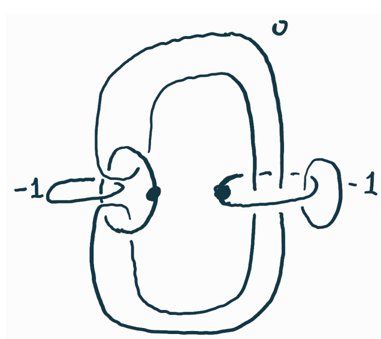
If we start from the Kirby diagram of $(S^3 \backslash K) \times S^1$ perform the handle slides as indicated in the figure on the left, we obtain the Kirby diagram of $T^2 \times B^2$.

Plots/surgery-process-no-cusp.png

Conversely, if we start from the Kirby diagram of $T^2 \times B^2$, and perform the handle slides indicated on the right figure, we obtain the Kirby diagram of $(S^3 \backslash K) \times S^1$. Note that the "circled 1/2 notation" means that a half twist is applied when doing the handle side.

Next we apply the above observation to the near-cusp situation to study the general case. The first question is:

Question. What does a near-cusp embedded torus look like?



Question. How does knot surgery change the Kirby diagram in the general case?

To sum up,

Example. Here is a Kirby diagram of the manifold $E(1)_K$, where K is a trefoil knot.

4.5 Proof of the Fintushel-Stern Theorem

In this section, we will follow [5] and prove the main theorem of this chapter.

Chapter 5

Zero Surgeries Constructions

5.1 Possible future studies

Appendices

Appendix A

A Note on Seiberg-Witten Invariants

A.1 Overview

The Seiberg-Witten invariant on a smooth 4-manifold X is a smooth structure invariant that takes the form of a function: $\mathcal{SW}_X : \{\text{Spin}^{\mathbb{C}} \text{ structures on } X\} \to \mathbb{Z}$. The invariant can be defined for any smooth 4-manifold X but only provides non-trivial information for spaces X such that $b^+(X) + b_1(X) + 1$ is odd. The Seiberg-Witten invariants play an important role for distinguishing exotic smooth structures of a closed 4-manifold.

Seiberg-Witten Theory is an easier gauge theory approach to the topology of 4-manifolds campared to Donaldson Theory. Seiberg-Witten was originally developed based on theoretical Physics ideas: the magnetic monopoles equations.

A.2 Backgrounds

To understand Seiberg-Witten theory, it is essential to understand:

- Almost complex structures.
- $\text{Spin}^{\mathbb{C}}$ structures.

A.2.1 Almost Complex Structures

A 4-manifold M is said to have an almost complex structure if its tangent bundle τ_M is a complex plane bundle. Equivalently, M has an almost complex structure if there exists a bundle automorphism $J : \tau_M \to \tau_M$ that is fibre-preserving and that $J^2 = -id$.

Definition A.1 (Integrable)**.** An almost complex structure is said to be integrable if it comes from an actual complex manifold.

Chern classes

The idea of an almost complex structure is to generalize the notion of a complex manifold. One important invariant about complex manifolds is the Chern classes. However, it is not necessary to have a complex structure to define a Chern class for M. Instead, an almost complex structure is enough.

Indeed, the second Chern class $c_2(\tau_M)$ is equal to the Euler class of the tangent bundle $e(\tau_M)$. Moreover, we define the first Chern class of the almost complex structure as the first Chern class of the tangent bundle $c_1(J) := c_1(\tau_M)$. Note that this first Chern class is exactly the the first Chern class corresponding to the determinant line bundle K_J^* of the tangent bundle.

J-invariant Curves

The idea of J-invariant curves in almost complex 4-manifolds is a generalization of the notion of complex curves in complex surfaces. A curve S is said to be J-invariant if its tangent subbundle is invariant under J, i.e. $J[\tau_S] = \tau_S$. (Of course we can similarly define a J-invariant normal bundle). For a J-invariant curve S, we alwasy have the following identity:

$$\chi(S) + S \cdot S = c_1(J) \cdot S$$

. This is called the adjunction formula for S.

Existence of Almost Complex Structures

We know that for an almost complex structure J, the identity $c_1(J) \cdot c_1(J) = 3sign(M) + 2\chi(M)$ holds.

The converse is in fact also true. An almost complex structure exists if and only if there exists an integral lift $\underline{w}$ of the second Stiefel-Whitney class $w_2(M)$ satisfying $\underline{w} \cdot \underline{w} = 3sign(M) + 2\chi(M)$. In this case, the integral lift $\underline{w}$ can be seen as the first Chern class of the corresponding almost complex structure.

Note that if the manifold M is simply connected or indefinite, such an integral lift exists if and only if $b^+ + b_1$ is odd.

Also note that an integral lift (satisfying the above identity or not) always define a partial almost complex structure on the three skeleton for the smooth manifold. This partial almost complex structure is in fact a Spin$^{\mathbb{C}}$-structure on M, which is discussed in the next section.

Two out of Three

For the three different structures on the smooth manifold M: almost complex structures, Riemannian metrics, and an exterior 2-form satisfy the two-out-of-three rule: whenever given two of the three structures compatible with each other, they together determines the third.

1) Almost complex + Riemannian $\Rightarrow$ 2-form:

2) 2-form + Riemannian $\Rightarrow$ Almost complex :

3) Almost-complex + 2-form $\Rightarrow$ Riemannian:

A.2.2 Spin$^{\mathbb{C}}$ structures.

A.3 Seiberg-Witten Invariants

We fix a smooth manifold M, a Riemannian structure g, and a $Spin^{\mathbb{C}}$ structure $\mathfrak{s}$ on M. Then we observe the following set of Seiberg-Witten Equations (following the notation from last section):

$$\begin{cases} \mathcal{D}^A \varphi = 0 \\ F_A^+ = \sigma(\varphi) \end{cases}$$

, where A is a $U(1)$ connection, $\varphi \in \Gamma(\mathcal{W}^+)$ is a self-dual spinor field, $\mathcal{D}^A$ is the Dirac operator mentioned earlier, and σ is the squaring map mentioned before.

The solutions of the set of PDEs are called **Seiberg-Witten monopoles**, with its name coming from the monopole equations in Physics. The space of monopoles is denoted as $\mathfrak{S}$, which can be embedded in the configuration space $\Gamma(\mathcal{W}^+) \times \mathcal{A}$, where $\mathcal{A}$ is the space of all connections (Following the notation of [2]). The gauge equivalence classes $\mathcal{M} := \mathfrak{S}/\mathcal{G}$ is called the Seiberg-Witten moduli space.

We wish to define a numerical invariant (depending only on the chosen $Spin^{\mathbb{C}}$ structure) on the 4-manifold using Seiberg-Witten moduli space. The main idea is to use the moduli space to represent a homology class in the configuration space, but we need a bit more restrictions, which goes as follows:

- We wish to avoid reducible solutionsin the moduli space $\mathcal{M}$ via transversality results. If so, for any two choices of Riemannian metrics in the first step, we have a cobordism in the configuration space. thus we have a numerical invariant independent of the choice of the Riemannian metric.

- The moduli space $\mathcal{M}$ has to be compact.

- The moduli space $\mathcal{M}$ should be orientable since we want to define a numerical invariant other than $\mathbb{Z}/2$.

A.3.1 Transversality

A.3.2 Compactness

A.3.3 Orientability

A.3.4 Definition of Seiberg-Witten Invariants

With the above results, we can finally finish with the definition of the Seberg-Witten invariant. We start with the simply-connected case, and a fact about the ambient space $\Gamma(\mathcal{W}^+) \times \mathcal{A}/\mathcal{G}$.

Fact. When the manifold M is simply connected, the ambient space of gauge equivalence classes of connection-spinor pairs $\Gamma(\mathcal{W}^+) \times \mathcal{A}/\mathcal{G}$ has the homotopy type of $\mathbb{CP}^2$.

Thus its homology ring is $\mathbb{Z}[u]$ with $|u| = 2$. Thus we can define the following:

Definition A.2. Seiberg-Witten Invariant For a simply commected 4-manifold X and a fixed $Spin^{\mathbb{C}}$ structure $\mathfrak{s}$, its Seiberg-Witten moduli space is denoted $\mathcal{M}$. Then the Seiberg-Witten invariant is defined by:

$$\mathcal{SW}_X(\mathfrak{s}) := \begin{cases} \int_{\mathcal{M}} u \wedge ... \wedge u & b_2^+ \text{ is odd } \Leftrightarrow dim(\mathcal{M}) \text{ is even} \\ 0 & b_2^+ \text{ is even } \Leftrightarrow dim(\mathcal{M}) \text{ is odd} \end{cases}$$

Note that for the non-simply connected case, the condition is similar. In this case, the parity of b_2^+ is modified to the parity of $b_1 + b_2^+$.

At last we define an important property called “simple type”.

Definition A.3. Simple Type A 4-manifold X is said to be of simple type if all non-empty moduli spaces either:

- are 0-dimensional.

- the associated $Spin^{\mathbb{C}}$ structure $\mathfrak{s}$ comes from almost complex structures.

- the associated $Spin^{\mathbb{C}}$ structure satisfies $c_1(\mathfrak{s}) \cdot c_1(\mathfrak{s}) = 2 \cdot \chi(X) + 3 \cdot sgn(X)$.

We finish this section by a conjecture that all simply-connected 4-manifolds of $b_2^+ \geq 2$ is of simple type.

A.4 Properties and Applications

A.4.1 General properties

We know from the previous section that for 4-manifolds with $b_2^+ \geq 2$, the Seiberg-Witten invariant can be seen as a function $\mathcal{SW}_M : \{Spin^{\mathbb{C}} \text{ structures }\} \to \mathbb{Z}$ (by counting the solutions of Seiberg-Witten equations).

If $H^2(M)$ has no 2-torsion, the set of $Spin^{\mathbb{C}}$ is uniquely characterized by the integral lifts of the second Stiefel-Whitney class w_2. In this sense, we can treat the Seiberg-Witten invariant as a function

$$\mathcal{SW}_M : \{\underline{w} \in H^2(M) | \underline{w} = w_2(M) \bmod 2\} \to \mathbb{Z}$$

.

Also note that if $H^2(M)$ has 2-torsion, then there might be more than one associated $Spin^{\mathbb{C}}$ structures. We can still get a function in the above form by summing up all $Spin^{\mathbb{C}}$ structures associated to $\underline{w}$.

Note. We can also study a parameterized version of the above function. We pick a distinguished $Spin^{\mathbb{C}}$ structure $\mathfrak{s}_0$ and write the function as $\mathcal{SW}_M(\mathfrak{s}_0 + \cdot) : H^2(M) \to \mathbb{Z}$

Note. There is another form of the Seiberg-Witten invariants taking the form of Laurent series, which is studied in the section involving knot surgeries.

Definition A.4. Basic classes An integral cohomology class $\kappa \in H^2(M)$ is called a basic class if $SW_M(\kappa) \neq 0$.

An immediate consequence is that any 4-manifold has at most a finite number of basic classes.

Next we list a number of general properties of Seiberg-Witten invariants with only a sketch of the proofs.

Theorem A.5 (Involution Lemma)**.** *If $b_2^+ \geq 2$, and κ is a basic class, then we have $\mathcal{SW}_M(-\kappa) = \pm\mathcal{SW}_M(\kappa)$*

Proof. □

Theorem A.6 (Vanishing Theorem for Scalar Curvature)**.** *If $b_2^+ \geq 2$, and if M admits a Riemannian metric with everywhere-positive scalar curvature, then $\mathcal{SW}_M \equiv 0$.*

Proof. The theorem follows directly from the curvature bound mentioned in the previous sections. □

Theorem A.7 (Vanishing Theorem for Connected Sums)**.** *If a smooth 4-manifold M is diffeomorphic to $N'\sharp N''$, where $b_2^+(N') \geq 1$ and $b_2^+(N'') \geq 1$, then $\mathcal{SW}_M \equiv 0$.*

Proof. The theorem follows from the following three facts.

- All Seiberg-Witten solutions vanish on the connecting cylinder $S^3 \times [0, 1]$.
- $\mathcal{M}_{N'\sharp N''} = \mathcal{M}_{N'} \times \mathcal{M}_{N''}$ for any fixed choice of $Spin^{\mathbb{C}}$ structures on N' and N''.
- $dim\mathcal{M}_{N'\sharp N''} = dim\mathcal{M}_{N'} + dim\mathcal{M}_{N''} + 1$ by the virtual dimension formula.

□

Theorem A.8 (Blow-up Formula)**.** *For a simply-connected 4-manifold M with $b_2^+(M) \geq 2$ and is of simple type. Let $\{\kappa_i\}$ be the set of basic classes of M, then the topological blow-up $M\sharp\overline{CP}^2$ has basic classes $\{\kappa_i \pm E\}$, where E is the class of $\overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2}$. Moreover, we have*

$$\mathcal{SW}_{M\sharp\overline{CP}^2}(\kappa_i \pm E) = \pm\mathcal{SW}_M(\kappa_i)$$

.

Proof. The formula follows from the same procedure as the last theorem. □

Theorem A.9 (Non-vanishing Theorem for Symplectic Manifolds). *If M is a simply connected 4-manifold such that $b_2^+(M) \geq 2$, and admits a symplectic structure ω, then $K_M^* = c_1(\omega)$ is a basic class and $\mathcal{SW}_M(\pm K_M^*) = \pm 1$.*

Proof. $\square$

Then the following corollary follows directly from the non-vanishing theorem for symplectic manifolds and the vanishing theorem for connected sums.

Corollary A.10. *A symplectic 4-manifold cannot split into $N' \sharp N''$ where $b_2^+(N') \geq 1$ and $b_2^+(N'') \geq 1$.*

Theorem A.11 (Adjunction Formula). *If M is a 4-manifold such that $b_2^+(M) \geq 2$, and if S is a connected surface embedded in M such that either of the following holds*

- *$S \cdot S \geq 0$ and S is homologically non-trivial.*
- *M is of simple type and S is not the 2-sphere.*

Then for any basic class κ, we have the following inequality:

$$\chi(S) + S \cdot S \leq -|\kappa \cdot S|$$

.

Proof. $\square$

Note. The adjunction formula gives a lower bound for the genus of S fixing a homology class represented by S.

Note. If we use the adjunction formula backwards, we can fix S and give restrictions to the basic class κ.

Corollary A.12. *If a 4-manifold M contains a homologically non-trivial embedded sphere S such that $S \cdot S \geq 0$, then $\mathcal{SW}_M \equiv 0$.*

A.4.2 Properties for Symplectic Manifolds

A.4.3 Properties for Complex Surfaces

Appendix B

A Note on Kirby Calculus

B.1 Basic ideas

B.2 Handle moves

B.2.1 Handle Slide

B.2.2 Handle Cancellation

B.2.3 Blow-up and Blow-down

B.3 Examples

B.3.1 Plumbing

B.3.2 Double of a 4-Manifold

B.3.3 Gluing Results

Appendix C

A Note on Elliptic Fibrations

C.1 temp

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