A NOTE ON STIEFEL-WHITNEY CLASSES

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1. Overview

This is a lecture note of the first talk of the seminar "Characteristic Classes", and we will focus on the definition (axioms) of Stiefel-Whitney Classes along with its first applications. Throughout this note, we will first give a quick review of vector bundles (definitions, examples, constructions and sections), since bundles are the foundation of the whole seminar. Then we will turn to the four axioms defining the Stiefel-Whitney Classes. We will also discuss some first consequences of the four axioms and do some calculations of various vector bundles as examples. Finally we will discuss two applications of Stiefel-Whitney Classes, namely the existence of Division Algebras in dimension only the powers of 2, and a necessary condition for a manifold immersing in Euclidean space $\mathbb{R}^n$.

2. Quick Review on Vector Bundles

Before we introduce the first characteristic classes, we need to keep in mind that characteristics classes are built for the classification of vector bundles up to bundle isomorphism. All characteristic classes are bundle invariants. Thus we shall start this note by a quick review of vector bundles.

2.1. Basic definitions and examples.

Definition 1 (vector bundles)**.** A rank k vector bundle $\xi : \mathbb{R}^k \hookrightarrow E \xrightarrow{\pi} B$ consists of the following data:

- a topological space E called total space.
- a topological space B called base space.
- a surjective continuous function $\pi : E \to B$ called projection.

satisfying locally triviality: there exists an open cover $\mathcal{A} = \{U_\alpha\}$ of B such that for each $U_\alpha \in \mathcal{A}$ there exists a homeomorphism $h_\alpha : U_\alpha \times \mathbb{R}^k \xrightarrow{\cong} \pi^{-1}(U)$ satisfying:

- $\pi \circ h_\alpha = pr_1$ where pr_1 means projection onto the first factor.
- $\forall U_\alpha, U_\beta \in \mathcal{A}, U \subset U_\alpha \cap U_\beta$, exists a continuous map $\theta_{\alpha\beta} : U \to \mathbb{GL}_k(\mathbb{R})$ called transition function such that $h_\alpha(b, v) = h_\beta(b, \theta_{\alpha\beta}(b) \cdot v)$.

Under this definition, each fibre $\pi^{-1}(b)$ has a k dimensional vector space structure and the map $x \mapsto h(b, x)$ defines a linear isomorphism between $\mathbb{R}^k$ and $\pi^{-1}(b)$.

Definition 2 (bundle map)**.** A bundle map from $\xi_1 : E_1 \xrightarrow{\pi_1} B_1$ to $\xi_2 : E_2 \xrightarrow{\pi_2} B_2$ is a pair of continuous maps (f, g) such that $f : E_1 \to E_2$ covers $g : B_1 \to B_2$, i.e. the following diagram commutes

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

, and $\forall x \in B_1, f : \pi_1^{-1}(x) \to \pi_2^{-1}(g(x))$ is a linear map.

Definition 3 (bundle isomorshism)**.** A bundle isomorphism is a bundle map with inverse.

Note that a bundle isomorphism induces linear isomorphisms on each fibre.

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Example (Trivial Bundle)**.** A vector bundle over B is called trivial if it is isomorphic to the product bundle $E = B \times \mathbb{R}^n \xrightarrow{pr_1} B$. In this case, the bundle isomorphism is called a global trivialization.
e.g. The annulus is a trivial bundle over S^1.
e.g. The Möbius bundle is a non-trivial bundle over S^1.

Example (Tangent Bundle)**.** For an n dimensional smooth manifold M, it has a tangent bundle $\tau_M : \mathbb{R}^n \hookrightarrow TM \xrightarrow{\pi} M$. (Note that Milnor uses the notation DM.) Its fibre at a point x is called the tangent space at x T_xM.

Definition 4. A manifold M is called parallelizable if its tangent bundle is trivial.

Example (Normal bundle)**.** If an n dimensional smooth manifold M^n is embedded or immersed in l dimensional Euclidean space $\mathbb{R}^l$, then we can define its normal bundle $\nu_M : \mathbb{R}^k \hookrightarrow E \xrightarrow{\pi} M$, with $l = n + k$. Its fibre space of a given point x being the orthogonal complement of T_xM.

Example (Canonical line bundle for $\mathbb{RP}^n$)**.** First recall that $\mathbb{RP}^n$ is defined as the set of lines in $\mathbb{R}^{n+1}$ or as S^n quotienting out antipodal points or $\mathbb{R}^{n+1}\backslash\{0\}$ quotienting out scalar multiplications. Thus we can define the canonical line bundle as $\gamma_n^1 : E \xrightarrow{\pi} \mathbb{RP}^n$, where $E := \{(L, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : L \text{ a line in } \mathbb{R}^{n+1}, v \in L\}$, and π is just projection onto the first factor.
e.g. For $n = 1$, the canonical line bundle is just the Möbius bundle.

Example (Canonical vector bundle for $Gr(k, \mathbb{R}^n)$)**.** First recall that Grasmannian $Gr(k, \mathbb{R}^n)$ is a generalization of projective spaces and is defined as the set of k-planes in the n dimensional vector space. Thus we can generalize the canonical line bundle for projective spaces to a canonical rank k bundle over $Gr(k, \mathbb{R}^n)$ as $\gamma_n^k : E \xrightarrow{\pi} Gr(k, \mathbb{R}^n)$, where $E := \{(V, v) \in Gr(k, \mathbb{R}^n) \times \mathbb{R}^n : V \text{ a k-plane in } \mathbb{R}^n, v \in V\}$, and π is again projection onto the first factor.
e.g. For $k = 1$, we have a canonical line bundle over $\mathbb{RP}^{n-1}$.

2.2. **Constructing new bundles out of existing ones.** Now we recall different ways to construct a vector bundle given existing ones. Here we list five basic ways of construction.

(1) **Restriction of base space:** we just restrict the base space to a subset and restrict the total space accordingly.
(2) **Pullback (induced bundles)** Given a vector bundle $\xi : E \xrightarrow{\pi} B$ and a map $f : B' \to B$, we can pull back the bundle ξ by $f^*\xi : E' \xrightarrow{\pi} B'$, where $E' := \{(b, e) \in B' \times E : f(b) = \pi(e)\}$ and π is the projection onto the first factor, i.e. we have the following commutative diagram:

$$\begin{array}{ccc} E' & \xrightarrow{pr_2} & E \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

e.g. if we pullback an inclusion map, we actually have a restriction of vector bundles.
(3) **Cartesian Product** Given two vector bundles $\xi_1 : E_1 \xrightarrow{\pi_1} B_1$ and $\xi_2 : E_2 \xrightarrow{\pi_2} B_2$, we can construct a product bundle $\xi_1 \times \xi_2 : E_1 \times E_2 \xrightarrow{\pi_1 \times \pi_2} B_1 \times B_2$.
e.g. For two smooth manifolds M_1 and M_2, we have $\tau_{M_1 \times M_2} \cong \tau_{M_1} \times \tau_{M_2}$.
(4) **Whitney Sum** Given two vector bundles over the same base space $\xi_1 : E_1 \xrightarrow{\pi_1} B$ and $\xi_2 : E_2 \xrightarrow{\pi_2} B$, we can construct their Whitney sum as the pullback of their product bundle by the diagonal map $\Delta : B \to B \times B$, i.e. $\xi_1 \oplus \xi_2 := \Delta^*(\xi_1 \times \xi_2)$.
Note: The fibre space of the direct sum is the direct sum of the fibre spaces.
With Whitney sum, we can define another equivalence relation slightly weaker than isomorphism, called stable isomorphism.

Definition 5 (stable isomorphism)**.** Two vector bundles ξ_1 and ξ_2 are called stably isomorphic if there exists natural numbers a, b such that $\xi_1 \oplus \mathbb{R}^a \cong \xi_2 \oplus \mathbb{R}^b$.

e.g. The tangent bundle of S^2 is not trivial, but stably trivial.

(5) **Orthogonal Complement** If we have a bundle $\eta : \mathbb{R}^b \hookrightarrow E_\eta \xrightarrow{\pi_\eta} B$ with a Euclidean metric (A Euclidean metric on η is a bundle map $d : \eta \oplus \eta \to \underline{\mathbb{R}^1}$, such that if we restrict d to each fibre, we have a nondegenerate bilinear map) and a subbundle $\xi : \mathbb{R}^a \hookrightarrow E_\xi \xrightarrow{\pi_\xi} B$ with $a \leq b$, we can construct the orthogonal complement of ξ, denoted $\xi^\perp$ as follows. By fixing any point b in the base space, the fibre of $\xi^\perp$ is the orthogonal complement (with respect to the inner product) of $\pi_\xi^{-1}(b) \hookrightarrow \pi_\eta^{-1}(b)$.

(6) **The Hom Functor and Dual Bundle** Given two vector bundles over the same base space $\xi_a : \mathbb{R}^a \hookrightarrow E_a \xrightarrow{\pi_a} B$ and $\xi_b : \mathbb{R}^b \hookrightarrow E_b \xrightarrow{\pi_b} B$, we can construct a bundle $\xi : E \xrightarrow{\pi} B$ where its fibre is the vector space $Hom(\mathbb{R}^a, \mathbb{R}^b)$. If ξ_b is a trivial line bundle, the bundle ξ is the dual bundle of ξ_a. And we have the following theorem:

Theorem 1. *If ξ is a vector bundle with Euclidean metric, then ξ is isomorphic to its dual bundle.*

It is worth to note that any vector bundle over a paracompact space has a Euclidean metric. An important case is any vector bundle over a manifold is isomorphic to its dual bundle.

2.3. **Sections of vector bundles.** Finally we recall on sections of vector bundles. The slogan is "the number of linear independent sections measures the triviality of the vector bundle".

Definition 6 (sections of a bundle)**.** A section of a vector bundle $\xi : E \to B$ is a continuous map $s : B \to E$ such that $\pi \circ s = id_B$.

The first example of a section is a zero section, where you map a point on the base space to the zero of the corresponding fibre space.

Also, we have the following theorem.

Theorem 2. *If a rank n vector bundle ξ with Euclidean metric has k (fibrewise) linear independent sections, then ξ splits as $\underline{\mathbb{R}^k} \oplus \xi'$ with ξ' is a rank $(n-k)$ vector bundle. Note that $\underline{\mathbb{R}^k}$ means a rank k trivial bundle.*

3. Definition of Stiefel-Whitney classes and its first Consequences

In this section we will define the first characteristic classes, namely the Stiefel-Whitney classes. But first, let focus on the question: what are characteristic classes?

For each fibre bundle, we can assign characteristics classes, which are in the cohomology of the base space. They are supposed to satisfy naturality with respect to pullback, thus are invariant under bundle isomorphism. The slogan is that characteristic classes are used to measure how twisted a bundle is.

Some examples of characteristic classes that we will discuss throughout the semester are Pontrjagin classes ($p_k \in H^{4k}(B, \mathbb{Z})$) and Euler classes ($e \in H^n(B, \mathbb{Z})$) for real vector bundles, and Chern classes ($c_k \in H^{2k}(B, \mathbb{Z})$) for complex vector bundles. Of course the list also includes Stiefel-Whitney classes, the topic of this note.

Stiefel-Whitney classes are defined only for real vector bundles, and were originally studied by Stiefel and Whitney in 1935, and are used to study obstructions of constructing linear independent sections. For manifolds, the first Stiefel-Whitney class w_1 measures the orientability of the total space, and the second Stiefel-Whitney classes measures whether a manifold has a spin stricture, which gave rise to the field of obstruction theory.

The definition of Stiefel-Whitney Classes is given by the following theorem:

Theorem 3. *For each vector bundle $\xi : E \to B$, there exist a unique sequence $w_0, w_1, ...$ such that $w_i \in H^i(B; \mathbb{Z}/2)$ and satisfies the following four axioms:*

(1) *For each vector bundle $\xi : E \to B$, $w_0(\xi) = 1 \in H^0(B; \mathbb{Z}/2)$, and $w_i(\xi) = 0$ for $i > k = rank(\xi)$.*
(2) *(Naturality) if we have a bundle map (f, g) from ξ to η, we have $w_i(\xi) = g^* w_i(\eta)$.*
(3) *(Whitney Sum Formula or Whitney Product Theorem) For two vector bundles ξ_1 and ξ_2, we have*

$$w_n(\xi_1 \oplus \xi_2) = \sum_{i+j=n} w_i(\xi_1) \smile w_j(\xi_2)$$

.

(4) *(Non-triviality) The first Stiefel-Whitney class for the canonical line bundle for $\mathbb{RP}^1$ is non-trivial, i.e. $w_1(\gamma_1^1) \neq 0$.*

Proof. The proof of this theorem is too long to fit in this talk. Different ways of constructions and uniqueness of Stiefel-Whitney classes will be introduced in the next talk. ☐

With the definition of Stiefel-Whitney classes, we have some immediate consequences.

Proposition 4. *If vector bundles ξ and η are isomorphic, $w_i(\xi) = w_i(\eta)$ for all i.*

Proof. Directly by naturality. ☐

Proposition 5. *If ξ is a trivial vector bundle, $w_i(\xi) = 0$ for $i > 0$.*

Proof. The following diagram commutes:

$$\begin{tikzcd} B \times \mathbb{R}^n \arrow[r, "pr_2"] \arrow[d, "pr_1"] & \mathbb{R}^n \arrow[d, "*"] \\ B \arrow[r, "*"] & * \end{tikzcd}$$

The proposition follows directly from naturality. ☐

Proposition 6 (stability)**.** $w_i(\xi \oplus \underline{\mathbb{R}^n}) = w_i(\xi)$ *for all i.*

Proof. Follows directly from the last proposition and Whitney sum formula. ☐

Proposition 7. *If ξ is a rank n vector bundle with Euclidean metric and has k linearly independent sections, $w_{n-k+1}(\xi) = ... = w_n(\xi) = 0$.*

Proof. ξ splits as the sum of a rank k trivial bundle and a rank $n-k$ vector bundle. The proposition then follows from the last proposition and axiom 1. ☐

Fact. Fix a topological space X, denote by $H^{\Pi}(X;\mathbb{Z}/2) = \prod_i H^i(X;\mathbb{Z}/2)$ the ring of all infinite series $a = a_0 + a_1 + a_2 + ...$ with $a_i \in H^i(X;\mathbb{Z}/2)$. Note that the cohomology ring $H^*(X;\mathbb{Z}/2) = \bigoplus_i H^i(X;\mathbb{Z}/2)$ is a subring of the above mentioned ring of infinite series. The subcollection with leading term $1 \in H^0(X;\mathbb{Z}/2)$, $\{a = 1 + a_1 + a_2 + ... \in H^{\Pi}(B;\mathbb{Z}/2)\}$ is the subgroup of multiplicative invertible elements.

Proof. Inversion comes from the following inductive algorithm:
Given $a = 1 + a_1 + a_2 + ...$, we can define its inverse by $\bar{a} = 1 + \bar{a}_1 + \bar{a}_1 = ...$, with $\bar{a}_n = a_1\bar{a}_{n-1} + a_2\bar{a}_{n-2} + ... + a_{n-1}\bar{a}_1 + a_n$.
☐

Some first instances are $\bar{a}_1 = a_1$, $\bar{a}_2 = a_1^2 + a_2$, $\bar{a}_3 = a_1^3 + a_3$ and $\bar{a}_4 = a_1^4 + a_1^2 a_2 + a_2^2 + a_4$.
e.g. $(1 + a_1)^{-1} = 1 + a_1 + a_1^2 + a_1^3 + ...$
With the above fact, we can now define the total Stiefel-Whitney classes.

Definition 7 (total Stiefel-Whitney classes)**.** Define the total Stiefel-Whitney class of a rank k vector bundle ξ as the element $w(\xi) := 1 + w_1(\xi) + ... + w_k(\xi) + 0 + ... \in H^{\Pi}(B;\mathbb{Z}/2)$.

Note that in this sense, we can rewrite the Whitney sum formula as $w(\xi_1 \oplus \xi_2) = w(\xi_1) \smile w(\xi_2)$.
Now it makes sense to talk about the inverse of a Stiefel-Whitney class.

Proposition 8. *If the Whitney sum of two bundles ξ and η is trivial, $w(\eta) = \bar{w}(\xi)$.*

One special case for the above proposition is the tagent/normal bundle pair, where $w_i(\nu_M) = \bar{w}_i(\tau_m)$ for any embedded or immersed smooth manifold M.

After some first consequences of Stiefel-Whitney classes, we calculate Stiefel-Whiney classes for the following three examples.

Example (Tangent bundle of S^n)**.** Suppose S^n is in the standard embedding in $\mathbb{R}^{n+1}$. The normal bundle is trivial since we can find a nowhere vanishing outward pointing vector field. Thus $w(\nu_{S^n}) = 1$. By the previous proposition, we know that $w(\tau_{S^n}) = \bar{w}(\nu_{S^n}) = 1$.

In other words, Stiefel-Whitney Classes cannot be used to distinguish the triviality of τ_{S^n}. In fact, all of them are stably trivial.

Example (Canonical line bundles of $\mathbb{RP}^n$)**.** Observe the inclusion $i : \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^n$. This inclusion can be covered by a bundle map:

$$\begin{tikzcd} E(\gamma_1^1) \arrow[r, hook, "j"] \arrow[d, "\pi"] & E(\gamma_n^1) \arrow[d, "\pi"] \\ \mathbb{RP}^1 \arrow[r, hook, "i"] & \mathbb{RP}^n \end{tikzcd}$$

Thus by naturality, we know that $w_1(\gamma_1^1) = i^*(w_1(\gamma_n^1))$ and is nontrivial by the non-triviality axiom. Thus $w_1(\gamma_n^1)$ cannot be trivial.

Now recall that the the cohomology ring of $\mathbb{RP}^n$ is $H^*(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/(\alpha^{n+1})$ with $|\alpha| = 1$.

Thus $w(\gamma_n^1) = 1 + \alpha$. The higher terms vanish following axiom 1 since γ_n^1 is a line bundle.

Note that by definition γ_n^1 is a sunbundle embedded in the trivial bundle $\underline{\mathbb{R}^{n+1}}$, thus we can calculate the Stiefel-Whitney classes of its orthogonal complement bundle, i.e. $w(\gamma^\perp) = 1 + \alpha + \alpha^2 + ... + \alpha^n$.

Example (Tangent bundles of $\mathbb{RP}^n$)**.** In this example we will see that $\mathbb{RP}^n$ is parallelizable only if $n+1$ is a power of 2. In fact $\mathbb{RP}^n$ is parallelizable iff $n = 1, 3, 7$, but that is out of the scope of Stiefel-Whitney classes. To prove this statement, we start by calculating the Stiefel-Whitney clasees for $\tau_{\mathbb{RP}^n}$. The calculation comes from the following two lemmas.

Lemma 9. $\tau_{\mathbb{RP}^n} \cong Hom(\gamma_n^1, \gamma^\perp)$.

Proof. First fix a line $L \subset \mathbb{R}^{n+1}$ passing through the origin. L intersects the sphere S^n at two points $\{\pm x\}$. We also denote the orthogonal complement plane as $L^\perp$.

Then observe the the quotient map $f : S^n \to \mathbb{RP}^n$ taking $x \mapsto [\pm x]$. Its differential $DF : TS^n \to T\mathbb{RP}^n$ takes (x, v) and $(-x, -v)$ to the same point..

Thus $T\mathbb{RP}^n$ can be identified with pairs $\{(x, v), (-x, -v)\}$ such that $\begin{cases} x \cdot x = 1 \\ x \cdot v = 0 \end{cases}$. However, each such pair detaermines and is determined by a linear mapping $l : L \to L^\perp$ generated by $x \mapsto v$.

Thus we know that $T_{[\pm x]}\mathbb{RP}^n \cong Hom(L, L^\perp)$.

Passing this isomorphism onto all fibres, we have $\tau_{\mathbb{RP}^n} \cong Hom(\gamma_n^1, \gamma^\perp)$. ☐

Lemma 10. $\tau_{\mathbb{RP}^n} \oplus \underline{\mathbb{R}^1} \cong \gamma_n^1 \oplus \gamma_n^1 \oplus ... \oplus \gamma_n^1 \cong \bigoplus_{n+1} \gamma_n^1$

Proof. Note that $Hom(\gamma_n^1, \gamma_n^1) \cong \underline{\mathbb{R}^1}$. Indeed, this bundle is a trivial line bundle since its fibres are 1×1 matrices (i.e. rank 1) and it has a nowhere vanishing section $x \mapsto id$.

Thus we have

$$\begin{aligned} \tau_{\mathbb{RP}^n} \oplus \underline{\mathbb{R}^1} &\cong Hom(\gamma_n^1, \gamma^\perp) \oplus Hom(\gamma_n^1, \gamma_n^1) \\ &\cong Hom(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) \\ &\cong Hom(\gamma_n^1, \underline{\mathbb{R}^{n+1}}) \\ &\cong \bigoplus_{n+1} Hom(\gamma_n^1, \underline{\mathbb{R}^1}) \end{aligned}$$

However, $Hom(\gamma_n^1, \underline{\mathbb{R}^1}) \cong \gamma_n^1$, and we conclude that $\tau_{\mathbb{RP}^n} \oplus \underline{\mathbb{R}^1} \cong \bigoplus_{n+1} \gamma_n^1$ ☐

With the above lemma, we can now calculate the Stiefel-Whitney classes for $\mathbb{RP}^n$:

$$\begin{aligned} w(\mathbb{RP}^n) &= (1 + \alpha)^{n+1} \\ &= 1 + \binom{n+1}{1} \alpha + ... + \binom{n+1}{n} \alpha^n \end{aligned}$$

Note that the higher terms vanish following from axiom 1 (too high rank).

The coefficients of the Stiefel-Whitney classes of the first few projective spaces can be read off the following mod 2 version of the Pascal triangle.

$$
\begin{array}{lccccccccccccccccc}
 & & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & & 1 & & 1 & & & & & & & \\
\mathbb{RP}^1: & & & & & & & 1 & & 0 & & 1 & & & & & & \\
\mathbb{RP}^2: & & & & & & 1 & & 1 & & 1 & & 1 & & & & & \\
\mathbb{RP}^3: & & & & & 1 & & 0 & & 0 & & 0 & & 1 & & & & \\
\mathbb{RP}^4: & & & & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & & \\
\mathbb{RP}^5: & & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & \\
\mathbb{RP}^6: & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\mathbb{RP}^7: & 1 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 1
\end{array}
$$

With the above calculation, we can study the parallelizability of $\mathbb{RP}^n$ with the following corollary.

Corollary 11. *$w(\mathbb{RP}^n) = 1$ iff $n+1$ is a power of 2.*

Proof. First prove ($\Leftarrow$).

Note that $(a+b)^2 \equiv a^2 + b^2 \pmod 2$, thus we have $(1+\alpha)^{2^r} \equiv 1 + \alpha^{2^r} \pmod 2$ by induction.

Thus if $n+1 = 2^r$, we have $w(\mathbb{RP}^n) = (1+\alpha)^{n+1} = 1 + \alpha^{n+1} = 1$.

Then prove ($\Rightarrow$).

If $n+1 = 2^r \cdot m$ for some odd $m > 1$, we have $w(\mathbb{RP}^n) = (1+\alpha)^{n+1} = (1+\alpha^{2^r})^m = 1 + m \cdot \alpha^{2^r} + ... \neq 1$. $\square$

4. Applications of Stiefel-Whitney Classes

In this section, we will discuss two applications of Stiefel-Whitney classes: existence of division algebras, and immersion of smooth manifolds inside Euclidean space $\mathbb{R}^n$.

4.1. **Existence of division algebras.** We start with the definition of a division algebra.

Definition 8 (division algebra)**.** A division algebra is an algebra D over a field (vector space with bilinear multiplication) such that $\forall \alpha \in D, \beta \neq 0 \in D$, exist unique $x \in D$ such that $\alpha = \beta x$ and unique $y \in D$ such that $\alpha = y\beta$.

Note that the bilinear multiplication in the definition is allowed to be non-associative.

Some examples of division algebras are the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$ (also called Cayley numbers). In fact these are all the examples of division algebras due to the following theorem.

Theorem 12 (Stiefel)**.** *If there exists a division algebra $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, then $\mathbb{RP}^{n-1}$ is parallelizable.*

Proof. Let $\{e_i\}_1^n$ be the standard basis for $\mathbb{R}^n$, then the map $y \mapsto p(y, e_1)$ defines an isomorphism. This follows directly from the definition of division algebras.

Thus we can define a set of n linear transformations $\{v_i\}_1^n$ by $v_i : \mathbb{R}^n \to \mathbb{R}^n$ such that $v_i(p(y, e_1)) = p(y, e_i)$.

Note that for $x \neq 0$, the set $\{v_i(x)\}_1^n$ actually defines a basis for $\mathbb{R}^n$ (this also follows from the definition of division algebra). This means that $v_2, ..., v_n$ give rise to $n-1$ linear independent sections of the vector bundle $Hom(\gamma_{n-1}^1, \gamma^\perp) \cong \tau_{\mathbb{RP}^{n-1}}$.

Thus $\tau_{\mathbb{RP}^{n-1}}$ is trivial. $\square$

4.2. **Immersion into Euclidean spaces.** The question we want to discuss is "can an n dimensional manifold M^n be immersed in $\mathbb{R}^{n+k}$?"

We start by recalling the definition of smooth immersions. An immersion is a smooth map whose differential is injective, or in short, an "embedding but with self-intersections". The consequence is we can talk about normal bundles of the tangent space.

Suppose M^n can be immersed into $\mathbb{R}^{n+k}$. Then τ_M is a rank n bundle, and ν_M is a rank k bundle. Thus $\overline{w_i}(\tau_M) = w_i(\nu_M) = 0$ for all $i > k$.

Note that $\overline{w_i}(\tau_M) = 0$ for all $i > k$ is a necessary condition for immersion, but sometimes it's far from being sufficient.

Example ($\mathbb{RP}^9$)**.** Recall that $w(\mathbb{RP}^9) = (1+\alpha)^{10} = 1 + \alpha^2 + \alpha^8$. If we calculate its inverse, we have $\overline{w}(\mathbb{RP}^9) = 1 + \alpha^2 + \alpha^4 + \alpha^6$. This means k should at least be 6, and $\mathbb{RP}^9$ cannot be immersed in $\mathbb{R}^{14}$ or lower dimensions.

Example ($\mathbb{RP}^{2^r}$)**.** Supposed $n = 2^r$. Then $w(\mathbb{RP}^{2^r}) = (1+\alpha)^{2^r+1} = 1 + \alpha + \alpha^{2^r}$. Its inverse is $\overline{w}(\mathbb{RP}^{2^r}) = 1 + \alpha + \alpha^2 + ... + \alpha^{2^r-1}$. This means k should at least be $2^r - 1$, and $\mathbb{RP}^{2^r}$ cannot be immersed in $\mathbb{R}^{2^{r+1}-2}$ or lower dimensions.

However, Whitney's Theorem provides a sufficient condition for immersion:

Theorem 13 (Whitney's Theorem)**.** *Every n dimensional smooth manifold with $n > 1$ can be immersed in $\mathbb{R}^{2n-1}$.*

Compare this with the necessary condition above, what we have is when n is a power of 2, $\mathbb{RP}^n$ can be immersed in $\mathbb{R}^{2n-1}$, and is actually the best estimate.

References

[1] John Milnor and James Stasheff, *Characteristic Classes*, 1974.
[2] Allen Hatcher, *Vector bundles and K-theory*, 2003.