A NOTE ON CONNECTIONS AND CURVATURES

JONAH EPSTEIN
YIKAI TENG

1. Overview

This is a lecture note for the 10^{th} talk of the seminar "Complex Geometry", and we will focus on the definitions of connections and curvatures, along with their compatibility with various structures on (complex) vector bundles, and their connections to sheaf cohomology.

For the first part of this note (before Christmas), we focus on basic definitions and properties of connections and curvatures. We will discuss their local presentations, and see that a curvature acts as the obstruction of a connection to be a differential. Finally if time permits, we will talk about constructions and calculations of induced connections and curvatures using bundle operations.

For the second part of this note (after Christmas), we will discuss their relation to complex geometry. We will discuss connections (as well as associated curvatures) compatible with Hermiantian structures and holomorphic structures. The connection compatible with both is called Chern connection. Finally we will define a more restrictive connection called holomorphic connection, the existence of which depends on a Čech cohomology class called Atiyah class. Conversely, we shall see that the Atiyah class can in turn be defined as the curvature of Chern connection.

Note that this note does **NOT** include knowledge about second fundamental form and positivity of forms and curvatures.

2. Definitions and basic properties

Exterior differentials are generally not defined for the sheaf of sections of a general complex vector bundle. A substitute for the differential is called "connection". A connection, sometimes also called covariant derivative, is linear and satisfies Leibniz rule, but unfortunately its square is generally not zero. The obstruction of a connection becoming a differential is called "curvature".

Before we start with the rigorous definition of a connection, we fix some notations. M is a real (smooth) manifold, and $\pi : E \to M$ is a complex vector bundle. Moreover, denote $\mathcal{A}^i(E)$ for the sheaf of bundle valued i-forms.

Definition 1 (Connection)**.** A connection on a complex vector bundle E is a $\mathbb{C}$-linear sheaf homomorphism $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ that satisfies the **Leibniz rule**: $\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$.

A section is called "flat" or "parallel" or "constant" if we have $\nabla s = 0$.

The curvature of a connection is defined as the obstruction of the connection actually becoming a differential. To define the curvature rigorously, we need to extend the definition of the connection a little bit:

There is a natural extension of a connection: $\nabla : \mathcal{A}^k(E) \to \mathcal{A}^{k+1}(E)$ satisfying the generalized Leibniz rule: $\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla(s)$.

With this natural extension, we can now define curvatures.

Definition 2 (Curvature)**.** The curvature F_∇ of a connection ∇ is the composite $F_\nabla := \nabla \circ \nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E) \to \mathcal{A}^2(E)$.

Example. 1) On the trivial bundle of a smooth manifold, the usual exterior differential is the easiest example of a connection. In this case, the curvature vanishes since the exterior differential is actually a differential operator.

Date: December, 2021.

2) On the tangent bundle of a Riemannian manifold, the Levi-Civita connection is the unique connection that is torsion-free and preserves Riemannian metric. In this case, the associated curvature is the Riemann curvature tensor from differential geometry.

Our next goal is to study local presentations for connections and curvatures. To do so, we start with the following two lemmas:

Proposition 1. *For two connections ∇ and ∇' on E, their difference $\nabla - \nabla'$ is $\mathcal{A}^0_M$-linear and thus can be seen as an element in $\mathcal{A}^1(M, End(E))$. Conversely, if ∇ is a connection, and there is an element $a \in \mathcal{A}^1(M, End(E))$, then $\nabla + a$ is also a connection.*

Proof. Linearity is direct: $(\nabla - \nabla')(f \cdot s) = f \cdot (\nabla - \nabla')(s)$.

For the converse, we first note that an element $a \in \mathcal{A}^1(M, End(E))$ acts on $\mathcal{A}^0(E)$ by multiplication on the form part and evaluation $End(E) \times E \to E$ on the bundle part. Then we know that $(\nabla + a)(f \cdot s) = \nabla(f \cdot s) + a(f \cdot s) = d(f) \otimes s + f \cdot \nabla(s) + fa(s) = d(f) \otimes s + f \cdot (\nabla + a)(s)$. Thus $\nabla + a$ satisfies the Leibniz rule, and thus is a connection. $\square$

Proposition 2. *The curvature homomorphism $F_\nabla : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$ is $\mathcal{A}^0-$linear, and thus can be seen as an element in $\mathcal{A}^2(M, End(E))$.*

Proof. $F_\nabla(f \cdot s) = \nabla(\nabla(f \cdot s)) = \nabla(df \otimes s + f \cdot \nabla(s)) = d^2(f) \otimes s - df \wedge \nabla(s) + df \wedge \nabla(s) + f \cdot \nabla(\nabla(s)) = f \cdot F_\nabla(s)$. $\square$

With these two lemmas, we can give a local presentation for any connection and its associated curvature on the bundle $E \to M$.

If E is the trivial product bundle $M \times \mathbb{C}^r$, then the sheaf $\mathcal{A}^\bullet(E) = \bigoplus_{i=1}^r \mathcal{A}^k_M$. Then we can define the "trivial connection" $d : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ by applying the usual exterior differential on each component. Any other connection on E can be locally written as $\nabla = d + A$, where $A \in \mathcal{A}^1(M, End(E))$ is a matrix valued 1-form in this trivial case. Under this setting, $F_\nabla(s) = (d+A)((d+A)(s)) = (d+A)(d(s)+A\cdot s) = d^2(s)+d(A\cdot s)+A\cdot d(s)+A(A(s)) = d(A)(s)+(A\wedge A)(s) = (d(A)+A\wedge A)(s)$. Thus $F_\nabla = d(A)+A\wedge A$.

Lemma 3. *More generally, $F_{\nabla+a} = F_\nabla + \nabla(a) + a \wedge a$, where $a \wedge a \in \mathcal{A}^2(M, End(E))$ is done by wedge product on the form part and composition on the endomorphism part.*

Proof. Exactly as before, left as an exercise. $\square$

Suppose now E is an arbitrary vector bundle, endowed with a trivialization $\psi : E|_U \to U \times \mathbb{C}^r$. We can, as before, write any connection in the form $\nabla = d + A$ (Explicitly, this means $\nabla = \psi^{-1} \circ (d+A) \circ \psi$.) If now the trivialization is changed by $\phi : U \to GL_r(\mathbb{C})$, then the connection can also be written as $\nabla = (\psi')^{-1} \circ (d + A') \circ \phi'$, where $\psi' = \phi \circ \psi$ is the new trivialization, and A' is the matrix-valued 1-form $\phi^{-1}d(\phi) + \phi^{-1}A\phi$.

2.1. Induced Connections and Curvatures.

(1) (Whitney sum of bundles) For connections ∇_1 on E_1, ∇_2 on E_2, there is a natural connection ∇ on $E_1 \oplus E_2$ such that $\nabla(s_1 \oplus s_2) = \nabla_1(s_1) \oplus \nabla_2(s_2)$. In this case, $F_\nabla = F_{\nabla_1} \oplus F_{\nabla_2}$.
(2) (Tensor of bundles) For connections ∇_1 on E_1, ∇_2 on E_2, there is a natural connection ∇ on $E_1 \otimes E_2$ such that $\nabla(s_1 \otimes s_2) = \nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla_2(s_2)$. In this case, $F_\nabla = F_{\nabla_1} \otimes 1 + 1 \otimes F_{\nabla_2}$.
(3) (Hom and dual construction of bundles) For connections ∇_1 on E_1, ∇_2 on E_2, there is a natural connection ∇ on $Hom(E_1, E_2)$ such that $\nabla(f)(s_1) = \nabla_2(f(s_1)) - f(\nabla_1(s_1))$.
    A **special case** for this is the dual bundle. $\nabla^*(f)(s) = d(f(s)) - f(\nabla(s))$. In this case, $F_{\nabla^*} = -F^t_*$.
(4) (Pullback of bundles) For a map $f : M \to N$ and a connection on a bundle over N locally written as $d + A$, then the connection can be pulled back to a local connection on the pullback bundle over M locally written as $f^*\nabla = d + f^*A$. In this case, $F_{f^*\nabla} = f^*F_\nabla$.

At last we finish this section with a remarkable result on curvatures as an application of induced bundles: the Bianchi identity. We already know that $F_\nabla \in \mathcal{A}^2(M, End(E))$. We can apply the induced connection ∇ (a slight abuse of notations) on the bundle $End(E)$, and we have:

Theorem 4 (Bianchi Identity)**.** $\nabla(F_\nabla) = 0 \in \mathcal{A}^3(M, End(E))$.

Proof. We know that the induced connection $\mathcal{A}^0(M, End(E)) \to \mathcal{A}^1(M, End(E))$ is $\nabla(f(-)) - f(\nabla(-))$. We additionally note that $\nabla^2 : \mathcal{A}^k \to \mathcal{A}^{k+2}$ for any bundle is given by taking the exterior product with the form part of the curvature and apply the endomorphsim to the bundle. Thus in this case, we have $\nabla(F_\nabla) : \mathcal{A}^2(M, End(E)) \to \mathcal{A}^3(M, End(E))$ such that $\nabla(F_\nabla(s)) - F_\nabla(\nabla(s))$.

Thus we have $\nabla(F_\nabla)(s) = \nabla(F_\nabla(s)) - F_\nabla(\nabla(s)) = \nabla(\nabla^2(s)) - \nabla^2(\nabla(s)) = 0$. □

If we write this identity in local coordinates, the Bianchi identity becomes $dF_\nabla = F_\nabla \wedge A - A \wedge F_\nabla$. The explicit calculation is left as an exercise.

Example. Let (M, g) be a Riemannian manifold with metric g. There is a unique connection ∇ on TM which is torsion free, e.g. satisfies $\nabla_X(Y) - \nabla_Y(X) = [X, Y]$, for vector fields $X, Y \in \mathcal{A}^0(TM)$. Here $\nabla(X)$ is a 1-form, so we can pair it with a vector field Y which we denote by $\nabla_Y(X)$. Additionally, it has to be compatible with the metric, i.e. $dg(X, Y) = g(\nabla X, Y) + g(X, \nabla Y)$. This connection is called the Levi-Civita connection and is determined by

$$g(\nabla_X Y, Z) = \frac{1}{2}(X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(Y, [X, Z]) - g(Z, [Y, X]) + g(X, [Z, Y])).$$

In local coordinates (e_i) we define the Christoffel-symbols by $\nabla_{\partial_i} e_j = \Gamma^k_{ij} e_k$. The Levi-Civita connection is given by

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l (g)^{-1}_{lk}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

Let us describe the connection 1-form and the curvature of $\nabla = d + A$. we obtain that $A = \sum_i \Gamma_i dx^i$ with the matrices $(\Gamma_i)^k_j = \Gamma^k_{ji}$. The curvature calculates to

$$F_\nabla = dA + A\wedge A = dA + \frac{1}{2}[A, A] = d\Gamma_j \wedge dx^j + \frac{1}{2}(\Gamma_i\Gamma_j - \Gamma_j\Gamma_i)dx^i \wedge dx^j = \frac{1}{2}(\partial_i\Gamma_j - \partial_j\Gamma_i + \Gamma_i\Gamma_j - \Gamma_j\Gamma_i)dx^i \wedge dx^j.$$

The last expression resembles the well known Riemannian curvature tensor,

$$F_\nabla = R(\partial_i, \partial_j)dx^i \wedge dx^j.$$

3. Connections in Complex geometry

Within this section, we will talk about the compatibility of connections and curvatures to various geometric structures. We will start with compatibility with Hermitian structures and holomorphic structures of a bundle. Combining the two structures, we have a unique connection called the Chern connection. In a way, this is the complex (hermitian) analogy for the two axioms of Levi-Civita connections.

For the second part of this section, we will talk about a complex holomorphic version of connections, called holomorphic sections. Unlike Chern connections, holomorphic connections do not always exist, the existence of which depends on a specific cohomology class called the Atiyah class.

3.1. **Hermitian Metrics and Connections.** Recall that a hermitian metric on a complex vector bundle $E \to M$ over a smooth manifold M is a bundle map (linear restricting to each fibre) $h : E \otimes \overline{E} \to \mathbb{C}$ which fibrewise restricts to a non-degenerate hermitian form, i.e.

$$h(ax, y) = ah(x, y) = a\overline{h(y, x)} = h(x, \overline{a}y),$$

for $x, y \in E_p$, $p \in M$ and $a \in \mathbb{C}$.

Note that for bundle valued forms $\alpha_1 \otimes s_1$ and $\alpha_2 \otimes s_2$, the hermitian metric is h is defined as

$$h(\alpha_1 \otimes s_1, \alpha_2 \otimes s_2) = (\alpha_1 \wedge \overline{\alpha_2})h(s_1, s_2).$$

With the definition of hermitian metrics, in analogy to Riemannian geometry it is natural to ask for a compatibility condition between a connection ∇ on E and h.

Definition 3 (Hermitian connection)**.** A connection ∇ on E is compatible with a hermitian metric h if for any sections $s_1, s_2 \in \mathcal{A}^0(E)$

$$d(h(s_1, s_2)) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2).$$

An alternate definition of hermitian connections is given by the following proposition.

Proposition 5. *A connection ∇ is compatible with h if and only if $\nabla(h) = 0$. Note that here ∇ is the induced connection on the bundle $(E \otimes \overline{E})^*$.*

Proof. By the formula on induced bundles (dual bundle and tensor of bundles), we have

$$\begin{aligned}\nabla(h)(s_1 \otimes s_2) &= d(h(s_1 \otimes s_2)) - h(\nabla(s_1 \otimes s_2)) \\ &= d(h(s_1 \otimes s_2)) - (h(\nabla(s_1) \otimes s_2) + h(s_1 \otimes \nabla(s_2)))\end{aligned}$$

Thus we have that $d(h(s_1, s_2)) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$ iff $\nabla(h) = 0$.

□

Note that given two hermitian forms ∇ and ∇', their difference 1-form $a \in \mathcal{A}^1(M, End(E))$ satisfies $h(a(s_1), s_2) + h(s_1, a(s_2)) = 0$. Therefore, the space of hermitian connections is modelled by the affine space of all forms satisfiyng this equation. We denote this subsheaf as $\mathcal{A}^1(M, End(E, h))$. It is worth to note that this subsheaf is merely a space of real forms.

Changing the degree of the forms, we can similarly define $\mathcal{A}^k(M, End(E, h))$. Then we have the following immediate consequence.

Proposition 6. *The curvature of a hermitian connection satisfies $F_\nabla \in \mathcal{A}^2(M, End(E, h))$, in other words*

$$h(F_\nabla(s_1), s_2) + h(s_1, F_\nabla(s_2)) = 0.$$

Proof. If s_1, s_2 are forms, $s_i \in \mathcal{A}^{k_i}(E)$, then note that

$$dh(s_1, s_2) = h(\nabla s_1, s_2) + (-1)^{k_1} h(s_1, \nabla s_2).$$

One obtains for $s_1, s_2 \in \mathcal{A}^0(E)$

$$\begin{aligned}dh(\nabla s_1, s_2) &= h(F_\nabla s_1, s_2) - h(\nabla s_1, \nabla s_2) \\ dh(s_1, \nabla s_2) &= h(s_1, F_\nabla s_2) + h(\nabla s_1, \nabla s_2).\end{aligned}$$

Summing both equations, we have $h(F_\nabla(s_1), s_2) + h(s_1, F_\nabla(s_2)) = d(d(h(s_1, s_2))) = 0$, which gives the desired result.

□

3.2. **Holomorphic Structures and Connections.** Assume that $E \to X$ is a holomorphic vector bundle. Recall that there is a differential

$$\overline{\partial}_E : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$$

which satisfies the Leibniz rule $\overline{\partial}_E(f\alpha) = \overline{\partial}(f) \wedge \alpha + f\overline{\partial}_E(\alpha)$. Let $\alpha \in \mathcal{A}^{p,q}(E)$, locally given as $\alpha = \sum \alpha_i \otimes s_i$. Then $\overline{\partial}_E \alpha := \sum \overline{\partial}(\alpha_i) \otimes s_i$. This expression is well defined because the transition functions are holomorphic. Since $\mathcal{A}^1(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$, a connection splits as $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$.

Definition 4. A connection is called compatible with the holomorphic structure if $\nabla^{0,1} = \overline{\partial}_E$.

The connections compatible with the complex structure form an affine space over $\mathcal{A}^{1,0}(X, End(E))$.

Proposition 7. *Let ∇ be compatible with the holomorphic structure. Then $F_\nabla \in \mathcal{A}^{2,0} \oplus \mathcal{A}^{1,1}(X, End(E))$*

Proof. We compute

$$\nabla^2 = (\nabla^{1,0})^2 + \nabla^{1,0} \circ \overline{\partial}_E + \overline{\partial}_E \circ \nabla^{1,0} + (\overline{\partial}_E)^2.$$

This is evidently in the claimed space since $(\overline{\partial}_E)^2 = 0$.

□

3.3. **The Chern Connection.** Given a holomorphic vector bundle with hermitian metric (E, h) there is a natural connection called Chern connection. As mentioned before, the Chern connection is an analogy or generalization of the Levi-Civita connection. In fact, in the case of tangent bundle over a Kahler manifold, the two notions coincide.

We begin this section with existence and uniqueness of Chern connections.

Theorem 8. *There is a unique connection ∇ which is both compatible with the hermitian metric and the holomorphic structure. It is called Chern Connection.*

Proof. We first proof uniqueness which can be evaluated locally. Locally, we assume that $E = X \times \mathbb{C}^r$, $\nabla = d + A$, with $A = (a_{ij})$ and $H(x) = (h_{ij}(x))$. Additionally let e_i be a local frame. Let us spell out compatibility with the hermitian structure:

$$dh(e_i, e_j) = h(\sum a_{ki}e_k, e_j) + h(e_i, \sum a_{lj}e_l) = \sum a_{ki}h_{kj} + \sum \overline{a_{lj}}h_{il},$$

or in matrix form $dH = A^t H + H\overline{A}$.

Compatibility with holomorphic structure implies $A \in \mathcal{A}^{1,0}(End(E))$. By considering composition in bidegrees we conclude that $\overline{\partial} H = H\overline{A}$. This gives $A = \overline{H}^{-1}\partial\overline{H}$. Therefore, A is uniquely determined by H. For existence we locally define A by the given formula and use uniqueness to show that it can globally glue to a connection.

□

Proposition 9. *The curvature of the Chern connection satisfies $F_\nabla \in \mathcal{A}^{1,1}_{\mathbb{R}}(X, End(E, h))$*

Proof. We combine the results of Propositions 6 and 7. Note that by proposition 7, $h(F_\nabla(s_1), s_2) \in \mathcal{A}^{2,0} \oplus \mathcal{A}^{1,1}$ and $h(s_1, F_\nabla(s_2)) \in \mathcal{A}^{0,2} \oplus \mathcal{A}^{1,1}$. By Proposition 6 they are equal and therefore in $\mathcal{A}^{1,1}$. □

This decomposition implies that locally $F_\nabla = \overline{\partial}(\overline{H}^{-1}\partial\overline{H})$. Globally by the Bianchi identity $\overline{\partial}(F_\nabla) = (\nabla(F_\nabla))^{1,2} = 0$, so F_∇ defines a cohomology class $[F_\nabla] \in H^1(X, \Omega_X \otimes End(E))$. Here we abuse notations a little bit and $End(E)$ here denotes the sheaf of holomorphic sections of the bundle.

Example.
- Consider the tangent bundle over a complex torus $\mathbb{C}^n/\Gamma$, equipped by the standard flat metric. Then the Chern connection is just the usual exterior differential.
- Let E be a line bundle then a hermitian connection is just a positive real valued function H. The Chern connection is given by $A = H^{-1}\partial(H) = \partial \log(H)$. The curvature is $F = \overline{\partial}(\partial \log(H))$.
- Consider the line bundle $\mathcal{O}(1)$ over $\mathbb{P}^n$ with the standard linear coordinates $z_0, ..., z_n$. There is a natural hermitian metric on this line bundle given on the standard local trivializations by

  $$h = (1 + \sum |w_i|^2)^{-1}.$$

  Then the associated curvature is given by $\frac{i}{2\pi}F = \omega_{FS}$, the Fubini Study metric. The cohomology class it represents will later turn out to be the first Chern class.

3.4. **Holomorphic Connections.** At last we will talk about holomorphic connections.

Definition 5 (Holomorphic connection)**.** A holomorphic connection on a holomorphic vector bundle E over a complex manifold X is a $\mathbb{C}$-linear sheaf homomorphism $D : E \to \Omega_X \otimes E$ that satisfies $D(f \cdot s) = \partial f \otimes s + f \cdot D(s)$. Here we abuse notations and denote the sheaf of holomorphic sections also by E.

Warning. A holomorphic connection is **NOT** the same as a connection compatible to the holomorphic structure (despite they have very similar names). Later we shall see that the holomorphic connection exists only on very special and restrictive bundles.

What we said earlier about "ordinary" connection can be extended to holomorphic connections, that is:

(1) If D and D' are two holomorphic connections on E, then their difference $D - D'$ is a holomorphic section of the bundle $\Omega_X \otimes End(E)$.
(2) Any holomorphic connection can be locally written as $\partial + A$, where A is a holomorphic section of $\Omega_X \otimes End(E)$.

3.4.1. *Existence of holomorphic connections.* As is mentioned in the warning above, the existence of holomorphic connections is very restrictive (on the bundle). For the rest of this note, we will discuss the obstruction of such existence.

Definition 6 (Atiyah class)**.** Let $E \to X$ be a holomorphic bundle over a complex manifold, and $X = \bigcup_{i\in I} U_i$ be an open covering, with $\psi_i : E|_{U_i} \cong U_i \times \mathbb{C}^r$ being the corresponding local trivializations. The Atiyah class of the vector bundle E is given by the Čech cocycle:

$$A(E) := \{U_{ij}, \psi_j^{-1} \circ (\psi_{ij}^{-1} d\psi_{ij}) \circ \psi_j\} \in H^1(X, \Omega_X \otimes End(E))$$

.

The following theorem then states that the Atiyah class acts as the obstruction of the existence of holomorphic connections

Theorem 10. *The bundle E admits a holomorphic connection iff its Atiyah class $A(E)$ is trivial.*

Proof. Note that since ψ_{ij}'s are holomorphic, $d\psi_{ij} = \partial\psi_{ij}$.

From before, we know that on each $U_i \times \mathbb{C}^r$, a holomorphic connection takes the form of $\partial + A_i$. These local connections glue into a global one if and only if on each U_{ij}, we have:

$$\psi_i^{-1} \circ (\partial + A_i) \circ \psi_i = \psi_j^{-1} \circ (\partial + A_j) \circ \psi_j$$

.

After rearranging terms, we have

$$\psi_i^{-1} \circ \partial \circ \psi_i - \psi_j^{-1} \circ \partial \circ \psi_j = \psi_j^{-1} A_j \psi_j - \psi_j^{-i} A_i \psi_i$$

.

Here the left hand side:

$$\begin{aligned} LHS &= \psi_j^{-1} \circ (\psi_{ij}^{-1} \circ \partial \circ \psi_{ij}) \circ \psi_j - \psi_j^{-1} \circ \partial \circ \psi_j \\ &= \psi_j^{-1} \circ (\psi_{ij}^{-1} \circ \partial \circ \psi_{ij} - \partial) \circ \psi_j \\ &= \psi_j^{-1} \circ (\psi_{ij}^{-1} \partial(\psi_{ij})) \circ \psi_j \\ &= \psi_j^{-1} \circ (\psi_{ij}^{-1} d\psi_{ij}) \circ \psi_j \end{aligned}$$

.

Also, the right hand side is the boundary of $\{\psi_i^{-1} A_i \psi_i \in \Gamma(U_i, \Omega \otimes End(E))\}$.

Thus we conclude that $A(E) = 0$ if and only if the local connections glue into a global holomorphic connection.

□

3.4.2. *Relations to Chern connections.* As it turns out, the Atiyah class can be seen as the curvature of a Chern connection. As already mentioned, F_∇ of the Chern connection defines a cohomology class $[F_\nabla] \in H^1(X, \Omega_X \otimes End(E))$.

Theorem 11. *The curvature of the Chern connection coincides with the Atiyah class, i.e. $[F_\nabla] = A(E)$.*

Before we prove this result, it is worth noticing that the Chern connection depends on the choice of the hermitian structure, yet the Atiyah class depends only on the bundle data. This shows that the cohomology class a Chern connection represents is actually independent of the chosen hermitian structure.

Proof. We need to compare Čech cohomology with Dolbeault cohomology. Recall that this is done calculating the spectral sequence of the bicomplex below:

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\check{C}^0(\mathcal{A}^{1,2}) & \xrightarrow{\delta} & \check{C}^1(\mathcal{A}^{1,2}) & \xrightarrow{\delta} & \check{C}^2(\mathcal{A}^{1,2}) & \longrightarrow & \cdots \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \\
\check{C}^0(\mathcal{A}^{1,1}) & \xrightarrow{\delta} & \check{C}^1(\mathcal{A}^{1,1}) & \xrightarrow{\delta} & \check{C}^2(\mathcal{A}^{1,1}) & \longrightarrow & \cdots \\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \\
\check{C}^0(\mathcal{A}^{1,0}) & \xrightarrow{\delta} & \check{C}^1(\mathcal{A}^{1,0}) & \xrightarrow{\delta} & \check{C}^2(\mathcal{A}^{1,0}) & \longrightarrow & \cdots
\end{array}$$

If we calculate the horizontal differential first, we obtain

$$\mathcal{A}^{1,0} \xrightarrow{\overline{\partial}} \mathcal{A}^{1,1} \xrightarrow{\overline{\partial}} \mathcal{A}^{1,2} \xrightarrow{\overline{\partial}} \ldots,$$

so the spectral sequence collapses on the E^2 page, where it gives the Dolbeault cohomology. If we take the vertical differential first, we get

$$\check{C}^0(\Omega^1) \xrightarrow{\delta} \check{C}^1(\Omega^1) \xrightarrow{\delta} \check{C}^2(\Omega^1) \xrightarrow{\delta} \ldots,$$

so one obtains the Čech cohomology of the $\Omega_X \otimes End(E)$-sheaf. Both collapse to the total cohomology of the total complex. We claim that the difference of the Atiyah class and the curvature class in the total complex is $(d = \overline{\partial} \pm \delta)$-exact. Indeed, the corresponding element to F_∇ is

$$\{U_i, \psi_i^{-1} \circ (\overline{\partial}(\overline{H}_i^{-1}\partial\overline{H}_i) \circ \psi_i\} = \overline{\partial}\{U_i, \psi_i^{-1} \circ ((\overline{H}_i^{-1}\partial\overline{H}_i) \circ \psi_i\}.$$

Denote $B := \{U_i, \psi_i^{-1} \circ ((\overline{H}_i^{-1}\partial\overline{H}_i) \circ \psi_i\}$.

Thus, we need to show that $\delta_1 B = A(E)$. By definition

$$\delta_1 B = \{U_{ij}, \psi_j^{-1} \circ (\overline{H}_j^{-1}\partial\overline{H}_j) \circ \psi_j - \psi_i^{-1} \circ (\overline{H}_i^{-1}\partial\overline{H}_i) \circ \psi_i\}$$

After a short manipulation, we see that

$$\overline{H}_j^{-1}\partial\overline{H}_j - \psi_{ij}^{-1} \circ (\overline{H}_i^{-1}\partial\overline{H}_i) \circ \psi_{ij} = \psi_{ij}^{-1}\partial\psi_{ij}$$

has to be satisfied. Note that $\overline{H}_j = \overline{\psi}_{ij}^t \overline{H}_i \psi_{ij}$ and thus

$$\overline{H}_j^{-1}\partial\overline{H}_j = \psi_{ij}^{-1}\overline{H}_i^{-1}(\overline{\psi}_{ij}^t)^{-1}(\overline{\psi}_{ij}^t(\partial\overline{H}_i)\psi_{ij} + \overline{\psi}_{ij}^t\overline{H}_i\partial\psi_{ij}) = \psi_{ij}^{-1}(\overline{H}_i^{-1}\partial\overline{H}_i)\psi_{ij} + \psi_{ij}^{-1}\partial\psi_{ij}.$$

In total we get that

$$dB = (\overline{\partial} - \delta)B = F_\nabla - A(E)$$

and therefore $[F_\nabla] = [A(E)]$.

□

We end this note by a rough conclusion of this section, namely a holomorphic connection on a vector bundle exists if and only if the Atiyah classes vanishes if and only if the bundle is flat. In fact, in the next talk we shall see that the Atiyah class in fact encodes all characteristic classes of E.

References

[1] Daniel Huybrechts, *Complex Geometry: an introduction*, 2004.