

Minimum Snap

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1 Modeling

We assume that we use 5^{th} polynomial to fit a trajectory.

$$p(t) = p_0 + p_1t + p_2t^2 + p_3t^3 + p_4t^4 + p_5t^5$$

There are $m_0, m_1, m_2, \dots, m_N$, $N + 1$ points in total. Denote position, velocity, acceleration, jerk, and snap of the n^{th} trajectory as:

$$\begin{aligned} p_n(t) &= p_{n0} + p_{n1}t + p_{n2}t^2 + p_{n3}t^3 + p_{n4}t^4 + p_{n5}t^5 \\ v_n(t) &= p_{n1} + 2p_{n2}t + 3p_{n3}t^2 + 4p_{n4}t^3 + 5p_{n5}t^4 \\ a_n(t) &= 2p_{n2} + 6p_{n3}t + 12p_{n4}t^2 + 20p_{n5}t^3 \\ j_n(t) &= 6p_{n3} + 24p_{n4}t + 60p_{n5}t^2 \\ s_n(t) &= 24p_{n4} + 120p_{n5}t \end{aligned} \tag{1}$$

Based on 1, we denote the coefficient matrix of position, velocity, acceleration, jerk, and snap as:

$$\begin{aligned} C_p &= [1 \quad t \quad t^2 \quad t^3 \quad t^4 \quad t^5] \\ C_v &= [0 \quad 1 \quad 2t \quad 3t^2 \quad 4t^3 \quad 5t^4] \\ C_a &= [0 \quad 0 \quad 2 \quad 6t \quad 12t^2 \quad 20t^3] \\ C_j &= [0 \quad 0 \quad 0 \quad 6 \quad 24t \quad 60t^2] \\ C_s &= [0 \quad 0 \quad 0 \quad 0 \quad 24 \quad 120t] \end{aligned} \tag{2}$$

, and

$$P_n = [p_{n0} \quad p_{n1} \quad p_{n2} \quad p_{n3} \quad p_{n4} \quad p_{n5}]^T \tag{3}$$

Then we have:

$$\begin{aligned} p_n(t) &= C_p P_n \\ v_n(t) &= C_v P_n \\ a_n(t) &= C_a P_n \\ j_n(t) &= C_j P_n \\ s_n(t) &= C_s P_n \end{aligned} \tag{4}$$

2 Convex Optimization

I choose package 'cvxopt' as the optimizer of this problem, and the standard formation is:

$$x^* = \min_x \frac{1}{2} x^T Q x + P^T x, s.t. Ax = B, Gx \leq h \tag{5}$$

3 Cost Function

Denote the cost function of the n^{th} polynomial as:

$$\begin{aligned}
 J_n &= \int_{T_n}^{T_{n+1}} s_n^2 dt \\
 &= \int_{T_n}^{T_{n+1}} s_n^T s_n dt \\
 &= \int_{T_n}^{T_{n+1}} P_n^T C_s^T C_s P_n dt \\
 &= P_n^T \left[\int_{T_n}^{T_{n+1}} C_s^T C_s dt \right] P_n \\
 &= P_n^T Q_n P_n
 \end{aligned} \tag{6}$$

, the cumulative cost is:

$$J = \sum_{n=0}^{N-1} J_n \tag{7}$$

, and the cost matrix as:

$$Q = \begin{bmatrix} Q_0 & & & \\ & Q_1 & & \\ & & \ddots & \\ & & & Q_{N-1} \end{bmatrix} \tag{8}$$

The P matrix is 0 for this problem.

4 Equality constraints

4.1 Boundary Conditions

Boundary conditions at initial node:

$$\begin{aligned}
 C_p(t_0) P_0 &= m_0 \\
 C_v(t_0) P_0 &= v_0 \\
 C_a(t_0) P_0 &= a_0
 \end{aligned} \tag{9}$$

Boundary conditions at terminal node:

$$\begin{aligned}
 C_p(t_N) P_0 &= m_N \\
 C_v(t_N) P_0 &= v_N \\
 C_a(t_N) P_0 &= a_N
 \end{aligned} \tag{10}$$

4.2 Other Equality constraints

The velocity and acceleration of each node on the two segments should be the same.

$$\begin{aligned}
 C_v(t_1) P_0 &= C_v(t_1) P_1 \\
 C_a(t_1) P_0 &= C_a(t_1) P_1 \\
 C_v(t_2) P_1 &= C_v(t_2) P_2 \\
 C_a(t_2) P_1 &= C_a(t_2) P_2 \\
 &\dots = \dots \\
 C_v(t_{n-1}) P_{n-2} &= C_v(t_{n-1}) P_{n-1} \\
 C_a(t_{n-1}) P_{n-2} &= C_a(t_{n-1}) P_{n-1}
 \end{aligned} \tag{11}$$

5 An example for an optimization problem with five nodes

Q and P matrices are the same as ones as mentioned above. This part I will illustrate how a A and B matrices are generated.

$$A = \begin{bmatrix} C_{pT_0} & 0_{1 \times 6} & 0_{1 \times 6} & 0_{1 \times 6} \\ C_{vT_0} & 0_{1 \times 6} & 0_{1 \times 6} & 0_{1 \times 6} \\ C_{aT_0} & 0_{1 \times 6} & 0_{1 \times 6} & 0_{1 \times 6} \\ C_{pT_1} & 0_{1 \times 6} & 0_{1 \times 6} & 0_{1 \times 6} \\ 0_{1 \times 6} & C_{pT_1} & 0_{1 \times 6} & 0_{1 \times 6} \\ 0_{1 \times 6} & C_{pT_2} & 0_{1 \times 6} & 0_{1 \times 6} \\ 0_{1 \times 6} & 0_{1 \times 6} & C_{pT_2} & 0_{1 \times 6} \\ 0_{1 \times 6} & 0_{1 \times 6} & C_{pT_3} & 0_{1 \times 6} \\ 0_{1 \times 6} & 0_{1 \times 6} & 0_{1 \times 6} & C_{pT_3} \\ 0_{1 \times 6} & 0_{1 \times 6} & 0_{1 \times 6} & C_{pT_4} \\ 0_{1 \times 6} & 0_{1 \times 6} & 0_{1 \times 6} & C_{vT_4} \\ 0_{1 \times 6} & 0_{1 \times 6} & 0_{1 \times 6} & C_{aT_4} \\ C_{vT_1} & -C_{vT_1} & 0_{1 \times 6} & 0_{1 \times 6} \\ C_{aT_1} & -C_{aT_1} & 0_{1 \times 6} & 0_{1 \times 6} \\ 0_{1 \times 6} & C_{vT_2} & -C_{vT_2} & 0_{1 \times 6} \\ 0_{1 \times 6} & C_{aT_2} & -C_{aT_2} & 0_{1 \times 6} \\ 0_{1 \times 6} & 0_{1 \times 6} & C_{vT_3} & -C_{vT_3} \\ 0_{1 \times 6} & 0_{1 \times 6} & C_{aT_3} & -C_{aT_3} \end{bmatrix}_{18 \times 4} \quad (12)$$

$$B = \begin{bmatrix} m_0 \\ v_0 \\ a_0 \\ m_1 \\ m_1 \\ m_2 \\ m_2 \\ m_3 \\ m_3 \\ m_4 \\ v_4 \\ a_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{18 \times 1} \quad (13)$$

$$X = P = \begin{bmatrix} P_0^T \\ P_1^T \\ P_2^T \\ P_3^T \end{bmatrix}_{18 \times 1} \quad (14)$$

Then the problem can be constructed, and we can solve it by using:

```
1 import cvxopt as cp
2
3 sol = cp.solvers.qp(Q, P, None, None, A, B)
4 printf(sol['x'])
```

6 Simulation

