Minimum Snap

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1 Modeling

We assume that we use 5^{th} polynomial to fit a trajectory.

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4 + p_5 t^5$$

There are $m_0, m_1, m_2, \cdots, m_N$, $N+1$ points in total. Denote position, velocity, acceleration, jerk, and snap of the n^{th} trajectory as:

$$\begin{aligned}
p_n(t) &= p_{n0} + p_{n1} t + p_{n2} t^2 + p_{n3} t^3 + p_{n4} t^4 + p_{n5} t^5 \\
v_n(t) &= p_{n1} + 2p_{n2} t + 3p_{n3} t^2 + 4p_{n4} t^3 + 5p_{n5} t^4 \\
a_n(t) &= 2p_{n2} + 6p_{n3} t + 12p_{n4} t^2 + 20p_{n5} t^3 \\
j_n(t) &= 6p_{n3} + 24p_{n4} t + 60p_{n5} t^2 \\
s_n(t) &= 24p_{n4} + 120p_{n5} t
\end{aligned} \tag{1}$$

Based on 1, we denote the coefficient matrix of position, velocity, acceleration, jerk, and snap as:

$$\begin{aligned}
C_p &= \begin{bmatrix} 1 & t & t^2 & t^3 & t^4 & t^5 \end{bmatrix} \\
C_v &= \begin{bmatrix} 0 & 1 & 2t & 3t^2 & 4t^3 & 5t^4 \end{bmatrix} \\
C_a &= \begin{bmatrix} 0 & 0 & 2 & 6t & 12t^2 & 20t^3 \end{bmatrix} \\
C_j &= \begin{bmatrix} 0 & 0 & 0 & 6 & 24t & 60t^2 \end{bmatrix} \\
C_s &= \begin{bmatrix} 0 & 0 & 0 & 0 & 24 & 120t \end{bmatrix}
\end{aligned} \tag{2}$$

, and

$$P_n = \begin{bmatrix} p_{n0} & p_{n1} & p_{n2} & p_{n3} & p_{n4} & p_{n5} \end{bmatrix}^T \tag{3}$$

Then we have:

$$\begin{aligned}
p_n(t) &= C_p P_n \\
v_n(t) &= C_v P_n \\
a_n(t) &= C_a P_n \\
j_n(t) &= C_j P_n \\
s_n(t) &= C_s P_n
\end{aligned} \tag{4}$$

2 Convex Optimization

I choose package 'cvxopt' as the optimizer of this problem, and the standard formation is:

$$x^* = \min_x \frac{1}{2} x^T Q x + P^T x, s.t. Ax = B, Gx \leq h \tag{5}$$

3 Cost Function

Denote the cost function of the n^{th} polynomial as:

$$
\begin{aligned}
J_n &= \int_{T_n}^{T_{n+1}} s_n^2 dt \\
&= \int_{T_n}^{T_{n+1}} s_n^T s_n dt \\
&= \int_{T_n}^{T_{n+1}} P_n^T C_s^T C_s P_n dt \\
&= P_n^T \left[\int_{T_n}^{T_{n+1}} C_s^T C_s dt \right] P_n \\
&= P_n^T Q_n P_n
\end{aligned} \tag{6}
$$

, the cumulative cost is:

$$
J = \sum_{n=0}^{N-1} J_n \tag{7}
$$

, and the cost matrix as:

$$
Q = \begin{bmatrix} Q_0 & & & \\ & Q_1 & & \\ & & \ddots & \\ & & & Q_4 \end{bmatrix} \tag{8}
$$

The P matrix is 0 for this problem.

4 Equality constraints

4.1 Boundary Conditions

Boundary conditions at initial node:

$$
\begin{aligned}
C_p(t_0) P_0 &= m_0 \\
C_v(t_0) P_0 &= v_0 \\
C_a(t_0) P_0 &= a_0
\end{aligned} \tag{9}
$$

Boundary conditions at terminal node:

$$
\begin{aligned}
C_p(t_N) P_0 &= m_N \\
C_v(t_N) P_0 &= v_N \\
C_a(t_N) P_0 &= a_N
\end{aligned} \tag{10}
$$

4.2 Other Equality constraints

The velocity and acceleration of each node on the two segments should be the same.

$$
\begin{aligned}
C_v(t_1) P_0 &= C_v(t_1) P_1 \\
C_a(t_1) P_0 &= C_a(t_1) P_1 \\
C_v(t_2) P_1 &= C_v(t_2) P_2 \\
C_a(t_2) P_1 &= C_a(t_2) P_2 \\
\ldots &= \ldots \\
C_v(t_{n-1}) P_{n-2} &= C_v(t_{n-1}) P_{n-1} \\
C_a(t_{n-1}) P_{n-2} &= C_a(t_{n-1}) P_{n-1}
\end{aligned} \tag{11}
$$

5 An example for an optimization problem with five nodes

Q and P matrices are the same as ones as mentioned above. This part I will illustrate how a A and B matrices are generated.

$$A = \begin{bmatrix} C_{pT_0} & 0_{1\times 6} & 0_{1\times 6} & 0_{1\times 6} \\ C_{vT_0} & 0_{1\times 6} & 0_{1\times 6} & 0_{1\times 6} \\ C_{aT_0} & 0_{1\times 6} & 0_{1\times 6} & 0_{1\times 6} \\ C_{pT_1} & 0_{1\times 6} & 0_{1\times 6} & 0_{1\times 6} \\ 0_{1\times 6} & C_{pT_1} & 0_{1\times 6} & 0_{1\times 6} \\ 0_{1\times 6} & C_{pT_2} & 0_{1\times 6} & 0_{1\times 6} \\ 0_{1\times 6} & 0_{1\times 6} & C_{pT_2} & 0_{1\times 6} \\ 0_{1\times 6} & 0_{1\times 6} & C_{pT_3} & 0_{1\times 6} \\ 0_{1\times 6} & 0_{1\times 6} & 0_{1\times 6} & C_{pT_3} \\ 0_{1\times 6} & 0_{1\times 6} & 0_{1\times 6} & C_{pT_4} \\ 0_{1\times 6} & 0_{1\times 6} & 0_{1\times 6} & C_{vT_4} \\ 0_{1\times 6} & 0_{1\times 6} & 0_{1\times 6} & C_{aT_4} \\ C_{vT_1} & -C_{vT_1} & 0_{1\times 6} & 0_{1\times 6} \\ C_{aT_1} & -C_{aT_1} & 0_{1\times 6} & 0_{1\times 6} \\ 0_{1\times 6} & C_{vT_2} & -C_{vT_2} & 0_{1\times 6} \\ 0_{1\times 6} & C_{aT_2} & -C_{aT_2} & 0_{1\times 6} \\ 0_{1\times 6} & 0_{1\times 6} & C_{vT_3} & -C_{vT_3} \\ 0_{1\times 6} & 0_{1\times 6} & C_{aT_3} & -C_{aT_3} \end{bmatrix}_{18\times 4} \tag{12}$$

$$B = \begin{bmatrix} m_0 \\ v_0 \\ a_0 \\ m_1 \\ m_1 \\ m_2 \\ m_2 \\ m_3 \\ m_3 \\ m_4 \\ v_4 \\ a_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{18\times 1} \tag{13}$$

$$X = P = \begin{bmatrix} P_0^T \\ P_1^T \\ P_2^T \\ P_3^T \end{bmatrix}_{18\times 1} \tag{14}$$

Then the problem can be constructed, and we can solve it by using:

```
import cvxopt as cp

sol = cp.solvers.qp(Q, P, None, None, A, B)
printf(sol['x'])
```

6 Simulation