Mathematical Model of a Dynamic Natural Gas System for the Submission Entitled: A Modified Rank Minimization Algorithm for Nonconvex Dynamic Natural Gas System Dispatch Problems

In this supplement, the detailed formulations are given for the nonconvex dynamic natural gas system (DNGS), novel rank minimization algorithm (NRMA), and primal, dual, and positive semidefinite (PSD) models, denoted as NC, NR, P, D, and PR models, respectively, in the submission entitled: *A modified rank minimization algorithm for nonconvex dynamic natural gas system dispatch problems*.

NOMENCLATURE

Variables

$d_{u,t}$	Served gas demand of gas-fired unit *u*
$m^t_{p,s}$	Mass flow rate through segmentation points of pipe *p*
$\rho^{(\cdot)}$	Gas density at nodes or segmentation points of pipes
$v^G_{g,t}$	Gas supply of well *g* at time *t*
$\gamma^t_{p,s}$	Lifting variable associated with segmentation points of pipe *p*
$\eta^{(\cdot)}_{(\cdot)},\zeta^{(\cdot)}_{(\cdot)},\lambda^{(\cdot)}_{(\cdot)}$	Dual variables for equality constraints
$\underline{\mu}^{(\cdot)}_{(\cdot)},\overline{\mu}^{(\cdot)}_{(\cdot)}$	Dual variables for inequality constraints
$d^C_{c,t}$	Gas demand of compressor *c* at time *t*

Sets

$j\in\mathbb{J}$	Set of nodes in gas network
$c\in\mathbb{C}$	Set of nodes with compressors
$g\in\mathbb{G}$	Set of gas wells
$h\in\mathbb{H}$	Set of hours
$u\in\mathbb{U}$	Set of gas-fired units
$p\in\mathbb{P}$	Set of pipes in gas network
$s\in\mathbb{S}_p$	Set of all segmentation points of pipe *p*
$t\in\mathbb{T}$	Set of time intervals

Parameters

α	Speed of sound (m/s)
D	Diameter of natural gas pipe (m)
$d^G_{l,t}$	Demand of natural gas load *l* at time *t*
f	Friction factor of natural gas pipes
F^G_u	Gas consumption function of gas-fired unit *u*
$\underline{\rho}^J_j,\overline{\rho}^J_j$	Lower/upper bounds for density at node *j*
$\underline{v}^G_g,\overline{v}^G_g$	Lower/upper bounds for natural gas well *g*
ρ_1	Gas density at reference node
Γ	Ratio of compressors in natural gas network
n^{seg}_p	Number of segmentation points in pipe *p*
k_G	Value of lost load in natural gas systems
ξ^t_g	Cost multiplier of the natural gas well *g*
a^C_c,b^C_c,q^C_c	Natural gas consumption coefficients of compressor *c*
σ	Small range for the system linepack level to vary
Δx_p	Length of natural gas pipe segments in pipe *p*
Δt	Time step duration

Gas flow dynamics:

The dynamic characteristics of the natural gas flow are described by partial differential equations (PDEs) below.

$$\frac{\partial}{\partial t}pr(x,t)+\frac{4\alpha^2}{\pi D^2}\frac{\partial}{\partial t}m(x,t)=0 \tag{a1}$$

$$\frac{\partial}{\partial x}pr(x,t)+\frac{4}{\pi D^2}\frac{\partial}{\partial t}m(x,t)+\frac{\partial}{\partial x}\rho(x,t)u^2(x,t)=-\frac{8f}{\pi^2D^5}\frac{m^2(x,t)}{\rho(x,t)} \tag{a2}$$

$$pr(x,t)=\alpha^2\rho(x,t) \tag{a3}$$

In (a), the gas pressure (pr), gas density (ρ), and mass flow (m) are related by the PDEs in terms of time and space. Generally, the term $\partial m/\partial t$ in (a2) can be omitted [1], so this simplification is also adopted here. Then, the above PDEs can be transformed into algebraic equations (i.e., (b12)-(b14)) by the finite difference method, and the following nonconvex Model NC is obtained.

Nonconvex *DNGS* model (Model NC):

$$\min\sum_{t\in\mathbb{T}}\sum_{g\in\mathbb{G}}\xi^t_g v^G_{g,t}+k_G\sum_{t\in\mathbb{T}}\sum_{u\in\mathbb{U}}(F^G_u(P^U_{u,t})-d_{u,t}) \tag{b1}$$

Subject to

$$\underline{\rho}^J_j\le\rho^J_{j,t}\le\overline{\rho}^J_j,\ \forall j\in\mathbb{J},\ t\in\mathbb{T}:\underline{\mu}^\rho_{j,t},\overline{\mu}^\rho_{j,t} \tag{b2}$$

$$\underline{v}^G_g\le v^G_{g,t}\le\overline{v}^G_g,\ \forall g\in\mathbb{G},\ t\in\mathbb{T}:\underline{\mu}^G_{g,t},\overline{\mu}^G_{g,t} \tag{b3}$$

$$\rho^J_{j,t}=\rho^t_{p,1},\ \forall j\in\mathbb{J}\setminus\mathbb{C},\ p\in\mathbb{P}^{fr}_j,t\in\mathbb{T}:\lambda^{fr}_{\rho,t} \tag{b4}$$

$$\rho^J_{j,t}=\rho^t_{p,n^{seg}_p},\ \forall j\in\mathbb{J},\ p\in\mathbb{P}^{to}_j,t\in\mathbb{T}:\lambda^{to}_{\rho,t} \tag{b5}$$

$$\rho^J_{1,t}=\rho_1,t\in\mathbb{T}:\lambda^F_{1,t} \tag{b6}$$

$$\rho^J_{j,t}\le\rho^t_{p,1}\le\Gamma\bullet\rho^J_{j,t},\ \forall j\in\mathbb{C},\ p\in\mathbb{P}^{fr}_j,\ t\in\mathbb{T}:\underline{\mu}^C_{\rho,t},\overline{\mu}^C_{\rho,t} \tag{b7}$$

$$d^C_{c,t}=a^C_c m^J_{c,1}+b^C_c\rho^J_{c,t}+q^C_c\rho^J_{c,1},\ \forall c\in\mathbb{C},p\in\mathbb{P}^{fr}_c,t\in\mathbb{T}:\lambda^C_{c,t} \tag{b8}$$

$$0\le d_{u,t}\le F^G_u(P^U_{u,t}),\forall u\in\mathbb{U},\ t\in\mathbb{T}:\underline{\mu}^U_{u,t},\overline{\mu}^U_{u,t} \tag{b9}$$

$$\sum_{g\in\mathbb{G}}v^G_{(g,t)}+\sum_{p\in\mathbb{P}^{to}_j}m^t_{p,n^{seg}_p}-\sum_{p\in\mathbb{P}^{fr}_j}m^t_{p,1}=\sum_{l\in\mathbb{L}_j}d^G_{l,t}+\sum_{u\in\mathbb{U}_j}d_{u,t}+\sum_{c\in\mathbb{C}_j}d^C_{c,t},\forall j\in\mathbb{J},\ t\in\mathbb{T}:\lambda^L_{j,t} \tag{b10}$$

$$-\sigma\le\sum_{t\in\mathbb{T}}\sum_{p\in\mathbb{P}}(m^t_{p,n^{seg}_p}-m^t_{p,1})\le\sigma:\underline{\mu}^P,\overline{\mu}^P \tag{b11}$$

$$\frac{\rho^{t+1}_{p,s+1}-\rho^t_{p,s+1}}{2\Delta t}+\frac{\rho^{t+1}_{p,s}-\rho^t_{p,s}}{2\Delta t}+\frac{m^{t+1}_{p,s+1}-m^{t+1}_{p,s}}{(\pi D^2/4)\Delta x}=0,\ \forall p\in\mathbb{P},\ s\in\mathbb{S}_p,t\in\mathbb{T}:\eta^t_{p,s} \tag{b12}$$

$$\frac{\rho^{t+1}_{p,s+1}-\rho^{t+1}_{p,s}}{\Delta x}+\frac{(\gamma^t_{p,s})^2+(\gamma^t_{p,s+1})^2}{(\pi^2D^5/4f)}=0,\ \forall p\in\mathbb{P},\ s\in\mathbb{S}_p,t\in\mathbb{T}:\zeta^t_{p,s} \tag{b13}$$

$$(m_{p,s}^t)^2 = \gamma_{p,s}^t \rho_{p,s}^t, \forall p \in \mathbb{P},\ s \in \mathbb{S}_p, t \in \mathbb{T} \quad \text{(b14)}$$

The objective function (b1) is composed of two elements. The first term is the operational cost of gas wells, and the second term is the penalty cost generated by the gas demand of gas-fired units that are not served. The output of natural gas wells and the gas density at each node, at time *t*, are limited by (b2) and (b3), respectively. The natural gas density relationship at each node is described in (b4) and (b5). The gas density $\rho_{1,t}^J$ at the reference node is constant, as shown in (b6). The compressor model is presented in (b7) and (b8). Constraint (b7) means that the density can be increased up to Γ times in the compressor nodes, which is a parameter greater than 1. Constraint (b8) describes the relationship between the gas consumption of the compressor and the gas flow through the compressor. Constraint (b9) limits the demand of gas-fired units, which means that the natural gas demand of gas-fired units can be shed. The natural gas supply-demand balance for each node is ensured by (b10). Constraint (b11) limits the change in linepack to ensure that the linepack storage should be recovered every 24 hours. Based on the time step Δt and the spatial step Δx_p, the PDEs in (a) are transformed into algebraic equations (b12), (b13), and (b14). The equations included in (b14) are nonlinear, which makes the whole problem (b) nonconvex.

The compact formulation of Model NC is given below.

Compact formulation of Model NC:

$$\min \boldsymbol{a}^T \boldsymbol{x} \quad \text{(c1)}$$

Subject to: (c2)

$$(m_{p,s}^t)^2 = \gamma_{p,s}^t \rho_{p,s}^t, \forall p \in \mathbb{P},\ s \in \mathbb{S}_p, t \in \mathbb{T} \quad \text{(c3)}$$

$$\mathbf{A}\boldsymbol{x} \le \boldsymbol{b} : \boldsymbol{\mu}, \mathbf{B}\boldsymbol{x} = \boldsymbol{d} : \boldsymbol{\lambda}, \boldsymbol{x} \ge \mathbf{0} \quad \text{(c4)}$$

where $\boldsymbol{x}$ is a vector of all variables $m_{p,s}^t, d_{u,t}, d_{c,t}^C\ , \rho_{p,s}^t, v_{p,t}^G$, and $\gamma_{p,s}^t$, $\boldsymbol{\mu}$ is a vector of dual variables $\eta_{(\cdot)}^{(\cdot)}, \zeta_{(\cdot)}^{(\cdot)}, \lambda_{(\cdot)}^{(\cdot)}$, and $\boldsymbol{\lambda}$ is a vector of dual variables $\underline{\mu}_{(\cdot)}^{(\cdot)}, \overline{\mu}_{(\cdot)}^{(\cdot)}$. $\mathbf{A}$ and $\boldsymbol{b}$ are the coefficient matrix and vector corresponding to the equation constraints (b4)-(b6), (b8), (b10), and (b12)-(b13), respectively. $\mathbf{B}$ and $\boldsymbol{d}$ are the coefficient matrix and vector corresponding to the inequality constraints (b2)-(b3), (b7), (b9), and (b11), respectively. $\boldsymbol{a}^T \boldsymbol{x}$ is the variable cost of the natural gas system (i.e., $\sum_{t\in\mathbb{T}}\sum_{g\in\mathbb{G}} \xi_g^t v_{g,t}^G - k_G \sum_{t\in\mathbb{T}}\sum_{u\in U} d_{u,t}$). The constant $k_G \sum_{t\in\mathbb{T}}\sum_{u\in U} (F_u^G(P_{u,t}^U)$ has no effect on the derivation and conclusion, so it is omitted in the compact formulation.

Compact formulation of the primal model (Model P):

$$\min \boldsymbol{a}^T \boldsymbol{x} \quad \text{(d1)}$$

Subject to: (d2)

$$\mathbf{A}\boldsymbol{x} \le \boldsymbol{b} : \boldsymbol{\mu}, \mathbf{B}\boldsymbol{x} = \boldsymbol{d} : \boldsymbol{\lambda}, \boldsymbol{x} \ge \mathbf{0} \quad \text{(d3)}$$

where the bold matrices and vectors $\mathbf{A}, \mathbf{B}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{x}, \boldsymbol{\lambda}$, and $\boldsymbol{\mu}$, have the same definitions as explained below in Model NC above.

Compared with Model NC, the nonconvex constraint (c3) is removed from Model P. Therefore, it is clear that Model P is linear and has the following dual form.

Compact formulation of the dual model (Model D):

$$\min -\boldsymbol{\mu}^T \boldsymbol{b} - \boldsymbol{\lambda}^T \boldsymbol{d} \quad \text{(e1)}$$

Subject to: (e2)

$$\boldsymbol{a} + \mathbf{A}^T \boldsymbol{\mu} + \mathbf{B}^T \boldsymbol{\lambda} \ge \mathbf{0},\ \boldsymbol{\mu} \ge \mathbf{0} \quad \text{(e3)}$$

where the bold matrices and vectors $\mathbf{A}, \mathbf{B}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{x}, \boldsymbol{\lambda}$, and $\boldsymbol{\mu}$, have the same definitions as explained below in Model NC above.

Compact formulation of the PSD model (Model PR):

The nonconvex constraint (c3) in Model NC is converted to the PSD constraint (f4); then, the convex Model PR is shown as follows:

$$\min \boldsymbol{a}^T \boldsymbol{x} \quad \text{(f1)}$$

Subject to: (f2)

$$\mathbf{A}\boldsymbol{x} \le \boldsymbol{b} : \boldsymbol{\mu}, \mathbf{B}\boldsymbol{x} = \boldsymbol{d} : \boldsymbol{\lambda}, \boldsymbol{x} \ge \mathbf{0} \quad \text{(f3)}$$

$$\begin{bmatrix} \gamma_{p,s}^t & m_{p,s}^t \\ m_{p,s}^t & \rho_{p,s}^t \end{bmatrix} \succeq 0, \forall p \in \mathbb{P},\ s \in \mathbb{S}_p, t \in \mathbb{T} \quad \text{(f4)}$$

where the bold matrices and vectors $\mathbf{A}, \mathbf{B}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{x}, \boldsymbol{\lambda}$, and $\boldsymbol{\mu}$, have the same definitions as explained in Model NC above.

Compact formulation of Model NR (what NRMA actually solves):

NRMA in [2] solves Model NR instead of the original Model NC. In Model NR, the objective function (g1), which minimizes the rank of the PSD matrices, is introduced to tighten this relaxation to obtain a feasible solution to Model NC. The dual constraint (g4) and primal-dual objective pairing constraint (g5) are introduced to ensure the optimality of dynamic natural gas system dispatch problems.

$$\min \sum_{t\in\mathbb{T}} \sum_{p\in\mathbb{P}} \sum_{s\in\mathbb{S}_p} \gamma_{p,s}^t \quad \text{(g1)}$$

Subject to:

$$\begin{bmatrix} \gamma_{p,s}^t & m_{p,s}^t \\ m_{p,s}^t & \rho_{p,s}^t \end{bmatrix} \succeq 0, \forall p \in \mathbb{P},\ s \in \mathbb{S}_p, t \in \mathbb{T} \quad \text{(g2)}$$

$$\mathbf{A}\boldsymbol{x} \le \boldsymbol{b} : \boldsymbol{\mu}, \mathbf{B}\boldsymbol{x} = \boldsymbol{d} : \boldsymbol{\lambda}, \boldsymbol{x} \ge \mathbf{0} \quad \text{(g3)}$$

$$\boldsymbol{a} + \mathbf{A}^T \boldsymbol{\mu} + \mathbf{B}^T \boldsymbol{\lambda} \ge \mathbf{0},\ \boldsymbol{\mu} \ge \mathbf{0} \quad \text{(g4)}$$

$$-\boldsymbol{\mu}^T \boldsymbol{b} - \boldsymbol{\lambda}^T \boldsymbol{d} = \boldsymbol{a}^T \boldsymbol{x} \quad \text{(g5)}$$

where the bold matrices and vectors $\mathbf{A}, \mathbf{B}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{x}, \boldsymbol{\lambda}$, and $\boldsymbol{\mu}$, have the same definitions as explained in Model NC above.

REFERENCES

[1] L. A. Roald, K. Sundar, A. Zlotnik, S. Misra and G. Andersson, "An Uncertainty Management Framework for Integrated Gas-Electric Energy Systems," *Proc. IEEE*, vol. 108, no. 9, pp. 1518-1540, Sept. 2020.

[2] R. Bayani and S. D. Manshadi, "Natural Gas Short-Term Operation Problem With Dynamics: A Rank Minimization Approach," *IEEE Trans. Smart Grid*, vol. 13, no. 4, pp. 2761-2773, July 2022.