312554021 鄭丞洋

Beta distribution:

$$Beta(\theta | a,b) = \theta^{a-1} \cdot (1-\theta)^{b-1} \cdot B(a,b)^{-1} \quad , \text{where } B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Beta-Binomial Conjugation:

likelihood (Binomial) prior (Beta distribution)

$$P(\theta|X) = \frac{\cancel{\binom{N}{m}}\, \theta^{m} \cdot (1-\theta)^{N-m} \cdot \theta^{a-1} \cdot (1-\theta)^{b-1} \cdot \cancel{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}}}{\int_0^1 \cancel{\binom{N}{m}}\, \theta^{m} \cdot (1-\theta)^{N-m} \cdot \theta^{a-1} \cdot (1-\theta)^{b-1} \cdot \cancel{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}}\, d\theta}$$

constant eliminated

$$= \frac{\theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1}}{\boxed{\int_0^1 \theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1}\, d\theta}}$$

Since $\int_0^1 Beta(\theta | m+a, N-m+b)\, d\theta$

$$= \int_0^1 \theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1} \cdot B(m+a, N-m+b)^{-1}\, d\theta$$

$$= 1$$

Therefore,

$$\boxed{\int_0^1 \theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1}\, d\theta} = B(m+a, N-m+b)$$

$$= \theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1} \cdot B(m+a, N-m+b)^{-1}$$

$$= Beta(\theta | m+a, N-m+b)$$

posterior

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Poission distribution:

$$P(X = x_1, x_2, \cdots x_n \mid \theta) = \frac{\theta^{\sum_{i=1}^{n} x_i} e^{-n\theta}}{\prod_{i=1}^{n} x_i!}$$

Gamma distribution:

$$g(\theta \mid a, b) = \theta^{a-1} \cdot e^{-b\theta} \cdot b^{a} \cdot \Gamma(a)^{-1}$$

Gamma - Poission Conjugation:

likelihood　　　prior

$$P(\theta \mid X) = \frac{\cancel{\frac{\theta^{\sum_{i=1}^{n} x_i} e^{-n\theta}}{\prod_{i=1}^{n} x_i!}} \cdot \theta^{a-1} \cdot e^{-b\theta} \cdot \cancel{b^{a}} \cdot \cancel{\Gamma(a)^{-1}}}{\int_0^1 \cancel{\frac{\theta^{\sum_{i=1}^{n} x_i} e^{-n\theta}}{\prod_{i=1}^{n} x_i!}} \cdot \theta^{a-1} \cdot e^{-b\theta} \cdot \cancel{b^{a}} \cdot \cancel{\Gamma(a)^{-1}} \, d\theta}$$

and let $\sum_{i=1}^{n} x_i = S$

$$= \frac{\theta^{S+a-1} \cdot e^{-(n+b)\theta}}{\boxed{\int_0^1 \theta^{S+a-1} \cdot e^{-(n+b)\theta} \, d\theta}}$$

Since $\int_0^1 g(\theta \mid S+a, n+b) \, d\theta$

$$= \int_0^1 \theta^{S+a-1} \cdot e^{-(n+b)\theta} \cdot (n+b)^{(S+a)} \, \Gamma(S+a)^{-1} \, d\theta$$

$$= 1$$

Therefore,

$$\boxed{\int_0^1 \theta^{S+a-1} \cdot e^{-(n+b)\theta} \, d\theta} = (n+b)^{-(S+a)} \cdot \Gamma(S+a)$$

$$= \theta^{S+a-1} \cdot e^{-(n+b)\theta} \cdot (n+b)^{(S+a)} \cdot \Gamma(S+a)^{-1}$$

$$= g(\theta \mid \underbrace{S+a}_{\text{n次結果相加}}, \underbrace{n+b}_{\text{做幾次.}})$$