Homework 3

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1

$\frac{1}{K_{TG}} = \lambda^{+}_{\min}(\tilde{M}^{-1}A(I-\Pi_A))$.

Proof

$$\begin{aligned}
X^{-1}\tilde{M}^{-1}A(I-\Pi_A)X =& X^{-1}\tilde{M}^{-1}AXX^{-1}(I-\Pi_A)X \\
==& \begin{pmatrix} \hat{X}_{11} & \hat{X}_{12} \\ \hat{X}_{21} & \hat{X}_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{n-n_c} \end{pmatrix} \\
=& \begin{pmatrix} 0 & \hat{X}_{12} \\ 0 & \hat{X}_{22} \end{pmatrix}
\end{aligned}$$

Thus, $\lambda^{+}_{\min}(\tilde{M}^{-1}A(I-\Pi_A)) = \lambda_{\min}(\hat{X}_{22}) = \frac{1}{K_{TG}}$

2

$A_k^{\frac{1}{2}} E^{(k)}_{\mathrm{IMG}} A_k^{-\frac{1}{2}}$ is symmetric and $\lambda(E^{(k)}_{\mathrm{IMG}}) \subset [0,1)$

Proof

$$\begin{aligned}
A_{L-1}^T =& (P_L^T A_L P_L)^T \\
=& P_L^T A_L^T P_L \\
=& P_L^T A^T P_L \\
=& P_L^T A P_L \\
=& P_L^T A_L P_L \\
=& A_{L-1}
\end{aligned}$$

so, we can assume that A_{k+1} is symmetric.

$$\begin{aligned}
A_k^T =& (P_{k+1}^T A_{k+1} P_{k+1})^T \\
=& P_{k+1}^T A_{k+1}^T P_{k+1} \\
=& P_{k+1}^T A_{k+1} P_{k+1} \\
=& A_k
\end{aligned}$$

Thus A_k is symmetric.

$$
\begin{aligned}
A_1^{\frac{1}{2}}E_{\mathrm{IMG}}^{(1)}A_1^{-\frac{1}{2}} =&A_1^{\frac{1}{2}}(I-M_1^{-T}A_1)A_1^{-\frac{1}{2}}A_1^{\frac{1}{2}}(I-P_1\hat{A}_0^{-1}P_1^TA_1)A_1^{-\frac{1}{2}}A_1^{\frac{1}{2}}(I-M_1^{-1}A_1)A_1^{-\frac{1}{2}}\\
=&(I-A_1^{\frac{1}{2}}M_1^{-T}A_1^{-\frac{1}{2}})(I-A_1^{\frac{1}{2}}P_1\hat{A}_0^{-1}P_1^TA_1^{\frac{1}{2}})(I-A_1^{\frac{1}{2}}M_1^{-1}A_1^{\frac{1}{2}})
\end{aligned}
$$

$$
\begin{aligned}
(A_1^{\frac{1}{2}}E_{\mathrm{IMG}}^{(1)}A_1^{-\frac{1}{2}})^T =&(I-A_1^{\frac{1}{2}}M_1^{-1}A_1^{\frac{1}{2}})^T(I-A_1^{\frac{1}{2}}P_1\hat{A}_0^{-1}P_1^TA_1^{\frac{1}{2}})^T(I-A_1^{\frac{1}{2}}M_1^{-T}A_1^{\frac{1}{2}})^T\\
=&(I-A_1^{\frac{1}{2}}M_1^{-T}A_1^{\frac{1}{2}})(I-A_1^{\frac{1}{2}}P_1\hat{A}_0^{-1}P_1^TA_1^{\frac{1}{2}})(I-A_1^{\frac{1}{2}}M_1^{-1}A_1^{\frac{1}{2}})\\
=&A_1^{\frac{1}{2}}E_{\mathrm{IMG}}^{(1)}A_1^{-\frac{1}{2}}
\end{aligned}
$$

So $A_1^{\frac{1}{2}}E_{\mathrm{IMG}}^{(1)}A_1^{-\frac{1}{2}}$ is symmetric. Thus we assume that $A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}}$ is SPD.

$$
\begin{aligned}
(A_k^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k)}A_k^{-\frac{1}{2}})^T =&(N_K^T[I-A_k^{\frac{1}{2}}P_kA_{k-1}^{-\frac{1}{2}}(I-(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})^\gamma)A_{K-1}^{-\frac{1}{2}}P_k^TA_k^{\frac{1}{2}}]N_k)^T\\
=&N_K^T[I-A_k^{\frac{1}{2}}P_kA_{k-1}^{-\frac{1}{2}}(I-(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})^\gamma)A_{K-1}^{-\frac{1}{2}}P_k^TA_k^{\frac{1}{2}}]N_k\\
=&A_k^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k)}A_k^{-\frac{1}{2}}
\end{aligned}
$$

So $A_k^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k)}A_k^{-\frac{1}{2}}$ is symmetric.

For all x,

$$
\begin{aligned}
x^TA_1^{\frac{1}{2}}E_{\mathrm{IMG}}^{(1)}A_1^{-\frac{1}{2}}x =&x^T(I-A_1^{\frac{1}{2}}M_1^{-T}A_1^{-\frac{1}{2}})(I-A_1^{\frac{1}{2}}P_1\hat{A}_0^{-1}P_1^TA_1^{\frac{1}{2}})(I-A_1^{\frac{1}{2}}M_1^{-1}A_1^{\frac{1}{2}})x\\
=&((I-A_1^{\frac{1}{2}}M_1^{-1}A_1^{\frac{1}{2}})x)^TA^{\frac{1}{2}}(I-P_1\hat{A}_0^{-1}P_1^TA)A^{\frac{1}{2}}((I-A_1^{\frac{1}{2}}M_1^{-1}A_1^{\frac{1}{2}})x)
\end{aligned}
$$

Assume $\lambda(E_{\mathrm{IMG}}^{(k)})=\lambda(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})\subset[0,1)$,

$$
\begin{aligned}
\lambda(I-(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})^\gamma) =&1-\lambda((A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})^\gamma)\\
=&1-\lambda(A_{k-1}^{\frac{1}{2}}(E_{\mathrm{IMG}}^{(k-1)})^\gamma A_{k-1}^{-\frac{1}{2}})\\
=&1-\lambda((E_{\mathrm{IMG}}^{(k-1)})^\gamma)
\end{aligned}
$$

Thus $\lambda(I-(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})^\gamma)\subset(0,1]$, $I-A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}}$ is a SPD matrix, $A_k^{\frac{1}{2}}P_kA_{k-1}^{-\frac{1}{2}}(I-(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})^\gamma)A_{K-1}^{-\frac{1}{2}}P_k^TA_k^{\frac{1}{2}}$ is a SPD matrix.

$$
\lambda(A_k^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k)}A_k^{-\frac{1}{2}}) =1-\lambda(A_k^{\frac{1}{2}}P_kA_{k-1}^{-\frac{1}{2}}(I-(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})^\gamma)A_{k-1}^{-\frac{1}{2}}P_k^TA_k^{\frac{1}{2}})<1
$$

$$
\begin{aligned}
&\lambda_{\max}(A_k^{\frac{1}{2}}P_kA_{k-1}^{-\frac{1}{2}}(I-(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})^\gamma)A_{k-1}^{-\frac{1}{2}}P_k^TA_k^{\frac{1}{2}})\\
=&\sigma_{\max}(A_k^{\frac{1}{2}}P_kA_{k-1}^{-\frac{1}{2}}(I-(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})^\gamma)A_{k-1}^{-\frac{1}{2}}P_k^TA_k^{\frac{1}{2}})\\
=&\|A_k^{\frac{1}{2}}P_kA_{k-1}^{-\frac{1}{2}}(I-(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})^\gamma)A_{k-1}^{-\frac{1}{2}}P_k^TA_k^{\frac{1}{2}}\|_2\\
\leqslant&\|A_k^{\frac{1}{2}}P_kA_{k-1}^{-\frac{1}{2}}\|_2^2\|I-(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})^\gamma\|_2\\
\leqslant&\lambda_{\max}(A_k^{\frac{1}{2}}P_kA_{k-1}^{-\frac{1}{2}}A_{k-1}^{-\frac{1}{2}}P_k^TA_k^{\frac{1}{2}})\\
=&\lambda_{\max}(A_k^{\frac{1}{2}}P_kA_{k-1}^{-1}P_k^TA_k^{\frac{1}{2}})
\end{aligned}
$$

$$
\begin{aligned}
(A_k^{\frac{1}{2}}P_kA_{k-1}^{-1}P_k^TA_k^{\frac{1}{2}})^2 =& A_k^{\frac{1}{2}}P_kA_{k-1}^{-1}P_k^TA_k^{\frac{1}{2}}A_k^{\frac{1}{2}}P_kA_{k-1}^{-1}P_k^TA_k^{\frac{1}{2}}\\
=& A_k^{\frac{1}{2}}P_kA_{k-1}^{-1}P_k^TA_kP_kA_{k-1}^{-1}P_k^TA_k^{\frac{1}{2}}\\
=& A_k^{\frac{1}{2}}P_kA_{k-1}^{-1}A_{k-1}A_{k-1}^{-1}P_k^TA_k^{\frac{1}{2}}\\
=& A_k^{\frac{1}{2}}P_kA_{k-1}^{-1}P_k^TA_k^{\frac{1}{2}}
\end{aligned}
$$

Thus, $A_k^{\frac{1}{2}}P_kA_{k-1}^{-1}P_k^TA_k^{\frac{1}{2}}$ is a projection, $\lambda_{\max}(A_k^{\frac{1}{2}}P_kA_{k-1}^{-1}P_k^TA_k^{\frac{1}{2}}) \leqslant 1, \lambda_{\max}(A_k^{\frac{1}{2}}P_kA_{k-1}^{-\frac{1}{2}}(I-(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})^{\gamma})A_{k-1}^{-\frac{1}{2}}P_k^TA_k^{\frac{1}{2}}) \leqslant 1$ $\lambda(A_k^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k)}A_k^{-\frac{1}{2}}) \geqslant 0$.

So, $\lambda(E_{\mathrm{IMG}}^{(k)}) = \lambda(A_k^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k)}A_k^{-\frac{1}{2}}) \subset [0,1)$.

3

$B_c = A_{k-1}(I-(E_{\mathrm{IMG}}^{(k-1)})^{\gamma})^{-1}$ is SPD.

Proof

$$
\begin{aligned}
B_c =& A_{k-1}(I-(E_{\mathrm{IMG}}^{(k-1)})^{\gamma})^{-1}\\
=& A_{k-1}^{\frac{1}{2}}(A_{k-1}^{-\frac{1}{2}}-(E_{\mathrm{IMG}}^{(k-1)})^{\gamma}A_{k-1}^{-\frac{1}{2}})^{-1}\\
=& A_{k-1}^{\frac{1}{2}}(A_{k-1}^{-\frac{1}{2}}-A_{k-1}^{-\frac{1}{2}}A_{k-1}^{\frac{1}{2}}(E_{\mathrm{IMG}}^{(k-1)})^{\gamma}A_{k-1}^{-\frac{1}{2}})^{-1}\\
=& A_{k-1}^{\frac{1}{2}}(I-A_{k-1}^{\frac{1}{2}}(E_{\mathrm{IMG}}^{(k-1)})^{\gamma}A_{k-1}^{-\frac{1}{2}})^{-1}A_{k-1}^{\frac{1}{2}}\\
=& A_{k-1}^{\frac{1}{2}}(I-(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}})^{\gamma})^{-1}A_{k-1}^{\frac{1}{2}}
\end{aligned}
$$

Due to the fact that $A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{k-1}A_{k-1}^{-\frac{1}{2}}$ is symmetric, B_c is symmetric. And $\lambda(E_{\mathrm{IMG}}^{(k-1)}) = \lambda(A_{k-1}^{\frac{1}{2}}E_{\mathrm{IMG}}^{(k-1)}A_{k-1}^{-\frac{1}{2}}) \subset [0,1)$, thus B_c is SPD.