# Homework 2

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## 1 HW3.2

If $\mathcal{A}, \mathcal{B} : V \to V$ and $u, v \in V$, we have the following results:

1. $\underline{\mathcal{A}\mathcal{B}} = \underline{\mathcal{A}}\,\underline{\mathcal{B}}$;

2. $\underline{\mathcal{A}v} = \underline{\mathcal{A}}\,\underline{v}$;

3. $\sigma(\mathcal{A}) = \sigma(\underline{\mathcal{A}}), \kappa(\mathcal{A}) = \kappa(\underline{\mathcal{A}})$;

4. $\vec{v} = M\underline{v}, \overrightarrow{\mathcal{A}v} = \hat{\mathcal{A}}\underline{v}$;

5. $\hat{A} = M\underline{\mathcal{A}}$;

6. $(u, v) = (M\underline{u}, \underline{v})$.

**Proof:**

Suppose that $\{\phi_i\}_{i=1,\cdots,N}$ is a basis of the finite-dimensional space $V$.Suppose $\{\psi_i^1\}_{i=1,\cdots,N}, \{\psi_i^2\}_{i=1,\cdots,N}$ are another bases of $V$. The matrix representation of $\mathcal{A}$ is $\underline{\mathcal{A}} \in \mathbb{R}^{N\times N}$, such that $\sum_{i=1}^{N}(\underline{\mathcal{A}})_{i,j}\psi_i^2 = \mathcal{A}\psi_j^1\,(j, 1, \cdots, N)$ and the matrix representation of $\mathcal{B}$ is $\underline{\mathcal{B}} \in \mathbb{R}^{N\times N}$, such that $\sum_{i=1}^{N}(\underline{\mathcal{B}})_{i,j}\psi_i^1 = \mathcal{B}\phi_j\,(j, 1, \cdots, N)$

1.

$$
\begin{aligned}
\mathcal{A}\mathcal{B}\phi_j &= \mathcal{A}(\mathcal{B}\phi_j) \\
&= \mathcal{A}\sum_{i=1}^{N}(\underline{\mathcal{B}})_{i,j}\psi_i^1 \\
&= \sum_{i=1}^{N}(\underline{\mathcal{B}})_{i,j}\mathcal{A}\psi_i^1 \\
&= \sum_{i=1}^{N}\sum_{k=1}^{N}(\underline{\mathcal{B}})_{i,j}(\underline{\mathcal{A}})_{k,i}\psi_k^2
\end{aligned}
$$

and $\mathcal{A}\mathcal{B}\phi_j = \sum_{k=1}^{N}(\underline{\mathcal{A}\mathcal{B}})_{k,j}\psi_k^2$, thus ,$\sum_{i=1}^{N}(\underline{\mathcal{B}})_{i,j}(\underline{\mathcal{A}})_{k,i} = (\underline{\mathcal{A}\mathcal{B}})_{k,j}$, so $\underline{\mathcal{A}\mathcal{B}} = \underline{\mathcal{A}}\,\underline{\mathcal{B}}$.

2.

$$\begin{aligned}\mathcal{A}v =& \mathcal{A}\sum_{i=1}^{N}\underline{v}_i\phi_i \\ =& \sum_{i=1}^{N}\underline{v}_i\mathcal{A}\phi_i \\ =& \sum_{j=1}^{N}\sum_{i=1}^{N}\underline{v}_i(\underline{\mathcal{A}})_{j,i}\phi_j\end{aligned}$$

and $\mathcal{A}v = \sum_{j=1}^{N}(\underline{\mathcal{A}v})_j v_j$, thus ,$(\underline{\mathcal{A}v})_j = \sum_{i=1}^{N}\underline{v}_i(\underline{\mathcal{A}})_{j,i}$, so $\underline{\mathcal{A}v} = \underline{\mathcal{A}}\,\underline{v}$.

3. If $\lambda$ is the eigenvalue of $\mathcal{A}$ such that $\mathcal{A}v = \lambda v$, then $\mathcal{A}v = \sum_{i=1}^{n}(\underline{\mathcal{A}v})_i\phi_i$ and $\mathcal{A}v = \sum_{i=1}^{n}\lambda(\underline{v})_i\phi_i$, so $(\underline{\mathcal{A}v})_i = \lambda(\underline{v})_i$, thus $\underline{\mathcal{A}}\,\underline{v} = \underline{\mathcal{A}v} = \lambda v$. $\lambda$ is also the eigenvalue of $\underline{\mathcal{A}}$.

   If $\lambda$ is the eigenvalue of $\underline{\mathcal{A}}$ such that $\underline{\mathcal{A}}\,\underline{v} = \lambda\underline{v}, \sum_{j=1}^{N}(\underline{\mathcal{A}})_{i,j}\,\underline{v}_j = \lambda\underline{v}_i$, then $\mathcal{A}v = \sum_{i=1}^{n}(\underline{\mathcal{A}v})_i\phi_i = \sum_{j=1}^{n}\sum_{i=1}^{n}(\underline{\mathcal{A}})_{i,j}\underline{v}_j\phi_i = \lambda\sum_{i=1}^{N}(\overline{v})_i\phi_i = \lambda v$, thus $\mathcal{A}v = \lambda v$. $\lambda$ is also the eigenvalue of $\mathcal{A}$.

   So $\sigma(\mathcal{A}) = \sigma(\underline{\mathcal{A}}), \kappa(\mathcal{A}) = \kappa(\underline{\mathcal{A}})$.

4.

$$\begin{aligned}\vec{v} =& \begin{pmatrix}(v,\phi_1)\\(v,\phi_2)\\\vdots\\(v,\phi_N)\end{pmatrix} \\ =& \begin{pmatrix}(\sum_{i=1}^{N}\underline{v}_i\phi_i,\phi_1)\\(\sum_{i=1}^{N}\underline{v}_i\phi_i,\phi_2)\\\vdots\\(\sum_{i=1}^{N}\underline{v}_i\phi_i,\phi_N)\end{pmatrix} \\ =& \begin{pmatrix}\sum_{i=1}^{N}\underline{v}_i(\phi_i,\phi_1)\\\sum_{i=1}^{N}\underline{v}_i(\phi_i,\phi_2)\\\vdots\\\sum_{i=1}^{N}\underline{v}_i(\phi_i,\phi_N)\end{pmatrix} \\ =& M\underline{v}\end{aligned}$$

$$
\begin{aligned}
\overrightarrow{\mathcal{A}v} &= \begin{pmatrix} (\mathcal{A}v, \phi_1) \\ (\mathcal{A}v, \phi_2) \\ \vdots \\ (\mathcal{A}v, \phi_N) \end{pmatrix} \\
&= \begin{pmatrix} (\mathcal{A} \sum_{i=1}^N \underline{v}_i \phi_i, \phi_1) \\ (\mathcal{A} \sum_{i=1}^N \underline{v}_i \phi_i, \phi_2) \\ \vdots \\ (\mathcal{A} \sum_{i=1}^N \underline{v}_i \phi_i, \phi_N) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i=1}^N \underline{v}_i (\mathcal{A}\phi_i, \phi_1) \\ \sum_{i=1}^N \underline{v}_i (\mathcal{A}\phi_i, \phi_2) \\ \vdots \\ \sum_{i=1}^N \underline{v}_i (\mathcal{A}\phi_i, \phi_N) \end{pmatrix} \\
&= \hat{\mathcal{A}}\underline{v}
\end{aligned}
$$

5. $\overrightarrow{\mathcal{A}v} = M\underline{\mathcal{A}v} = M\underline{\mathcal{A}}\,\underline{v}$, so $\hat{\mathcal{A}} = M\underline{\mathcal{A}}$.

6. $(u, v) = (\sum_{i=1}^N \underline{u}_i \phi_i, \sum_{i=j}^N \underline{v}_j \phi_j) = \sum_j^N (\sum_{i=1}^N (\phi_i, \phi_j)\underline{u}_i)\underline{v}_j = (M\underline{u}, \underline{v})$.