# Homework 1

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## 1 HW2.5

Given $A \in \mathbb{R}^{N\times N}$ and $0 \neq v \in \mathbb{R}^N$. Then the Krylov subspace satisfy the following properties:

(1) $\mathcal{K}_m(A, v) \subset \mathcal{K}_{m+1}(A, v)$;

(2) $A\mathcal{K}_m(A, v) \subset \mathcal{K}_{m+1}(A, v)$;

(3) $\mathcal{K}_m(A, v) = \mathcal{K}_m(\alpha A, v) = \mathcal{K}_m(A, \alpha v)$, for any $0 \neq \alpha \in \mathbb{R}$;

(4) $\mathcal{K}_m(A, v) = \mathcal{K}_m(A - \alpha I, v)$, for any $\alpha \in \mathbb{R}$;

(5) $\mathcal{K}_m(Q^{-1}AQ, Q^{-1}v) = Q^{-1}\mathcal{K}_m(A, v)$, for any nonsingular matrix $Q \in \mathbb{R}^{N\times N}$;

(6) $\mathcal{K}_m(A, v) = \{p(A)v : p \in \mathcal{P}_{m-1}\}$, where $\mathcal{P}_{m-1}$ is the real polynomials of degree less than $m$.

**Proof:**

(1) If $x \in \mathcal{K}_m(A, v)$, then

$$x = \sum_{i=1}^{m} k_i A^{i-1} v \in \mathcal{K}_{m+1}(A, v), \quad k_i \in \mathbb{R},$$

so $\mathcal{K}_m(A, v) \subset \mathcal{K}_{m+1}(A, v)$.

(2) If $x \in A\mathcal{K}_m(A, v)$, then

$$\begin{aligned} x =& A \sum_{i=1}^{m} k_i A^{i-1} v \\ =& \sum_{i=1}^{m} k_i A^{i} v \in \mathcal{K}_{m+1}(A, v), \quad k_i \in \mathbb{R}, \end{aligned}$$

so $A\mathcal{K}_m(A, v) \subset \mathcal{K}_{m+1}(A, v)$.

(3) If $x \in \mathcal{K}_m(A, v)$, then

$$\begin{aligned} x =& \sum_{i=1}^{m} k_i A^{i-1} v \\ =& \sum_{i=1}^{m} \frac{k_i}{\alpha^{i-1}} (\alpha A)^{i-1} v \in \mathcal{K}_m(\alpha A, v), \quad k_i \in \mathbb{R}, \end{aligned}$$

so $\mathcal{K}_m(A, v) \subset \mathcal{K}_m(\alpha A, v)$, and in the same way,$\mathcal{K}_m(\alpha A, v) \subset \mathcal{K}_m(\frac{1}{\alpha}\alpha A, v) = \mathcal{K}_m(A, v)$, so $\mathcal{K}_m(A, v) = \mathcal{K}_m(\alpha A, v)$;

If $x \in \mathcal{K}_m(A, v)$, then

$$\begin{aligned} x =& \sum_{i=1}^{m} k_i A^{i-1} v \\ =& \sum_{i=1}^{m} \frac{k_i}{\alpha} A^{i-1} (\alpha v) \in \mathcal{K}_m(A, \alpha v), \quad k_i \in \mathbb{R}, \end{aligned}$$

so $\mathcal{K}_m(A, v) \subset \mathcal{K}_m(A, \alpha v)$, and in the same way,$\mathcal{K}_m(A, \alpha v) \subset \mathcal{K}_m(A, \frac{1}{\alpha}\alpha v) = \mathcal{K}_m(A, v)$, so $\mathcal{K}_m(A, v) = \mathcal{K}_m(A, \alpha v)$.

(4) If $x \in \mathcal{K}_m(A - \alpha I, v)$, then

$$\begin{aligned} x =& \sum_{i=1}^{m} k_i (A - \alpha I)^{i-1} v \\ =& \sum_{i=1}^{m} k_i \sum_{j=0}^{i-1} \binom{i-1}{j} (-\alpha)^{i-1-j} A^j v \in \mathcal{K}_m(A, v), \quad k_i \in \mathbb{R}, \end{aligned}$$

so $\mathcal{K}_m(A - \alpha I, v) \subset \mathcal{K}_m(A, \alpha v)$, and in the same way,$\mathcal{K}_m(A - \alpha I, v) \subset \mathcal{K}_m(A - \alpha I + \alpha I, v) = \mathcal{K}_m(A, v)$, so $\mathcal{K}_m(A, v) = \mathcal{K}_m(A - \alpha I, v)$.

(5) If $x \in \mathcal{K}_m(Q^{-1}AQ, Q^{-1}v)$, then

$$\begin{aligned} x =& \sum_{i=1}^{m} k_i (Q^{-1}AQ)^{i-1} Q^{-1} v \\ =& \sum_{i=1}^{m} k_i Q^{-1} A^{i-1} v \\ =& Q^{-1} \sum_{i=1}^{m} k_i A^{i-1} v \in Q^{-1}\mathcal{K}_m(A, v), \quad k_i \in \mathbb{R}, \end{aligned}$$

so $\mathcal{K}_m(Q^{-1}AQ, Q^{-1}v) \subset Q^{-1}\mathcal{K}_m(A, v)$.

If $x \in Q^{-1}\mathcal{K}_m(A, v)$, then

$$\begin{aligned} x =& Q^{-1}\sum_{i=1}^{m} k_i A^{i-1} v \\ =& \sum_{i=1}^{m} k_i Q A^{i-1} v \\ =& \sum_{i=1}^{m} k_i (Q^{-1}AQ)^{i-1} Q^{-1} v \in \mathcal{K}_m(Q^{-1}AQ, Q^{-1}v), \quad k_i \in \mathbb{R}, \end{aligned}$$

so $\mathcal{K}_m(Q^{-1}AQ, Q^{-1}v) \subset Q\mathcal{K}_m(A, v)$, so $\mathcal{K}_m(Q^{-1}AQ, Q^{-1}v) = Q\mathcal{K}_m(A, v)$.

(6) If $x \in \mathcal{K}_m(A, v)$, then

$$\begin{aligned} x =& \sum_{i=1}^{m} k_i A^{i-1} v \\ =& p(A)v \end{aligned}$$

where $p(A) = \sum_{i=1}^{m} k_i A^{i-1}$. So, $\mathcal{K}_m(A, v) \subset \{p(A)v : p \in \mathcal{P}_{m-1}\}$.

If $x \in \{p(A)v : p \in \mathcal{P}_{m-1}\}$, there exists a polynomial $p(y)$ of degree less than m that statisfies $x = p(A)v$. Assume $p(y) = \sum_{i=1}^{m} k_i y^{i-1}, k_i \in \mathbb{R}$,

$$\begin{aligned} x =& p(A)v \\ =& \sum_{i=1}^{m} k_i A^{i-1} v \in \mathcal{K}_m(A, v) \end{aligned}$$

So $\{p(A)v : p \in \mathcal{P}_{m-1}\} \subset \mathcal{K}_m(A, v)$, so $\{p(A)v : p \in \mathcal{P}_{m-1}\} = \mathcal{K}_m(A, v)$.

# 2 HW2.6

Suppose that $\mathcal{A}$ is SPD and $\mathcal{A}u = f$. Show that the following conditions are equivalent to each other:

(1) Vector $u_m \in \mathcal{K}_m(\mathcal{A}, f)$ satisfies that $\|u_m - u\|_{\mathcal{A}} = \min\{\|v - u\|_{\mathcal{A}} : v \in \mathcal{K}_m(\mathcal{A}, f)\}$;

(2) Vector $u_m \in \mathcal{K}_m(\mathcal{A}, f)$ satisfies that $\|f - \mathcal{A}u_m\|_{\mathcal{A}^{-1}} = \min\{\|f - \mathcal{A}u_m\|_{\mathcal{A}^{-1}} : v \in \mathcal{K}_m(\mathcal{A}, f)\}$;

(3) Vector $u_m \in \mathcal{K}_m(\mathcal{A}, f)$ satisfies that $v^T(f - \mathcal{A}u_m) = 0$ for any $v \in \mathcal{K}_m(\mathcal{A}, f)$.

**Proof:**

(1) $\Leftrightarrow$ (2):

$$\begin{aligned} \|u_m - u\|_{\mathcal{A}}^2 &= (\mathcal{A}(u_m - u), u_m - u) \\ &= (\mathcal{A}^{-1}\mathcal{A}(u_m - u), \mathcal{A}(u_m - u)) \\ &= (\mathcal{A}^{-1}(f - \mathcal{A}u_m), f - \mathcal{A}u_m) \\ &= \|f - \mathcal{A}u_m\|_{\mathcal{A}^{-1}}^2 \end{aligned}$$

So, $\|u_m - u\|_{\mathcal{A}} = \|f - \mathcal{A}u_m\|_{\mathcal{A}^{-1}}$, and then (1) and (2) are equivalent.

(1) $\Leftrightarrow$ (3): If $v \in \mathcal{K}_m(\mathcal{A}, v)$, thus $v + u_m \in \mathcal{K}_m(\mathcal{A}, v)$,

$$\begin{aligned}\|u_m + v - u\|_{\mathcal{A}}^2 &= (\mathcal{A}(u_m + v - u), u_m + v - u) \\ &= (\mathcal{A}v, v) + (\mathcal{A}(u_m - u), u_m - u) + 2(\mathcal{A}(u_m - u), v) \\ &= \|v\|_{\mathcal{A}}^2 + \|u_m - u\|_{\mathcal{A}}^2 + 2(\mathcal{A}(u_m - u), v)\end{aligned}$$

(1) $\Rightarrow$ (3): If vector $u_m \in \mathcal{K}_m(\mathcal{A}, f)$ satisfies that $\|u_m - u\|_{\mathcal{A}} = \min\{\|v - u\|_{\mathcal{A}} : v \in \mathcal{K}_m(\mathcal{A}, f)\}$,then $\|v\|_{\mathcal{A}}^2 + 2(\mathcal{A}(u_m - u), v) \geqslant 0$, substitute $tv \in \mathcal{K}_m(\mathcal{A}, v), t \in \mathbb{R}$ for $v$, where $(\mathcal{A}(u_m - u), v) = r$. And this yields $2tr \leqslant t^2\|v\|_{\mathcal{A}}^2$, in the same way $-2tr \leqslant t^2\|v\|_{\mathcal{A}}^2$. So $2t|r| \leqslant t^2\|v\|_{\mathcal{A}}^2$. Letting $t \to 0$, we see that $r = 0$. So $v^T(f - \mathcal{A}u_m) = -(\mathcal{A}(u_m - u), v) = 0$

(3) $\Rightarrow$ (1): If vector $u_m \in \mathcal{K}_m(\mathcal{A}, f)$ satisfies that $v^T(f - \mathcal{A}u_m) = 0$ for any $v \in \mathcal{K}_m(\mathcal{A}, f)$, then

$$\begin{aligned}\|v - u\|_{\mathcal{A}}^2 &= (\mathcal{A}(v - u_m + u_m - u), v - u_m + u_m - u) \\ &= \|v - u_m\|_{\mathcal{A}}^2 + \|u_m - u\|_{\mathcal{A}}^2 + 2(\mathcal{A}(u_m - u), v - u_m)\end{aligned}$$

$v - u_m \in \mathcal{K}_m(A, v)$, so $0 = (v - u_m)^T(f - \mathcal{A}u_m) = (\mathcal{A}(u_m - u), v - u_m)$. So

$$\begin{aligned}\|v - u\|_{\mathcal{A}}^2 &= \|v - u_m\|_{\mathcal{A}}^2 + \|u_m - u\|_{\mathcal{A}}^2 \\ &\geqslant \|u_m - u\|_{\mathcal{A}}^2.\end{aligned}$$