# Solution #1

① (a)

$$p_H(H) = \prod_{ij=1}^{2} \frac{1}{\sqrt{2\pi}} e^{-h_{ij}^2/2} = \frac{1}{4\pi^2} e^{-Tr(HH^T)/2}$$

(b) $H \triangleq LQ \xRightarrow{\text{Hint}}$

$$H = \begin{bmatrix} a\cos(\psi) & a\sin(\psi) \\ b\cos(\psi) - c\sin(\psi) & b\sin(\psi) + c\cos(\psi) \end{bmatrix}$$

Hence

$$J = \det \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 & 0 \\ 0 & 0 & \cos(\psi) & \sin(\psi) \\ 0 & 0 & -\sin(\psi) & \cos(\psi) \\ -a\sin(\psi) & a\cos(\psi) & -b\sin(\psi) - c\cos(\psi) & b\cos(\psi) - c\sin(\psi) \end{pmatrix}$$

$$= a$$

Thus,

$$p_{L,Q}(a,b,c,\psi) = \frac{a}{4\pi^2} e^{-((a^2+2b^2+c^2))/2}$$

$\psi$ is independent and uniform in $[0, 2\pi)$, hence

$$p_L(a,b,c) = \frac{a}{2\pi} e^{-((a^2+2b^2+c^2))/2}$$

(c) $W = LL^T = \begin{bmatrix} a^2 & ab \\ 0 & b^2+c^2 \end{bmatrix}$

$$J = \det \begin{pmatrix} 2a & b & 0 \\ 0 & a & 2b \\ 0 & 0 & 2c \end{pmatrix} = 4a^2c$$

$$p_W(W) = \frac{a}{2\pi} \cdot \frac{1}{4a^2c} e^{-(a^2+2b^2+c^2)/2}$$

$$= \frac{1}{8\pi ac} e^{-(a^2+2b^2+c^2)/2}$$

$$= \frac{1}{8\pi \det(L)} e^{-\mathrm{Tr}(LL^T)/2}$$

$$= \frac{1}{8\pi\sqrt{\det W}} e^{-\mathrm{Tr}(W)/2}$$

2. (a) $Q$ is p.s.d. $\Rightarrow x^H Q x \geq 0 \;\forall x \in \mathbb{C}^n$

so in particular choose $x = H^H y$, where $y \in \mathbb{C}^n$.

Thus $y^H \underbrace{HQH^H}y \geq 0 \;\forall y \in \mathbb{C}^n$, which means

$HQH^H \triangleq W$ is also p.s.d.

(b) Let $Q = VMV^H$ be the eigenvalue decomposition of $Q$.

Hint $\Rightarrow$ $W = (HV)M(HV)^H$ and $HMH^H$ have the same distribution.

(c) $\tilde{h}_{ij} = h_{ij}\sqrt{M_j}$ $\Rightarrow$

$$p_{\tilde{h}_{ij}}(z) = \frac{1}{M_j}\, p_{h_{ij}}\!\left(\frac{z}{\sqrt{M_j}}\right) = \frac{1}{\pi M_j} e^{-\frac{|z|^2}{M_j}}$$

and
$$p_{\tilde{H}}(\tilde{H}) = \prod_{i,j=1}^{n} \frac{1}{\pi M_j} e^{-\frac{|\tilde{h}_{ij}|^2}{M_j}}$$

$$= \frac{C_n}{(\det M)^n} e^{-Tr(\tilde{H} M^{-1} \tilde{H}^H)}$$

with $C_n \triangleq \frac{1}{\pi^{n^2}}$

(d) Hint $\Rightarrow$ $p_{\tilde{W}}(\tilde{W}) = \frac{C_n}{(\det M)^n} e^{-Tr(M^{-1}\tilde{W})}$

(e) ~~Hint~~ The joint eigenvalue distribution of $W$ and $\tilde{W}$ is identical.

~~Let $\tilde{W} = V\Lambda V^H$ be the eigenvalue decomposition of $\tilde{W}$.~~