ELEN E6880: RMT with Applications
Homework #3

Chenye Yang cy2540@columbia.edu

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P1

(a)

Ans:
Example of a regular sparse graph:

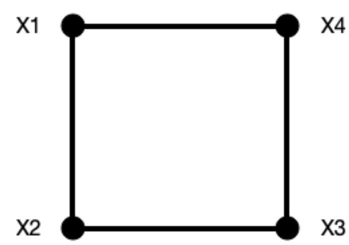


Figure 1: Regular sparse graph

Its corresponding matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(b)

Ans:
The self-consistency equations for the cavity variances $\Delta_i^{(j)}$:

$$\Delta_i^{(j)}(z) = \frac{1}{z - \sum_{l \in \partial i / j} A_{il}^2 \Delta_l^{(j)}(z)}$$
$$\Delta_i(z) = \frac{1}{z - \sum_{l \in \partial i} A_{il}^2 \Delta_l^{(j)}(z)}$$

Since the matrix is regular, homogeneous, the cavity equations bail down to:

$$\Delta^{(j)} = \frac{1}{z-(k-1)\Delta^{(j)}} \tag{1}$$

(c)

Ans:
Equation (1) is a quadratic equation, solve and get:

$$\Delta^{(j)} = \frac{z \pm \sqrt{z^2-4(k-1)}}{2(k-1)}$$

The solution of $\Delta^{(j)}$ should let the $p(\lambda)$ be on the upper side of real axis on complex plane. So we choose the "$-$":

$$\Delta^{(j)} = \frac{z - \sqrt{z^2-4(k-1)}}{2(k-1)}$$

E.g. in question **(d)**, to calculate $p(\lambda)$, we choose "$-$".

(d)

Ans:
Because

$$\begin{aligned} p(\lambda) &= \lim_{\varepsilon\to 0^+} \frac{1}{\pi N} \cdot N I_m[\Delta(z)]_{z=\lambda-j\varepsilon} \\ \Delta(z) &= \frac{1}{z-k\Delta^{(j)}} = \frac{1}{z-k\cdot\frac{z\pm\sqrt{z^2-4(k-1)}}{2(k-1)}} \end{aligned}$$

Thus

$$p(\lambda) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi} I_m \left[\frac{1}{z-\frac{kz\pm k\sqrt{z^2-4(k-1)}}{2(k-1)}} \right]_{z=\lambda-j\varepsilon} \tag{2}$$

Consider the expression in $Im[\cdot]$ in Equation (2), which equals

$$\frac{2(k-1)}{z\cdot 2(k-1)-kz\pm k\sqrt{z^2-4(k-1)}} = \frac{2(k-1)}{2\left(\lambda-j\varepsilon\right)\left(k-1\right)-k(\lambda-j\varepsilon)-k\sqrt{(\lambda-j\varepsilon)^2-4(k-1)}}$$

See $\varepsilon\to 0^+$, we get the expression equals

$$\frac{2(k-1)}{2\lambda(k-1)-k\lambda-jk\sqrt{-\lambda^2+4(k-1)}} = \frac{2(k-1)\left[2(k-1)\lambda-k\lambda+jk\sqrt{-\lambda^2+4(k-1)}\right]}{[2\lambda(k-1)-k\lambda]^2+k^2\left[4(k-1)-\lambda^2\right]}$$

Take the $Im[\cdot]$, we get

$$\begin{aligned}
p(\lambda) &= \frac{1}{\pi}\frac{2(k-1)k\sqrt{-\lambda^2+4(k-1)}}{4\lambda^2(k-1)^2-2k\lambda^2 2(k-1)+k^2\lambda^2+4k^2(k-1)-k^2\lambda^2} \\
&= \frac{1}{\pi}\frac{2(k-1)k\sqrt{-\lambda^2+4(k-1)}}{4\lambda^2(k-1)^2-4k\lambda^2(k-1)+4k^2(k-1)} \\
&= \frac{1}{\pi}\frac{2k\sqrt{4(k-1)-\lambda^2}}{4\lambda^2(k-1)-4k\lambda^2+4k^2} \\
&= \frac{k\sqrt{4(k-1)-\lambda^2}}{2\pi\left(k^2-\lambda^2\right)}\ , \quad |\lambda| \leq 2\sqrt{k-1}
\end{aligned}$$

The McKay Law for regular sparse matrices is derived.

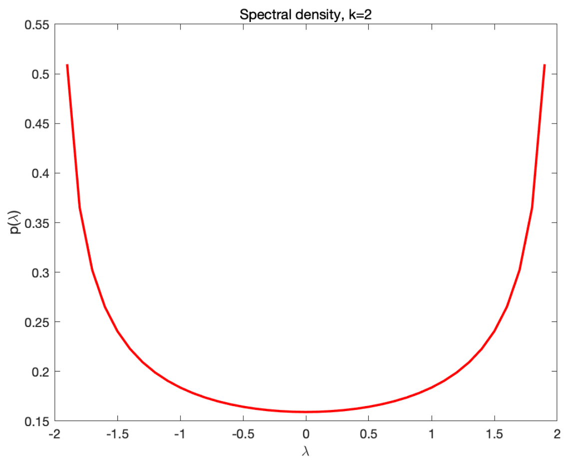
(e)

Code:

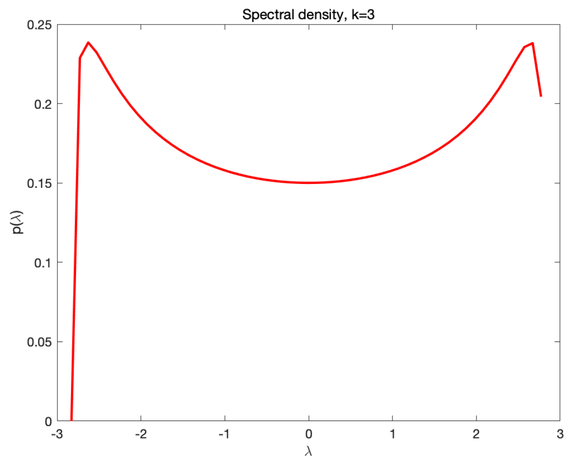
```
function [p] = SpectralDensity(lambda,k)
%the spectral density according to the McKay Law for regular sparse matrices
p = (k.*sqrt(4.*(k-1)-lambda.^2))./(2*pi.*(k^2-lambda.^2));
end
```

```
k = 10;
lambda = [-2*sqrt(k-1):0.1:2*sqrt(k-1)];
plot(lambda, SpectralDensity(lambda,k), '-r', 'linewidth', 2);
ylabel('p(\lambda)');
xlabel('\lambda');
title(['Spectral density, k=',num2str(k)]);
saveas(gcf,['/Users/yangchenye/Downloads/HW3_P1_e_',num2str(k),'.png'])
```

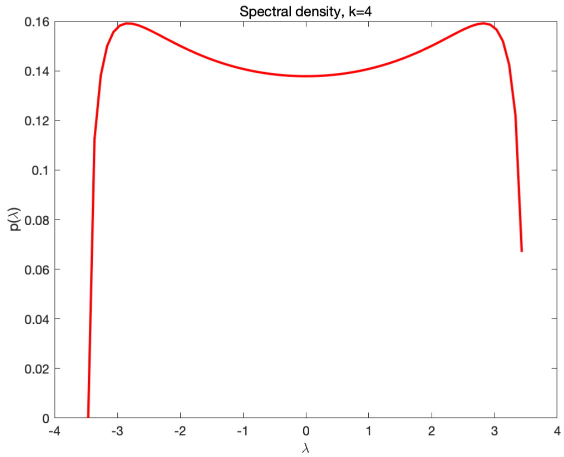
Result:



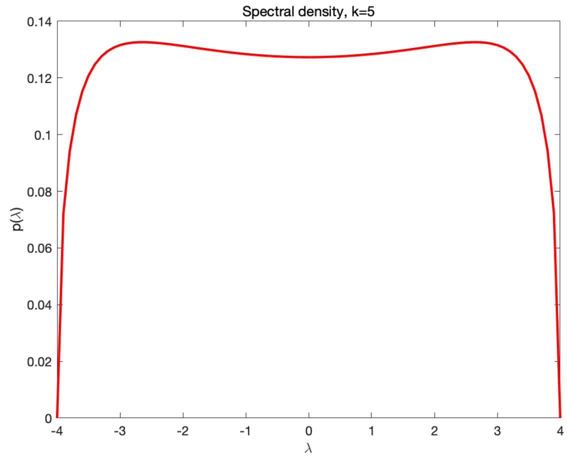
(a) $k = 2$



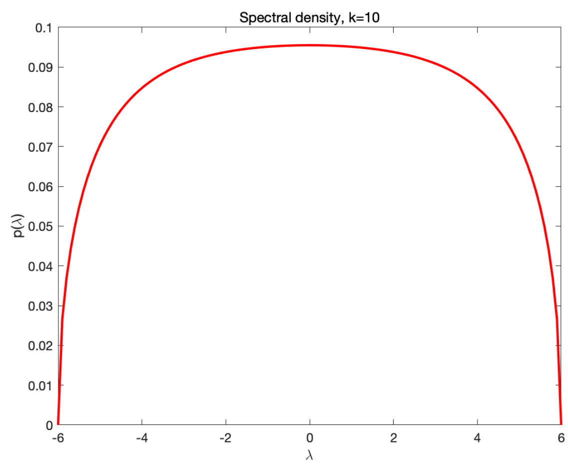
(b) $k = 3$



(c) $k = 4$



(d) $k = 5$



(e) $k = 10$

Figure 2: The spectral density $p(\lambda)$ for $k = 2, 3, 4, 5, 10$