ELEN E6880

Solution #2

① (a) $\sum_{j,k=0}^{n} c_j c_k m_{j+k} = \sum_{j,k=0}^{n} c_j c_k \int_{\mathbb{R}} x^{j+k} d\mu(x) = \int_{\mathbb{R}} \left(\sum_{j=0}^{n} c_j x^j\right)^2 d\mu(x) \geq 0$

(b) i, iii and iv are sequences of moments.

(i) corresponds to the uniform distribution on [0,1].

(iii) corresponds to $\mu = \delta(x-e)$

(iv) corresponds to the log-normal distribution

$$p_\mu(x) = \frac{1}{\sqrt{2\pi}\, x} e^{-\frac{\log^2 x}{2}} \quad x>0$$

However since (iv) does not satisfy Carleman's condition, there are other distributions with the same sequence of moments.

(ii) does not correspond to a distribution, as the corresponding matrices $A^{(n)}$ are not p.s.d. for all values of n. For instance,

$$A^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix} \text{ has eigenvalues } \lambda_\pm = 2 \pm \sqrt{5}$$

therefore is not p.s.d.

2 (a) by the change of variable

$$m_k = \int_0^4 x^{2k} \frac{1}{\pi}\sqrt{\frac{1}{x}-\frac{1}{4}}\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 4^k \sin(t)^{2k} \sqrt{\frac{1}{4\sin(t)^2}-\frac{1}{4}}\cdot 8\sin(t)\cos(t)\,dt$$

$$= \frac{4^{k+1}}{\pi}\int_0^{\frac{\pi}{2}} \sin(t)^{2k}\cos(t)^2\,dt = \frac{4^{k+1}}{\pi}\left(\int_0^{\pi/2}\sin(t)^{2k}\,dt - \int_0^{\frac{\pi}{2}}\sin(t)^{2(k+1)}\,dt\right)$$

By integration by parts with $u(t) \triangleq \sin^{2k+1}(t)$ and $v(t) \triangleq \sin(t)$ we get:

$$a_{k+1} \triangleq \int_0^{\frac{\pi}{2}} \sin(t)^{2(k+1)}\,dt = (2k+1)\int_0^{\frac{\pi}{2}} \sin(t)^{2k}\cos^2(t)\,dt =$$

$(2k+1)(a_k - a_{k+1})$, hence

$$a_{k+1} = \frac{2k+1}{2k+2}a_k = \dots = \frac{(2k+1)\cdots 3\cdot 1}{(2k+2)\cdots 4\cdot 2}\cdot a_0$$

$$= \frac{(2k+1)!/(2^k k!)}{2^{k+1}(k+1)!}a_0 = \frac{(2k+1)!}{2^{2k+1}k!(k+1)!}a_0$$

$$a_0 = \frac{\pi}{2} \Rightarrow m_k = \frac{4^{k+1}}{\pi}\frac{a_{k+1}}{2k+1} = \frac{(2k)!}{k!(k+1)!} = \frac{1}{k+1}\binom{2k}{k}$$

Carleman's condition is satisfied, since

$$m_k \le \frac{(2k)!}{(k!)^2} \le \frac{((2k)(2k-2)\cdots 2)^2}{(k!)^2} = \frac{(2^k k!)^2}{(k!)^2} \le 4^k$$

alternatively one can say μ has a compact support $[0,4]$

(b) define $y \triangleq x^{\lambda}$

$$\int_{\mathbb{R}} x^{k} \, d\mu(x) = c_{\lambda} \int_{0}^{\infty} x^{k} e^{-x^{\lambda}} \, dx = c_{\lambda} \int_{0}^{\infty} y^{k/\lambda} e^{-y} y^{\frac{1}{\lambda}-1} \, dy =$$

$$c_{\lambda} \lambda \, \Gamma((k+1)/\lambda)$$

↑

Euler Gamma function

using the approximation $\Gamma(x+1) \approx [x]!$

$$m_k = \int_{\mathbb{R}} x^{k} \, d\mu(x) \sim \left[\frac{k}{\lambda}\right]!$$

by Stirling's formula $\log(k!) \approx k \log k$

$$\limsup_{k\to\infty} \frac{1}{2k} (m_{2k})^{\frac{1}{2k}} \sim \limsup_{k\to\infty} \frac{1}{2k} e^{\frac{1}{\lambda}\log\left(\frac{2k}{\lambda}\right)}$$

$$\sim \limsup_{k\to\infty} \frac{1}{2k} \left(\frac{2k}{\lambda}\right)^{\frac{1}{\lambda}} < 0 \quad \text{iff } \lambda \geq 1.$$

Remember: a distribution is uniquely determined by its moments as long as its tail is not heavier than the exponential e^{-x}.

3 (a)

$$\operatorname{Re}\{g_\mu(u+jv)\} = \int_{\mathbb{R}} \frac{x-u}{(x-u)^2+v^2}\, d\mu(x)$$

$$\operatorname{Im}\{g_\mu(u+jv)\} = \int_{\mathbb{R}} \frac{v}{(x-u)^2+v^2}\, d\mu(x)$$

(b) g_μ is analytic on $\mathbb{C}\setminus\mathbb{R}$ since $z \mapsto \frac{1}{x-z}$ is analytic and the convergence theorem.

(c) If $v>0$, then $\operatorname{Im} g_\mu(u+jv)$ is clearly positive by the above formula.

(d)
$$\underbrace{v^2 |g_\mu(jv)|^2}_{\to\, 1} = \Bigg(\underbrace{\int_{\mathbb{R}} \frac{v(x-u)}{(x-u)^2+v^2}\, d\mu(x)}_{\xrightarrow{v\to\infty}\, 0}\Bigg)^2 + \Bigg(\underbrace{\int_{\mathbb{R}} \frac{v^2}{(x-u)^2+v^2}\, d\mu(x)}_{\xrightarrow{v\to\infty}\, 1}\Bigg)^2$$

(e) by the definition of S. transform.