ELEN E6880: RMT with Applications
Homework #2

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November 9, 2019

P1

(a)

Ans:
Because μ is a distribution and m_k is the sequence of its moments.

$$m_{j+k} = \int_R x^{j+k} d\mu(x)$$

Thus

$$\begin{aligned}\sum_{j,k=0}^{n} c_j c_k a_{jk}^{(n)} &= \sum_{j,k=0}^{n} c_j c_k m_{j+k} = \sum_{j,k=0}^{n} c_j c_k \int_R x^{j+k} d\mu(x) \\ &= \int_R \sum_{j,k=0}^{n} c_j c_k x^{j+k} d\mu(x)\end{aligned}$$

Because

$$\begin{aligned}\sum_{j,k=0}^{n} c_j c_k x^{j+k} =& (c_0 x^0)^2 + (c_1 x^1)^2 + (c_2 x^2)^2 + \cdots + (c_n x^n)^2 + \\ & 2(c_0 x^0 c_1 x^1 + c_0 x^0 c_2 x^2 + \cdots + c_0 x^0 c_n x^n + \\ & \quad c_1 x^1 c_2 x^2 + c_1 x^1 c_3 x^3 + \cdots + c_1 x^1 c_n x^n + \\ & \quad c_2 x^2 c_3 x^3 + c_2 x^2 c_4 x^4 + \cdots + c_2 x^2 c_n x^n + \\ & \quad \cdots + c_{n-1} x^{n-1} c_n x^n) \\ =& \left(c_0 x^0 + c_1 x^1 + \cdots + c_n x^n\right)^2 \\ =& \left(\sum_{j=0}^{n} x^j\right)^2\end{aligned}$$

Therefore

$$\sum_{j,k=0}^{n} c_j c_k a_{jk}^{(n)} = \int_R \left(\sum_{j=0}^{n} x^j\right)^2 d\mu(x) \geqslant 0$$

(b)

Ans:

i Yes

$$m_k = \frac{1}{k+1} = \int_R x^k d\mu(x)$$

Because

$$\int_0^1 x^k dx = \frac{1}{k+1}$$

Thus the p.d.f. of μ is

$$f(x) = \begin{cases} 1 & , \ x \in [0,1] \\ 0 & , \ \text{otherwise} \end{cases}$$

μ is uniform distribution on $[0,1]$

ii No

$$m_k = k^2 = \int_R x^k d\mu(x)$$

The corresponding matrix $A^{(1)}$

$$A^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}$$

has eigenvalues of $2 \pm \sqrt{5}$. The matrix $A^{(1)}$ is not Positive semi-definite. Therefore, the moments are not correspond to a distribution.

iii Yes

$$m_k = e^k = \int_R x^k d\mu(x)$$

Because

$$\int_R x^k \delta(x-e) dx = e^k$$

Thus the p.d.f. of μ is

$$f(x) = \delta(x-e)$$

iv Yes

$$m_k = e^{k^2/2} = \int_R x^k d\mu(x)$$

Because for log-normal distribution, $\ln X \sim N(\mu, \sigma^2)$, the n-th moment of a log-normally distributed variable X is given by

$$\mathrm{E}\left[X^n\right] = e^{n\mu + \frac{1}{2}n^2\sigma^2}$$

Thus, μ is log-normal distribution with 0 mean and 1 variance, $\ln X \sim N(0,1)$. The p.d.f. of μ is

$$f(x) = \frac{1}{x} \cdot \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{(\ln x)^2}{2}\right)$$

P2

(a)

Ans:
The moments

$$\begin{aligned} m_k &= \int_R x^k d\mu(x) = \int_R x^k \frac{1}{\pi}\sqrt{\frac{1}{x}-\frac{1}{4}}\mathbf{1}\{0<x<4\}dx \\ &= \int_0^4 x^k \frac{1}{\pi}\sqrt{\frac{1}{x}-\frac{1}{4}}dx \end{aligned}$$

Let $x = 4\sin^2(t)$

$$\begin{aligned} m_k &= \int_0^{\frac{\pi}{2}} \left[4\sin^2(t)\right]^k \frac{1}{\pi}\sqrt{\frac{1}{4\sin^2(t)}-\frac{1}{4}}\cdot d\left(4\sin^2(t)\right) \\ &= \int_0^{\frac{\pi}{2}} \frac{4^k\sin^{2k}(t)}{2\pi}\sqrt{\frac{1}{\sin^2(t)}-1}\cdot 8\sin(t)\cos(t)dt \\ &= \frac{4^{k+1}}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2k+1}(t)\sqrt{\frac{\cos^2(t)}{\sin^2(t)}}\cos(t)dt \\ &= \frac{4^{k+1}}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2k+1}(t)\frac{\cos(t)}{\sin(t)}\cos(t)dt \\ &= \frac{4^{k+1}}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2k}(t)\cos^2(t)dt \\ &= \frac{4^{k+1}}{\pi}\left[\int_0^{\frac{\pi}{2}} \sin^{2k}(t)dt - \int_0^{\frac{\pi}{2}} \sin^{2k+2}(t)dt\right] \end{aligned}$$

Let $a_{k+1} \triangleq \int_0^{\frac{\pi}{2}} \sin^{2(k+1)}(t)dt = (2k+1)\int_0^{\frac{\pi}{2}} \sin^{2k}(t)\cos^2(t)dt = (2k+1)(a_k - a_{k+1})$
$\Rightarrow$

$$a_{k+1} = \frac{2k+1}{2k+2}a_k$$

Thus

$$\begin{aligned} a_{k+1} &= \frac{2k+1}{2k+2}a_k = \cdots = \frac{(2k+1)(2k-1)(2k-3)\cdots 3\cdot 1}{(2k+2)(2k)(2k-2)\cdots 4\cdot 2}\cdot a_0 \\ &= \frac{(2k+1)!/\left(2^k\cdot k!\right)}{2^{k+1}\cdot(k+1)!}a_0 = \frac{(2k+1)!}{2^{2k+1}k!(k+1)!}a_0 \end{aligned}$$

Because $a_0 = \int_0^{\frac{\pi}{2}} \sin^2(t)dt = \int_0^{\frac{\pi}{2}} \frac{1-\cos(2t)}{2} dt = \frac{\pi}{4} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2t)dt = \frac{\pi}{4}$
Thus

$$
\begin{aligned}
m_k &= \frac{4^{k+1}}{\pi}\left(a_k - a_{k+1}\right) \\
&= \frac{4^{k+1}}{\pi}\left[\frac{(2k-1)!}{2^{2k-1}(k-1)!k!}a_0 - \frac{(2k+1)!}{2^{2k+1}k!(k+1)!}a_0\right] \\
&= \frac{4^{k+1}}{\pi}\frac{(2k-1)!2^2\cdot k\cdot(k+1) - (2k+1)!}{2^{2k+1}k!(k+1)!}a_0 \\
&= \frac{4^{k+1}}{\pi}\frac{(2k)!\cdot(2k+2) - (2k+1)!}{2^{2k+1}k!(k+1)!}a_0 \\
&= \frac{4^{k+1}}{\pi}\frac{(2k+2)! - (2k+1)(2k+1)!}{(2k+1)2^{2k+1}k!(k+1)!}a_0 \\
&= \frac{4^{k+1}}{\pi}\frac{(2k+1)!}{(2k+1)2^{2k+1}k!(k+1)!}a_0 \\
&= \frac{4^{k+1}}{\pi}\frac{(2k)!}{2^{2k+1}k!(k+1)!}\frac{\pi}{4} \\
&= 4^k\cdot\frac{(2k)!}{2^{2k+1}k!(k+1)!} \\
&= \frac{(2k)!}{2k!(k+1)!}
\end{aligned}
$$

Because

$$
\begin{aligned}
m_k \leqslant \frac{(2k)!}{(k!)^2} &\leqslant \frac{[(2k)(2k-2)\cdots 2]^2}{(k!)^2} \\
&= \frac{\left(2^k k!\right)^2}{(k!)^2} \\
&= 4^k
\end{aligned}
$$

Thus

$$
m_{2k}^{-\frac{1}{2k}} \geqslant \frac{1}{\sqrt[2k]{4^{2k}}} = \frac{1}{4}
$$

So

$$
\sum_{k\geqslant 1}\left(m_{2k}\right)^{-\frac{1}{2k}} = \infty
$$

Therefore, according to Carleman's condition, μ is the unique distribution with the sequence of moments.

(b)

Ans:
The moments

$$
m_k = \int_R x^k d\mu(x) = \int_0^{+\infty} x^k C_\lambda e^{-x^\lambda} dx = C_\lambda \int_0^{+\infty} x^k e^{-x^\lambda} dx
$$

Let $y = x^\lambda$, $x = y^{1/\lambda}$, $\Gamma(k) = \int_0^{+\infty} y^{k-1} e^{-y} dy$
We have

$$\begin{aligned} m_k &= C_\lambda \int_0^{+\infty} y^{\frac{k}{\lambda}} e^{-y} \frac{1}{\lambda} y^{\frac{1}{\lambda}-1} dy = \frac{C_\lambda}{\lambda} \int_0^{+\infty} y^{\frac{k}{\lambda}+\frac{1}{\lambda}-1} e^{-y} dy \\ &= \frac{C_\lambda}{\lambda} \Gamma(\frac{k+1}{\lambda}) \end{aligned}$$

Because as $x \to \infty$

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy \approx [x-1]!$$

Thus as $k \to \infty$

$$m_k \approx \frac{C_\lambda}{\lambda} [\frac{k+1}{\lambda} - 1]!$$

Thus

$$\begin{aligned} (m_{2k})^{-\frac{1}{2k}} &= \exp\left\{-\frac{1}{2k} \ln\left(m_{2k}\right)\right\} \\ &\approx \exp\left\{-\frac{1}{2k} \ln\left(\frac{C_\lambda}{\lambda}\left[\frac{2k+1}{\lambda} - 1\right]!\right)\right\} \\ &= \exp\left\{-\frac{1}{2k} \ln\frac{C_\lambda}{\lambda} - \frac{1}{2k} \ln\left(\left[\frac{2k+1}{\lambda} - 1\right]!\right)\right\} \\ &\approx \exp\left\{-\frac{1}{2k} \ln\frac{C_\lambda}{\lambda} - \frac{1}{2k}\left(\frac{2k+1}{\lambda} - 1\right) \ln\left(\frac{2k+1}{\lambda} - 1\right)\right\} \\ &= \exp\left\{-\frac{1}{2k} \ln\frac{C_\lambda}{\lambda}\right\} \exp\left\{-\frac{1}{2k}\left(\frac{2k+1}{\lambda} - 1\right) \ln\left(\frac{2k+1}{\lambda} - 1\right)\right\} \\ &= \left(\frac{C_\lambda}{\lambda}\right)^{-\frac{1}{2k}} \left(\frac{2k+1}{\lambda} - 1\right)^{-\frac{1}{2k}\left(\frac{2k+1}{\lambda}-1\right)} \\ &\geqslant \left(\frac{C_\lambda}{\lambda}\right)^{-\frac{1}{2k}} \left(\frac{2k+1}{\lambda}\right)^{-\frac{1}{2k}\left(\frac{2k+1}{\lambda}-1\right)} \\ &= \left(\frac{\lambda}{C_\lambda}\right)^{\frac{1}{2k}} \left(\frac{\lambda}{2k+1}\right)^{\frac{1}{\lambda}+\frac{1}{2k\lambda}-\frac{1}{2k}} \end{aligned}$$

Because only if $\lambda \to \infty$ as $k \to \infty$,

$$\lim_{k\to\infty} \left(\frac{\lambda}{C_\lambda}\right)^{\frac{1}{2k}} \left(\frac{\lambda}{2k+1}\right)^{\frac{1}{\lambda}+\frac{1}{2k\lambda}-\frac{1}{2k}} \neq 0$$

Thus λ can be chosen as $\lambda = k$, and

$$\lim_{k\to\infty} \left(\frac{\lambda}{C_\lambda}\right)^{\frac{1}{2k}} \left(\frac{\lambda}{2k+1}\right)^{\frac{1}{\lambda}+\frac{1}{2k\lambda}-\frac{1}{2k}} = 1$$

So

$$\lim_{k\to\infty} (m_{2k})^{-\frac{1}{2k}} \geqslant 1$$

Thus

$$\sum_{k\geqslant 1} (m_{2k})^{-\frac{1}{2k}} = \infty$$

Therefore, according to Carleman's condition, μ is the unique distribution with the sequence of moments.

P3

(a)

Ans:

$$
\begin{aligned}
g_\mu(z) &= \int_R \frac{1}{x-z} d\mu(x) = \int_R \frac{1}{x-u-jv} d\mu(x) \\
&= \int_R \frac{(x-u)+jv}{[(x-\mu)-jv][(x-u)+jv]} d\mu(x) \\
&= \int_R \frac{(x-u)+jv}{(x-u)^2+v^2} d\mu(x)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{Re}\{g_\mu(u+jv)\} &= \int_R \frac{x-u}{(x-u)^2+v^2} d\mu(x) \\
\mathrm{Im}\{g_\mu(u+jv)\} &= \int_R \frac{v}{(x-u)^2+v^2} d\mu(x)
\end{aligned}
$$

(b)

Ans:
Because $\frac{1}{x-z}$ is analytic on C/R, thus

$$
g_\mu(z) = \int_R \frac{1}{x-z} d\mu(x), \quad z \in C/R
$$

is analytic.

(c)

Ans:
According to **P3 (a)**,

$$
\mathrm{Im}\{g_\mu(z)\} = \int_R \frac{v}{(x-u)^2+v^2} d\mu(x)
$$

If $\mathrm{Im}\{z\} > 0$, i.e. $v > 0$, we have

$$
\frac{v}{(x-u)^2+v^2} > 0
$$

Thus

$$
\int_R \frac{v}{(x-u)^2+v^2} d\mu(x) > 0
$$

Therefore, if $\mathrm{Im}\{z\} > 0$, $\mathrm{Im}\{g_\mu(z)\} > 0$

(d)

Ans:

$$
\begin{aligned}
v\left|g_\mu(iv)\right| &= v\left|\int_R \frac{1}{x-iv}d\mu(x)\right| = v\left|\int_R \frac{x+iv}{x^2+v^2}d\mu(x)\right| \\
&= v\int_R \frac{|x+iv|}{x^2+v^2}d\mu(x) = v\int_R \frac{\sqrt{x^2+v^2}}{x^2+v^2}d\mu(x) \\
&= \int_R \frac{v}{\sqrt{x^2+v^2}}d\mu(x) = \int_R \sqrt{\frac{v^2}{x^2+v^2}}d\mu(x) \\
&= \int_R \frac{1}{\sqrt{x^2/v^2+1}}d\mu(x)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\lim_{v\to\infty} v\left|g_\mu(iv)\right| &= \lim_{v\to\infty}\int_R \frac{1}{\sqrt{x^2/v^2+1}}d\mu(x) \\
&= \int_R \lim_{v\to\infty}\frac{1}{\sqrt{x^2/v^2+1}}d\mu(x) \\
&= \int_R d\mu(x)
\end{aligned}
$$

Because μ is a probability distribution on R,

$$
\int_R d\mu(x) = 1
$$

Therefore, $\lim_{v\to\infty} v\left|g_\mu(iv)\right| = 1$

(e)

Ans:
Let $z = u + jv$

$$
\begin{aligned}
g_\mu(z^*) &= g_\mu(u-jv) \\
&= \int_R \frac{1}{x-u+jv}d\mu(x) \\
&= \int_R \frac{(x-u)-jv}{[(x-\mu)-jv][(x-u)+jv]}d\mu(x) \\
&= \int_R \frac{(x-u)-jv}{(x-u)^2+v^2}d\mu(x)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathrm{Re}\{g_\mu(z^*)\} &= \int_R \frac{x-u}{(x-u)^2+v^2}d\mu(x) = \mathrm{Re}\{g_\mu(z)\} \\
\mathrm{Im}\{g_\mu(z^*)\} &= -\int_R \frac{v}{(x-u)^2+v^2}d\mu(x) = -\mathrm{Im}\{g_\mu(z)\}
\end{aligned}
$$

Therefore, $g_\mu(z^*) = \{g_\mu(z)\}^*$