ELEN E6880: RMT with Applications
Homework #1

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October 19, 2019

P1

(a)

Ans:

$$\begin{aligned} P_{\boldsymbol{H}}(\boldsymbol{H}) &= \prod_{j,k=1}^{n,m} \frac{1}{\pi} e^{-|h_{jk}|^2} = \frac{1}{\pi^{nm}} e^{-\sum_{j,k=1}^{n,m} |h_{jk}|^2} = \frac{1}{\pi^{nm}} e^{-Tr(\boldsymbol{H}\boldsymbol{H}^H)} \\ &= \frac{1}{\pi^4} e^{-Tr(\boldsymbol{H}\boldsymbol{H}^H)} \end{aligned} \tag{1}$$

(b)

Ans:

$$\boldsymbol{H} \triangleq \boldsymbol{L}\boldsymbol{Q} = \begin{bmatrix} a\cos u & a\sin u \\ b\cos u - c\sin u & b\sin u + c\cos u \end{bmatrix} \tag{2}$$

Hence,

$$\begin{aligned} J &= \det \begin{bmatrix} \cos u & \sin u & 0 & 0 \\ 0 & 0 & \cos u & \sin u \\ 0 & 0 & -\sin u & \cos u \\ -a\sin u & a\cos u & -b\sin u - c\cos u & b\cos u - c\sin u \end{bmatrix} \\ &= a \end{aligned} \tag{3}$$

Therefore,

$$P_{\boldsymbol{L},\boldsymbol{Q}}(a,b,c,u) = \frac{J}{(2\pi)^2} e^{-Tr(\boldsymbol{H}\boldsymbol{H}^T)/2} = \frac{a}{4\pi^2} e^{-(a^2+b^2+c^2)/2} \tag{4}$$

u is independent and uniform in $[0, 2\pi)$, hence,

$$P_{\boldsymbol{L}}(a,b,c) = \int_0^{2\pi} P_{\boldsymbol{L},\boldsymbol{Q}}(a,b,c,u)du = \frac{a}{2\pi} e^{-(a^2+b^2+c^2)/2} \tag{5}$$

(c)

Ans:

$$\boldsymbol{W} = \boldsymbol{L}\boldsymbol{L}^T = \begin{bmatrix} a^2 & ab \\ 0 & b^2 + c^2 \end{bmatrix} \tag{6}$$

$$\begin{aligned} J &= \det \begin{bmatrix} 2a & b & 0 \\ 0 & a & 2b \\ 0 & 0 & 2c \end{bmatrix} \\ &= 4a^2 c \end{aligned} \tag{7}$$

$$\begin{aligned} P_{\boldsymbol{W}}(\boldsymbol{W}) &= \frac{a}{2\pi} \frac{1}{4a^2 c} e^{-(a^2+b^2+c^2)/2} \\ &= \frac{1}{8\pi ac} e^{-(a^2+b^2+c^2)/2} \\ &= \frac{1}{8\pi det(\boldsymbol{L})} e^{-Tr(\boldsymbol{H}\boldsymbol{H}^T)/2} \\ &= \frac{1}{8\pi \sqrt{det(\boldsymbol{W})}} e^{-Tr(\boldsymbol{W})/2} \end{aligned} \tag{8}$$

P2

(a)

Ans:
Because $\boldsymbol{Q}$ is positive semi-definite matrix, as a result $x^H \boldsymbol{Q} x \geq 0, \ \forall x \in \mathbb{C}^n$.
So in particular choose $x = \boldsymbol{H}^H y$, where $y \in \mathbb{C}^n$.
Thus, $y^H \boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^H y \geq 0, \ \forall y \in \mathbb{C}^n$.
Therefore, $\boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^H \triangleq \boldsymbol{W}$ is also positive semi-definite matrix.

(b)

Ans:
Let $\boldsymbol{Q} = \boldsymbol{V}\boldsymbol{M}\boldsymbol{V}^H$ be the eigenvalue decomposition of $\boldsymbol{Q}$. Because $\boldsymbol{H}$ and $\boldsymbol{H}\boldsymbol{V}$ have the same distribution for any unitary matrix $\boldsymbol{V}$, thus $\boldsymbol{W} = (\boldsymbol{H}\boldsymbol{V})\boldsymbol{M}(\boldsymbol{H}\boldsymbol{V})^H$ and $\boldsymbol{H}\boldsymbol{M}\boldsymbol{H}^H$ have the same distribution.

(c)

Ans:

$$\widetilde{h_{ij}} = h_{ij}\sqrt{\mu_j}$$

$\Rightarrow$

$$P_{\widetilde{h_{ij}}}(Z) = \frac{1}{\mu_j} P_{h_{ij}}\left(\frac{Z}{\sqrt{\mu_j}}\right) = \frac{1}{\pi \mu_j} e^{\frac{-|Z|^2}{\mu_j}}$$

And

$$\begin{aligned} P_{\widetilde{\boldsymbol{H}}}(\widetilde{\boldsymbol{H}}) &= \prod_{i,j=1}^{n} \frac{1}{\pi \mu_j} e^{\frac{-|\widetilde{h}_{ij}|^2}{\mu_j}} \\ &= \frac{C_n}{(det(\boldsymbol{M}))^n} e^{-Tr(\widetilde{\boldsymbol{H}}\boldsymbol{M}^{-1}\widetilde{\boldsymbol{H}}^H)} \end{aligned}$$

with $C_n \triangleq \frac{1}{\pi^{n^2}}$

(d)

Ans:
Because the Jacobian of $\widetilde{\boldsymbol{H}} \mapsto \widetilde{\boldsymbol{W}}$ is a constant.
Therefore,

$$P_{\widetilde{\boldsymbol{W}}}(\widetilde{\boldsymbol{W}}) = \frac{C_n}{(det(\boldsymbol{M}))^n} e^{-Tr(\boldsymbol{M}^{-1}\widetilde{\boldsymbol{W}})}$$

(e)

Ans:
The joint eigenvalue distribution of $\boldsymbol{W}$ and $\widetilde{\boldsymbol{W}}$ is identical.

P3

Ans:
Kernel:

$$K(\lambda, \mu) \triangleq e^{-\frac{\lambda+\mu}{2}} \sum_{\ell=0}^{n-1} L_\ell(\lambda) L_\ell(\mu) \tag{9}$$

Properties of the kernel K:
(a) $K(\mu, \lambda) = K(\lambda, \mu)$
(b) $\int_0^\infty d\lambda K(\lambda, \lambda) = n$
(c) Self-reproducing property: $\int_0^\infty d\mu K(\lambda, \mu) K(\mu, v) = K(\lambda, v)$

The Laguerre polynomials maintain the orthogonality relations:

$$\int_0^\infty d\lambda e^{-\lambda} L_k(\lambda) L_l(\lambda) = \delta_{k,l} = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases} \tag{10}$$

To prove property (b):

$$\int_0^\infty d\lambda K(\lambda, \lambda) = \sum_{l=0}^{n-1} \int_0^\infty d\lambda e^{-\lambda} L_l(\lambda)^2 = n \tag{11}$$

To prove property (c):

$$\begin{aligned} \int_0^\infty d\mu K(\lambda, \mu) K(\mu, v) &= e^{-\frac{\lambda+v}{2}} \sum_{l,m=0}^{n-1} L_l(\lambda) L_m(v) \cdot \int_0^\infty L_l(\mu) L_m(\mu) e^{-\mu} d\mu \\ &= e^{-\frac{\lambda+v}{2}} \sum_{l=0}^{n-1} L_l(\lambda) L_l(v) \triangleq K(\lambda, v) \end{aligned} \tag{12}$$