Homework #1: Finite Analysis

ELEN E6880: RMT with Applications

1. Let $\mathbf{H}$ be a 2×2 Gaussian matrix with i.i.d. $h_{ij} \sim \mathcal{N}(0,1)$ entries and $\mathbf{W} \triangleq \mathbf{H}\mathbf{H}^T$ be a Wishart matrix.

   (a) Explicitly write the joint distribution of the entries of the matrix $\mathbf{H}$.

   Let $\mathbf{H} \triangleq \mathbf{L}\mathbf{Q}$, where $\mathbf{L}$ is a 2×2 lower-triangular matrix and $\mathbf{Q}$ is a 2×2 orthogonal matrix.

   (b) Compute the Jacobian of the transformation $\mathbf{H} \mapsto (L, Q)$ and use it for writing down the joint distribution of $\mathbf{L}$ and $\mathbf{Q}$.
   Hint: You may assume that $\mathbf{L}$ and $\mathbf{Q}$ are of the form:

   $$\mathbf{L} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

   and that

   $$\mathbf{Q} = \begin{pmatrix} \cos(u) & \sin(u) \\ -\sin(u) & \cos(u) \end{pmatrix},$$

   where $a, c \geq 0$, $b \in \mathbb{R}$ and $u \in [0, 2\pi]$.

   From the LQ decomposition, we obtain $\mathbf{W} = \mathbf{H}\mathbf{H}^T = \mathbf{L}\mathbf{L}^T$.

   (c) Compute the Jacobian of the transformation $\mathbf{L} \mapsto \mathbf{W}$ and infer the joint distribution of the entries of $\mathbf{W}$.

2. Let $\mathbf{H}$ be an $n \times n$ complex matrix with i.i.d. $h_{ij} \sim \mathcal{CN}(0,1)$ entries and $\mathbf{Q}$ be an $n \times n$ deterministic and positive semi-definite matrix. We would like to determine the joint distribution of the eigenvalues of the $n \times n$ matrix $\mathbf{W} \triangleq \mathbf{H}\mathbf{Q}\mathbf{H}^H$.

   (a) Show that $\mathbf{W}$ is positive semi-definite.

(b) Let $\mathbf{M} \triangleq \mathrm{diag}(\mu_1, \ldots, \mu_n)$, where $\mu_1, \ldots, \mu_n$ are the positive eigenvalues of $\mathbf{Q}$. Show that $\mathbf{W}$ and $\mathbf{H}\mathbf{M}\mathbf{H}^H$ have the same distribution.
Hint: Recall $\mathbf{H}$ and $\mathbf{H}\mathbf{V}$ have the same distribution for any unitary matrix $\mathbf{V}$.

(c) Compute the joint distribution of the entries of $\tilde{\mathbf{H}} \triangleq \mathbf{H}\mathbf{M}^{1/2}$.
Hint: $\mathbf{M}^{1/2} = \mathrm{diag}(\sqrt{\mu_1}, \ldots, \sqrt{\mu_n})$.

(d) Compute the joint distribution of the entries of the matrix $\tilde{\mathbf{W}} \triangleq \tilde{\mathbf{H}}^H\tilde{\mathbf{H}}$.
Hint: Recall the Jacobian of $\tilde{\mathbf{H}} \mapsto \tilde{\mathbf{W}}$ is a constant.

(e) What can you say about the joint eigenvalue distributions of the matrices $\mathbf{W}$ and $\tilde{\mathbf{W}}$?

3. Prove the Kernel Lemma's properties (b) and (c) ((a) is obvious) based on the orthogonality relations of the Laguerre polynomials[1].

[1] For solving this you may need to wait to the class about the marginal eigenvalue distribution of Wishart random matrices.