Homework #2: Infinite Analysis

ELEN E6880: RMT with Applications

1. Let μ be a distribution and $(m_k, k \geq 0)$ the sequence of its moments. For a given $n \geq 1$, let also $A^{(n)}$ be the $(n+1)\times(n+1)$ matrix defined as $a^{(n)}_{jk} = m_{j+k}, 0 \leq j, k \leq n$.

   (a) Show that for all $n \geq 1$, the matrix $A^{(n)}$ is positive semi-definite, i.e. that for all $c_0, \ldots, c_n \in \mathbb{R}$,

   $$\sum_{j,k=0}^{n} c_j c_k a^{(n)}_{jk} \geq 0.$$

   hint: Note that $A^{(n)}$ is symmetric.

   (b) Among the following sequences of numbers, which are sequences of moments of a given distribution? and of which distribution?

      i. $(m_k = \frac{1}{k+1}, k \geq 0)$,
      ii. $(m_k = k^2, k \geq 0)$,
      iii. $(m_k = e^k, k \geq 0)$,
      iv. $(m_k = e^{k^2/2}, k \geq 0)$.

2. Compute the moments of the following distributions and tell which of them are uniquely determined by their moments, using Carleman's conition.

   (a) Let μ be the "quarter-circle law" whose pdf is given by

   $$p_\mu(x) = \frac{1}{\pi}\sqrt{\frac{1}{x} - \frac{1}{4}}1_{\{0<x<4\}}.$$

   hint: Use induction and the change of variables $x = 4\sin^2(t)$.

   (b) Let $\lambda > 0$ and μ be the distribution whose pdf is given by

   $$p_\mu(x) = C_\lambda \exp\big\{(-x^\lambda)\big\}, \quad x > 0,$$

   with C_λ an appropriate normalization constant. For which values of λ is the distribution μ_λ uniquely determined by its moments? (no need to exactly compute the moments.) **hint:** Use the approximation $\Gamma(x) = \int_0^\infty y^{x-1}e^{-y}dy \sim [x-1]!$ as $x \to \infty$.

3. Let μ be a probability distribution on $\mathbb{R}$ and $g_\mu : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ be its Stieltjes transform, defined as

   $$g_\mu(z) = \int_{\mathbb{R}} \frac{1}{x-z} d\mu(x), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

(a) Writing $z = u + jv$, decompose $g_\mu(z)$ into its real and imaginary parts.

(b) Show g_μ is analytic on $\mathbb{C}\backslash\mathbb{R}$.

(c) Show that $\text{Im}\{g_\mu(z)\} > 0$, if $\text{Im}\{z\} > 0$.

(d) Show that $\lim_{v\to\infty} v|g_\mu(iv)| = 1$.

(e) Show that $g_\mu(z^*) = \{g_\mu(z)\}^*$.