Lecture 8: Policy Gradient

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Outline

- Policy Gradient RL
- Actor-Critic Methods
- Policy Gradient w/ Advantage Function

*materials are modified from David Silver's RL lecture notes

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Introduction

- In the last lecture we approximated the value or action-value function using parameters θ,

$$V_\theta(s) \approx V^\pi(s)$$
$$Q_\theta(s,a) \approx Q^\pi(s,a)$$

- A policy was generated directly from the value function
  - e.g. using ϵ-greedy
- In this lecture we will directly parametrise the policy

$$\pi_\theta(s,a) = \mathbb{P}\left[a \mid s, \theta\right]$$

- We will focus again on model-free reinforcement learning

Why Policy-Based RL?

- Advantages:

  Better convergence properties

  Effective in high-dimensional or continuous action spaces

  Can learn stochastic policies

- Disadvantages:

  Typically converge to a local rather than global optimum

  Evaluating a policy is typically inefficient and high variance

Policy Objective Functions

- Goal: given policy $\pi_\theta(s, a)$ with parameters θ, find best θ
- But how do we measure the quality of a policy π_θ?
- In episodic environments we can use the start value

$$J_1(\theta) = V^{\pi_\theta}(s_1) = \mathbb{E}_{\pi_\theta}[v_1]$$

- In continuing environments we can use the average value

$$J_{avV}(\theta) = \sum_s d^{\pi_\theta}(s) V^{\pi_\theta}(s)$$

- Or the average reward per time-step

$$J_{avR}(\theta) = \sum_s d^{\pi_\theta}(s) \sum_a \pi_\theta(s, a) \mathcal{R}_s^a$$

where $d^{\pi_\theta}(s)$ is stationary distribution of Markov chain for π_θ

Policy Optimization

- Policy based reinforcement learning is an optimisation problem
- Find θ that maximises $J(\theta)$

- Similar to the value based function approximation, we focus on gradient descent method
- Gradient is a key to connect neural network with RL algorithms
- Other approaches are possible

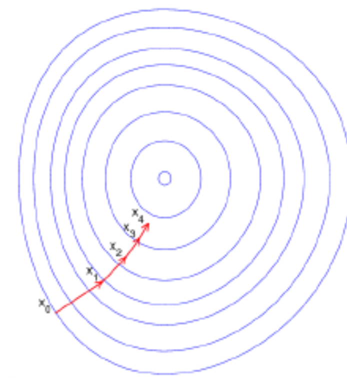
Gradient Descent (recap)

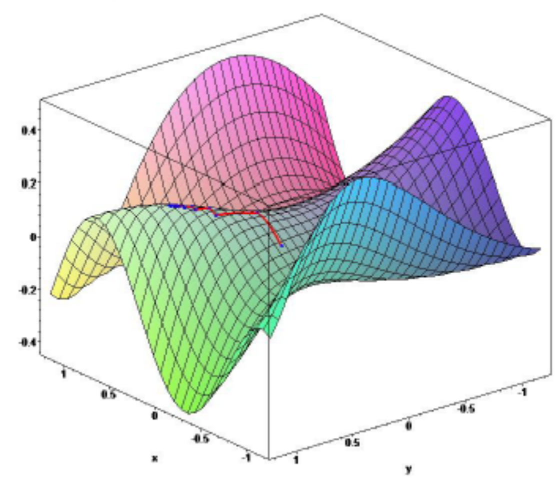
- Let $J(\mathbf{w})$ be a differentiable function of parameter vector $\mathbf{w}$
- Define the *gradient* of $J(\mathbf{w})$ to be

$$\nabla_{\mathbf{w}} J(\mathbf{w}) = \begin{pmatrix} \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}_1} \\ \vdots \\ \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}_n} \end{pmatrix}$$

- To find a local minimum of $J(\mathbf{w})$
- Adjust $\mathbf{w}$ in direction of -ve gradient

$$\Delta \mathbf{w} = -\frac{1}{2} \alpha \nabla_{\mathbf{w}} J(\mathbf{w})$$





Score Function

- We now compute the policy gradient *analytically*
- Assume policy π_θ is differentiable whenever it is non-zero and we know the gradient $\nabla_\theta \pi_\theta(s, a)$
- Likelihood ratios exploit the following identity

$$\begin{aligned} \nabla_\theta \pi_\theta(s, a) &= \pi_\theta(s, a) \frac{\nabla_\theta \pi_\theta(s, a)}{\pi_\theta(s, a)} \\ &= \pi_\theta(s, a) \nabla_\theta \log \pi_\theta(s, a) \end{aligned}$$

- The score function is $\nabla_\theta \log \pi_\theta(s, a)$

Example: Softmax Policy

- We will use a softmax policy as a running example
- Weight actions using linear combination of features $\phi(s, a)^\top \theta$
- Probability of action is proportional to exponentiated weight

$$\pi_\theta(s,a) = p(a|s,\theta) = \frac{e^{\phi(s,a)^T\theta}}{\sum_a e^{\phi(s,a)^T\theta}}$$

- The score function is

$$\nabla_\theta \log \pi_\theta(s, a) = \phi(s, a) - \mathbb{E}_{\pi_\theta}\left[\phi(s, \cdot)\right]$$

Example: Softmax Policy

- Proof:

$$
\begin{aligned}
\nabla \log \pi_\theta(s,a) &= \nabla \log \frac{e^{\phi(s,a)^T\theta}}{\sum_a e^{\phi(s,a)^T\theta}} \\
&= \nabla \phi(s,a)^T\theta - \nabla \log(\sum_a e^{\phi(s,a)^T\theta}) \\
&= \phi(s,a) - \frac{\sum_a (e^{\phi(s,a)^T\theta}\phi(s,a))}{\sum_a e^{\phi(s,a)^T\theta}} \\
&= \phi(s,a) - \sum_a \frac{e^{\phi(s,a)^T\theta}}{\sum_a e^{\phi(s,a)^T\theta}}\phi(s,a) \\
&= \phi(s,a) - \sum_a \pi_\theta(s,a)\phi(s,a) \\
&= \phi(s,a) - \mathbb{E}_{\pi_\theta}[\phi(s,\cdot)]
\end{aligned}
$$

Example: Gaussian Policy

- In continuous action spaces, a Gaussian policy is natural
- Mean is a linear combination of state features $\mu(s) = \phi(s)^\top \theta$
- Variance may be fixed σ^2, or can also parametrised
- Policy is Gaussian, $a \sim \mathcal{N}(\mu(s), \sigma^2)$
- The score function is

$$\nabla_\theta \log \pi_\theta(s, a) = \frac{(a - \mu(s))\phi(s)}{\sigma^2}$$

One-Step MDP

- Consider a simple class of one-step MDPs

  Starting in state $s \sim d(s)$

  Terminating after one time-step with reward $r = \mathcal{R}_{s,a}$
- Use likelihood ratios to compute the policy gradient

$$
\begin{aligned}
J(\theta) &= \mathbb{E}_{\pi_\theta}[r] \\
&= \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_\theta(s,a) \mathcal{R}_{s,a} \\
\nabla_\theta J(\theta) &= \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_\theta(s,a) \nabla_\theta \log \pi_\theta(s,a) \mathcal{R}_{s,a} \\
&= \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s,a) r]
\end{aligned}
$$

Generalized MDP

- The policy gradient theorem generalises the likelihood ratio approach to multi-step MDPs
- Replaces instantaneous reward r with long-term value $Q^{\pi}(s, a)$
- Policy gradient theorem applies to start state objective, average reward and average value objective

For any differentiable policy $\pi_\theta(s, a)$,
for any of the policy objective functions $J = J_1, J_{avR},$ *or* $\frac{1}{1-\gamma} J_{avV}$,
the policy gradient is

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta} \left[\nabla_\theta \log \pi_\theta(s, a) \; Q^{\pi_\theta}(s, a) \right]$$

Monte-Carlo Policy Gradient (REINFORCE)

- Update parameters by stochastic gradient ascent
- Using policy gradient theorem
- Using return v_t as an unbiased sample of $Q^{\pi_\theta}(s_t, a_t)$

$$\Delta\theta_t = \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) v_t$$

function REINFORCE
    Initialise θ arbitrarily
    for each episode $\{s_1, a_1, r_2, ..., s_{T-1}, a_{T-1}, r_T\} \sim \pi_\theta$ **do**
        for $t = 1$ to $T - 1$ **do**
            $\theta \leftarrow \theta + \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) v_t$
        end for
    end for
    return θ
end function

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Reducing Variance Using a Critic

- Monte-Carlo policy gradient still has high variance
- We use a critic to estimate the action-value function,

$$Q_w(s, a) \approx Q^{\pi_\theta}(s, a)$$

- Actor-critic algorithms maintain *two* sets of parameters
  - **Critic** Updates action-value function parameters w
  - **Actor** Updates policy parameters θ, in direction suggested by critic
- Actor-critic algorithms follow an *approximate* policy gradient

$$\nabla_\theta J(\theta) \approx \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s, a)\; Q_w(s, a)\right]$$
$$\Delta\theta = \alpha \nabla_\theta \log \pi_\theta(s, a)\; Q_w(s, a)$$

Example: TD(0)-based Actor-Critic Algorithm

- Simple actor-critic algorithm based on action-value critic
- Using linear value fn approx. $Q_w(s,a) = \phi(s,a)^\top w$

  Critic Updates w by linear TD(0)

  Actor Updates θ by policy gradient

function QAC
  Initialise s, θ
  Sample $a \sim \pi_\theta$
  for each step **do**
    Sample reward $r = \mathcal{R}_s^a$; sample transition $s' \sim \mathcal{P}_{s,\cdot}^a$
    Sample action $a' \sim \pi_\theta(s', a')$
    $\delta = r + \gamma Q_w(s', a') - Q_w(s, a)$
    $\theta = \theta + \alpha \nabla_\theta \log \pi_\theta(s, a) Q_w(s, a)$
    $w \leftarrow w + \beta \delta \phi(s, a)$
    $a \leftarrow a', s \leftarrow s'$
  end for
end function

Problem: Bias in Actor-Critic Algorithms

- Approximating the policy gradient introduces bias
- A biased policy gradient may not find the right solution

- Luckily, if we choose value function approximation carefully
- Then we can avoid introducing any bias

  i.e. We can still follow the *exact* policy gradient

Compatible Function Approximation Theorem*

If the following two conditions are satisfied:

Value function approximator is compatible to the policy

$$\nabla_w Q_w(s,a) = \nabla_\theta \log \pi_\theta(s,a)$$

Value function parameters w minimise the mean-squared error

$$\varepsilon = \mathbb{E}_{\pi_\theta}\left[\left(Q^{\pi_\theta}(s,a) - Q_w(s,a)\right)^2\right]$$

Then the policy gradient is exact,

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\ Q_w(s,a)\right]$$

*R. Sutton, et al. "Policy gradient methods for reinforcement learning with function approximation", 2000

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Reducing Variance Using a Baseline

- We subtract a baseline function $B(s)$ from the policy gradient
- This can reduce variance, without changing expectation

$$\begin{aligned}
\mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a) B(s)\right] &= \sum_{s \in \mathcal{S}} d^{\pi_\theta}(s) \sum_a \nabla_\theta \pi_\theta(s,a) B(s) \\
&= \sum_{s \in \mathcal{S}} d^{\pi_\theta} B(s) \nabla_\theta \sum_{a \in \mathcal{A}} \pi_\theta(s,a) \\
&= 0
\end{aligned}$$

- A good baseline is the state value function $B(s) = V^{\pi_\theta}(s)$
- So we can rewrite the policy gradient using the advantage function $A^{\pi_\theta}(s,a)$

$$A^{\pi_\theta}(s,a) = Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s)$$

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\; A^{\pi_\theta}(s,a)\right]$$

Estimating the Advantage Function

- The advantage function can significantly reduce variance of policy gradient
- So the critic should really estimate the advantage function

  For example, by estimating *both* $V^{\pi_\theta}(s)$ *and* $Q^{\pi_\theta}(s, a)$
- Using two function approximators and two parameter vectors,

$$
\begin{aligned}
V_v(s) &\approx V^{\pi_\theta}(s) \\
Q_w(s, a) &\approx Q^{\pi_\theta}(s, a) \\
A(s, a) &= Q_w(s, a) - V_v(s)
\end{aligned}
$$

- And updating *both* value functions by e.g. TD learning

Estimating the Advantage Function (cont.)

- For the true value function $V^{\pi_\theta}(s)$, the TD error δ^{π_θ}

  $$\delta^{\pi_\theta} = r + \gamma V^{\pi_\theta}(s') - V^{\pi_\theta}(s)$$

  is an unbiased estimate of the advantage function

  $$\begin{aligned}
  \mathbb{E}_{\pi_\theta}\left[\delta^{\pi_\theta} | s, a\right] &= \mathbb{E}_{\pi_\theta}\left[r + \gamma V^{\pi_\theta}(s') | s, a\right] - V^{\pi_\theta}(s) \\
  &= Q^{\pi_\theta}(s, a) - V^{\pi_\theta}(s) \\
  &= A^{\pi_\theta}(s, a)
  \end{aligned}$$

- So we can use the TD error to compute the policy gradient

  $$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s, a)\ \delta^{\pi_\theta}\right]$$

- In practice we can use an approximate TD error

  $$\delta_v = r + \gamma V_v(s') - V_v(s)$$

- This approach only requires one set of critic parameters v

Critic at Different Time-Scales

- Critic can estimate value function $V_\theta(s)$ from many targets at different time-scales

For MC, the target is the return v_t

linear approximation

$$\Delta\theta = \alpha(v_t - V_\theta(s))\boxed{\phi(s)}$$

For TD(0), the target is the TD target $r + \gamma V(s')$

$$\Delta\theta = \alpha(r + \gamma V(s') - V_\theta(s))\phi(s)$$

For forward-view TD(λ), the target is the λ-return v_t^λ

$$\Delta\theta = \alpha(v_t^\lambda - V_\theta(s))\phi(s)$$

For backward-view TD(λ), we use eligibility traces

$$\delta_t = r_{t+1} + \gamma V(s_{t+1}) - V(s_t)$$
$$e_t = \gamma\lambda e_{t-1} + \phi(s_t)$$
$$\Delta\theta = \alpha\delta_t e_t$$

Actor at Different Time-Scales

- The policy gradient can also be estimated at many time-scales

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta} \left[\nabla_\theta \log \pi_\theta(s, a) \; A^{\pi_\theta}(s, a)\right]$$

- Monte-Carlo policy gradient uses error from complete return

$$\Delta\theta = \alpha(v_t - V_v(s_t))\nabla_\theta \log \pi_\theta(s_t, a_t)$$

- Actor-critic policy gradient uses the one-step TD error

$$\Delta\theta = \alpha(r + \gamma V_v(s_{t+1}) - V_v(s_t))\nabla_\theta \log \pi_\theta(s_t, a_t)$$

Policy Gradient with Eligibility Traces

- Just like forward-view TD(λ), we can mix over time-scales

$$\Delta\theta = \alpha(v_t^\lambda - V_v(s_t))\nabla_\theta \log \pi_\theta(s_t, a_t)$$

where $v_t^\lambda - V_v(s_t)$ is a biased estimate of advantage fn

- Like backward-view TD(λ), we can also use eligibility traces
  - By equivalence with TD(λ), substituting $\phi(s) = \nabla_\theta \log \pi_\theta(s, a)$

$$\begin{aligned}
\delta &= r_{t+1} + \gamma V_v(s_{t+1}) - V_v(s_t) \\
e_{t+1} &= \lambda e_t + \nabla_\theta \log \pi_\theta(s, a) \\
\Delta\theta &= \alpha\delta e_t
\end{aligned}$$

- This update can be applied online, to incomplete sequences

Summary of Policy Gradient Algorithms

- The policy gradient has many equivalent forms

$$
\begin{aligned}
\nabla_\theta J(\theta) &= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\ v_t\right] && \text{REINFORCE} \\
&= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\ Q^w(s,a)\right] && \text{Q Actor-Critic} \\
&= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\ A^w(s,a)\right] && \text{Advantage Actor-Critic} \\
&= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\ \delta\right] && \text{TD Actor-Critic} \\
&= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\ \delta e\right] && \text{TD}(\lambda)\text{ Actor-Critic}
\end{aligned}
$$

- Each leads to a stochastic gradient ascent algorithm.