Review of Discrete Mathematics

Sets

Definition: A set is a collection of distinguisable, unordered objects, called members or elements

- If x is a member of a set S, we write $x \in S$.
- If x is not a member of a set S, we write $x \notin S$.

Examples: The set V of all the vowels in the English alphabet.

$$V = \{a, e, i, o, u\}$$

Definition: Two sets are equal if and only if they contain exactly the same elements

If A & B are sets, then $A = B$ if and only if $\forall x (x \in A \leftrightarrow x \in B)$.

Some special sets

- ϕ is the set w/ no elements. Empty set
- Z is the set of integer elements
- R is the set of real number elements
- N is the set of natural number elements.

SET OPERATORS

Let A & B be sets

Subset : A is a subset of B if and only if every element of A is also an element of B.

$A \subseteq B$ if and only if $\forall x (x \in A \rightarrow x \in B)$

Proper subset : if $A \subseteq B$ and $A \neq B$ then $A \subset B$

Intersection : The intersections of A & B, denoted by $A \cap B$, is the set containing those elements both in A & B

$A \cap B = \{x \mid x \in A \overset{\text{and}}{\wedge} x \in B\}$

Union : The union of A and B, denoted by $A \cup B$ is the set that contains ~~tb~~ those ~~$~~elements that are in either A or in B or in both

$A \cup B = \{x \mid x \in A \overset{\text{or}}{\vee} x \in B\}$

Difference : The difference of A and B, denoted by A − B, is the set that contains those elements that are in A but not in B. The difference of A & B is also called the complement of B with respect to A

$A - B = \{x \mid x \in A \wedge x \notin B\}$

Cartecian Product : The cartesian product of A & B, denoted ~~B~~ by $A \times B$ is the set of all ordered pairs $\{a, b\}$, where $a \in A$ & $b \in B$

$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

RELATIONS

Let A and B be sets

Binary Relations: A binary relation ~~R~~ ~~on~~ from A to B is a subset of $A \times B$.

A binary relation from A to B is a set R of ordered pairs where the first element comes from A and the second element comes from B.

$a R b$ denotes $(a, b) \in R$ — a is said to be related to b by R.

$a \not R b$ denotes $(a, b) \notin R$

Properties of Relations

Reflexive: Relation R on a set A is reflexive if $(a, a) \in R$ for every element ~~of A~~ $a \in A$.

Symmetric: Relation R on a set A is symmetric if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$

Using quantifiers $\forall a \forall b \, ((a, b) \in R \rightarrow (b, a) \in R)$

Antisymmetric: Relation R on a set A is antisymmetric if for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.

$\forall a \forall b \, ((a, b) \in R \wedge (b, a) \in R \rightarrow (a = b))$

Transitive: R is transitive if ~~a~~ whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ for all $a, b, c \in A$.

Example

$A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1)\}$

$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$

$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), \cancel{a}(3,3), (3,4), (4,4)\}$

$R_6 = \{(3,4)\}$

Reflexive: R_3, R_5

Symmetric: R_2, R_3

Antisymmetric: R_4, R_5, R_6

Hint: No pair of element a and b with $a \neq b$ such that (a,b) & (b,a) belong to the relation

Transitive: $\cancel{R_3}$ R_4, R_5

Equivalence relation: A relation on set A is called an equivalence relation if it is reflexive, symmetric and transitive.

e.g. $R = \{(a, b) : a, b \in N \wedge a+b \text{ is even}\}$

Let R be an equivalence relation on set A. The set of elements that are related to an element a of A is called the equivalence class of a.

$$[a]_R = \{s \mid (a, s) \in R\}$$

Partial Order: A relation R on a set S is called partial order if it is reflexive, antisymmetric & transitive

e.g. Relation $\geq$ is a partial ordering on the set of Integers.

(i) $a \geq a$ for every integer a, $\geq$ is reflexive

(ii) If $a \geq b$ and $b \geq a$, the $a = b$
$\geq$ is antisymmetric

(iii) If $a \geq b$ and $b \geq c \Rightarrow a \geq c$
$\therefore \geq$ is transitive

Total Order: A partial order on A is a total order if for ~~every~~ ever $a, b \in A$ aRb or bRa hold

Functions

Given non-empty sets A and B, a function f is a binary relation on $A \times B$ such that $\forall a \in A$, there exists exactly one $b \in B$ such that $(a, b) \in f$

- A is the domain of f ($a \in A$ is an argument to the function)
- B is the co-domain of f ($b \in B$ is the value of the function)

Common notation

- $f: A \to B$
- if $(a, b) \in f$, $b = f(a)$

f assigns an element of B to each element of A. No element of A is assigned to two different elements of B, but same element of B can be assigned to two different elements of A.

Definitions

A finite sequence is a function whose domain is $\{0, 1, \dots, n-1\}$, often written as $\langle f(0), f(1), \dots, f(n-1) \rangle$

An infinite sequence is a function whose domain is a set of $\mathbb{N}$ (natural numbers) ($\{0, 1, \dots\}$).

When the domain of f is a Cartesian product e.g. $A = A_1 \times A_2 \times \ldots \times A_n$, we write $f(a_1, a_2, \ldots, a_n)$ Instead of $f((a_1, a_2, \ldots, a_n))$.

We call each a_i an argument of f even though the argument is really the n-tuple $(a_1, a_2, \ldots, a_n)$

Image: If $f: A \to B$ is a function and $b = f(a)$ then we say that b is the image of a under f.

Range: The range of f is the set of all images of elements of A.

Surjection: A function f from A to B is called onto or surjection, if and only if for every element $b \in B$ there is an element $a \in A$ w/ $f(a) = b$.

- A function f is called surjective if it is onto.
- In other words, a function is a surjection if its range is its codomain

$f(n) = \lfloor n/2 \rfloor$ is a surjection $f: N$ to N

$f(n) = 2n$ is not a surjection $f:$ from N to N

$f(n) = 2n$ is a surjection $f:$ from N to even numbers

Injection : A function is an injection if distinct arguements to f produce distinct values.

i.e. $a \neq a' \Rightarrow f(a) \neq f(a')$

Also referred to as a one-to-one $f^{=}$

$f(n) = \lfloor n/2 \rfloor$ is not an injective $f^{=}$ from N to N

$f(n) = 2n$ is an injective $f^{=}$ from N to N

Bijection : A function is a bijection if it is both injective and surjective.

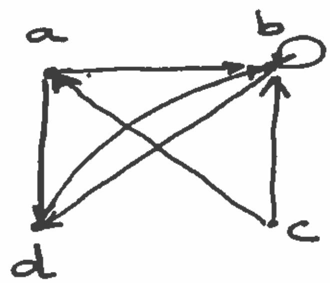
Also referred to a one-to-one correspondance

GRAPHS

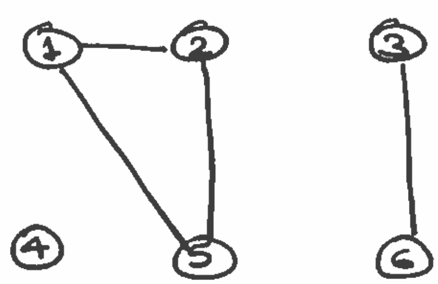
A directed graph (digraph) G is a pair (V, E) where V is a finite set of vertices and

E is a set of ordered pairs of elements of V called edges
(E is a subset of $V \times V$)

Example $V = \{a, b, c, d\}$
$E = \{(a,b), (a,d), (b,b), (b,d), (c,a)$
$(c,b), (d,b)\}$



An undirected graph G is a pair (V, E) where V is a finite set of ~~edges~~ (vertices) and
E (edges) is a set of unordered pairs of edges $\{u, v\}$ where $u \neq v$.



Properties of Edges

- If (u,v) is an edge in digraph G, then (u,v) is incident from or leaves u and is incident to or enters v.

- If (u,v) is an ~~and~~ edge in an undirected graph G, then (u,v) is ~~inc~~ incident to both u, and v.

- In both cases, v is adjacent to u; in a digraph adjacency is not necessary symmetric

- The degree of a vertex in an undirected graph is the ~~same~~ number of edges incident to it (which is the same as the number of vertices adjacent to it.

- The out-degree of a vertex in a digraph is the number of edges leaving it

- The in-degree of a vertex ~~is~~ in a digraph is the number of edges entering it.

PATHS IN GRAPHS

A path from u to v is a sequence of vertices $\langle v_0, v_1, \dots, v_k \rangle$ such that $u = v_0$, $v = v_k$ and $(v_{i-1}, v_i) \in E$ for $i = 1, 2, \dots k$

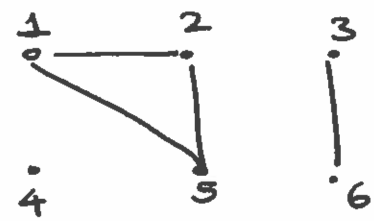
- The length of a path is the number of edges
- The path contains the vertices $v_0, v_1, \dots v_k$ and the edges $(v_0, v_1), (v_1, v_2) \dots (v_{k-1}, v_k)$
- v is reachable from u if there is a path from u to v.
- A path is simple if all vertices are distinct
- A subpath of a path p is any $\langle v_i, v_{i+1}, \dots, v_j \rangle$ where $0 \leq i \leq j \leq k$. p is a subpath of itself
- In a digraph, $\langle v_0, v_1, \dots, v_k \rangle$ is a cycle if $v_0 = v_k$ and $k \geq 1$. A cycle is simple if all vertices except $v_0 = v_k$ are distinct
- In an undirected graph, a path $\langle v_0, v_1 \dots v_k \rangle$ forms a cycle if $v_0 = v_k$, $k \geq 3$ & $v_1, v_2, \dots v_k$ are distinct
- An acyclic graph has no cycles

Connectivity in Graphs

An undirected graph is connected if each pair of vertices is connected by a path

The connected components are the equivalence classes of vertices under the "is reachable from" relation

E.g.



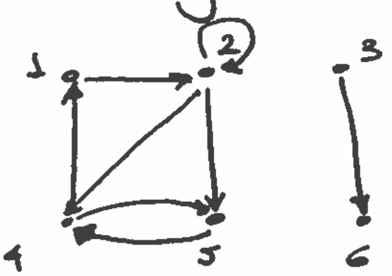
three connected components $\{1,2,5\}$, $\{3,6\}$ $\{4\}$. Every vertex in $\{1,2,5\}$ is reachable from every other vertex in $\{1,2,5\}$

An undirected graph is connected if it has exactly one connected component. i.e. every vertex is reachable from every other vertex.

A directed graph is strongly connected if every two vertices are reachable from one another.

The strongly connected components of a graph are the equivalence classes of vertices under the "are mutually reachable" relation.

A directed graph is strongly connected if it has only one strongly connected component.



$\{1,2,4,5\}$, $\{3\}$, $\{6\}$ Strongly connected components

$\{1,2,4,5\}$ are mutually reachable

Two graphs $G=(V,E)$ and $G'=(V',E')$ are isomorphic if there exists a bijection $f: V \to V'$ such that $(u,v) \in E$ if and only if $(f(u), f(v)) \in E'$.

In other words, we can relabel the vertices of G to be the vertices of G' maintaining the corresponding edges in G and G'

- The graph $G' = (V', E')$ is a subgraph of $G=(V,E)$ if $V' \subseteq V$ and $E' \subseteq E$

- Given a set $V' \subseteq V$, the subgraph G' induced by V' is

$$G' = (V', (V' \times V') \cap E) \quad \text{or}$$

$$E' = \{(u,v) \in E : u,v \in V'\}$$

- Given an undirected graph $G = \{V, E\}$ the directed version of G is the graph $G' = (V, E']$ where $(u,v) \in E'$ if and only if $(u,v) \in E$

  i.e. each undirect edge (u,v) in E is repkced in the directed version by the two directed edges (u,v) and (v,u).

Given a directed Graph $G=(V,E)$, the undirected version of G is the graph $G'=(V,E')$ where $(u,v) \in E'$ if $u \neq v$ and $(u,v) \in E$

i.e. the undirected version contains the ~~egs~~ edges of G with their direction removed and w/ self loops eliminated.

~~Special Graphs:~~

~~complete graph:~~

In a directed graph $G=(V,E)$, a neighbor of a vertex u is any vertex that is adjacent to u in the undirected version of G.

i.e. v is a neighbor of u if $u \neq v$ and either $(u,v) \in E$ or $(v,u) \in E$.

In an undirected graph, u and v are neighbors if they are adjacent.

Special Graphs:

complete graph: an undirected graph in which every pair of vertices is adjacent.

bipartite graph: an undirected graph in which the vertex set can be partitioned into two sets V_1 & V_2 such that every edge in the graph is of the form (x,y) ~~∈~~ where $x \in V_1$ & $y \in V_2$

i.e. all edges go between two sets V_1 and V_2.

forest : an acyclic undirected graph

tree : a connected, acyclic undirected graph

dag : directed acyclic graph

multigraph : like an undirected graph but can have multiple edges bet= vertices & self-loops.

hypergraph : Like an undirected graph but each hyper edge can connect an arbitrary number of vertices.

TREES

Trees: is a connected, acyclic, undirected graph.

Theorem (Properties of Trees)

Let $G = (V, E)$ be an undirected graph. Then the following are equivalent statements

1. G is a tree
2. Any two vertices of G are connected by a unique simple path.
   (each pair of vertices is connected by a path)
3. G is connected, but if any edge is removed from E, the resulting graph will not be connected
4. G is connected and $|E| = |V| - 1$
5. G is acyclic and $|E| = |V| - 1$
6. G is acyclic, but if any edge is added to E, the resulting graph contains a cycle.

Rooted Trees

A rooted trees is a tree in which on vertex is distinguished from the others.

- The distinguised vertex is called the root.
- A vertex of a routed tree is often called a node

Let r be the root of a rooted tree T. For any node x, there is a unique path from r to x.

- any node y on a path from r to x is an ancestor of x
- if y is an ancestor of x, then x is a descendant of y.
- every node is its own ancestor & descendant
- a proper ancestor (descendant) is an ancestor (descendant) that is not the node itself.
- the subtree rooted at x is the tree induced by the descendants of x.

- If the last edge of the path from r to x is (y, x), then y is the parent of x and x is the child of y.
  - The root is the only node w/ no parent
  - Siblings : two nodes that share the same parent
  - leaf : a node w/ no childern (external node)
  - internal node : a non-leaf node

The number of children of a node x in a rooted tree T is called the degree of x.

The length of a path from r to x is called the depth of x.

The largest depth of any node in T is the height of T.

An ordered tree is a rooted tree in which the children at each node are ordered.

Binary Trees

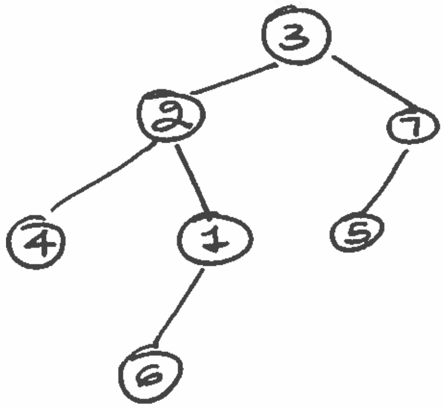
Binary trees are defined recursively.
A binary tree T is a structure defined on a finite set of nodes that either:

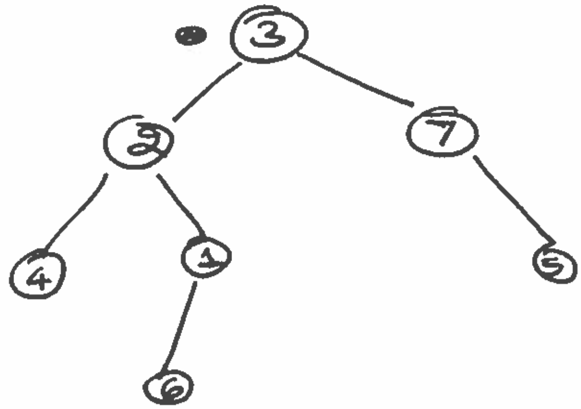
1. Contains no nodes (empty tree, null or NIL) or
2. composed of three disjoint ~~sets~~ sets of nodes: a root node, a left subtree & a right subtree

If the left subtree of a binary tree is non empty its root is called the ~~test~~ left child of the root of the entire tree. Similar definition of the right child.

A full binary tree is a binary tree in which each node is either a leaf or has degree 2.

A binary tree is not just an ordered tree in which each node has degree at most 2. left & right children matter.





As ordered trees these trees are the same. As binary trees these are distinct