Linear and Nonlinear Models

A statistical estimation problem is nonlinear if the estimating equations—the equations whose solution yields the parameter estimates—depend on the parameters in a nonlinear fashion. Such estimation problems typically have no closed-form solution and must be solved by iterative, numerical techniques.

Nonlinearity in the mean function is often used to distinguish between linear and nonlinear models. A model has a nonlinear mean function if the derivative of the mean function with respect to the parameters depends on at least one other parameter. Consider, for example, the following models that relate a response variable Y to a single regressor variable x:

$$
\begin{aligned}
\mathrm{E}[Y|x] &= \beta_0 + \beta_1 x \\
\mathrm{E}[Y|x] &= \beta_0 + \beta_1 x + \beta_2 x^2 \\
\mathrm{E}[Y|x] &= \beta + x/\alpha
\end{aligned}
$$

In these expressions, $\mathrm{E}[Y|x]$ denotes the expected value of the response variable Y at the fixed value of x. (The conditioning on x simply indicates that the predictor variables are assumed to be non-random. Conditioning is often omitted for brevity in this and subsequent chapters.)

The first model in the previous list is a simple linear regression (SLR) model. It is linear in the parameters β_0 and β_1 since the model derivatives do not depend on unknowns:

$$
\begin{aligned}
\frac{\partial}{\beta_0}(\beta_0 + \beta_1 x) &= 1 \\
\frac{\partial}{\beta_1}(\beta_0 + \beta_1 x) &= x
\end{aligned}
$$

The model is also linear in its relationship with x (a straight line). The second model is also linear in the parameters, since

$$
\begin{aligned}
\frac{\partial}{\beta_0}\left(\beta_0 + \beta_1 x + \beta_2 x^2\right) &= 1 \\
\frac{\partial}{\beta_1}\left(\beta_0 + \beta_1 x + \beta_2 x^2\right) &= x \\
\frac{\partial}{\beta_2}\left(\beta_0 + \beta_1 x + \beta_2 x^2\right) &= x^2
\end{aligned}
$$

However, this second model is *curvilinear*, since it exhibits a curved relationship when plotted against x. The third model, finally, is a nonlinear model since

$$
\begin{aligned}
\frac{\partial}{\beta}(\beta + x/\alpha) &= 1 \\
\frac{\partial}{\alpha}(\beta + x/\alpha) &= -\frac{x}{\alpha^2}
\end{aligned}
$$