# 矩阵论

## First Assignment

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1. **Question 2.2.2:**

   **Solve:** 因为
   $$Ax = \lambda x$$
   所以有
   $$A^m x = \lambda^m x$$
   即有
   $$\begin{aligned} ||A^m x|| &= ||\lambda^m x|| \\ &= |\lambda|^m ||x|| \end{aligned}$$
   且有 $||A^m x|| \leq ||A^m|| ||x||$ 所以
   $$|\lambda| \leq \sqrt[m]{||A^m||}$$

2. **Question 2.2.5:**

   **Solve**
   $$\begin{aligned} ||A||_S &= \max \frac{||Ax||_S}{||x||_S} \\ &= \max \frac{||SAx||_2}{||Sx||_2} \\ &= \max \frac{||SAS^{-1}y||_2}{||y||_2} \\ &= ||SAS^{-1}||_2 \end{aligned}$$

3. **Question 2.2.7: Solve** 先计算基2到基1的过渡矩阵
   $$C^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$
   由此可以计算T在基2下的矩阵为
   $$B = C^{-1}AC = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

4. **Question 1.2.14: Solve** 矩阵A的特征多项式为
   $$det(\lambda I - A) = (\lambda - 1)(\lambda + 1)(\lambda) - (\lambda - 1) = \lambda^3 - 2\lambda + 1 = 0$$
   所以
   $$A^3 - 2A + 1 = 0$$

$$
\begin{aligned}
&2A^9-3A^5+A^4+A^2-4I\\
&=24A^2-37A+10I\\
&=\begin{bmatrix}-3 & 48 & -26\\ 0 & 95 & -61\\ 0 & -61 & 34\end{bmatrix}
\end{aligned}
$$

5. **Question 1.2.16: Solve** 矩阵A的特征多项式为

$$det(\lambda I-A)=(\lambda-9)(\lambda+9)^2=\lambda^3+9\lambda^2-81\lambda-729$$

   由H-K定理得最小多项式为

$$(\lambda-9)(\lambda+9)$$

6. **Question 1.2.18: Solve** 因为$\lambda_0$为$T_1$特征值所以

$$T_1x=\lambda_0x$$

$$
\begin{aligned}
T_1T_2x&=T_2\lambda_0x\\
&=\lambda_0(T_2x)
\end{aligned}
$$

   即$T_2x\in V_\lambda 0$ 所以$V_\lambda 0$为$T_2$不变子空间

7. **Question 1.2.19: Solve**

$$det(\lambda I-A)=(\lambda-1)(\lambda-2)(\lambda+1)$$

   所以若当标准型为

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -1\end{bmatrix}$$

8. **Question 1.3.7: Solve** 先证充分性，若$x_1......x_n$线性相关,则有

$$\sum k_ix_i=0$$

   使之与$x_m$做内积得

$$\sum k_i(x_i,x_m)=0$$

   故detB=0 再证必要性, 由detB!=0可知,

$$\sum k_i(x_i,x_m)=0$$

   即$x_1......x_n$线性无关。

9. **Question 1.3.9:**

$$
\begin{aligned}
(Tx,Tx)&=(x-2(y,x)y,x-2(y,x)y)\\
&=(x,x)-4(y,x)(x,y)+4(y,x)(y,x)(y,y)\\
&=(x,x)
\end{aligned}
$$

   所以为正交变换。

10. **Question 1.3.11: Solve** 先求得该矩阵的特征值

$$det(\lambda I - A) = 0$$

解得

$$\lambda_1 = 0, \lambda_2 = \sqrt{2}, \lambda_3 = -\sqrt{2}$$

对应的特征向量为

$$x_1 = (0, \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T, x_2 = (\frac{1}{\sqrt{2}}, -\frac{i}{2}, \frac{1}{2})^T, x_3 = (-\frac{1}{\sqrt{2}}, -\frac{i}{2}, \frac{1}{2})^T$$

所以可求得

$$P = (x_1, x_2, x3)$$

11. **Question 1.3.15: Solve** (1)其中的一个标准正交基为

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

且$X_1 \odot X_2 = 0$所以这是一对正交基,将他们标准化得标准正交基为

$$a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, a_2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix},$$

(2)

$$\begin{aligned} \langle TX, Y \rangle &= \left\langle XP + X^T, Y \right\rangle \\ &= \langle XP, Y \rangle + \left\langle X^T, Y \right\rangle \quad = \langle X, Y \rangle P + \left\langle X, Y^T \right\rangle = \langle X, TY \rangle \end{aligned}$$

所以为对称变换。

(3)

$$T[a_1, a_2] = [a_1, a_2]A$$

由此可以求得T在基$(a_1, a_2)$下的矩阵

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

设C为基$(a_1, a_2)$到目标基$(b_1, b_2)$的过渡矩阵则对所要求的基下T对应的矩阵B应满足

$$B = C^{-1}AC$$

即

$$A = CBC^{-1}$$

且B为对角阵,即A可进行特征值分解得到B 解得

$$\lambda_1 = 2, \lambda_2 = 0$$

$$C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

由此可以得到

$$b_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, b_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix},$$