# Contents

# 1 Eigenvalues

Every linear mapping has a unique transformation matrix with respect to an ordered basis. Such matrix describes the linear mapping if the input and output are written relative to the coordinate system of the ordered basis. For instance, consider the linear transformation T from V to V, and let $\boldsymbol{A}$ be the matrix of T with respect to an ordered basis $\boldsymbol{A} = (f_1, f_2, ..., f_n)$ of V. Then the unique transformation matrix with respect to ordered basis $\boldsymbol{A}$ is as follows:

$$\boldsymbol{A} = \begin{bmatrix} | & & | \\ [T(f_1)]_{\boldsymbol{A}} & \cdots & [T(f_n)]_{\boldsymbol{A}} \\ | & & | \end{bmatrix}$$

The columns of the $\boldsymbol{A}$-matrix of the linear transformation T are the $\boldsymbol{A}$-coordinate vectors of the transforms of the basis elements $f_1, ..., f_n$ of V. We can interpret linear mappings and their associated transformation matrices by performing an "eigen" analysis. The **eigenvalues** of a linear mapping will tell us how a special set of vectors, the **eigenvectors**, is transformed by the linear mapping.

## 1.1 Definition

**Definition 1.1.** Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a square matrix. Then $\lambda \in \mathbb{R}$ is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x} \in \mathbb{R}^n \backslash \{\boldsymbol{0}\}$ (non-zero) is the corresponding eigenvector of $\boldsymbol{A}$ if

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}.$$

This is the eigenvalue equation.

*Remark.* In the linear algebra literature and software, it is often a convention that eigenvalues are sorted in descending order, so that the largest eigenvalue and associated eigenvector are called the first eigenvalue and its associated eigenvector, and the second largest called the second eigenvalue and its associated eigenvector, and so on.

## 1.2 Characteristic Equation

From the eigenvalue equation, we see that if there exists a nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$,

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \lambda \boldsymbol{x} \\ \boldsymbol{A}\boldsymbol{x} - \lambda \boldsymbol{x} &= \vec{0} \\ \boldsymbol{A}\boldsymbol{x} - \lambda I_n \boldsymbol{x} &= \vec{0} \\ (\boldsymbol{A} - \lambda I_n)\boldsymbol{x} &= \vec{0} \end{aligned}$$

Then by the definition of the kernel (null-space):

$$\ker(\boldsymbol{A} - \lambda I_n) \neq \{\vec{0}\}$$

In other words, there are other vectors in the kernel or null-space of the matrix $\boldsymbol{A} - \lambda I_n$ besides the zero vector. This is the case if (and only if) the matrix $\boldsymbol{A} - \lambda I_n$ **fails to be invertible**, which means that $\det(\boldsymbol{A} - \lambda I_n) = 0$. The logic can be summarized as follows:

$$\lambda \text{ is an eigenvalue of } \boldsymbol{A}$$
$$\Updownarrow$$
$$\text{There exists a nonzero vector } \vec{v} \text{ such that } \boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \text{ or } (\boldsymbol{A} - \lambda I_n)\, \boldsymbol{x} = \vec{0}$$
$$\Updownarrow$$
$$\ker(\boldsymbol{A} - \lambda I_n) \neq \{\vec{0}\}$$
$$\Updownarrow$$
$$\text{Matrix } \boldsymbol{A} - \lambda I_n \text{ fails to be invertible}$$
$$\Updownarrow$$
$$rank(\boldsymbol{A} - \lambda I_n) < n$$
$$\Updownarrow$$
$$\det(\boldsymbol{A} - \lambda I_n) = 0$$

## 1.3 Characteristic Polynomial

**Definition 1.2.** If $\boldsymbol{A}$ is an $n \times n$ matrix, then $\det(A - \lambda I_n)$ is a polynomial of degree $n$, of the form

$$\begin{aligned} &(-\lambda)^n + (\operatorname{tr} A)(-\lambda)^{n-1} + \cdots + \det A \\ &= (-1)^n \lambda^n + (-1)^{n-1} (\operatorname{tr} A) \lambda^{n-1} + \cdots + \det A \end{aligned}$$

This is called the characteristic polynomial of $\boldsymbol{A}$, denoted by $f_A(\lambda)$. Therefore, $\lambda \in \mathbb{R}$ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$ if and only if $\lambda$ is a root of the characteristic polynomial $f_A(\lambda)$ of $\boldsymbol{A}$.

## 1.4 Algebraic Multiplicity

**Definition 1.3.** Let a square matrix $\boldsymbol{A}$ have an eigenvalue $\lambda_i$. The algebraic multiplicity of $\lambda_i$ is the number of times the root appears in the characteristic polynomial.

## 1.5 Geometric Multiplicity

**Definition 1.4.** Consider an eigenvalue $\lambda$ of an $n \times n$ matrix $\boldsymbol{A}$. The dimension of eigenspace $E_\lambda = \ker(A - \lambda I_n)$ is called the geometric multiplicity of eigenvalue $\lambda$, denoted $\operatorname{gemu}(\lambda)$. Thus,

$$\operatorname{gemu}(\lambda) = \operatorname{nullity}(A - \lambda I_n) = n - \operatorname{rank}(A - \lambda I_n)$$

Another definition for geometric multiplicity is as follows.

**Definition 1.5.** Let $\lambda_i$ be an eigenvalue of a square matrix $\boldsymbol{A}$. Then the geometric multiplicity of $\lambda_i$ is the number of linearly independent eigenvectors associated with $\lambda_i$. In other words, it is the dimensionality of the eigenspace spanned by the eigenvectors associated with $\lambda_i$.

*Remark.* A specific eigenvalue's geometric multiplicity must be at least one because every eigenvalue has at least one associated eigenvector. An eigenvalue's geometric multiplicity cannot exceed its algebraic multiplicity, but it may be lower.

### 1.5.1 Example

The matrix $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ has two repeated eigenvalues $\lambda_1 = \lambda_2 = 2$ and an algebraic multiplicity of 2 . The eigenvalue has, however, only one distinct unit eigenvector $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and, thus, geometric multiplicity 1 .

## 1.6 Useful Properties

1. An $n \times n$ matrix has at most $n$ real eigenvalues, even if they are counted with their algebraic multiplicities. If $n$ is odd, then an $n \times n$ matrix has at least one real eigenvalue.

2. The eigenvalues of a triangular matrix are its diagonal entries.

3. If an $n \times n$ matrix $\boldsymbol{A}$ has the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, listed with their algebraic multiplicities, then

$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n$$

is the product of the (possibly repeated) eigenvalues and

$$\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

is the sum of the eigenvalues.

4. If $\lambda$ is an eigenvalue of a square matrix $\boldsymbol{A}$, then

$$\operatorname{gemu}(\lambda) \leq \operatorname{almu}(\lambda)$$

5. A matrix $\boldsymbol{A}$ and its transpose $\boldsymbol{A}^\top$ possess the same eigenvalues, but not necessarily the same eigenvectors.

6. Symmetric, positive definite matrices always have positive, real eigenvalues.

7. Similar matrices possess the same eigenvalues. Therefore, a linear mapping $\Phi$ has eigenvalues that are independent of the choice of basis of its transformation matrix. This makes eigenvalues, together with the determinant and the trace, key characteristic parameters of a linear mapping as they are all invariant under basis change. This last property deserves further attention.

### 1.7 Eigenvalues and Similarity

**Definition 1.6.** Consider two $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. We say that $\boldsymbol{A}$ is similar to $\boldsymbol{B}$ if there exists an invertible matrix $S$ (the change of basis matrix) such that

$$AS = SB, \quad \text{or} \quad B = S^{-1}AS.$$

In other words, matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ represent the same linear transformation with respective to different bases.

In such cases, the relationship between the eigenvalues of A and B can be described as follow:

a. Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same characteristic polynomial; that is, $f_A(\lambda) = f_B(\lambda)$.

b. $\operatorname{rank} A = \operatorname{rank} B$ and nullity $A =$ nullity $\boldsymbol{B}$.

c. Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same eigenvalues, with the same algebraic and geometric multiplicities. (However, the eigenvectors need not be the same.)

d. Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same determinant and the same trace: $\det A = \det B$ and $\operatorname{tr} A = \operatorname{tr} B$.

# 2 Eigenvectors

**Definition 2.1.** Consider an eigenvalue $\lambda$ of an $n \times n$ matrix $\boldsymbol{A}$. Then the kernel of the matrix $\boldsymbol{A} - \lambda I_n$ is called the eigenspace associated with $\lambda$, denoted by $E_\lambda$ :

$$E_\lambda = \ker\left(\boldsymbol{A} - \lambda I_n\right) = \left\{\boldsymbol{x} \text{ in } \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}\right\}.$$

Note that the eigenvectors associated with the eigenvalue $\lambda$ are the nonzero vectors in the eigenspace $E_\lambda$.

Another definition of the eigenspace is as follow.

**Definition 2.2.** For $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, the set of all eigenvectors of $\boldsymbol{A}$ associated with an eigenvalue $\lambda$ spans a subspace of $\mathbb{R}^n$, which is called the **eigenspace** of $\boldsymbol{A}$ with respect to $\lambda$ and is denoted by $E_\lambda$. The set of all eigenvalues of $\boldsymbol{A}$ is called the **eigenspectrum**, or just spectrum, of $\boldsymbol{A}$.

In other words, if $\lambda$ is an eigenvalue of $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, then the corresponding eigenspace $E_\lambda$ is the solution space of the homogeneous system of linear equations $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \vec{0}$.

### 2.1 Collinearity and Codirection

Two vectors that point in the same direction are called codirected. Two vectors are collinear if they point in the same or the opposite direction.

*Remark.* (Non-uniqueness of eigenvectors). If $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ associated with eigenvalue $\lambda$, then for any $c \in \mathbb{R}\backslash\{0\}$ (non-zero) it holds that $c\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ with the same eigenvalue since

$$\boldsymbol{A}(c\boldsymbol{x}) = c\boldsymbol{A}\boldsymbol{x} = c\lambda\boldsymbol{x} = \lambda(c\boldsymbol{x}).$$

Thus, all vectors that are collinear (they point in the same or the opposite direction) to $\boldsymbol{x}$ are also eigenvectors of $\boldsymbol{A}$.

### 2.2 Useful Properties and Theorems

1. The eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ of a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ with $n$ distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ are linearly independent. This theorem states that eigenvectors of a matrix with $n$ distinct eigenvalues form a basis of $\mathbb{R}^n$.

2. A square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is **defective** if it possesses fewer than $n$ linearly independent eigenvectors. A non-defective matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ does not necessarily require $n$ distinct eigenvalues (as eigenvalues may be repeated), but it does require that the eigenvectors form a basis of $\mathbb{R}^n$.

   Looking at the eigenspaces of a defective matrix, it follows that the sum of the dimensions of the eigenspaces is less than $n$. Specifically, a defective matrix has at least one eigenvalue $\lambda_i$ with an algebraic multiplicity $m > 1$ and a geometric multiplicity of less than $m$.

   *Remark.* A defective matrix cannot have $n$ distinct eigenvalues, as distinct eigenvalues have linearly independent eigenvectors.

## 3 Application: Eigen-decomposition and Diagonalization

A diagonal matrix is a matrix that has value zero on all off-diagonal elements, i.e., they are of the form

$$\boldsymbol{D} = \begin{bmatrix} c_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_n \end{bmatrix}$$

They allow fast computation of determinants, powers, and inverses.

- The determinant is the product of its diagonal entries.
- Matrix power $\boldsymbol{D}^k$ is given by each diagonal element raised to the power $k$
- The inverse $\boldsymbol{D}^{-1}$ is the reciprocal of its diagonal elements if all of them are nonzero.

Transforming matrix into a diagonal matrix is an application of change of basis. Two matrices $\boldsymbol{A}, \boldsymbol{D}$ are similar if there exists an invertible matrix $\boldsymbol{P}$, such that $\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$.

## 3.1 Diagonalization

**Definition 3.1.** (Diagonalizable). A matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is diagonalizable if it is similar to a diagonal matrix, i.e., if there exists an invertible matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ such that $\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$.

**In its essence, diagonalizing a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is a way of expressing the same linear mapping but in another basis, which will turn out to be a basis that consists of the eigenvectors of $\boldsymbol{A}$.**

### 3.1.1 Derivation

Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, let $\lambda_1, \dots, \lambda_n$ be a set of scalars, and let $\boldsymbol{p}_1, \dots, \boldsymbol{p}_n$ be a set of vectors in $\mathbb{R}^n$. We define $\boldsymbol{P} := [\boldsymbol{p}_1, \dots, \boldsymbol{p}_n]$ and let $\boldsymbol{D} \in \mathbb{R}^{n\times n}$ be a diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$. Then we can show that

$$\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{D}$$

if and only if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{p}_1, \dots, \boldsymbol{p}_n$ are corresponding eigenvectors of $\boldsymbol{A}$. This statement holds because

$$\boldsymbol{A}\boldsymbol{P} = \boldsymbol{A}\left[\boldsymbol{p}_1, \dots, \boldsymbol{p}_n\right] = \left[\boldsymbol{A}\boldsymbol{p}_1, \dots, \boldsymbol{A}\boldsymbol{p}_n\right]$$

$$\boldsymbol{P}\boldsymbol{D} = \underbrace{[\boldsymbol{p}_1, \dots, \boldsymbol{p}_n]}_{1\times n} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{n\times n} = \underbrace{[\lambda_1\boldsymbol{p}_1, \dots, \lambda_n\boldsymbol{p}_n]}_{1\times n}$$

Thus, $\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{D}$ implies that

$$\begin{aligned} \boldsymbol{A}\boldsymbol{p}_1 &= \lambda_1\boldsymbol{p}_1 \\ &\vdots \\ \boldsymbol{A}\boldsymbol{p}_n &= \lambda_n\boldsymbol{p}_n \end{aligned}$$

Since $\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{D}$ is true, we see that a square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ can be factored into

$$\begin{aligned} \boldsymbol{A}\boldsymbol{P} &= \boldsymbol{P}\boldsymbol{D} \\ \boldsymbol{A}\boldsymbol{P}\boldsymbol{P}^{-1} &= \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1} \\ \boldsymbol{A}\boldsymbol{I} &= \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1} \\ \boldsymbol{A} &= \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1} \end{aligned}$$

where $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{D}$ is a diagonal matrix whose diagonal entries are the eigenvalues of $\boldsymbol{A}$, if and only if the eigenvectors of $\boldsymbol{A}$ form a basis of $\mathbb{R}^n$. This further implies that only non-defective matrices ((with n linearly independent eigenvectors) can be diagonalized and that the columns of P are the n eigenvectors of A.

## 3.2 Symmetric Matrices

A symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ can always be diagonalized. This follows directly from the spectral theorem, which states that a matrix $\boldsymbol{A}$ is orthgonally diagonalizable (i.e., there exists an orthgonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{D}\boldsymbol{P} = \boldsymbol{P}^{\top}\boldsymbol{A}\boldsymbol{P}$) if and only if $\boldsymbol{A}$ is symmetric ($\boldsymbol{A} = \boldsymbol{A}^{\top}$). Moreover, the spectral theorem states that we can find an ONB (orthnormal basis) of eigenvectors of $\mathbb{R}^n$. This makes $\boldsymbol{P}$ an orthogonal matrix so that $\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{D}\boldsymbol{P} = \boldsymbol{P}^{\top}\boldsymbol{A}\boldsymbol{P}$. For defective matrices, the Jordan normal form of a matrix offers a decomposition that works (Lang, 1987).

## 3.3 Geometric Intuition Diagonalization

The matrix $\boldsymbol{A}$ represents a linear transformation with respect to some basis, and diagonalizing it by decomposition boils down to expressing this same linear mapping with respect to another basis, namely, the eigenbasis. Whatever the linear transformation may be with respect to the original basis— be it scaling or rotating— if it can be expressed with respect to the eigenbasis, it becomes a simple scaling in the coordinates of the eigenbasis. This is why $D$, the transformation matrix relative to the eigenbasis, is a diagonal, which is a scaling operation in this basis.

### 3.3.1 Example

Consider two $2 \times 2$ matrices $\boldsymbol{A}$ and $D$. Let $\boldsymbol{A}$ be the transformation matrix of a linear mapping with respect to the standard basis $e_i$ (blue arrows). We say that $\boldsymbol{A}$ is similar to $D$ if there exists an invertible matrix $P$ (the change of basis matrix) such that

$$AP = PD, \quad \text{or} \quad D = P^{-1}AP$$

The following diagram summarizes the change of basis nature of eigen-decomposition:

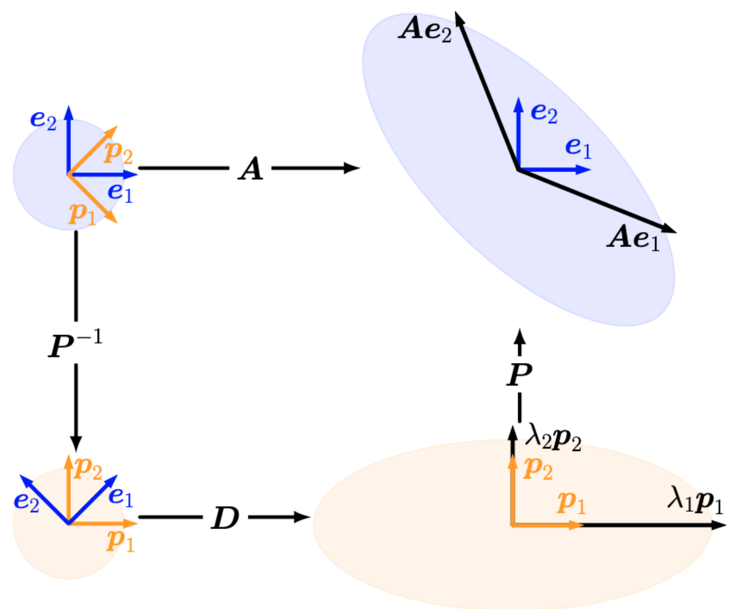
1. The matrix $\boldsymbol{A}$ has columns that are the transformed standard basis vectors:

$$A = \begin{bmatrix} | & | \\ Ae_1 & Ae_2 \\ | & | \end{bmatrix}$$

   That is, matrix $\boldsymbol{A}$ represents the linear mapping with respective to the standard basis vectors $e_i$; it scales both the basis vectors and rotates them. This results in the unit circle spanned by the blue vectors in the top left corner becoming a tilted ellipse spanned by the black vectors in the top right corner.

2. Matrix $P$ is the change of basis matrix from the eigenbasis to the standard basis with columns

$$P = \begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix}$$




Recall that columns of the change of basis matrix $P$ are the original basis expressed with respect to the new basis into which we are changing. In our case, the original basis is the eigenbasis and the basis we are change to is the standard basis. Indeed, vectors $p_1$ and $p_2$ are the eigenbasis vectors (orange in the top left corner) expressed with resepect the standard basis coordinates. So, the matrix $P$ translates vectors from the eigenbasis coordinate into the standard basis coordinates (arrow point from down to up).

3. The inverse $P^{-1}$ therefore translates a vector relative to the standard basis to be relative to the eigenbasis (arrow pointing from up to down).

4. In the eignbasis coordinates, the same linear transformation represented by $\boldsymbol{A}$ is a scaling transformation represented by diagonal matrix $D$. The bottom left and right figures show that $D$ simply scales the eigenbasis vectors $p_1$ and $p_2$ by $\lambda_1$ and $\lambda_2$, which are the eigenvalues of $\boldsymbol{A}$. Notice that the transformation no longer involves any sort of rotation in the eigenbasis coordinates, which simplifies greatly.

## 3.4 Implication of Diagonalization and Eigendecomposition

What diagonalization allows us to to do is, if there is a linear mapping $T$ with matrix $\boldsymbol{A}$ that is computationally costly in the coordinates of the current basis, then we can bypass this and achieve this same linear transformation via a change of basis. To summarize, if there is a vector $\boldsymbol{x}$ and a linear transformation $\boldsymbol{A}$ and computing $\boldsymbol{A}\boldsymbol{x}$ is computationally costly. We can avoid this via the following steps:

1. Translate the vector $\boldsymbol{x}$ relative to the current basis $A$ to be relative to the eigenbasis, $[\boldsymbol{x}]_P = P^{-1}\boldsymbol{x}$

where $[\boldsymbol{x}]_P$ is the vector $\boldsymbol{x}$ in the eigenbasis coordinate.

2. Scale the vector $[\boldsymbol{x}]_P$ by the eigenvalues of $\boldsymbol{A}$: $[T(x)]_P = D[\boldsymbol{x}]_P$

3. Undo the basis change by premultiplying the resultant vector above by matrix $P$ which receives vectors with respect to the eigenbasis and outputs vectors with respect to the original coordinate.

4. Therefore, we see that the linear transformation represented by $A$ is equivalent to the matrix $PDP^{-1}$:

$$A = PDP^{-1}$$

5. In the diagram above, starting from the top left corner, we go down, right, and then up.

## 3.5 Useful Properties of Diagonal Matrix

- Diagonal matrices $\boldsymbol{D}$ can efficiently be raised to a power. Therefore, we can find a matrix power for a matrix $\boldsymbol{A} \in \mathrm{R}^{n \times n}$ via the eigenvalue decomposition (if it exists) so that

$$\boldsymbol{A}^k = \left(\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}\right)^k = \boldsymbol{P}\boldsymbol{D}^k\boldsymbol{P}^{-1}$$

  Computing $\boldsymbol{D}^k$ is efficient because we apply this operation individually to any diagonal element.

- Assume that the eigendecomposition $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ exists. Then,

$$\begin{aligned} \det(\boldsymbol{A}) = \det\left(\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}\right) &= \det(\boldsymbol{P})\det(\boldsymbol{D})\det\left(\boldsymbol{P}^{-1}\right) \\ &= \det(\boldsymbol{D}) = \prod_i d_{ii} \end{aligned}$$

  allows for an efficient computation of the determinant of $\boldsymbol{A}$.

- **The eigenvalue decomposition requires square matrices**. It would be useful to perform a decomposition on general matrices. In the next section, we introduce a more general matrix decomposition technique, the singular value decomposition.