On elliptesque and hyperbolesque curves

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Five Points Determine A Conic Section

- Any five points in the plane (no four on a line) determine an unique conic.

- Five points determine a coinc. Formally, give any five points in the plane in general linear position (meaning no three collinear, there is an unique conic passing through them, which will be non-degenerate, this is true over both the affine plane and projective plane) Indeed, give any five points there is a conic passing through them, but if three points are collinear the conic will be degenerate (because it contains a line), and may not be unique.

- $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$
  Need five equations to solve a linear equation in five unknowns (A, B, C, D, E, F).
  It seems there are A, B, C, D, E, F six unknowns.
  But the equation can be written as
  $(Ax^2+Bxy+Cy^2+Dx+Ey)/F=-1$
  So in this equation has five unknowns if we treat A/F, B/F C/F, D/F, E/F as five unknowns.

Our goal

Simple example: The set { $y = x^3$; $x>0$ } is hyperbolesque and set { $y=x^{1.5}$; $1<x<38$ } is elliptesque.

• A set S is hyberbolesque if the conic determined by any five points from S is a hyperbola.

• A set S is elliptesque if the conic determined by any five points from S is an ellipse (or circle).

• And for now we tried some more examples.

Our goals are finding examples of these two sets.

Examples

- Example 1

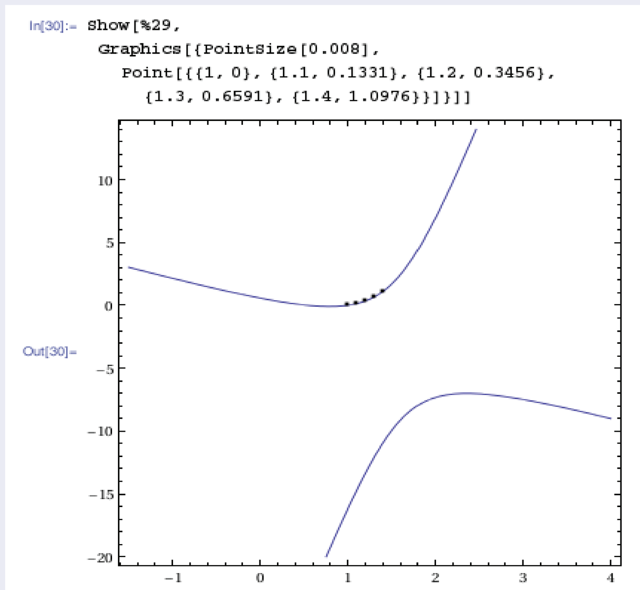


Figure 1

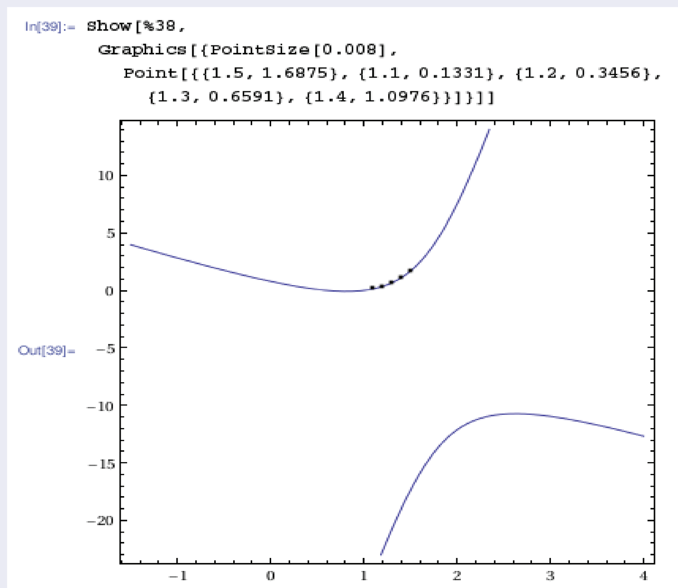


Figure 2

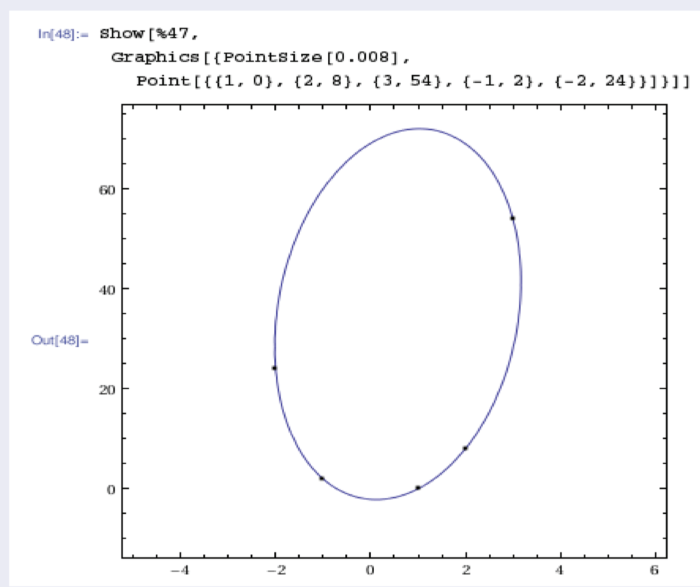
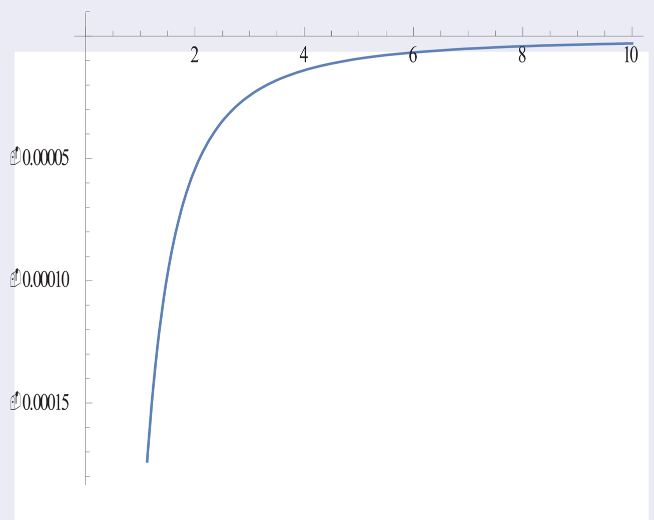


Figure 3

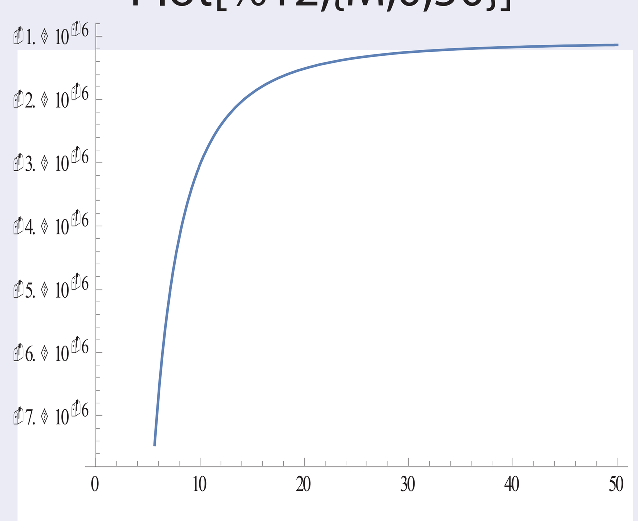
The example above is $y = x^4 - x^3$ and we choose five points from this equation then find the conic section goes through this five points
And for Figure 1 and Figure 2 we find that the equation seems to be a hyperbola most time. But in Figure 3 which is the special case when 5 points far apart, it is an ellipse.

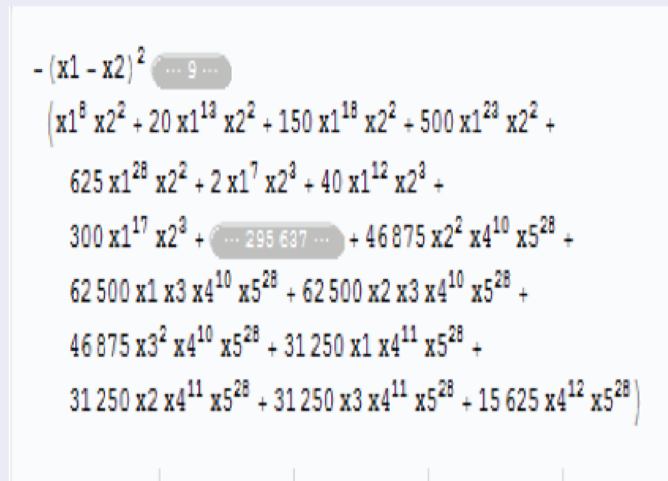
- Example 2

Plot[%12,{M,0,10}]



Plot[%12,{M,0,50}]





This is an example of $y=x^4+5x^9$. Firstly we pick up 4 points randomly and get our fifth point as M from Mathematica. Once we get M, Mathematica helps us get the sign of discriminant, if it is larger than 0, we will get an ellipse, otherwise it is hyperbola.
Then we pick up five points (x1,y1) (x2,y2) (x3,y3) (x4,y4) (x5,y5), and make our (x , y) goes to them individually, we solve the equation as c6 = 1 by Mathematica and get the sign of the coefficients, as shown in the third picture, there is no negative coefficient, which means for a >= 0 and xi >= 0, we will get a hyperbolesque.

Examples

$y=x^3$ $x=1,2,3,4,5$
$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 16 & 64 & 2 & 8 & 1 \\ 9 & 81 & 729 & 3 & 27 & 1 \\ 16 & 256 & 4096 & 4 & 64 & 1 \\ 25 & 625 & 15625 & 5 & 125 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = 0$$

A= -3101/1800
B= -7/90
C= 1/1800
D= 133/60
E= 7/12
F= 1

```
In[3]:= F[x_, y_] := {x^2, x*y, y^2, x, y, 1}
In[4]:= {F[1, 1], F[2, 8], F[3, 27], F[4, 64], F[5, 125]}
Out[4]= {{1, 1, 1, 1, 1, 1}, {4, 16, 64, 2, 8, 1}, {9, 81, 729, 3, 27, 1},
  {16, 256, 4096, 4, 64, 1}, {25, 625, 15625, 5, 125, 1}}
In[7]:= NullSpace[%4]
Out[7]= {{3101, 140, -1, -3990, -1050, 1800}}
In[8]:= %7[[1]]
Out[8]= {3101, 140, -1, -3990, -1050, 1800}
In[9]:= %8.F[x, y]
Out[9]= 1800 - 3990 x + 3101 x^2 - 1050 y + 140 x y - y^2
In[13]:= ContourPlot[%9 == 0, {x, -20, 20}, {y, -800, 200}]
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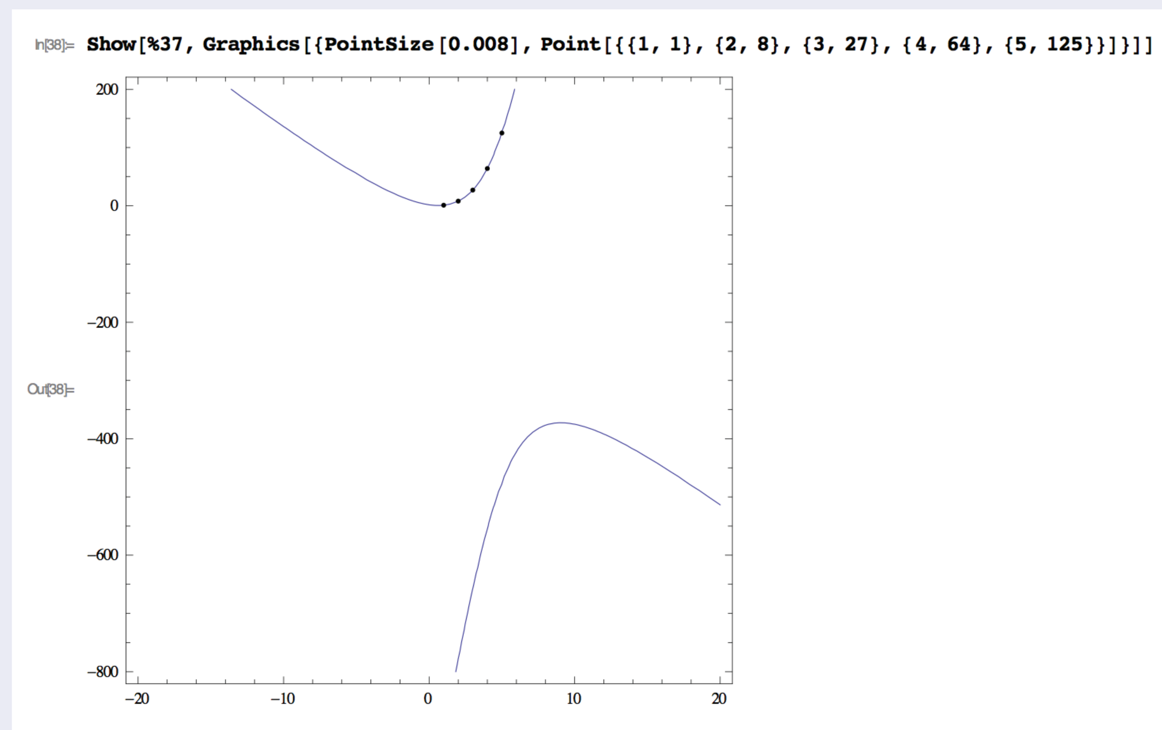

This is an image of hyperbola. We pick five points from the function $y=x^3$, and find that using these five points we can draw a hyperbola.

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