A Numerical Solution to the Partial Differential Equation of the Risk Based Capital for Guaranteed Minimum Withdrawal Benefit

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Abstract

This research project is dedicated to implement an algorithm for the partial differential equation solution to a risk management problem of the GMWB variable annuity developed in Feng & Vecer [1]. In the first part of the report, we give an introduction of the insurance product and the mathematical dynamics behind it. In the second part of the report, we introduce the finite difference method in solving partial differential equation with the illustration of an example. Finally, we discuss the implementation of the algorithm in Matlab.

1 Problem Introduction

Before introducing the concept of guaranteed minimum withdrawal benefit (GMWB), we first give a brief explanation of variable annuity, which is the base contract of GMWBs. From policyholders' perspective, variable annuity is a kind of insurance contract which allows the insured to gain financial returns from investing their insurance premium. Usually policyholders have a variety of options to invest their premium payments. Then with the capital that policyholders invest, the insurers transfer the funds to third-party vendors who manage and service the accounts of the policyholders' choice. The relatiohship between the insurer, the policyholder and the vendor can be represented in such a graph.[Figure 1]

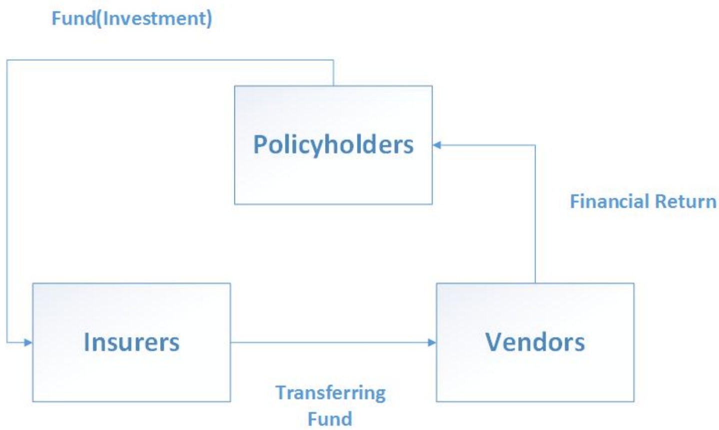

Figure 1: Variable Annuity Explanation

Then, what is the guaranteed minimum withdrawal benefit (GMWBs)? The GMWBs are usually sold as add-ons to variable annuity base contracts, which allow annual withdrawals of a certain percentage of the benefit base until the base is exhausted, even if the policyholder's account value itself had already fallen to zero before the ending date. Hence with GMWBs, policyholders receive two parts of return: one is a certain percent of the benefit base of which rate is predetermined and the other is partial financial return from capital gains of investment based on the present value of the benefit base(if policyholder's account value falls to zero, then this part becomes zero). Hence, we could view GMWB feature as a kind of insurance mechanism to protect policyholders from certain loss of benefit base during the whole investment.[Figure 2]

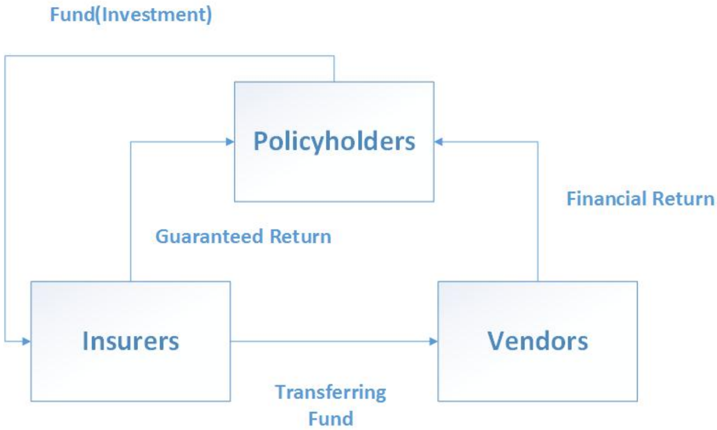

Figure 2: Variable Annuity With GMWB

Since GMWBs are only seen as add-ons to variable annuity base contract, they would not affect the profit of the whole investment if the benefit base value is a positive value. In this project, we are only interested in the situation which the benefit base value falls below zero. For example, the policyholder makes a purchase payment of F_0 and elects to withdraw the maximum amount of 5% of purchase amount without penalty each year. Then, we know the withdrawl can last for 20 years. Due to the poor performance of the fund in which the policyholders invest, F_0 may oscillating up and down along with the time and ended up with a total amount of F_{20} which is less than F_0.[Figure 3] Thereby, with certain percentage withdrawal each year, the account value is depleted at t_1, which is less than 20.[Figure 4] By this time, the policyholders would have only withdrawn $5t_1\%$ of F_0 in total. Then the guarantee feature kicks in to sustain the annual withdrawal of 5% of F_0 until the entire purchase payment is recovered, which means it pays until the maturity at the end of 20 years. In this case, the insurer has the responsibility to pay for the rest and thus generates liability to the policyholder.

From the insurer's perspective, we are interested in finding a certain number K such that the insures' liability L would be less than K with a probability that we predetermined. Then we could find the number K to determine the price of this specific financial derivative. In this case, we would assume a known K first and figure out the probability of $L \leq K$.

In the illustrative example, we set the initial fund F_0 equals to 1, net liability K equals to 0.1, mortality and expense fee m equals to 0.01, withdrawal rate w equals to 0.07, portion of fees that are used to fund the GMWB riders m_w equals to 0.0035, interest rate r equals to 0.05, mean rate of return of Brownian motion μ equals to 0.09 and constant volatility θ equals to 0.3. According to our calculation, the probability of the net liability less than K is around to 0.75.

2 Mathematical Derivation

2.1 Equity-linking mechanism

We now formulate the risk management problem of the aforementioned GMWB mathematically. Suppose that the underlying equity index/fund of the policyholder's choosing is driven by a geometric Brownian motion. By definition of Geometric Brownian motion:

$$S_t = S_0 e^{\sigma W_t + (\alpha - \frac{1}{2}\sigma^2)t}$$

where

$$W_t, \qquad t > 0.$$

is a Brownian Motion.

Recall the definition of Itô process:

Definition 2.1 *([3]Pg.143) Let W_t, $t \geq 0$, be a Brownian motion, and let F_t, $t \geq 0$, be an associated filtration. An Itô process is a stochastic process of the form*

$$X_t = X_0 + \int_0^t \Delta_u dW_u + \int_0^t \Theta_u du,$$

where X_0 is non-random and Δ_u and Θ_u are adapted stochastic processes. And the first derivative and second derivative are:

$$\begin{aligned} dX_t &= \Delta_t dW_t + \Theta_t dt, \\ dX_t dX_t &= \Delta_t^2 dW_t dW_t + 2\Delta_t \Theta_t dW_t dt + \Theta_t^2 dt dt \\ &= \Delta_t^2 dt. \end{aligned}$$

Just as the definition of Itô process, we define the process for geometric Brownian motion ([3]Pg.147):

$$X_t = \int_0^t \sigma dW_s + \int_0^t (\alpha - \frac{1}{2}\sigma^2) ds.$$

Then

$$\begin{aligned} dX_t &= \sigma dW_t + (\alpha - \frac{1}{2}\sigma^2) dt, \\ dX_t dX_t &= \sigma^2 dt. \end{aligned}$$

The asset price process we have could be rewritten as the following([3]Pg.148):

$$S_t = S_0 e^{X_t} = S_0 \exp(\int_0^t \sigma dW_s + \int_0^t (\alpha - \frac{1}{2}\sigma^2) ds),$$

where S_0 is non-random and positive. We may write $S_t = f(X_t)$, where $f(x) = S_0 e^x$, $f'(x) = S_0 e^x$ and $f''(x) = S_0 e^x$. In order to solve the It$\hat{o}$'s process for Geometric Brownian Motion, Let's recall Itô-Doeblin formula for Brownian motion.

Theorem 2.2 (Itô-Doeblin formula for Brownian motion) *([3]Pg.138) Let f(t,x) be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous, and let W_t be a Brownian motion. Then for every $T \geq 0$,*

$$f(T, W_T) = f(0, W_0) + \int_0^T f_t(t, W_t) dt + \int_0^T f_x(t, W_t) dW_t + \frac{1}{2} \int_0^T f_{xx}(t, W_t) dt.$$

Dropping the term respect to T, we have:

$$f(W_T) - f(W_0) = \int_0^T f'(W_t) dW_t + \frac{1}{2} \int_0^T f''(W_t) dt.$$

According to the Theorem 2.2([3]Pg.148):

$$\begin{aligned} dS_t &= df(X_t) \\ &= f'(X_t) dX_t + \frac{1}{2} f''(X_t) dX_t dX_t \\ &= S_0 e^{X_t} dX_t + \frac{1}{2} S_0 e^{X_t} dX_t dX_t \\ &= S_t dX_t + \frac{1}{2} S_t dX_t dX_t \\ &= \alpha S_t dt + \sigma S_t dW_t. \end{aligned}$$

The asset price S_t has instantaneous mean rate of return α_t and volatility σ_t. Both the instantaneous mean rate of return and the volatility are allowed to be time-varying and random. But in our case σ and α are fixed rate and we substitute α with μ and W_t with B_t to simplify. Hence we have:

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \qquad t > 0.$$

The rider charges are usually made on a daily basis as a percentage of account values, called mortality and expenses fee (M&E) , say m per time unit. Hence it is natural to model the fees and charges by continuous deduction from the asset value. Without any withdrawals, the policyholder's investment account evolves in proportion to the equity prices. Hence the dynamics of the fund values $\{F_t, t \geq 0\}$ is determined by

$$F_t = F_0 \frac{S_t}{S_0} e^{-mt}, \qquad t \geq 0,$$

which satisfies the stochastic differential equation

$$\begin{aligned} dF_t &= \frac{F_0}{S_0} d(f(X_t)e^{-mt}), \qquad f(X_t) = S_t. \\ &= \frac{F_0}{S_0}(d(f(X_t))e^{-mt} + f(X_t)d(e^{-mt})), \\ &= \frac{F_0}{S_0}((\mu S_t dt + \sigma S_t dB_t)e^{-mt} + S_t(-m)e^{-mt}), \\ &= (\mu - m)F_t dt + \sigma F_t dB_t, \qquad F_0 > 0. \end{aligned}$$

We assume that the policyholder chooses to withdraw only at the maximum rate without penalty, which is the majority of cases in practice and has been a commonly used assumption in practitioners' simulations. Denote the withdrawal rate per time unit by w. Thus, the dynamics of account value is given by

$$dF_t = (\mu - m)F_t dt - w dt + \sigma F_t dB_t, \quad F_0 > 0. \tag{2.1}$$

Note that the withdrawal rate is typically a fixed percentage of the benefit base G. In this work, the benefit base G is equal to F_0, meaning that the policyholder is guaranteed to receive a full refund of his/her premium payments. Bear in mind that withdrawals are often taken monthly or quarterly in practice. Nevertheless, the continuous-time model can serve as a good approximation.

2.2 Insurer's liabilities

The amount of time needed for the policyholder to recoup the original premium payment $G = F_0$ by withdrawing w per time unit is $T = G/w$. Clearly, there is no financial obligation to the insurer until the time at which the account value hits zero, i.e.

$$\tau_0 = \inf\{t : F_t \leq 0\}. \tag{2.2}$$

It is only when the account value is depleted prior to the maturity T that the maximum withdrawal rate w is paid at the cost of the insurer. Denote $x \wedge y = \min\{x, y\}$. Therefore, the present value of the cost to an insurer of the GMWB rider (gross liability) is thus given by

$$\int_{\tau_0 \wedge T}^{T} e^{-rs} w ds, \tag{2.3}$$

where r is the yield rate on the assets backing up the liability.

Recall that to compensate for its liability for the GMWB rider, the insurer receives the distribution of fees from the third-party fund manager, which are often a fixed percentage of the policyholder's subaccount until the account value hits zero. We denote the portion of fees that are used to fund the GMWB rider by m_w. Note that the total fee m is in general larger than m_w, as the rest goes to cover overheads, and other expenses. Thus the accumulated present value of the fee income is given by

$$\int_0^{\tau_0 \wedge T} m_w e^{-rs} F_s ds. \tag{2.4}$$

From the perspective of risk analysis, we are interested in the insurer's net liability (gross liability less fee income)

$$L = \int_{\tau_0 \wedge T}^{T} e^{-rs} w ds - m_w \int_0^{\tau_0 \wedge T} e^{-rs} F_s ds. \tag{2.5}$$

The net liability is in general expected to be negative in most cases, as the product is designed to be profitable. Nevertheless, it is the likelihood and severity of the positive side of net liability (loss) that is of the most interest from the viewpoint of risk management.

2.3 Risk measures

The general principle of risk management is to determine sufficient funds to set aside in order to absorb unexpected losses in adverse economic scenarios. Here we model the essence of a risk management strategy that determines risk capitals by certain measures of insurance net liabilities. The two most common risk measures used by practitioners and regulatory bodies around the world are quantile risk measure, also known as value at risk in finance, and conditional tail expectation risk measure, also known as expected shortfall.

In the US market, the National Association of Insurance Commissioners (NAIC) published model regulations, which are essentially adopted by all insurance regulators at the state level, that requires variable annuity writers to use 70% CTE to determine the reserves and 90% CTE to determine the risk-based capital (RBC).

The quantile risk measure for L is defined for $\alpha\,(0 \le \alpha \le 1)$ as

$$\mathrm{VaR}_\alpha := \inf\{y : \mathbb{P}[L \le y] \ge \alpha\}. \tag{2.6}$$

Since L is modeled by continuous random variables in this model, we can compute the quantile risk measure by V_α using a root search algorithm such that

$$\mathbb{P}[L > V_\alpha] = 1 - \alpha. \tag{2.7}$$

The quantile risk measure V_α is interpreted as the minimum capital required to ensure that there is sufficient fund to cover future liability with the probability of at least α.

The conditional tail expectation risk measure for L is also defined for $\alpha\,(0 \le \alpha \le 1)$ as

$$\mathrm{CTE}_\alpha := \mathbb{E}[L | L > V_\alpha]. \tag{2.8}$$

It is the capital required to cover the average amount of liabilities when they exceed the quantile measure with the probability of at most $1 - \alpha$.

3 Distribution of the net liability L

Recall that the net liability is a path-dependent functional of the underlying equity index/fund process

$$L = \int_{\tau_0 \wedge T}^{T} e^{-rs} w ds - m_w \int_0^{\tau_0 \wedge T} e^{-rs} F_s ds.$$

The net liability L depends on two random processes, the value of the equity index/fund F_t and the present value of the accumulated capital $A_t = \int_0^t e^{-rs} F_s ds$. We denote τ_0 as the first time when the process F_t hits zero, after which the insurers's liability becomes deterministic. The stochastic dynamics are the following:

$$\begin{aligned} dF_t &= (\mu - m)F_t dt - w dt + \sigma F_t dB_t, \quad F_0 = G, \\ dA_t &= e^{-rt} F_t dt. \end{aligned}$$

Let us define function $v(t, x, y)$ for a fixed number K as

$$\begin{aligned} v(t, x, y) &= \mathbb{E}[\mathbb{I}(L \leq K) | F_t = x, A_t = y] \\ &= \mathbb{P}(L \leq K | F_t = x, A_t = y). \end{aligned}$$

We denote the partial derivatives of v with respect to t, x, y by v_t, v_x, v_y respectively and the second partial derivative of v with respect to x by v_{xx}.In particular, the loss distribution at time $t = 0$ corresponds to $v(0, G, 0) = \mathbb{P}(L \leq K)$. In order to solve this equation, let's recall the theorem for an Itô process:

Theorem 3.1 (Itô-Doeblin formula for an Itô process) *([3]Pg.146) Let X_t, $t \geq 0$, be an Itô process as described above, and let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x)$,$f_x(t, x)$, and $f_{xx}(t, x)$ are defined and continuous. Then for every $T \geq 0$,*

$$\begin{aligned} f(T, X_T) &= f(0, X_0) + \int_0^T f_t(t, X_t) dt + \int_0^T f_x(t, X_t) dX_t \\ &\quad + \frac{1}{2} \int_0^T f_{xx}(t, X_t) dX_t dX_t \\ &= f(0, X_0) + \int_0^T f_t(t, X_t) dt + \int_0^T f_x(t, X_t) \Delta_t dW_t \\ &\quad + \int_0^T f_x(t, X_t) \Theta_t dt + \frac{1}{2} \int_0^T f_{xx}(t, X_t) \Delta_t^2 dt. \end{aligned}$$

Hence by taking derivative we would get differential equation as following:

$$\begin{aligned} df(t, X_t) &= f_t(t, X_t) dt + f_x(t, X_t) dX_t + \frac{1}{2} f_{xx}(t, X_t) dX_t dX_t, \\ &= f_t(t, X_t) dt + f_x(t, X_t) \Delta_t dW_t \\ &\quad + f_x(t, X_t) \Theta_t dt + \frac{1}{2} f_{xx}(t, X_t) dX_t dX_t. \end{aligned}$$

Theorem 3.2 (Computation of $\mathbb{P}(L < K)$) *The function v is a solution of the following partial differential equation for $t \in (0, T), x \in (0, \infty), y \in (0, \infty)$*

$$v_t + ((\mu - m)x - w)v_x + e^{-rt} x v_y + \tfrac{1}{2} \sigma^2 x^2 v_{xx} = 0 \tag{3.1}$$

with few boundary and terminal conditions labelled below.

Working on the process of $v(t, x, y)$, We take the derivative respect to each element: t,F_t and A_t. Since

$$\begin{aligned} (dA_t)^2 &= (e^{-rt} F_t dt)^2 = 0 \\ dA_t dF_t &= ((\mu - m)F_t dt - w dt + \sigma F_t dB_t)(e^{-rt} F_t dt) = 0 \end{aligned}$$

Then following the theorem,we have the equation as following:

$$\begin{aligned} dv(t, F_t, A_t) &= v_t dt + v_x dF_t + v_y dA_t + \tfrac{1}{2} v_{xx} (dF_t)^2 \\ &= v_t dt + ((\mu - m)F_t - w) v_x dt + \sigma F_t v_x dW_t \\ &\quad + v_y e^{-rt} F_t dt + \tfrac{1}{2} \sigma^2 F_t^2 v_{xx} dt, \end{aligned}$$

Recalling continuous-time martingale respect to the stochastic process X_t is a stochastic process Y_t such that for all t:

$$\begin{aligned} \mathbb{E}(|Y_t|) &< \infty, \\ \mathbb{E}(Y_t|X_\tau, \tau \le s) &= Y_s, \forall s \le t. \end{aligned}$$

This expresses the property that the conditional expectation of an observation at time t, given all the observations up to time s , is equal to the observation at time s (of course, provided that $s \le t$). Since $v(t, F_t, A_t)$ is a martingale, the net dt term in the differential $dv(t, F_t, A_t)$ must be zero. If it were positive at any time, then $v(t, F_t, A_t)$ would have a tendency to rise at that time; if it were negative, $v(t, F_t, A_t)$ would have a trendy to fall.Hence the d_t terms must vanish. So the function v must satisfy the partial differential equation.

$$v_t(t, F_t, A_t) + ((\mu - m)F_t - w)v_x(t, F_t, A_t) + e^{-rt}F_t v_y(t, F_t, A_t) + \tfrac{1}{2}\sigma^2 F_t^2 v_{xx}(t, F_t, A_t) = 0$$

along every path of F. Therefore,

$$v_t(t, x, y) + ((\mu - m)x - w)v_x(t, x, y) + e^{-rt}x v_y(t, x, y) + \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t, x, y) = 0$$

Then we starting to define the terminal condition and boundary conditions for partial differential equation mentioned above. Since L represents the net liability, at the end of the period L should be less than K, otherwise people would not buy such a market derivative. Hence we get the terminal condition:

$$v(T, x, y) = 1, \tag{3.2}$$

Then we want to focus on the boundary conditions. Recall that F_t represents the filtration for the Brownian motion, also could be understood as all the information available at time t to determine the value of Brownian motion. Hence at x=0, we have no information about the whole process. $v(t, x, y)$ simply becomes an indictor whether the liability account is smaller than the present value of the accumulated capital, which is A_t. Deriving the formula:

$$L = \int_{\tau_0 \wedge T}^{T} e^{-rs} w ds - m_w \int_0^{\tau_0 \wedge T} e^{-rs} F_s ds.$$

since

$$dA_t = e^{-rt} F_t dt$$

We have:

$$\begin{aligned} L &= \int_{\tau_0 \wedge T}^{T} e^{-rs} w ds - m_w \int_0^t dA_t \\ &= \int_{\tau_0 \wedge T}^{T} e^{-rs} w ds - m_w y \\ &= \int_t^T e^{-rs} w ds - m_w y \\ &= \tfrac{w}{r}(e^{-rt} - e^{-rT}) - m_w y. \end{aligned}$$

Hence

$$\begin{aligned} v(t, 0, y) &= \mathbb{I}(L \le K | F_t = 0, A_t = y) \\ &= \mathbb{I}(f(t, K) \le y) \end{aligned} \tag{3.3}$$

where

$$f(t,K) := \frac{w}{rm_w}(e^{-rt} - e^{-rT}) - \frac{K}{m_w}.$$

Since as x approaches to infinity, $\mathbb{P}(L \le K|F_t = x, A_t = y)$ should converge, therefore the derivative respect to x should be zero. Hence, we get the second boundary condition:

$$\lim_{x\to\infty} v_x(t,x,y) = 0. \tag{3.4}$$

Last but not the least, as y approaches to infinity, by formula $L = \int_{\tau_0 \wedge T}^{T} e^{-rs} w ds - m_w y$, L approaches to negative infinity resulting in $\mathbb{P}(L \le K|F_t = x, A_t = y)$ converges to 1, which is :

$$\lim_{y\to\infty} v(t,x,y) = 1. \tag{3.5}$$

4 Finite Difference Method and Introductory Example

In this section, we gives a brief introduction of the numerical method we are about to use to solve the partial differential equation in the prior section.

4.1 Finite Difference Method

Finite Difference Method is a common method to solve partial differential equations based on the idea of approximating partial derivatives by a difference quotient. This transforms the partial differential equation into a set of algebraic equations([2]Pg.293). According to Taylor Theorem, the function $f(x)$ may be represented as:

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(x) + \cdots \tag{4.1}$$

Neglecting the terms of order h^2 and higher, we get:

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h). \tag{4.2}$$

which is called forward approximation.

Similarily, we may get backward approximation from the equation:

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{6}h^3 f'''(x) + \cdots \tag{4.3}$$

Neglecting the terms of order h^2 and higher, we get:

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h) \tag{4.4}$$

In order to solve the problem in our case, we also need to approximate second order derivatives. It is obtained by adding up (4.1) and (4.3) which yields:

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^4) \tag{4.5}$$

Then,

$$f''(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} + O(h^2) \tag{4.6}$$

4.2 Brownian Motion Heat Equation

With the knowledge of converting a derivative into difference quotient in mind, we are able to discretize a partial differential equation in a similar manner.

Introducing a heat equation:

$$\frac{\partial}{\partial s}h(s,z) + \mu\frac{\partial}{\partial z}h(s,z) = \frac{1}{2}\frac{\partial^2}{\partial z^2}h(s,z), \qquad 0 < s < T$$

subjects to the following conditions:

$$\lim_{z\to\infty} h(s,z) = 1, \qquad 0 \le s \le T \tag{4.7}$$

$$\lim_{z\to 0} h(s,z) = 0, \qquad 0 \le s \le T \tag{4.8}$$

$$h(0,z) = 1, \qquad z > 0 \tag{4.9}$$

We are interested in finding the numerical value of h given any combinations of s and z. The first step is to create an s - z grid to represent the coordinates of points to apply the idea of discretization. The next step is to convert the original partial differential equation into the difference quotient form to explore the relationship between adjacent points. Iterating along one dimension, we can find the value of each points in this s - z grid. As a result, the collection of points determined by the partial differetial equation forms a surface in a 3 - dimensional space. For computational simplicity, we truncate the z dimension to $(0,b)$ for large enough b.

First, we set up the grids (x_i, y_i) and denote the point $h(x_i, y_i)$ on the grid by $h^{i,j}$. We get

$$x_i = i\Delta x, \qquad \text{for } i = 1,\cdots,Nx+1, \qquad N_s = T/\Delta x;$$
$$y_j = j\Delta y, \qquad \text{for } j = 1,\cdots,Ny+1, \qquad N_z = b/\Delta y;$$

In Matlab, the following code executes this step:

```
function sol = Brownian(ds,dz,T,b,mu)
Ns = round(T/ds); % time variable, indexed by i
Nz = round(b/dz); % position variable, indexed by j
h = zeros(Nz+1,Ns+1);%set up grid
```

The round() function rounds inputs to integers and this step sets up a grid of $(N_z+1)\times(N_s+1)$ with each point in this grid has a value of 0.

Next, we assign values to points on the boundary and at terminal time. According to equation (4.7), (4.8) and (4.9), we use the following to execute this step:

```
for i = 1:Ns+1
    h(Nz+1,i) = 1;
    h(1,i) = 0;
end

for j = 1:Nz+1
    h(j,1) = 1;
end
```

According to (4.2) (4.4) and (4.6),the discretization of the heat equation gives

$$\frac{h_{i,j}-h_{i-1,j}}{\Delta s} + \mu\frac{h_{i,j}-h_{i,j-1}}{\Delta z} = \frac{1}{2}\frac{h_{i,j+1}-2h_{i,j}+h_{i,j-1}}{\Delta z^2}$$

It is not hard to find that $h_{i-1,j}$ can be represented by $h_{i,j-1}$, $h_{i,j}$ and $h_{i,j+1}$. It can be rewritten as:

$$\frac{h_{i-1,j}}{\Delta s} = \frac{1}{2}\frac{-h_{i,j+1}+2h_{i,j}-h_{i,j-1}}{\Delta z^2} + \mu\frac{h_{i,j}-h_{i,j-1}}{\Delta z} + \frac{h_{i,j}}{\Delta s}$$

Collecting terms we get:

$$\frac{h_{i-1,j}}{\Delta s} = -(\frac{1}{2\Delta z^2} + \frac{\mu}{\Delta z})h_{i,j-1} + (\frac{1}{\Delta z^2} + \frac{\mu}{\Delta z} + \frac{1}{\Delta s})h_{i,j} - (\frac{1}{2\Delta z^2})h_{i,j+1}$$

Moving the Δs term to the right hand side:

$$h_{i-1,j} = -(\frac{\Delta s}{2\Delta z^2} + \mu\frac{\Delta s}{\Delta z})h_{i,j-1} + (\frac{\Delta s}{\Delta z^2} + \mu\frac{\Delta s}{\Delta z} + 1)h_{i,j} - (\frac{\Delta s}{2\Delta z^2})h_{i,j+1}$$

For simplicity, we use the notation:

$$\alpha = \frac{\Delta s}{\Delta z^2} \qquad \beta = \frac{\Delta s}{\Delta z}$$

In Matlab, the step goes as follows:

```
alpha = ds\(dz^2);
beta = ds\dz;
```

We have to solve a system of linear equations for each time layer. Since boundary conditions are given, we have $N_z - 1$ equations in $N_z - 1$ unknowns. Let $\mathbf{h} = h_{N_z}$ be an vector, we have

$$\mathbf{h}^{i-1} = \mathbf{B}\mathbf{h}^i + (0, \cdots, 0, -\frac{\alpha}{2})^\top, \qquad i = 2, \cdots, Ns + 1$$

where $\mathbf{B}$ is a tridiagonal matrix:

$$\mathbf{B} = \begin{pmatrix} 1+\alpha+\mu\beta & -\frac{\alpha}{2} & 0 & \cdots & 0 & 0 \\ -(\frac{\alpha}{2}+\mu\beta) & 1+\alpha+\mu\beta & -\frac{\alpha}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(\frac{\alpha}{2}+\mu\beta) & 1+\alpha+\mu\beta \end{pmatrix}$$

looping through each i, we are able to get the value of each point in the s - z grid with the assist of boundary conditions and terminal condition.

```
B1 = diag((1+alpha+mu*beta)*ones(Nz-1,1));
B2 = diag((-alpha/2)*ones(Nz-2,1),1);
B3 = diag(-(mu*beta+alpha/2)*ones(Nz-2,1),-1);
B = B1+B2+B3;
rem = zeros((Nz-1),1);
rem(Nz-1,1) = -alpha/2;

for i = 1:Ns
    h(2:Nz,i+1) = B \ (h(2:Nz,i)-rem);
end
```

In the above step, we create the elements of the matrix by generating **B1**, **B2** and **B3**. The matrix **B** is generated by adding up the three matrices. The step

```
for i = 1:Ns
    h(2:Nz,i+1) = B \ (h(2:Nz,i)-rem);
end
```

allow Matlab to perform a multiplication of the inverse of an matrix.

Let $\mu = 1$, $b = 5$, $T = 10$, we can obtain all the values of the matrix h and generate a surface made up of all the points of the PDE in [Figure 3]. Thus, it is easy to find the value of a point given a combination of (s, z) coordinate.

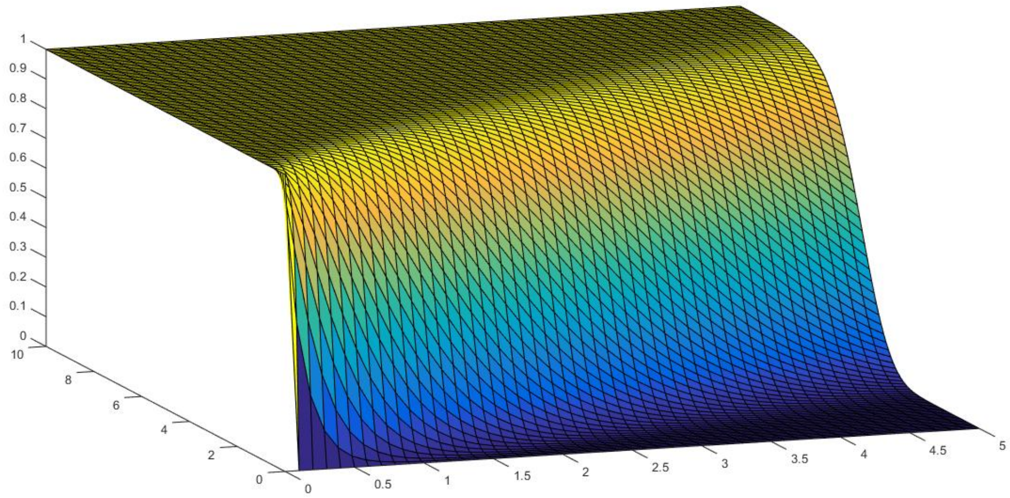

Figure 3: Heat Equation

5 Algorithm and Actual Example

In this section, we begin to solve the actual problem that we are interested in using the finite difference method by Matlab. We are interested in finding the solution of the partial differential equation:

$$v_t + [(\mu - m)x - w]v_x + e^{-rt}xv_y + \frac{1}{2}\sigma^2 x^2 v_{xx} = 0, \qquad t \in (0,T), x \in (0,\infty), y \in (-\infty,\infty)$$

We first truncate the state space of X to $(0,b)$ and the state space of Y to $(0,c)$ to replace the infinity condition of x dimension and y dimension for b,c large enough.

Therefore, the partial differential equation subjects to the following conditions:

$$v(T,x,y) = 1, \qquad x \in (0,b), y \in (0,c) \tag{5.1}$$

$$v(t,x,c) = 1, \qquad x \in (0,b), t \in (0,T) \tag{5.2}$$

$$v(t,0,y) = I(f(t,K) \leq y), \qquad t \in (0,T), y \in (0,c) \tag{5.3}$$

$$v_x(t,b,y) = 0, \qquad t \in (0,T), y \in (0,c) \tag{5.4}$$

where

$$f(t,K) := \frac{w}{rm_w}(e^{-rt} - e^{-rT}) - \frac{K}{m_w}$$

For the ease of computation, we do a change of variable and let $s(t,x,y) = v(T-t,x,c-y)$ to satisfies the following equation

$$s_t + e^{-r(T-t)}xs_y = [(\mu - m)x - w]s_x + \frac{1}{2}\sigma^2 x^2 s_{xx}$$

subject to conditions

$$h(0,x,y) = 1, \qquad x \in (0,b), y \in (0,c); \tag{5.5}$$

$$h(t,x,0) = 1, \qquad x \in (0,b), t \in (0,T); \tag{5.6}$$

$$h(t,0,y) = I(f(T-t,K) \leq c-y), \qquad t \in (0,T), y \in (0,c); \tag{5.7}$$

$$h_x(t,b,y) = 0, \qquad t \in (0,T), y \in (0,c). \tag{5.8}$$

Then, we set up the grids (x_i, y_j, t_k) and denote the solution $s(x_i, y_j, t_k)$on the grid by $h_i^{j,k}$ where

$$\begin{aligned} x_i &= i\Delta x, & &\text{for } i = 1, \cdots, Nx+1, & Nx &= b/\Delta x; \\ y_j &= j\Delta y, & &\text{for } j = 1, \cdots, Ny+1, & Ny &= c/\Delta y; \\ t_k &= k\Delta t, & &\text{for } k = 1, \cdots, Nt+1, & Nt &= T/\Delta t; \end{aligned}$$

In Matlab, the above steps can be represented as:

```
function s = Risk(dt,dx,dy,T,b,c,K,w,r,mw,sig,mu,m)
%set up grid
Nt = round(T/dt);
Nx = round(b/dx);
Ny = round(c/dy);
s = zeros(Nx+1,Ny+1,Nt+1);
```

The round() function round inputs to integers. The function zeros(a,b,c) creates a matrix with dimension $a \times b \times c$. Matrix s is a three-dimension matrix to store the value of each point in this PDE.

The following block of code sets up our initial condition $h(0, x, y) = 1$, and the boundary conditions $h(t, x, 0) = 1$

```
for i = 1:(Nx+1);
    for j = 1:(Ny+1);
        s(i,j,1) = 1;
    end
end

for i = 1:(Nx+1);
    for k = 1:(Nt+1);
        s(i,1,k) = 1;
    end
end
```

At the initial moment, the partial differential equation obtains a value of 1 across the whole x-y plane. Similarly, the value of x-z plane is also 1 at initial value of y.

The partial differential equation can be expressed using $s_i^{j,k}$

$$\begin{aligned} &\frac{s_i^{j+1,k+1} - s_i^{j+1,k}}{\Delta t} + g^{k-1} i\Delta x \frac{s_i^{j+1,k+1} - s_i^{j,k+1}}{\Delta y} \\ &= [(\mu - m)k\Delta x - w]\frac{s_{i+1}^{j+1,k+1} - s_{i-1}^{j+1,k+1}}{2\Delta x} + \frac{\sigma^2}{2}(k\Delta x)^2 \frac{s_{i+1}^{j+1,k+1} + s_{i-1}^{j+1,k+1} - 2s_i^{j+1,k+1}}{(\Delta x)^2} \end{aligned} \tag{5.9}$$

where

$$g^k := e^{-r(T-k\Delta t)}$$

Collecting terms we have:

$$\begin{aligned} s_i^{j+1,k+1}[\frac{1}{\Delta t} + g^{k-1} i \frac{\Delta x}{\Delta y} + \sigma^2 k^2] + s_{i-1}^{j+1,k+1}[\frac{(\mu - m)k\Delta x - w}{2\Delta x} - \frac{\sigma^2 (k\Delta x)^2}{2(\Delta x)^2}] \\ + s_{i+1}^{j+1,k+1}[-\frac{(\mu - m)k\Delta x - w}{2\Delta x} - \frac{\sigma^2 (k\Delta x)^2}{2(\Delta x)^2}] = s_i^{j+1,k}\frac{1}{\Delta t} + s_i^{j,k+1} g^{k-1} i \frac{\Delta x}{\Delta y} \end{aligned} \tag{5.10}$$

It can be further rewritten as

$$s_i^{j+1,k+1}[\frac{1}{\Delta t}+g^{k-1}i\frac{\Delta x}{\Delta y}+\sigma^2k^2]+s_{i-1}^{j+1,k+1}[\frac{(\mu-m)k}{2}-\frac{w}{2\Delta x}-\frac{\sigma^2(k)^2}{2}]$$
$$+s_{i+1}^{j+1,k+1}[-\frac{(\mu-m)k}{2}+\frac{w}{2\Delta x}-\frac{\sigma^2(k)^2}{2}]=\frac{s_i^{j+1,k}}{\Delta t}+g^{k-1}i\frac{\Delta x}{\Delta y}s_i^{j,k+1} \quad (5.11)$$

The equation gives

$$-\frac{1}{2}[\sigma^2k^2-(\mu-m)k+\frac{w}{\Delta x}]s_{i-1}^{j+1,k+1}+[\frac{1}{\Delta t}+g^{k-1}i\frac{\Delta x}{\Delta y}+\sigma^2k^2]s_i^{j+1,k+1}$$
$$-\frac{1}{2}[\sigma^2k^2+(\mu-m)k-\frac{w}{\Delta x}]s_{i+1}^{j+1,k+1}=\frac{1}{\Delta t}s_i^{j+1,k}+g^{k-1}i\frac{\Delta x}{\Delta y}s_i^{j,k+1}, \qquad i=1,\cdots,Nx \quad (5.12)$$

For simplicity, we use the notation:

$$\alpha=\frac{1}{\Delta t}, \qquad \beta=\frac{\Delta x}{\Delta y}, \qquad \gamma=\frac{w}{\Delta x}$$

In Matlab, it can be represented as follows:

```
alpha = 1/dt;
beta = dx/dy;
gamma = w/dx;
```

We also need to discretize boundary conditions, the Dirichlet condition gives:

$$s_1^{j,k}=\mathbb{I}(j\Delta y\le c+\frac{K}{m_w}-\frac{w}{rm_w})e^{-rT}(e^{rk\Delta t}-1)$$

The Neumann condition yields:

$$s_{N_x+1}^{j,k}=s_{N_x-1}^{j,k}$$

Further exploiting the fact that $s_{N_x+1}^{j,k}=s_{N_x-1}^{j,k}$, equation (39) is given as:

$$-\sigma^2N_x^2s_{N_x-1}^{j+1,k+1}+[\sigma^2N_x^2+\beta N_xg^i+\alpha]s_{N_x}^{j+1,k+1}=\alpha s_{N_x}^{j+1,k}+g^{k-1}\beta N_xs_{N_x}^{j,k+1}$$

Let $\mathbf{s}=s_{N_x}$ be a vector, Then the above equation can be represented in matrix form:

$$\mathbf{B}^k\mathbf{s}^{j+1,k+1}=\alpha\mathbf{s}^{j+1,k}+\mathbf{C}\mathbf{s}^{j,k+1}+(\frac{1}{2}(\sigma^2-\mu+m+\gamma)s_1^{j+1,k+1}),0,\cdots,0)^\top$$

where $\mathbf{C}$ is a diagonal matrix with diagonal elements $(g^{k-1}\beta, g^{k-1}2\beta,\cdot,g^{k-1}N_x\beta)$ and $\mathbf{B}^k$ is a tridiagonal matrix

$$\mathbf{B}^k=\begin{pmatrix} \sigma^2+\beta g^k+\alpha & -\frac{1}{2}(\sigma^2+\mu-m-\gamma) & 0 & \cdots & 0 & 0\\ -\frac{1}{2}(4\sigma^2-2(\mu-m)+\gamma) & 4\sigma^2+2\beta g^k+\alpha & -\frac{1}{2}(4\sigma^2+2(\mu-m)-r) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -\sigma^2N_x^2 & \sigma^2N_x^2+\beta N_xg^k+\alpha \end{pmatrix}$$

The general form of tridiagonal elements excluding the first row and the last row are:

$$-\frac{1}{2}[\sigma^2i^2-(\mu-m)i+\gamma],\sigma^2i^2+\beta ig^k+\alpha,-\frac{1}{2}[\sigma^2i^2+(\mu-m)i-\gamma]$$

Thus we can determine each $h_i^{j,k}$by

$$\mathbf{h}^{j+1,k+1}=(\mathbf{B}^k)^{-1}(\alpha\mathbf{s}^{j+1,k}+\mathbf{C}\mathbf{s}^{j,k+1}+(\frac{1}{2}(\sigma^2-\mu+m+\gamma)s_1^{j+1,k+1}),0,\cdots,0)^\top)$$

starting from $j=1$ and $k=1$

With the initial conditions $\mathbf{s}^{j,1}=\mathbf{1}$ and $\mathbf{s}^{1,k}=\mathbf{1}$, we can determine all the value of $\mathbf{s}$ by marching in j and k direction alternatively.

```
%loop to get solution
for k = 1:Nt;
    C = diag((1:Nx+1)*beta*exp(-r*(T-(k-1)*dt)));
    B1 = diag(((1:Nx+1).^2)*sig*sig+(1:Nx+1)*beta*(exp(-r*(T-(k-1)*dt)))+alpha,0);
    B2 = diag(-(((1:Nx).^2)*sig*sig+(1:Nx)*(mu-m)-gamma)/2,1);
    B3 = diag(-(((2:Nx+1).^2)*sig*sig-(2:Nx+1)*(mu-m)+gamma)/2,-1);
    B = B1+B2+B3;
    B(Nx+1,Nx) = -(sig*sig*(Nx+1)*(Nx+1));
    [L,U] = lu(B);

    for j = 1:Ny;
        rhs3 = zeros((Nx+1),1);
        rhs3(1,1) = (sig*sig-mu+m+gamma)*((j+1)*dy <=
            (c+K/mw-w*exp(-r*T)*(exp(r*k*dt)-1)/(r*mw)))/2;
        rhs = alpha*s(1:(Nx+1),j+1,k) + C*s(1:(Nx+1),j,k+1) + rhs3;
        s(1:(Nx+1),j+1,k+1) = U\(L\rhs);
    end
end

sol = s(round(Nx/5)+1,Ny+1,Nt+1);
```

In the above block of code, we created a loop first about T to generate instances of matrices of B and C, then about Y to obtain values of the PDE. Initially, we set up diagonal matrix C and B. The values of entries of these two matrices are constantly changing with the change of time. In other words,new matrices B and C are generated for each iteration of the outside loop. Therefore, it is necessary to set up the loop about time variable. Matrix B is a tridiagonal matrix. In this case, three matrices are combined to get the tridiagonal matrix. Since the second last entry of the last row are different from the standard expression, an extra step, B(Nx,Nx-1) = -(sig*sig*Nx*Nx);, is needed in order to assign value to that entry. To make the computation more efficient, LU factorization is being employed to improve the performance of the algorithm. The factorization method makes a significance difference in the speed of computation as reflected:

The following graph represent the performance of the algorithm employing LU factorization:

Function Name	Calls	Total Time	Self Time*	Total Time Plot (dark band = self time)
Risk	1	241.150 s	241.150 s	
ExecuteRisk	1	0 s	0.000 s	

In comparison to the one without LU factorization:

Function Name	Calls	Total Time	Self Time*	Total Time Plot (dark band = self time)
Risk	1	276.215 s	276.215 s	
ExecuteRisk	1	0 s	0.000 s	

Looping through a 500×1000 matrix, the LU factorization is able to improve the speed by 35 seconds.

Subsequently, we loop through the y variable to find values for each s(x,y,t). "rhs3" is a vector with a special element. As it has been mentioned before, an identity function is used to represent the Dirichlet condition. The part

```
((j+1)*dy <= (c+K/mw-w*exp(-r*T)*(exp(r*k*dt)-1)/(r*mw)))
```

returns 1 if the $\leq$ holds and 0 otherwise.

Since we already did an LU factorization before hand, we use

```
s(1:Nx,j+1,k+1) = U\(L\rhs);
```

to executue the multiplication of B inverse.

Finally, the matrix corresponding to the last time layer is stored in the matrix "graph"

```
graph = s(1:(Nx+1),1:(Ny+1),Nt+1);
end
```

Bringing in the example in the opening paragraph, we are able to calculate the solution:

```
%ExecuteRisk.m
function ExecuteRisk
dt = 0.1;
dx = 0.01;
dy = 0.01;
T = 1/0.07;
b= 5;
c = 10;
K =0.1;
w = 0.07;
r = 0.05;
mw = 0.0035;
sig = 0.3;
mu = 0.09;
m = 0.01;
Risk(dt,dx,dy,T,b,c,K,w,r,mw,sig,mu,m);
```

The value we have obtained through this algorithm is 0.753785277611327 and it is close to the solution generated by Monte Carlo Method which is 0.75055. Since the solution is stored in a 3 dimensional matrix, we need to fix one variable in order to visualize it. We fix the time dimension and make a x-y graph at the terminal time point, namely, $t = T$.

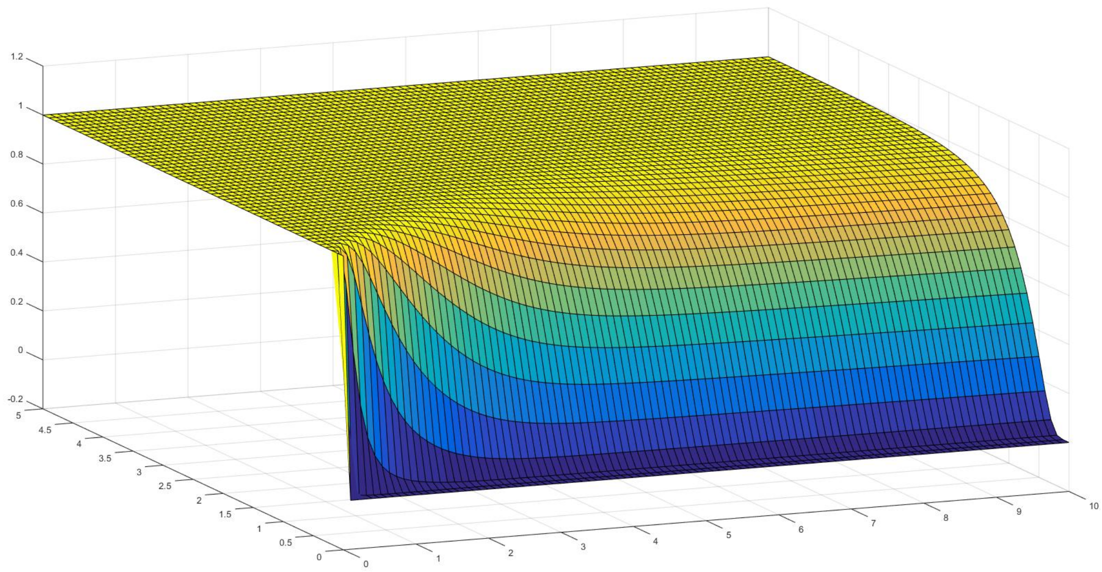

Figure 4: Graph at Terminal time

Appendices

We use Laplace Transform method to solve the equation.
As a background knowledge, the Laplace Transform method is a close relative of the Fourier Transform, using the method of Laplace Transform could always allow us to solve some PDE problems in a very simple way. For a function $f(t)$, we define its Laplace Transform as $F(s) = \int_0^{+\infty} f(t)e^{-st}\,dt$. Since for equation $(A.1)$ we have $f(t) = h(s,z)$, then we could write Laplace Transform as:

$$\tilde{h}(z) = \tilde{h}(r,z) = \int_0^{+\infty} e^{-rs}h(s,z)\,ds$$

Integrate by parts and get:

$$\begin{aligned}\int_0^{+\infty} e^{-rs}h(s,z)\,ds = \int_0^{+\infty} e^{-rs}\,dh(s,z)& \\ &= e^{-rs}h(s,z)\mid_0^{+\infty} + r\int_0^{+\infty} e^{-rs}h(s,z)\,ds \\ &= h(0,z) + r\tilde{h}(r,z)\end{aligned}$$

Taking Laplace Transform on both sides of (A.1), we get:

$$1 + r\tilde{h}(z) + \mu\tilde{h'}(z) = \frac{1}{2}\tilde{h''}(z)$$

Reforming the equation and we have,

$$\frac{1}{2}\tilde{h''}(z) - \mu\tilde{h'}(z) - r\tilde{h}(z) = 1 \tag{.13}$$

with boundary conditions:

$$\lim_{z\to\infty}\tilde{h}(s,z) = \frac{1}{r}$$

$$\lim_{z\to 0}\tilde{h}(s,z)=0$$

Since the number of solution is determined by the order of the equation, and we have the order for $(A.5)$ is two, thus there are at most two solutions for the equation and we write them in the form as:

$$\tilde{h_1}(z)=e^{\rho_1 z}$$
$$\tilde{h_2}(z)=e^{\rho_2 z}$$

where ρ_1,ρ_2 are solutions of the ODE function we transformed from the PDE function based on the initial condition $h(0,z)=1,(z>0)$:

$$\frac{1}{2}\rho^2-\mu\rho-r=0$$

We solve this ODE function and get: $\rho=\mu\pm\sqrt{\mu^2+2r}$

$$\Rightarrow \rho_1=\mu+\sqrt{\mu^2+2r},\rho_2=\mu-\sqrt{\mu^2+2r}$$

Since $\rho_1=\mu+\sqrt{\mu^2+2r}>0$, and $\rho_2=\mu-\sqrt{\mu^2+2r}<\mu-\sqrt{\mu^2}=\mu-\mu=0$, then we have:

$$\Rightarrow \rho_1>0,\rho_2<0$$

Therefore, we rewrite the equation we want to solve as:

$$\tilde{h}(z)=\frac{1}{r}+c_1e^{\rho_1 z}+c_2e^{\rho_2 z}\qquad (z>0) \tag{.14}$$

In order to get the specific value for c_1 and c_2, we need to use the boundary condition:

$$\lim_{z\to\infty}\tilde{h}(s,z)=\frac{1}{r}\Rightarrow c_1=0$$
$$\lim_{z\to 0}\tilde{h}(s,z)=0\Rightarrow c_2=-\frac{1}{r}$$

Plug the values for c_1 and c_2 into the equation $(A.6)$, we have a new equation in the form:

$$\tilde{h}(z)=\frac{1}{r}-\frac{1}{r}e^{(\mu-\sqrt{\mu^2+2r})z}\qquad (z>0) \tag{.15}$$

By the method of Laplace Transform, we want to prove that:

$$h(z)=\Phi(\frac{z-\mu s}{\sqrt{s}})-e^{2\mu z}\Phi(\frac{-z-\mu s}{\sqrt{s}})=1-\Phi(\frac{-z+\mu s}{\sqrt{s}})-e^{2\mu z}\Phi(\frac{-z-\mu s}{\sqrt{s}})$$

Where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{y^2}{2}}\,dy$$

Since we already have $\tilde{h}(z)=\tilde{h}(r,z)=\int_0^{+\infty}e^{-rs}h(s,z)\,ds$, substitute $h(s,z)$ by $[\Phi(\frac{-z+\mu s}{\sqrt{s}})+e^{2\mu z}\Phi(\frac{-z-\mu s}{\sqrt{s}})]$, the equation we need to prove becomes:

$$\int_0^{+\infty}e^{-rs}[\Phi(\frac{-z+\mu s}{\sqrt{s}})+e^{2\mu z}\Phi(\frac{-z-\mu s}{\sqrt{s}})]\,ds=\frac{1}{r}e^{(\mu-\sqrt{\mu^2+2r})z} \tag{.16}$$

We write the left hand side of the equation $(A.8)$ into two parts as $\int_0^{+\infty} e^{-rs}\Phi(\frac{-z+\mu s}{\sqrt{s}})\,ds$ and $\int_0^{+\infty} e^{-rs}e^{2\mu z}\Phi(\frac{-z-\mu s}{\sqrt{s}})\,ds$ and use the method of Intergration by Parts to them individually, we have:

$$
\begin{aligned}
&\int_0^{+\infty} e^{-rs}\Phi(\frac{-z+\mu s}{\sqrt{s}})\,ds \\
&= -\frac{1}{r}e^{-rs}\Phi(\frac{-z+\mu s}{\sqrt{s}})\,|_0^{+\infty} + \frac{1}{r\sqrt{2\pi}}\int_0^{+\infty} e^{-rs}e^{-\frac{(z-\mu s)^2}{2s}}\frac{\mu\sqrt{s}-\frac{1}{2\sqrt{s}}(-z+\mu s)}{s}\,ds \\
&= \frac{1}{r}\int_0^{+\infty} e^{-rs}e^{-\frac{(z-\mu s)^2}{2s}}\frac{\mu\sqrt{s}+\frac{z}{\sqrt{s}}}{2s}\,ds
\end{aligned}
$$

By using the same method, we also have:

$$
\begin{aligned}
&\int_0^{+\infty} e^{-rs}e^{2\mu z}\Phi(\frac{-z-\mu s}{\sqrt{s}})\,ds \\
&= -\frac{1}{r}e^{-rs}e^{2\mu z}\Phi(\frac{-z-\mu s}{\sqrt{s}})\,|_0^{+\infty} + \frac{1}{r\sqrt{2\pi}}\int_0^{+\infty} e^{-rs}e^{2\mu z}e^{-\frac{(z+\mu s)^2}{2s}}\frac{-\mu\sqrt{s}-\frac{(-z-\mu s)}{2\sqrt{s}}}{s}\,ds \\
&= \frac{1}{r\sqrt{2\pi}}\int_0^{+\infty} e^{-rs}e^{-\frac{(z-\mu s)^2}{2s}}\frac{-\mu\sqrt{s}+\frac{z}{\sqrt{s}}}{2s}\,ds
\end{aligned}
$$

Therefore we have the left hand side of equation $(A.8)$ becomes:

$$
\begin{aligned}
&\int_0^{+\infty} e^{-rs}\Phi(\frac{-z+\mu s}{\sqrt{s}})\,ds + \int_0^{+\infty} e^{-rs}e^{2\mu z}\Phi(\frac{-z-\mu s}{\sqrt{s}})\,ds \\
&= \frac{1}{r}\int_0^{+\infty} e^{-rs}e^{-\frac{(z-\mu s)^2}{2s}}\frac{\mu\sqrt{s}+\frac{z}{\sqrt{s}}}{2s}\,ds + \frac{1}{r\sqrt{2\pi}}\int_0^{+\infty} e^{-rs}e^{-\frac{(z-\mu s)^2}{2s}}\frac{-\mu\sqrt{s}+\frac{z}{\sqrt{s}}}{2s}\,ds \\
&= \frac{1}{r\sqrt{2\pi}}\int_0^{+\infty} e^{-rs}e^{-\frac{(z-\mu s)^2}{2s}}\frac{z}{s^{\frac{3}{2}}}\,ds
\end{aligned}
$$

Thus the equation $(A.8)$ remains to show that:

$$
\frac{1}{r\sqrt{2\pi}}\int_0^{+\infty} e^{-rs}e^{-\frac{(z-\mu s)^2}{2s}}\frac{z}{s^{\frac{3}{2}}}\,ds = \frac{1}{r}e^{(\mu-\sqrt{\mu^2+2r})z}
$$

muitiply $r\sqrt{2\pi}$ on both sides and it becomes:

$$
\int_0^{+\infty} e^{-rs}e^{-\frac{(z-\mu s)^2}{2s}}\frac{z}{s^{\frac{3}{2}}}\,ds = \sqrt{2\pi}e^{(\mu-\sqrt{\mu^2+2r})z} \tag{.17}
$$

Since the left hand side of $(A.9)$ could also be written as:

$$
\int_0^{+\infty} e^{-rs-\frac{(z-\mu s)^2}{2s}}\frac{z}{s^{\frac{3}{2}}}\,ds = -2z\int_0^{+\infty} e^{-rs-\frac{(z-\mu s)^2}{2s}}\,d(\frac{1}{\sqrt{s}})
$$

Make the change of variable:

$$
Let \qquad \frac{1}{\sqrt{s}} = t, \qquad s = \frac{1}{t^2}
$$

Then we have:

$$
\begin{aligned}
& 2z\int_0^{+\infty} e^{-rs-\frac{(z-\mu s)^2}{2s}}\, d(\frac{1}{\sqrt{s}}) \\
&= 2z\int_0^{+\infty} e^{-\frac{r}{t^2}-\frac{t^2(z-\frac{\mu}{t^2})^2}{2}}\, dt \\
&= 2z\int_0^{+\infty} e^{-\frac{r}{t^2}-\frac{t^2(z^2-\frac{2\mu z}{t^2}+\frac{\mu^2}{t^4})}{2}}\, dt \\
&= 2ze^{\mu z}\int_0^{+\infty} e^{-\frac{r+\frac{\mu^2}{2}}{t^2}-\frac{t^2z^2}{2}}\, dt \\
&= 2ze^{\mu z}\int_0^{+\infty} e^{-a^2t^2-\frac{b^2}{t^2}}\, dt
\end{aligned}
$$

$$
Where \qquad a = \frac{z}{\sqrt{2}}, \qquad b = \sqrt{r+\frac{\mu^2}{2}}.
$$

Since a and b are positive numbers, we define:

$$
I(a,b) = \int_0^{+\infty} e^{-a^2x^2-\frac{b^2}{x^2}}\, dx. \qquad (1)
$$

Make the change of variable:

$$
Let \qquad y = \frac{b}{ax} \qquad and \qquad dx = \frac{b}{ay^2}
$$

We have:

$$
I(a,b) = \frac{b}{a}\int_0^{+\infty} \frac{1}{y^2}e^{-a^2y^2-\frac{b^2}{y^2}}\, dy = \frac{b}{a}\int_0^{+\infty} \frac{1}{x^2}e^{-a^2x^2-\frac{b^2}{x^2}}\, dx. \qquad (2)
$$

Since we have (1) and (2) both equal to $I(a,b)$, so we could get the value for $I(a,b)$ by dividing the summation of (1) and (2) by 2 to obtain:

$$
I(a,b) = \frac{1}{2a}\int_0^{+\infty} (a+\frac{b}{x^2})e^{-a^2x^2-\frac{b^2}{x^2}}\, dx. \tag{.18}
$$

In order to simplify $I(a,b)$, we need to make the change of variable again:

$$
t = ax - \frac{b}{x}
$$

plug the value for t into the right side of $(A.10)$ and we get:

$$
I(a,b) = \int_0^{+\infty} e^{-a^2t^2-\frac{b^2}{t^2}}\, dt
$$

Using the method for integration, we can get $I(a,b)$ as:

$$
I(a,b) = e^{-2ab}\frac{\sqrt{\pi}}{2a}
$$

Plug the value for a and b into $I(a,b)$ and get:

$$\begin{aligned} & 2z\int_0^{+\infty} e^{-rs-\frac{(z-\mu s)^2}{2s}}\, d(\frac{1}{\sqrt{s}}) \\ &= 2ze^{\mu z}\int_0^{+\infty} e^{-a^2t^2-\frac{b^2}{t^2}}\, dt \\ &= 2ze^{\mu z}\frac{\sqrt{\pi}}{z\sqrt{2}}e^{-z\sqrt{\mu^2+2r}} \\ &= \sqrt{2\pi}e^{z(\mu-\sqrt{\mu^2+2r})} \end{aligned}$$

Therefore we have the equation $(A.8)$ and $(A.9)$ hold and

$$h(z) = \Phi(\frac{z-\mu s}{\sqrt{s}}) - e^{2\mu z}\Phi(\frac{-z-\mu s}{\sqrt{s}}) = 1-\Phi(\frac{-z+\mu s}{\sqrt{s}}) - e^{2\mu z}\Phi(\frac{-z-\mu s}{\sqrt{s}})$$

Where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy$$

has been proved.

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