Eigenvalues and Eigenvectors

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1 Eigenvalues and Eigenvectors

Consider this equation

$$(A - \lambda I)x = 0$$

It is called **eigenvalue equation** of matrix A, λ and x are called **eigenvalues** and **eigenvectors** respectively, we are interested only in those particular values λ for which there is a nonzero eigenvector x, so $A - \lambda I$ **must be singular**. We emphasize the steps in solving $(A - \lambda I)x = 0$:

- **Compute the determinant of $A - \lambda I$.** With λ subtracted along the diagonal, this determinant is a polynomial of degree n. It starts with $(-\lambda)^n$.
- **Find the roots of this polynomial.** The n roots are the eigenvalues of A.
- **For each eigenvalues solve the equation $(A - \lambda I)x = 0$.** Since the determinant is zero, there are solutions other than $x = 0$. Those are eigenvectors.

There are some properties about λ and x:

- Trace of $A = \lambda_1 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn}$
- $\lambda_1 \cdot \ \cdots \ \cdot \lambda_n = |A|$
- If $x_1, \cdots, x_k$ correspond to **different** sigenvalues $\lambda_1, \cdots, \lambda_k$, then those eigenvectors are linearly independent.

Suppose the n by n matrix A has n linearly independent eigenvectors. If these eigenvectors are the columns of a matrix S, then $S^{-1}AS$ is a diagonal matrix Λ. The eigenvalues of A are on the diagonal of Λ:

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

We call S the **eigenvector matrix** and Λ the **eigenvalue matrix**. According to diagonalization, we can calculate powers and products easily:

$$\Lambda^k = (S^{-1}AS)(S^{-1}AS)\cdots(S^{-1}AS) = S^{-1}A^kS$$

Generally, the matrix A and $M^{-1}AM$ (M is any invertible matrix) are **similar**, Going form one to the other is a **similarity transformation.** Λ is a special similar form. Similar matrices share the same eigenvalues.

If columns of Q contain orthonormal eigenvectors of A, we can get:

$$\Lambda = Q^T AQ = Q^{-1}AQ$$

2 Complex Matrices

2.1 Basics

A review of complex numbers is easy to give:

$$i^2 = -1$$

$$a + ib = \overline{a - ib} = re^{i\theta}$$

$$r = |a + ib| = \sqrt{a^2 + b^2}$$

$$\theta = arctan(\frac{b}{a})$$

2.2 Transposes in the Complex Case

By definition, the **complex vector space** C^n contains all vectors x with n complex components:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{with components } x_j = a_j + ib_j$$

For complex matrices, a superscript H (or a star) combines both conjugate and tranpose, and it is called A **Hermitian**.

$$\overline{A}^T = A^H = A^*$$

2.3 Hermitian Matrices

Hermitian matrices si defined:

$$A = A^H$$

If $A = A^H$:

- For all complex vector x, the number $x^H Ax$ is real.
- Every eigenvalue is real.
- Two eigenvectors of a real symmetric matrix or a Hermitian matrix, if they come from different eigenvalues. are orthogonal to one another.

2.3.1 Unitary Matrices

If $U^H U = I$(or $UU^H = I,\ U^H = U^{-1}$), the complex matrix U with orthonormal columns is called a **unitary matrix**, and unitary matrix U has following property:

- $(Ux)^H(Uy) = x^H U^H U y = x^H y$ and lengths are preserved by U:

$$||Ux||^2 = x^H U^H U x = ||x||^2$$

- Every eigenvalue of U has absolute value $|\lambda| = 1$
- Eigenvectors corresponding to different eigenvalues are orthonormal.