Lecture 10: Lumped Nonlinear Systems

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Contents

1 Lumped Nonlinear Systems

Lumped continuous-time nonlinear system:

$$\frac{d}{dt}x(t) = f(x, u, t), \quad y(t) = g(x, u, t)$$

Lumped discrete-time nonlinear system:

$$x[k+1] = f(x[k], u[k], k), \quad y[k] = g(x[k], u[k], k)$$

- For m-input, n-state, p-output systems,

$$f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n, \quad g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p$$

2 Autonomous Time-lnvariant Nonlinear System

$$\frac{d}{dt}x(t) = f(x(t)), \qquad \text{where } f : \mathbb{R}^n \to \mathbb{R}^n$$

- **Equilibrium points**: solutions x_e to $f(x) = 0$

$$f\left(x_{\mathrm{e}}\right) = 0$$

- Let $\delta x(t) = x(t) - x_{\mathrm{e}}$. The dynamics of $\delta x(t)$ is approximated by

$$\frac{d}{dt}\delta x(t) \approx \underbrace{\mathrm{Df}\left(x_{\mathrm{e}}\right)}_{\text{Jacobian}} \delta x(t)$$

3 Example: Simple Pendulum

Dynamics: $\ddot{\theta} = -mg\ell \sin\theta - \eta\dot{\theta}$

Define state as $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

State equation: $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -mg\ell \sin x_1 - \eta x_2 \end{bmatrix}$

4 Stability of Autonomous Nonlinear Systems

Definition:
$\frac{d}{dt}x(t) = f(x(t))$ is locally asymptotically stable near the equilibrium point x_e if there exists some $r > 0$ such that

$$\|x(0) - x_e\| < r \Rightarrow \lim_{t\to\infty} \|x(t) - x_e\| \to 0$$

- All solutions starting in a ball of radius r around x_e converge to it
- If r can be chosen to be ∞, we get global asymptotic stability

5 Sufficient Condition for (In)stability

Theorem (Hartman-Gorbman Theorem):
For a nonlinear system $\dot{x} = f(x)$ with an equilibrium point x_{e}, let $\frac{d}{dt}\delta x = Df\left(x_e\right)\delta x$ be its linearization around x_e

- If $Df\left(x_e\right)$ has all eigenvalues with negative real part, then nonlinear system $\dot{x} = f(x)$ is asy. stable around the equilibrium point x_e
- If $Df\left(x_e\right)$ has all eigenvalues with positive real part, then nonlinear system $\dot{x} = f(x)$ is unstable around the equilibrium point x_e

6 Inconclusive Results from Linearization

- When sufficient stability/instability conditions fail, can have either stability or instability
- Example: $\dot{x} = -x^3$
- Example: $\dot{x} = x^3$
- Simple pendulum at $x_{\mathrm{e}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

7 Example: Lotka-Volterra Model

Population model of two species:

- x_1: population of prey
- x_2: population of predator

Assumptions:

- Unlimited food supply
- Predator total dependence on prey

$$\begin{cases} \frac{dx_1}{dt} = 4x_1 - 2x_1x_2 \\ \frac{dx_2}{dt} = -x_2 + x_1x_2 \end{cases}$$

Equilibrium points: $x_{e,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x_{e,2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

8 Phase Plot of Lotka-Volterra Model

9 Linearization of Controlled NLTI Systems

A controlled nonlinear time-invaraint system

$$\dot{x}(t) = f(x(t), u(t))$$

- x_e is an equilibrium if it satisfies

$$f(x_e, 0) = 0$$

- For small initial deviation $\delta x = x - x_e$, and small input u:

$$\frac{d}{dt}\delta x(t) \simeq \underbrace{D_x f\left(x_e, 0\right)}_{A} \delta x(t) + \underbrace{D_u f\left(x_e, 0\right)}_{B} u(t)$$

10 Linearization of NLTV Systems around a Trajectory

Suppose the controlled nonlinar time-varying system

$$\frac{d}{dt}x(t) = f(x, u, t), \quad x(0) = x_0, \quad y(t) = g(x, u, t)$$

has a solution $x^*(t)$ and $y^*(t)$ under the input $u^*(t)$
Suppose input is perturbed by a small perturbation: $u(t) = u^*(t) + \delta u(t)$
The resulting $x(t) = x^*(t) + \delta x(t)$ and $y(t) = y^*(t) + \delta y(t)$ satisfy