Lecture 14: Observability I

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1 Motivation: State Estimation of D-T LTI Systems

A discrete-time n-state m-input p-output LTI system

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k], \quad x[0] = x_0 \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

- Matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$ are known
- Input and output are observed over times $0, 1, \ldots, k-1$:

$$u[0], \ldots, u[k-1], y[0], \ldots, y[k-1]$$

- **Objective**: Estimate unknown initial condition $x[0]$

Definition (Observability):
The D-T LTI system is **observable at time** k if the initial condition $x[0]$ can be uniquely determined from any given $u[0], ..., u[k-1]$ and $y[0], ..., y[k-1]$. It is **observable** if observable at a large enough time k.

2 Characterizing Observability

From system dynamics, we have

$$\begin{bmatrix} y[0] \\ \vdots \\ y[k-1] \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix}}_{\mathcal{O}_k} x[0] + \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ CA^{k-2}B & & CB & D \end{bmatrix}}_{\mathcal{T}_k} \begin{bmatrix} u[0] \\ \vdots \\ u[k-1] \end{bmatrix}$$

Therefore, **system is observable at time k if $\mathcal{O}_k$ is injective**

- Null space $\mathcal{N}(\mathcal{O}_k)$ gives ambiguity in determining $x[0]$
- If $x[0] \in \mathcal{N}(\mathcal{O}_k)$ and $u \equiv 0$, output is zero over times $0, \ldots, k-1$
- Input u does not affect observability: it can be subtracted out

Hence, we can assume $u \equiv 0$, and consider simplified system (C, A):

$$\begin{aligned} x[k+1] &= Ax[k] \\ y[k] &= Cx[k] \end{aligned}$$

3 Observability Matrix

Proposition:
The null space $\mathcal{N}(\mathcal{O}_k)$ of matrices $\mathcal{O}_k$ satisfy:

$$\mathcal{N}(\mathcal{O}_1) \supseteq \mathcal{N}(\mathcal{O}_2) \supseteq \cdots \supseteq \mathcal{N}(\mathcal{O}_n) = \mathcal{N}(\mathcal{O}_{n+1}) = \cdots$$

The **observability matrix** is defined as $\mathcal{O} := \mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

Theorem:
The D-T LTI system (A, B, C, D), or simply (C, A), is observable if the observability matrix $\mathcal{O}$ is injective, or equivalently, full (column) rank n.

4 Unobservable Subspace

Unobservable subspace is the null space $\mathcal{N}(\mathcal{O})$ of observability matrix

- Describes the ambiguity in determining state from input and output
- System is observable if the unobsevable subspace is trivial $\{0\}$

Fact:
Unobservable subspace $\mathcal{N}(\mathcal{O})$ is A-invariant: $A\mathcal{V}(\mathcal{O}) \subseteq \mathcal{N}(\mathcal{O})$

Example:

- $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}$
- $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

5 Observability-Controllability Duality

Definition (Dual System):
The system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with $\tilde{A} = A^T$, $\tilde{B} = C^T$, $\tilde{C} = B^T$, and $\tilde{D} = D^T$ is called the **dual** of system (A, B, C, D)

- Controllability matrix $\tilde{\mathcal{C}} = \mathcal{O}^T$
- Observability matrix $\tilde{\mathcal{O}} = \mathcal{C}^T$

Theorem:
System (A, B, C, D) is observable (resp. controllable) if and only if its dual system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is controllable (resp. observable).

• Furthermore, $\mathcal{N}(\mathcal{O}) = \mathcal{R}(\tilde{\mathcal{C}})^{\perp}$, and $\mathcal{R}(\mathcal{C}) = \mathcal{N}(\tilde{\mathcal{O}})^{\perp}$

6 Observability Conditions

Theorem:
For the system (A, B, C, D), the following are equivalent:

- The system is observable
- The observability matrix $\mathcal{O}$ is full rank

- **PBH Rank Test**: For any $\lambda \in \mathbb{C}$,

$$\operatorname{rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$$

- **PBH Eigenvector Test**: for any right eigenvector $v \in \mathbb{C}^n$ of A,

$$Cv \neq 0$$

- The matrix $\sum_{i=0}^{n-1}\left(A^T\right)^i C^T C A^i$ is nonsingular

7 Example

$$x[k+1] = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 \end{bmatrix} x[k]$$

8 Quantitative Observability

Suppose system (A, B, C, D) observable, and $u \equiv 0$.
Starting from initial condition $x[0]$, the output energy over time $[0, k-1]$ is

$$\sum_{i=0}^{k-1} \|y[i]\|^2 = x[0]^T \underbrace{\left(\sum_{i=0}^{k-1}\left(A^T\right)^i C^T C A^i\right)}_{W_o(k)} x[0]$$

If further A is stable, the following limit exists:

$$\sum_{i=0}^{\infty} \|y[i]\|^2 = x[0]^T \underbrace{\left(\sum_{i=0}^{\infty}\left(A^T\right)^i C^T C A^i\right)}_{\text{Observability Gramian } W_o} x[0] \quad (as \quad k \to \infty)$$

• The larger the output energy, the "easier" it is to estimate $x[0]$

9 State Observer with Perfect Measurements

Suppose system (A, B, C, D) is observable at time k : $\operatorname{rank}(\mathcal{O}_k) = n$
Under zero input $u \equiv 0$, we can deduce $x[0]$ from $y[0], \ldots, y[k-1]$:

$$\begin{bmatrix} y[0] \\ \vdots \\ y[k-1] \end{bmatrix} = \mathcal{O}_k x[0] \Rightarrow x[0] = \mathcal{O}_k^{\dagger} \begin{bmatrix} y[0] \\ \vdots \\ y[k-1] \end{bmatrix}$$

• Since $\mathcal{O}_k$ is one-to-one, $\mathcal{O}_k^{\dagger} = \left(\mathcal{O}_k^T \mathcal{O}_k\right)^{-1} \mathcal{O}_k^T$

10 State Observer with Noisy Measurements

Suppose now output measurements are corrupted by noises $w[k]$:

$$\begin{aligned} x[k+1] &= Ax[k] \\ \hat{y}[k] &= Cx[k] + w[k] \end{aligned}$$

We have: $\mathcal{O}_k x[0] = \begin{bmatrix} y[0] \\ \vdots \\ y[k-1] \end{bmatrix} = \begin{bmatrix} \hat{y}[0] \\ \vdots \\ \hat{y}[k-1] \end{bmatrix} - \begin{bmatrix} w[0] \\ \vdots \\ w[k-1] \end{bmatrix}$

11 Observer Error

Suppose sensor noises $w[k]$ are Gaussian white noises:

- $w[k]$ and $w[\ell]$ are independent for different k and ℓ
- $\mathrm{E}\left(w[k]w[k]^T\right) = I_p$, i.e., $w[k] \sim \mathcal{N}(0, l)$

Then, covariance of estimation error e is

12 Infinite Horizon Error Covariance

Let the time horizon $k \to \infty$. Then matrix

$$P = \left(\sum_{i=0}^{\infty} \left(A^T\right)^i C^T C A^i \right)^{-1} = W_o^{-1}$$

gives error covariance in estimating $x[0]$ from u, y over ∞ time horizon

- $\mathrm{E}\, \|\hat{x}_{\mathrm{ls}}[0] - x[0]\|^2 = \operatorname{tr} P$
- If A is stable, $P \succ 0$, i.e., can't estimate $x[0]$ perfectly even with infinite number of measurements $u[k]$, $y[k]$ (memory of $x[0]$ fades)
- If A is not stable. P has nonzero null space: $Pv = 0$ for $v \neq 0$. Hence projection of $x[0]$ along v can be exactly determined eventually:

$$E\left[\left(\hat{x}_{1s}[0] - x[0]\right)^T v \right]^2 \to 0 \quad \text{as } k \to \infty$$

- Eigenvectors of P divide state-space into directions with varying degrees of "ease" of observability