Singular Value Decomposition

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The singular value decomposition (SVD) is closely associated with eigenvalue-eigenvector factorization $Q\Lambda Q^T$ of a positive definite matrix. The eigenvalues are in the diagonal matrix Λ and the eigenvector matrix Q is orthohonal ($Q^TQ = I$) because eigenvectors matrix can be chosen to be orthogonal.

However,for most matrices that is not true. Now we allow the Q on the left and the Q^T on the right to be any orthogonal matrices U and V^T. Then every matrix will be split into $A = U\Sigma V^T$.

Any m by n matrix A can be factored into

$$A = U\Sigma V^T = (m \times m\ orthogonal)(m \times n\ diagonal)(n \times n\ orthogonal).$$

The diagonal (but rectangular) matrix Σ has eigenvalues from AA^T. Those positive entries will be $\sigma_1, ..., \sigma_r$. They are the singluar values of A. They fill the first r places on the main diagonal of Σ — when A has rank r. The rest of Σ is zero.

- To get U, Σ and V of A, we first calculate AA^T and A^TA.

  $$AA^T = (U\Sigma V^T)(V\Sigma^T U^T) = U\Sigma\Sigma^T U^T\ (m \times m)$$
  $$A^TA = (V\Sigma^T U^T)(U\Sigma V^T) = V\Sigma^T\Sigma V^T\ (n \times n)$$

  U must be the eigenvector matrix for AA^T and V for A^TA. The eigenvalue matrix in the middle is $\Sigma\Sigma^T$ — which is m by m with $\sigma_1^2, ..., \sigma_r^2$ on the diagonal.

- The eigenvalues λ_n of AA^T can be calcuated by

  $$\left|AA^T - \lambda E\right| = 0,$$

  and Σ is solved by $\sigma_n = \sqrt{\lambda_n}$.

- The eigenvectors x_m that corresponding to the eigenvalues λ_m can be obtained by

  $$(AA^T - \lambda I)x = 0,$$

  and U is set to Schmidt orthogonalization of $[x_1, ..., x_m]$.

- The eigenvectors x_n that corresponding to the eigenvalues λ_n can be obtained by

  $$(A^TA - \lambda I)x = 0,$$

  and V is set to Schmidt orthogonalization of $[x_1, ..., x_n]$.

For now, we complete the SVD to A.