Positive Definite Matrices

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For any symmetric matrix A, the product $x^T Ax$ is a pure quadratic form $f(x_1, ..., x_n)$:

$$\begin{bmatrix} x_1 & x_2 & \cdot & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & \cdot & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

Each of the following tests is a **necessary and sufficient** condition for the real symmetric matrix A to be **positive definite**:

- $x^T Ax > 0$ for all nonzero real vectors x
- All the eigenvalues of A satisfy $\lambda_i > 0$
- All the upper left submatrices A_k have positive determinants
- All the pivots (without row exchanges) satisfy $d_k > 0$
- There is a matrix R with independent columns such that $A = R^T R$

Each of the following tests is a **necessary and sufficient** condition for the real symmetric matrix A to be **positive semidefinite**:

- $x^T Ax \geqslant 0$ for all vectors x
- All the eigenvalues of A satisfy $\lambda_i \geqslant 0$
- No principal submatrices have negative determinants
- No pivots are negative
- There is a matrix R, possibly with dependent columns, such that $A = R^T R$