AAE 568 Applied Optimal Control and Estimation Problem Set 2

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Remark

You can use MATLAB but you need to explain the results (numbers and plot) as clear as possible.

Problem 1

Consider the optimal control problem with:

$$\begin{array}{ll} \text{Dynamics} & : \quad \dot{x}(t) = f(x(t), u(t), t), x(t_0) = x_0 \\ \text{Cost} & : \min J(u) = \phi\left(x\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} L(x(t), u(t), t) dt \end{array}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$. Derive the necessary conditions for optimality of the above optimal control problem when:

1. t_f is fixed and the final state lying on the surface defined by

$$m(x(t)) = 0$$

   where $m(\cdot) \in \mathbb{R}^k$

2. t_f is free and the final state lying on the moving point $\Theta(t) \in \mathbb{R}^n$

Solution

Define $H \equiv L(x(t), u(t), t) + \lambda^T(t) f(x(t), u(t), t)$. Then, augmented cost J_a is

$$J_a = \phi\left(x\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} \left[H(x(t), \lambda(t), u(t), t) - \lambda^T(t)\dot{x}(t)\right] dt$$

1. The variation of the cost $\Delta J_a = J_a(x(t), \lambda(t), u(t), t) - J_a\left(x^*(t), \lambda^*(t), u^*(t), t\right)$ is computed as:

$$\begin{aligned} \Delta J_a = {} & \phi\left(x\left(t_f\right), t_f\right) - \phi\left(x^*\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} \left[H(x(t), \lambda(t), u(t), t) - \lambda^T(t)\dot{x}(t)\right] dt \\ & - \int_{t_0}^{t_f} \left[H\left(x^*(t), \lambda^*(t), u^*(t), t\right) - \lambda^{*T}(t)\dot{x}^*(t)\right] dt \end{aligned}$$

where superscript * means optimal value. The first order approximation of the variation is then

$$\delta J_a = \left[\left.\frac{\partial \phi}{\partial x}\right|_{x^*(t_f)} - \lambda^{*T}(t_f) \right] \delta x(t_f) + \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial \lambda}\delta\lambda + \frac{\partial H}{\partial u}\delta u - \delta\lambda^T(t)\dot{x}^*(t) + \dot{\lambda}^{*T}(t)\delta x \right] dt$$

From the condition $m(x(t_f)) = 0$,

$$\left[\left.\frac{\partial m}{\partial x}\right|_{x^*(t_f)} \right]^T \delta x(t_f) = 0$$

This means that the final state lies on the hyper-plane spanned by $\left.\frac{\partial m}{\partial x}\right|_{x^*(t_f)}$ to which $\delta x(t_f)$ is orthogonal. Combine this with $\left[\left.\frac{\partial \phi}{\partial x}\right|_{x^*(t_f)} - \lambda^{*T}(t_f) \right] \delta x(t_f) = 0$,

$$\left[\left.\frac{\partial \phi}{\partial x}\right|_{x^*(t_f)} - \lambda^{*T}(t_f) \right] = \mu^T \left[\left.\frac{\partial m}{\partial x}\right|_{x^*(t_f)} \right], \qquad \mu \in \mathbb{R}^k$$

For optimality, $\delta J_a = 0$ for arbitrary $\delta x(t), \delta\lambda(t), \delta u(t)$, the necessary and boundary conditions are:

$$\frac{\partial H}{\partial u} = 0$$

$$\dot{x}(t) = \left(\frac{\partial H}{\partial \lambda}\right)^T$$

$$\dot{\lambda}(t) = -\left(\frac{\partial H}{\partial x}\right)^T$$

$$\left[\left.\frac{\partial \phi}{\partial x}\right|_{x^*(t_f)} - \lambda^{*T}(t_f) \right] = \mu^T \left[\left.\frac{\partial m}{\partial x}\right|_{x^*(t_f)} \right], \quad \mu \in \mathbb{R}^k$$

$$x^*(t_0) = x_0$$

$$m(x(t_f)) = 0$$

2. The variation of $\Delta J_a = J_a(x(t), \lambda(t), u(t), t) - J_a(x^*(t), \lambda^*(t), u^*(t), t)$ is computed as:

$$\begin{aligned}
\Delta J_a = {} & \phi(x(t_f), t_f) - \phi\left(x^*\left(t_f^*\right), t_f^*\right) + \int_{t_0}^{t_f^*+\delta t_f} \left[H(x(t), \lambda(t), u(t), t) - \lambda^T(t)\dot{x}(t)\right] dt \\
& - \int_{t_0}^{t_f^*} \left[H(x^*(t), \lambda^*(t), u^*(t), t) - \lambda^{*T}(t)\dot{x}^*(t)\right] dt \\
& = \phi(x(t_f), t_f) - \phi\left(x^*\left(t_f^*\right), t_j^*\right) \\
& + \int_{t_0}^{t_f^*} \left[H(x(t), \lambda(t), u(t), t) - \lambda^T(t)\dot{x}(t) - H(x^*(t), \lambda^*(t), u^*(t), t) + \lambda^{*T}(t)\dot{x}^*(t)\right] dt \\
& + \int_{t_f^*}^{t_f^*+\delta t_f} \left[H(x(t), \lambda(t), u(t), t) - \lambda^T(t)\dot{x}(t)\right] dt
\end{aligned}$$

where superscript * means optimal value. The first order approximation of the variation is then

$$
\begin{aligned}
\delta J_a = & \left.\frac{\partial \phi}{\partial x\left(t_f\right)}\right|_{x^*\left(t_f^*\right), t_f^*} \delta x\left(t_f\right)+\left.\frac{\partial \phi}{\partial t_f}\right|_{x^*\left(t_f^*\right), t_f^*} \delta t_f \\
& +\int_{t_0}^{t_f^*}\left[\frac{\partial H}{\partial x} \delta x(t)+\frac{\partial H}{\partial \lambda} \delta \lambda(t)+\frac{\partial H}{\partial u} \delta u(t)-\delta \lambda^T(t) \dot{x}(t)+\dot{\lambda}^T(t) \delta x(t)\right] d t \\
& -\lambda^{* T}\left(t_f^*\right) \delta x\left(t_f^*\right)+\left[H\left(x^*\left(t_f\right), \lambda^*\left(t_f\right), u^*\left(t_f\right), t_f^*\right)-\lambda^{* T}\left(t_f\right) \dot{x}^*\left(t_f\right)\right] \delta t_f
\end{aligned}
$$

Note that, the following approximation can be made for the relationship between $\delta x\left(t_f\right)$ and $\delta x\left(t_f^*\right)$:

$$
\delta x\left(t_f^*\right) \approx \delta x\left(t_f\right)-\dot{x}^*\left(t_f^*\right) \delta t_f
$$

Also, from the fact that the final state must lie on the moving point $\theta(t) \in \mathbb{R}^n$, the condition between $\delta x\left(t_f\right)$ and δt_f is approximated as:

$$
\delta x\left(t_f\right) \approx \dot{\theta}\left(t_f\right) \delta t_f
$$

From the above two relations the first order approximation δJ_a is rewritten as:

$$
\begin{aligned}
\delta J_a = & \left[\left.\frac{\partial \phi}{\partial x\left(t_f\right)}\right|_{x^*\left(t_f^*\right), t_f^*}-\lambda^{* T}\left(t_f^*\right)\right] \delta x\left(t_f\right) \\
& +\left[\left.\frac{\partial \phi}{\partial t_f}\right|_{x^*\left(t_f^*\right), t_f^*}+\lambda^{* T}\left(t_f^*\right) \dot{x}^*\left(t_f^*\right)+H\left(x^*\left(t_f\right), \lambda^*\left(t_f\right), u^*\left(t_f\right), t_f^*\right)-\lambda^{* T}\left(t_f\right) \dot{x}^*\left(t_f\right)\right] \delta t_f \\
& +\int_{t_0}^{t_f^*}\left[\frac{\partial H}{\partial x} \delta x(t)+\frac{\partial H}{\partial \lambda} \delta \lambda(t)+\frac{\partial H}{\partial u} \delta u(t)-\delta \lambda^T(t) \dot{x}(t)+\dot{\lambda}^T(t) \delta x(t)\right] d t \\
= & \left[\left.\frac{\partial \phi}{\partial x\left(t_f\right)}\right|_{x^*\left(t_f^*\right), t_f^*}-\lambda^{* T}\left(t_f^*\right)\right] \dot{\theta}\left(t_f\right) \delta t_f+\left[\left.\frac{\partial \phi}{\partial t_f}\right|_{x^*\left(t_f^*\right), t_f^*}+H\left(x^*\left(t_f\right), \lambda^*\left(t_f\right), u^*\left(t_f\right), t_f^*\right)\right] \delta t_f \\
& +\int_{t_0}^{t_f^*}\left[\frac{\partial H}{\partial x} \delta x(t)+\frac{\partial H}{\partial \lambda} \delta \lambda(t)+\frac{\partial H}{\partial u} \delta u(t)-\delta \lambda^T(t) \dot{x}(t)+\dot{\lambda}^T(t) \delta x(t)\right] d t \\
= & \left[\left(\left.\frac{\partial \phi}{\partial x\left(t_f\right)}\right|_{x^*\left(t_f^*\right), t_f^*}\right) \dot{\theta}\left(t_f\right)+\left.\frac{\partial \phi}{\partial t_f}\right|_{x^*\left(t_f^*\right), t_f^*}+H\left(x^*\left(t_f\right), \lambda^*\left(t_f\right), u^*\left(t_f\right), t_f^*\right)\right] \delta t_f \\
& +\int_{t_0}^{t_f^*}\left[\frac{\partial H}{\partial x} \delta x(t)+\frac{\partial H}{\partial \lambda} \delta \lambda(t)+\frac{\partial H}{\partial u} \delta u(t)-\delta \lambda^T(t) \dot{x}(t)+\dot{\lambda}^T(t) \delta x(t)\right] d t
\end{aligned}
$$

For optimality, $\delta J_a=0$ for arbitrary $\delta x(t), \delta \lambda(t), \delta u(t)$, and δt_f . These yield the following necessary conditions and boundary conditions:

$$
\frac{\partial H}{\partial u}=0
$$

$$
\dot{x}(t)=\left(\frac{\partial H}{\partial \lambda}\right)^T
$$

$$\dot{\lambda}(t) = -\left(\frac{\partial H}{\partial x}\right)^T$$

$$\left(\left.\frac{\partial \phi}{\partial x\left(t_f\right)}\right|_{x^*\left(t_f^*\right), t_f^*} - \lambda^{*T}\left(t_f^*\right)\right) \dot{\theta}\left(t_f\right) + \left.\frac{\partial \phi}{\partial t_f}\right|_{x^*\left(t_f^*\right), t_f^*} + H\left(x^*\left(t_f\right), \lambda^*\left(t_f\right), u^*\left(t_f\right), t_f^*\right) = 0$$

$$x^*\left(t_0\right) = x_0$$

$$x\left(t_f^*\right) = \theta\left(t_f^*\right)$$

Problem 2

Consider a double integral system:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = u(t), \quad \forall u(t) > 0$$

with the cost function:

$$J(u) = \frac{1}{2} \int_0^{t_f} u(t)^2 dt$$

Given the boundary conditions as:

$$x(0) = \left[\begin{array}{l} 1 \\ 2 \end{array}\right], \quad x_1\left(t_f\right) = 3, \quad x_2\left(t_f\right) = \text{ free}, \quad t_f = \text{ free}$$

Compute the optimal control that minimizes the cost function. (You can use the MATLAB function byp4c but need to carefully design state and co-state dynamics since the given problem is with free-final-time.)

Solution

For the optimal control problems with free-final-time, the final time t_f itself is a variable. Thus, we have one additional boundary condition from:

$$H\left(x^*\left(t_f\right), u^*\left(t_f\right), \lambda^*\left(t_f\right), t_f\right) + \left.\frac{\partial \phi}{\partial t}\right|_{x^*\left(t_f\right), t_f} = 0$$

Since $\phi\left(x^*\left(t_f\right), t_f\right) \equiv 0$ in this problem, the above condition yields:

$$\lambda_1\left(t_f\right) x_2\left(t_f\right) - 0.5 \lambda_2^2\left(t_f\right) = 0$$

Now we have 5 algebraic equations with 5 unknowns.
One common treatment for the unknown time interval problem is to change the independent variable t to $= \frac{t}{T}$, the augmented state and co-state equations will then become:

$$\left[\begin{array}{c} \dot{x} \\ \dot{\lambda} \end{array}\right] = T f(x, \lambda, \tau)$$

where the function f denotes the ODEs for states and co-states. Then, the problem is posed on fixed interval $[0,1]$ and this can be implemented in **bvp4c** by treating T as an auxiliary variable. The following MATLAB code shows the detail.

```
solint = bvpinit(linspace(0,1),[2;3;1;1;2]);
sol = bvp4c(@ode, @bc, solinit);
y = sol.y;
time = y(5) * sol.x;

% ODE"s of augmanted states
function dydt = ode(t,y)
dydt = y(5) * [y(2); -y(4); 0; -y(3); 0];
end

% boundary conditions: x1(0)=1; x2(0)=2, x1(tf)=3, lambda2(tf)=0;
% lambda1(tf)*x2(tf)-0.5*lambda2(2)^2
function res = bc(ya,yb)
res = [ya(1) - 1; ya(2) - 2; yb(1) - 3; yb(4); ...
    yb(3)*yb(2)-0.5*yb(4)^2];
end
```

From the numerical simulation, the computed optimal final time is $t_f = 3$ and the resulting optimal state, co-state, and control trajectories are shown as the figures below:

Problem 3

Consider a non-linear system:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -x_1(t) + x_2(t)\left(1.4 - 0.14x_2(t)^2\right) + 4u(t)$$

with the cost function to be minimized:

$$J(x,u) = \int_0^{t_f} u(t)^2 + x_1(t)^2 dt$$

where $x_1(0) = -5, x_2(0) = -5$, and the final time is fixed as $t_f = 4.5$. Plot the optimal control, state, and co-state trajectories for the following three types of constraints:

1. No constraints
2. Control constraint $-1 \leq u(t) \leq 1$
3. Mixed control-state constraint $-1 \leq u(t) + \frac{x_1(t)}{6} \leq 0$

Solution

1. The corresponding Hamiltonian is given by:

$$H(x,\lambda,u) = u^2 + x_1^2 + \lambda_1 x_2 + \lambda_2\left(-x_1 + x_2\left(1.4 - 0.14x_2^2\right) + 4u\right)$$

From the optimality condition $\dot{\lambda} = -\frac{\partial H}{\partial x}$, the co-state equations are:

$$\dot{\lambda}_1 = -2x_1 + \lambda_2$$

$$\dot{\lambda}_2 = -\lambda_1 - \lambda_2\left(1.4 - 0.42x_2^2\right)$$

where the boundary condition is $\lambda(t_f) = 0$. And the optimal control input is:

$$\frac{\partial H}{\partial u} = 2u + 4\lambda_2 = 0 \rightarrow u = -2\lambda_2$$

2. The corresponding Hamiltonian is given by:

$$H(x,\lambda,u) = u^2 + x_1^2 + \lambda_1 x_2 + \lambda_2\left(-x_1 + x_2\left(1.4 - 0.14x_2^2\right) + 4u\right) + \mu_1(u-1) + \mu_2(-u-1)$$

where the multipliers for inequality conditions $\mu_1, \mu_2 \geq 0$. The co-state equations are same as the case 1 and the optimal control input is represented as

$$\frac{\partial H}{\partial u} = 2u + 4\lambda_2 + \mu_1 - \mu_2 = 0 \rightarrow u = \begin{cases} 1 & \text{if } u > 1 \\ -2\lambda_2 & \text{if } -1 \leq u \leq 1 \\ -1 & \text{if } u < -1 \end{cases}$$

3. The corresponding Hamiltonian is described as:

$$H(x,\lambda,u) = u^2 + x_1^2 + \lambda_1 x_2 + \lambda_2\left(-x_1 + x_2\left(1.4 - 0.14x_2^2\right) + 4u\right) + \mu_1\left(u + \frac{x_1}{6}\right) + \mu_2\left(-u - \frac{x_1}{6} - 1\right)$$

Then, the co-state equations are:

$$\dot{\lambda}_1 = -2x_1 + \lambda_2 - \frac{\mu_1}{6} + \frac{\mu_2}{6}$$

$$\dot{\lambda}_2 = -\lambda_1 - \lambda_2\left(1.4 - 0.42x_2^2\right)$$

where the multipliers for inequality condition $\mu_1 \geq 0, \mu_2 \geq 0$ and the optimal control input is:

$$u = \begin{cases} -\frac{x_1}{6} & \text{if } u + \frac{x_1}{6} > 0 \\ -2\lambda_2 & \text{if } -1 \leq u + \frac{x_1}{6} \leq 0 \\ -1 - \frac{x_1}{6} & \text{if } u + \frac{x_1}{6} < -1 \end{cases}$$

Furthermore, from the condition $\frac{\partial H}{\partial u} = 2u + 4\lambda_2 + \mu_1 - \mu_2 = 0$, we can obtain

$$\mu_1 = \begin{cases} -\frac{x_1}{3} + 4\lambda_2 & \text{if } u + \frac{x_1}{6} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_2 = \begin{cases} -\frac{x_1}{3} - 2 + 4\lambda_2 & \text{if } u + \frac{x_1}{6} \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

```
% P1. ODE"s for states and costates
function dydt =BVP_ode(t, y)
u = -2 * y(4);
dydt = [y(2);
         -y(1) + y(2) * (1.4-0.14*y(2)^2)+4*u;
         -2*y(1)+y(4);
         -y(3)-y(4)*(1.4-0.42*y(2)^2)];
end
```

```matlab
% P2. ODE"s for states and costates
function dydt =BVP_ode(t, y)
if -2*y(4) > 1
    u = 1;
elseif -2*y(4)<-1
    u=-1;
else
    u=-2*y(4);
end

dydt = [y(2);
        -y(1) + y(2) * (1.4-0.14*y(2)^2)+4*u;
        -2*y(1)+y(4);
        -y(3)-y(4)*(1.4-0.42*y(2)^2)];
end
```

```matlab
% P3. ODE"s for states and costates
function dydt =BVP_ode_(t, y)
if -2*y(4)+y(1)/6 > 0
    u = -y(1)/6;
    mu1 = -y(1)/3 + 4*y(4);
    mu2 = 0;
elseif -2*y(4)+y(1)/6 < -1
    u = -1-y(1)/6;
    mu1 = 0;
    mu2 = -y(1)/3-2+4*y(4);
else
    u = -2*y(4);
    mu1 = 0;
    mu2 = 0;
end

dydt = [y(2);
        -y(1) + y(2) * (1.4-0.14*y(2)^2)+4*u;
        -2*y(1)+y(4)-mu1/6+mu2/6;
        -y(3)-y(4)*(1.4-0.42*y(2)^2)];
end
```

Problem 4

Consider a first-order dynamic system having the state equations

$$x(k+1) = -0.5x(k) + u(k)$$

The cost functional to be minimized is

$$J(u) = \sum_{k=0}^{2} |x(k)|$$

and the admissible states and controls are constrained as:

$$-0.2 \le x(k) \le 0.2 \quad , \quad k = 0, 1, 2$$

$$-0.1 \le u(k) \le 0.1 \quad , \quad k = 0, 1$$

1. Show the computational steps required to determine the optimal control law by using dynamic programming. Quantize both $u(k)$ and $x(k)$ in steps of 0.1 about zero and use linear interpola- tion, if necessary.
2. What is the optimal sequence for an initial state value of 0.2?

Solution

1. At $k = 1$,

Current state $x(1)$	Control $u(1)$	Next state $x(2)$	Cost $\|x(2)\|$	Optimal cost $J_{12}^*(x(1))$	Optimal control $u^*(1)$
-0.2	-0.1	0	0	$J_{12}^*(-0.2) = 0$	-0.1
	0	0.1	0.1		
	+0.1	0.2	0.2		
-0.1	-0.1	-0.05	0.05	$J_{12}^*(-0.1) = 0.05$	-0.1 or 1
	0	0.05	0.05		
	+0.1	0.15	0.15		
0	-0.1	-0.1	0.1	$J_{12}^*(0) = 0$	0
	0	0	0		
	+0.1	0.1	0.1		
0.1	-0.1	-0.15	0.15	$J_{12}^*(0.1) = 0.05$	+0.1 or 0
	0	-0.05	0.05		
	+0.1	0.05	0.05		
0.2	-0.1	-0.2	0.2	$J_{12}^*(0.2) = 0$	+0.1
	0	-0.1	0.1		
	+0.1	0	0		

At $k = 0$,

Current state $x(0)$	Control $u(0)$	Next state $x(1)$	Cost $\|x(1)\|+$ $J_{12}^*(x(1))$	Optimal cost $J_{02}^*(x(0))$	Optimal control $u^*(0)$
-0.2	-0.1	0	0+0=0	$J_{02}^*(-0.2) = 0$	-0.1
	0	0.1	0.1+0.05=0.15		
	+0.1	0.2	0.2+0=0.2		
-0.1	-0.1	-0.05	0.05+0.025=0.075	$J_{02}^*(-0.1) = 0.075$	-0.1 or 1
	0	0.05	0.05+0.025=0.075		
	+0.1	0.15	0.15+0.025=0.175		
0	-0.1	-0.1	0.1+0.05=0.15	$J_{02}^*(0) = 0$	0
	0	0	0+0=0		
	+0.1	0.1	0.1+0.05=0.15		
0.1	-0.1	-0.15	0.15+0.025=0.175	$J_{02}^*(0.1) = 0.075$	+0.1 or 0
	0	-0.05	0.05+0.025=0.075		
	+0.1	0.05	0.05+0.025=0.075		
0.2	-0.1	-0.2	0.2+0=0.2	$J_{02}^*(0.2) = 0$	+0.1
	0	-0.1	0.1+0.05=0.15		
	+0.1	0	0+0=0		

where the values for $J_{12}^*(-0.05) \approx 0.025, J_{12}^*(0.05) \approx 0.025, J_{12}^*(-0.15) \approx 0.025, J_{12}^*(0.15) \approx 0.025$ are computed by using linear interpolation.

2. When $x(0) = 0.2$, the optimal sequence is

$$\begin{array}{ccccc} x(0) = 0.2 & \rightarrow & x(1) = 0 & \rightarrow & x(2) = 0 \\ u(0) = +0.1 & \rightarrow & u(1) = 0 & & \end{array}$$

Problem 5

Consider the optimal control problem with:

$$\min J(u) = \int_0^{t_f} \frac{au(t)^2 + bx(t)^2}{2} \exp[-ct]dt$$
$$\text{s.t.} \quad \dot{x}(t) = u, \quad x(0) = x_0$$

where $a, b, c > 0$

1. Formulate the Hamilton-Jacobi-Bellman (HJB) equation of the above optimal control problem.
2. Find the cost-to-go function $V(x, t)$ which is the solution to the HJB equation. (In general the HJB equation is difficult to solve but, we can obtain a significant simplification if we can separate the time dependent part in $V(x, t)$ from the state dependent part.)

Solution

1. The associated HJB equation can be represented as

$$\frac{\partial V}{\partial t} = -\min_u \left[\frac{au^2 + bx^2}{2} \exp[-ct] + \frac{\partial V}{\partial x} u \right]$$

2. The above HJB equation can be rewritten as

$$\frac{\partial V}{\partial t} \exp[ct] = -\min_u \left[\frac{au^2 + bx^2}{2} + \frac{\partial V}{\partial x} \exp[ct]u \right]$$

We observe that the only time dependence in the above partial differential equation is the exponential adjacent to the value function. Thus, we try the form $V(x, t) = \mathcal{V}(x) \exp[-ct]$. Then, the original HJB equation is equivalent to the following equation:

$$-c\mathcal{V}(x) \exp[-ct] \exp[ct] = -\min_u \left[\frac{au^2 + bx^2}{2} + \frac{\partial \mathcal{V}}{\partial x} \exp[-ct] \exp[ct]u \right]$$
$$\Leftrightarrow c\mathcal{V}(x) = \min_u \left[\frac{au^2 + bx^2}{2} + \frac{\partial \mathcal{V}}{\partial x} u \right]$$

The result is an ordinary differential equation in the state variable. Due to the minimization operator, the right-hand-side of the above equation is converted into

$$\min_u \left[\frac{au^2 + bx^2}{2} + \frac{\partial \mathcal{V}}{\partial x} u \right] \Rightarrow au + \frac{\partial \mathcal{V}}{\partial x} = 0 \Rightarrow u^* = -\frac{1}{a} \frac{\partial \mathcal{V}}{\partial x}$$

$$\Rightarrow \min_u \left[\frac{au^2+bx^2}{2} + \frac{\partial \mathcal{V}}{\partial x}u\right] = -\frac{1}{2a}\left(\frac{\partial \mathcal{V}}{\partial x}\right)^2 + \frac{bx^2}{2} - \frac{1}{a}\left(\frac{\partial \mathcal{V}}{\partial x}\right)^2$$

Therefore,

$$c\mathcal{V}(x) = -\frac{1}{2a}\left(\frac{\partial \mathcal{V}}{\partial x}\right)^2 + \frac{bx^2}{2} - \frac{1}{a}\left(\frac{\partial \mathcal{V}}{\partial x}\right)^2 = -\frac{3}{2a}\left(\frac{\partial \mathcal{V}}{\partial x}\right)^2 + \frac{bx^2}{2}$$

This equation is called a Ricatti differential equation and the easiest approach to solve it is a good guess. Recall that we already know that a quadratic form of the cost function (in this problem, without exponential adjacent) has a quadratic value function, let $\mathcal{V}(x) = \alpha x^2$. Then, the above equation can be rewritten as:

$$c\alpha x^2 + \frac{6\alpha^2 x^2}{a} = \frac{bx^2}{2} \Leftrightarrow \left(c\alpha + \frac{6\alpha^2}{a} - \frac{b}{2}\right)x^2$$

which is satisfied for all x whenever

$$\frac{6\alpha^2}{a} + c\alpha - \frac{b}{2} = 0 \Leftrightarrow \alpha = \frac{-ac \pm \sqrt{a^2c^2 + 12ab}}{12}$$

So our trail solution worked and

$$\mathcal{V}(x) = \alpha x^2 = \frac{-ac \pm \sqrt{a^2c^2 + 12ab}}{12}x^2$$

Given that our cost-to-go value function has to be non-negative, we know that the positive root is the correct one and we have

$$V(x,t) = \mathcal{V}(x)\exp[-ct] = \frac{-ac + \sqrt{a^2c^2 + 12ab}}{12}x^2\exp[-ct]$$

Problem 6

Consider the systems $x(k+1) = Ax(k) + Bu(k)$ and $y(k) = Cx(k)$, with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

We use LQR cost function

$$J = \sum_{k=0}^{N-1} u(k)^2 + \sum_{k=0}^{N} y(k)^2$$

with $N = 50$

1. Construct the associated Riccati equation and show that the Riccati recursion converges to a steady-state value in fewer than 10 steps. Find the optimal time-varying state feedback gain $K(k)$ and plot its components versus k .

2. Find the initial condition $x(0)$, with norm not exceeding one, that maximizes J .

3. Is there a choice of C for which $K(k)$ is constant, i.e., $K(0) = \cdots = K(N-1)$?

Solution

1. From the cost function,

$$Q_f = Q = C^T C, \quad R = 1$$

With the system matrices, A, B and the weighting matrices, Q and R, we can constuct the following Riccati recursion equation:

$$S(N) = Q$$
$$S(k) = A^T S(k+1)A + Q - A^T S(k+1)B\left(R + B^T S(k+1)B\right)^{-1} B^T S(k+1)A$$

Using the following MATLAB code, we can solve the Riccati equation:

```
A = [1 0 0; 1 1 0; 0 1 1];
B = [1 0 0]";
C = [0 0 1];

Q = C"*C;
R = 1;

S(:,:,51)=Q;

for i = 1: 50
Sc = S(:,:,51-(i-1));
Sn = A"*Sc*A + Q - A"*Sc*B*inv(R+B"*Sc*B)*B"*Sc*A;
S(:,:,51-i) = Sn;
K(:,:,51-i) = inv(R+B"*Sc*B)*B"*Sc*A;
end
```

From the numerical solution we can show that the solution to the Riccati recursion equation converges to a steady-state value in fewer than 10 steps as in the following figure. Also, corresponding optimal feedback gain $K(k)$ is computed as

$$K(k) = \left[R + B^T S(k+1)B\right]^{-1} B^T S(k+1)A$$

The behavior of $K(k)$ is shown in the following figure.

2. Note that the cost is expressed as the following quadratic form:

$$J\left(x_0\right) = x_0^T S(0) x_0$$

where $S(0)$ is the solution to the Riccati equation at the initial step and computed from the numerical simulation as follows:

$$S(0) = \begin{bmatrix} 6.764 & 7.689 & 2.786 \\ 7.689 & 11.527 & 5.187 \\ 2.786 & 5.187 & 3.759 \end{bmatrix}$$

Because $S(0)$ is a symmetric matrix, the following inequality holds:

$$\lambda_{\min} \cdot \left\|x_0\right\|^2 \leq x_0^T S(0) x_0 \leq \lambda_{\max} \cdot \left\|x_0\right\|^2$$

where $\lambda_{\min}$ and $\lambda_{\max}$ denote the minimum and maximum eigenvalues of $S(0)$, respectively. Therefore, the initial value maximizing $J\left(x_0\right)$ is the normalized eigenvector corresponding to $\lambda_{\max}$. Note that $\lambda_{\max}$ is 19.375 and the corresponding eigenvector $V_{\max}$ (or x_0) is $[-0.5428 \quad -0.7633 \quad -0.3504]^T$.

3. Since $K(k)$ is defined by $K(k) := -\left(R + B^T S(k+1) B\right)^{-1} B^T S(k+1) A$,

$$S(1) = \cdots = S(N) \Leftrightarrow K(0) = \cdots = K(N-1)$$

Also, we know that $C^T C = Q = Q_f = S(N)$. The question now is how to pick $S(N)$ so that $S(N) = \cdots = S(1)$. Such a $S(N)$ is the solution to the following algebraic Riccati equation:

$$S(N) = A^T S(N) A + Q - A^T S(N) B \left(R + B^T S(N) B\right)^{-1} B^T S(N) A$$

Therefore, the proper choice of C is the one satisfying the following algebraic Riccati equation:

$$C^T C = A^T C^T C A + C^T C - A^T C^T C B \left(R + B^T C^T C B\right)^{-1} B^T C^T C A$$