Lecture 6: Autonomous LTI Systems

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Contents

1 Continuous-Time Autonomous LTI Systems

The autenomous linear time-invariant (LTI) system

$$\dot{x} = Ax, \quad t \geq 0$$

with the initial condition $x(0)$ has the solution:

$$x(t) = e^{At}x(0), \quad t \geq 0$$

2 State Transition Matrix

State transition matrix $\Phi(t)$ of the LTI system:

$$\Phi(t) := e^{At}$$

- $\Phi(t)$ propagates an initial state along the LTI solution t time forward
- $\Phi\left(t_1 + t_2\right) = \Phi\left(t_1\right)\Phi\left(t_2\right) = \Phi\left(t_2\right)\Phi\left(t_1\right), \forall t_1, t_2$

3 Solution Space

Solution space $\mathbb{X}$ of the LTI system is the set of all its solutions:

$$\mathbb{X} := \{x(t), t \geq 0 | \dot{x} = Ax\}$$

- $\mathbb{X}$ is a vector space
- Dimension of $\mathbb{X}$ is n (state space dimension) due to the bijection

$$x(t) \in \mathbb{X} \quad \leftrightarrow \quad x(0) \in \mathbb{R}^n$$

- Basis of $\mathbb{X}$

4 System Modes (Diagonalizable A)

Suppose $A \in \mathbb{R}^{n \times n}$ is diagonalizable: $A = T\Lambda T^{-1}$

- Diagonal entries of $\Lambda = \operatorname{diag}\left(\lambda_1, \ldots, \lambda_n\right)$ are the eigenvalues of A
- Column of $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ are (right) eigenvectors of A
- Rows of $T^{-1} = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^T$ are left eigenvectors of A

A **mode** of the LTI system $\dot{x} = Ax$ is its solution from an eigenvector of A:

$$x(t) = e^{At}v_i = e^{\lambda_i t}v_i$$

• The n modes (possibly repeat) from a basis of the solution space $\mathbb{X}$

5 Decomposition of Solution into Modes

Assume $A = T\Lambda T^{-1}$ is diagonalizble.
Solution to $\dot{x} = Ax$ from an arbitrary $x(0)$

6 Real and Complex Modes

The mode $e^{\lambda_i t} v_i$ corresponding to a real eigenvalue λ_i as $t \to \infty$

- If $\lambda_i < 0$, the mode converges to zero along v_i (stable)
- If $\lambda_i > 0$, the mode diverges to infinity along v_i (unstable)
- If $\lambda_i = 0$, the mode is stationary (marginally stable)

For a complex $\lambda_i = \sigma_i + j\omega_i \in \mathbb{C}$ with $v_i = p_i + jq_i \in \mathbb{C}^n$

- The mode $e^{\lambda_i t} v_i$ is complex, and there is another mode $e^{\overline{\lambda}_i t} \overline{v}_i$
- Suppose $a = \langle x(0), w_i \rangle = \alpha + j\beta$. Then a real solution in $\mathbb{X}$ is

$$2 \cdot \mathrm{Re}\left[a e^{\lambda_i t} v_i \right] = \begin{bmatrix} p_i & q_i \end{bmatrix} e^{\sigma_i t} \begin{bmatrix} \cos(\omega_i t) & \sin(\omega_i t) \\ -\sin(\omega_i t) & \cos(\omega_i t) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

with the initial condition $x(0) = \alpha p_i + \beta q_i$

7 LTI System After a Change of Coordinates

Change of coordinates by a nonsingular $T \in \mathbb{R}^{n \times n}$: $\tilde{x} = T^{-1} x$ coordinates

- Columns $t_1, \ldots, t_n$ form the new basis of $\mathbb{R}^n$
- $\tilde{x}$ is the coordinate of the vector x in this new basis

LTI system $\dot{x} = Ax$ in the new coordinate system:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} := \left(T^{-1} A T\right) \tilde{x}, \quad \tilde{x}(0) = T^{-1} x(0)$$

8 Decoupled Form for Diagonalizable A

LTI system $\dot{x} = Ax$ with diagonalizable $A = T \Lambda T^{-1}$

9 General Case

In general, A has the Jordan canonical form

$$A = TJT^{-1} = \begin{bmatrix} T_1 & \cdots & T_r \end{bmatrix} \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{bmatrix} \begin{bmatrix} S_1^T \\ \vdots \\ S_r^T \end{bmatrix}$$

- $J_i \in \mathbb{R}^{n_i \times n_i}$ is a Jordan block of size n_i for the eigenvalue λ_i
- Columns of T_i: generalized eigenvectors of A corresponding to λ_i

10 System Modes for General A

The solution to $\dot{x} = Ax$ with any initial condition $x(0)$ is

$$x(t) = e^{At}x(0) = Te^{Jt}T^{-1}x(0) = \sum_{i=1}^{r} T_i e^{J_i t} \left(S_i^T x(0) \right)$$

- Solution are linear combinations of the columns of $T_i e^{J_i t}$
- Columns of $T_i e^{J_i t}$ are **modes** corresponding to eigenvalue λ_i

11 Example

$$A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$$

Modes of $\dot{x} = Ax$:

12 Decoupled Form for General LTI Systems

If A has the Jordan canonical form: $A = T \operatorname{diag}(J_1, \ldots, J_r) T^{-1}$, after a change of coordinates $\tilde{x} = T^{-1}x$, the LTI system becomes:

$$\begin{cases} \dot{\tilde{x}}_1 = J_1 \tilde{x}_1 \\ \vdots \\ \dot{\tilde{x}}_r = J_r \tilde{x}_r \end{cases}$$

13 Discrete-Time Autonomous LTI Systems

Discrete-time LTI system

$$x[k+1] = Ax[k]$$

with initial condition $x[0]$ has the solution

$$x[k] = A^k x[0] := \Phi[k]x[0], \quad k = 0, 1, \ldots$$

where the **state transition matrix** $\Phi[k] := A^k$ has the property

$$\Phi[k+\ell] = \Phi[k] \cdot \Phi[\ell] = \Phi[\ell] \cdot \phi[k], \quad k, \ell \in \mathbb{N}$$

- The solution space also has dimension n

14 Modes of Discrete-Time LTI Systems

For diagonalizable $A = T\Lambda T^{-1}$, the solution with initial condition $x[0]$ is

$$x[k] = A^k x[0] = T\Lambda^k T^{-1} x[0] = \langle x[0], w_1 \rangle \lambda_1^k v_1 + \cdots + \langle x[0], w_n \rangle \lambda_n^k v_n$$

- $\lambda_1^k v_1, \ldots, \lambda_n^k v_n$ are the n **modes** of the system
- Any solution is a linear combination of these n modes

For general A with JCF $A = TJT^{-1}$, the modes are: