State-Space Model vs. Input-Output Model

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Contents

1 Internal vs. External Models

- Transfer function models are external (or I/O) models
  - Describe only how the input affects the output
  - System viewed as a black box
- State-space models are internal models
  - Describe how the input affects not only the output, but also all the internal state variables
  - More complete models suitable for complicated system

Many state-space model may correspond to one transfer function.

2 IO Model

2.1 IO Models of CT LTI System

Consider a continuous-time LTI system with **zero initial state**, let $h(t)$, $t > 0$ be the system's **impulse response**, then, under any $u(t), t \geq 0$, system has the output

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau = h(t) * u(t), t \geq 0$$

Taking Laplace transform, we obtain the **transfer function model**:

$$\frac{Y(s)}{U(s)} = H(s) = \mathcal{L}[h(t)]$$

2.2 IO Models of DT LTI System

For a discrete-time LTI system with zero initial state, **transfer function model** can be obtained by taking z-transform:

$$\frac{Y(z)}{U(z)} = H(z)$$

recall that $ay[k-n]$ has z-transform $az^{-n}Y(z)$

2.3 IO Models from CT State-Space Models

A continuous LTI system with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and zero initial condition $x(0) = 0$. Its transfer function (or matrix) can be deduced by taking Laplace transformation:

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

which lead to

$$Y(s) = \underbrace{[C(sI-A)^{-1}B + D]}_{H(s)} U(s)$$

2.4 IO Models from DT State-Space Models

A discrete-time LTI system with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and zero initial condition $x[0] = 0$. Its transfer function (or matrix) can be deduced by taking z-transformation:

$$\begin{aligned} zX(z) &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z) \end{aligned}$$

which lead to

$$Y(z) = \underbrace{[C(sI - A)^{-1}B + D]}_{H(z)} U(z)$$

Example:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -2 & 1 \end{bmatrix} x \end{aligned}$$

has transfer function

$$H(s) = -\frac{2}{s+1} + \frac{1}{s+2}$$

3 State-Space Realization

- A continuous-time state-space model (A, B, C, D) is called a realization of the transfer function $H(s)$ if $C(sI - A)^{-1}B + D = H(s)$
- A discrete-time state-space model (A, B, C, D) is called a realization of the transfer function $H(z)$ if $C(zI - A)^{-1}B + D = H(z)$

There are **infinitely many** realizations of a transfer function.

3.1 Obtaining State-Space Realizations from IO Model

IO model of a single-input single-output (SISO) system can be written in difference equation form:

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{i=0}^{m} b_i u^{(i)}(t)$$

Transfer function can be directly obtained by Laplace transformation.

3.2 Controller Canonical Form

3.3 Controllability Canonical Form

3.4 Observer Canonical Form

3.5 Observability Canonical Form

3.6 Diagonal Realization

Suppose $H(s)$ has distinct poles:

$$\begin{aligned} H(s) &= \frac{b_1 s^2 + b_2 s + b_3}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} \\ &= \frac{\gamma_1}{s-\lambda_1} + \frac{\gamma_2}{s-\lambda_2} + \frac{\gamma_3}{s-\lambda_3} \end{aligned}$$

Diagonal realization will be:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$