Integrals

Yangang Cao

August 2, 2019

Contents

1 The Definite Integral

When we compute an area, a limit of form is arised

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_i^*)\Delta x]$$

If f is a function defined for $a \geqslant x \geqslant b$, we divide the interval $[a,b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0(=a), x_1, x_2, ..., x_n(=b)$ be the endpoints of thses subintervals and we let $x_1^*, x_2^*, ..., x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a,b]$.

If f is continuous on $[a,b]$, or if f has only a finite number of jump discontinuities, then f is intergrable on $[a,b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.

2 Properties of the Definite Integral

2.1 Properties of the Integral

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

If we reverse a and b

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

If $a=b$

$$\int_a^b f(x)dx = \int_a^a f(x)dx = 0$$

If c is any constant number

$$\int_a^b cdx = c(b-a)$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

2.2 Comparison Properties of the Integral

If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$

$$\int_a^b f(x)dx \geqslant 0$$

If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$

$$\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$$

If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)$$

3 The Fundamental Theorem of Calculus

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \quad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

4 Indefinite Integrals and the Net Change Theorem

4.1 Indefinite Integrals

We need a convenient notation for antiderivative that makes them easy to work with, the notation $\int f(x)dx$ is traditionally used for an antiderivative of f and is called an **indefinite Integral**. Thus

$$\int f(x)dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

and

$$\int_a^b f(x)dx = \int f(x)dx]_a^b$$

4.2 Table of Indefinite Integrals

$$\int cf(x)dx = c\int f(x)dx \qquad \int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int kdx = kx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}dx = \ln|x| + C$$

$$\int e^x dx = e^x + C \qquad \int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x dx = -\cos x + C \qquad \int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C \qquad \int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C \qquad \int \csc x \cot x dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$$

$$\int \sinh x dx = \cosh x + C \qquad \int \cosh x dx = \sinh x + C$$

$$\int \tan x dx = \ln|\sec x| + C$$

5 The Substitution Rule

5.1 The Substitution Rule for Definite Integrals

If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

5.2 Integrals of Symmetry Functions

Suppose f is continuous on $[-a,a]$.
If f is even $[f(-x) = f(x)]$, then

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$

If f is odd $[f(-x) = -f(x)]$, then

$$\int_{-a}^a f(x)dx = 0$$