Lecture 13: Controllability II

Baboo J. Cui, Yangang Cao

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Contents

1 Controllability of C-T LTI Systems

A continuous-time n-state m-input LTI system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{1}$$

Definition:

- The LTI system is called **controllable at time** $t_f > 0$ if for any initial state $x_0 \in \mathbb{R}^n$ and any target state $x_f \in \mathbb{R}^n$, a control $u(t)$ exists that can steer the system from x_0 to x_f over the time interval $[0, t_f]$
- It is called **reachable** at time $t_f > 0$ if $x_0 = 0$ in the above definition

• Above two definition are equivalent
• Reachable subspace at time t_f:

$$\mathcal{R}_{t_f} = \left\{ \int_0^{t_f} e^{A(t_f - \tau)} B u(\tau) d\tau | u : [0, t_f] \to \mathbb{R}^m \right\}$$

2 Continuous-Time Reachability

Proposition:
At any $t_f > 0$, the reachable subspace is $\mathcal{R}_{tf} = \mathcal{R} = \mathcal{R}(\mathcal{C})$, where

$$\mathcal{C} = \left[\begin{array}{ccc} B & AB & \cdots A^{n-1}B \end{array} \right]$$

is the controllability matrix of the system (A, B)

Theorem:
The continuous-time system (A, B) is reachable/controllable (at any time t_f) if and only if its controllability matrix $\mathcal{C}$ is onto (full rank).

3 Proof

4 Equivalent Conditions of C-T Controllability

Theorem:
The continuous-time LTI system (A, B) is controllable if and only if

1. The controllability matrix $\mathcal{C} = \left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array} \right]$ is full rank
2. **PBH Rank Test**: For any $\lambda \in \mathbb{C}$, rank $[\lambda I - A \quad B] = n$
3. **Eigenvector Test**: For any left eigenvector $w \in \mathbb{C}^n$ of A, $w^T B \neq 0$
4. For any $t_f > 0$, the following matrix is nonsingular:

$$W_r\left(t_f\right) := \int_0^{t_f} e^{A\tau} B B^T e^{A^T \tau} d\tau$$

5 C-T Reachability Gramian

Definition (Reachability Garmian):
Given a C-T system $\dot{x} = Ax + Bu$ with A stable, its reachability (or controllability) Gramian is the matrix

$$W_r := \lim_{t_f \to \infty} W_r(t_f) = \int_0^\infty e^{A\tau} BB^T e^{A^T \tau} d\tau \in \mathbb{R}^{n \times n}$$

6 Minimum-Energy Input for Reachability

Suppose the system $\dot{x} = Ax + Bu$ is controllable
Minimun-energy input is the input u^* that steers the system from $x(0) = 0$ to $x(t_f) = x_d$ with minimal energy $\int_0^{t_f} \|u(\tau)\|^2 d\tau$
The minimum-energy input is given by

$$u^*(\tau) = B^T e^{A^T(t_f - \tau)} \left(\int_0^{t_f} e^{A\tau} BB^T e^{A^T \tau} d\tau \right)^{-1} x_d, \quad \tau \in [0, t_f]$$

and the minimum energy needed is

$$\mathcal{E}_{\min} = \int_0^{t_f} \|u^*(\tau)\|^2 \, d\tau = x_d^T \underbrace{\left(\int_0^{t_f} e^{A\tau} BB^T e^{A^T \tau} d\tau \right)^{-1}}_{W_r(t_f)} x_d$$

As $t_f \to \infty$, minimum energy required over infinite horizon is

$$\mathcal{E}_{\min}^{\infty} = x_d^T W_r^{-1} x_d$$

7 D-T Reachability Gramian

Definition (Reachability Gramian):
Given a D-T system $x[k+1] = Ax[k] + Bu[k]$ with A stable. Its reachability (or controllability) Gramian is the matrix

$$W_r := \sum_{i=0}^{\infty} A^i BB^T \left(A^T\right)^i \in \mathbb{R}^{n \times n}$$

8 Controllability Under Coordinate Transformations

Original (continuous-time or discrete-time) system (A, B)
New system $(\tilde{A}, \tilde{B})$ after a coordination transform $\tilde{x} = T^{-1}x$:

$$\check{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B$$

Fact:
(A, B) is controllable if and only if $\left(T^{-1}AT, T^{-1}B\right)$ is controllable

9 Kalman Controllable Form

Fact (Kalman Controllable Form):
For any C-T system $\dot{x} = Ax + Bu$, there exists a coordinate transform T such that the transformed system $(\tilde{A}, \tilde{B})$ is of the form:

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

$$\tilde{B} = T^{-1}B = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}$$

where $\tilde{A}_{11} \in \mathbb{R}^{r \times r}$ with $r = \text{rand}(\mathcal{C})$, and $\left(\tilde{A}_{11}, \tilde{B}_1\right)$ is controllable

10 Proof of Kalman Controllable Form

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where $\tilde{A}_{11} \in \mathbb{R}^{r \times r}$ with $r = \text{rank}(\mathcal{C})$, and $\left(\tilde{A}_{11}, \tilde{B}_1\right)$ is controllable.

11 Proof of Kalman Controllable Form