# Lecture 9: Solution of Controlled Linear Systems

Baboo J. Cui, Yangang Cao

July 16, 2019

## Contents


**1 Continuous-Time Controlled LTI Systems 2**

**2 Solutions of C-T Controlled LTI Systems 2**

**3 LTI Solutions by Laplace Transform 2**

**4 Examples 2**

**5 Transfer Function of LTI-System 2**

**6 I/O Equivalent Systems 3**

**7 Equivalent C-T LTI Systems 3**

**8 Discrete-Time Controlled LTI System 3**

**9 Solutions of D-T Controlled LTI Systems 3**

**10 Solutions by $z$-Transform 4**

**11 Equivalent D-T LTI System 4**

**12 Discretization of C-T LTI Systems 4**

**13 Continuous-Time Controlled LTV Systems 4**

**14 Solutions of C-T Controlled LTV Systems 5**

**15 Discrete-Time Controlled LTV Systems 5**

# 1 Continuous-Time Controlled LTI Systems

A continuous-time LTI system with input:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output.

- $A \in \mathbb{R}^{n\times n}$ is the state dynamics matrix
- $B \in \mathbb{R}^{n\times m}$ is the input matrix
- $C \in \mathbb{R}^{p\times n}$ is the output matrix
- $D \in \mathbb{R}^{p\times m}$

# 2 Solutions of C-T Controlled LTI Systems

When $u \neq 0$, tje system output is:

$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{zero-input response}} \underbrace{+C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t),}_{\text{zero-state response}} \quad t \geq 0$$

- Zero-input response: the response due to the initial state $x(0)$
- Zero-output response: the response due to the input $u(t)$

# 3 LTI Solutions by Laplace Transform

Take the Laplace transform of the LTI system equations:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \Rightarrow \begin{cases} sX(s) - x(0) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

# 4 Examples

$$\begin{cases} \dot{x} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & -1 \end{bmatrix} x \end{cases} \quad e^{At} = \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}$$

$$\begin{cases} \dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{cases} \quad e^{At} = \begin{bmatrix} e^{-t} & 0 \\ \frac{e^t - e^{-t}}{2} & e^t \end{bmatrix}$$

# 5 Transfer Function of LTI-System

Assume zero initial state $x(0) = 0$. The zero-state response is

$$Y(s) = C\left(sl_n - A\right)^{-1}BU(s) + DU(s) = H(s)U(s)$$

where $H(s)$ is the transfer function (or matrix)

$$H(s) = C\left(sl_n - A\right)^{-1}B + D$$

# 6 I/O Equivalent Systems

Theorem:
Two LTI system $(A, B, C, D)$ and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ have the same transfer function (hence the same zero-state response) if and only if

$$\begin{aligned} D &= \tilde{D} \\ CB &= \tilde{C}\tilde{B} \\ CAB &= \tilde{C}\tilde{A}\tilde{B} \\ &\vdots \\ CA^kB &= \tilde{C}\tilde{A}^k\tilde{B} \\ &\vdots \end{aligned}$$

# 7 Equivalent C-T LTI Systems

An LTI system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ after a change of coordinates $x = T\tilde{x}$:

$$\begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u = \left(T^{-1}AT\right)\tilde{x} + \left(T^{-1}B\right)u \\ y = \tilde{C}\tilde{x} + \tilde{D}u = (CT)\tilde{x} + Du \end{cases}$$

- LTI systems $(A, B, C, D)$ and $\left(T^{-1}AT, T^{-1}B, CT, D\right)$ are (algebraically equivalent)
- Matlab command $[\mathbf{Ap}, \mathbf{Bp}, \mathbf{Cp}, \mathbf{Dp}] = \mathrm{ss}\,\mathbf{2}\,\mathrm{sec}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{P})$
- Two equivalent systems have the same transfer function $H(s) = \tilde{H}(s)$

# 8 Discrete-Time Controlled LTI System

A discrete-time LTI system with input:

$$x[k+1] = Ax[k] + Bu[k]$$

$$y[k] = Cx[k] + Du[k]$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p, A, B, C, D$ are of proper dimensions

# 9 Solutions of D-T Controlled LTI Systems

For general input $u \neq 0$:

$$y[k] = \underbrace{CA^kx[0]}_{\text{zero-input response}} + \underbrace{C\sum_{i=0}^{k-1} k^{k-1-i}Bu[i] + Du[k]}_{\text{zero-state response}}, \quad k = 0, 1, 2, ...$$

- Zero-input response: the response due to the initial state $x[0]$
- Zero-state response: the response due to the input $u[k]$

# 10 Solutions by $z$-Transform

Take the $z$ -transform of the D-T LTI system equations to obtain

$$X(z) = (zI_n - A)^{-1} zx[0] + (zI_n - A)^{-1} BU(z)$$

$$Y(z) = \underbrace{C(zI_n - A)^{-1} zx[0]}_{\text{zero-input response}} + \underbrace{C(zI_n - A)^{-1} BU(z) + DU(z)}_{\text{zero-state response}}$$

In particular, assuming $x[0] = 0$, the zero-state response is

$$Y(z) = \underbrace{\left[C(zI_n - A)^{-1} B + D\right]}_{\text{Transfer (matrix) function } H(z)} U(z)$$

Theorem:
Two systems $(A, B, C, D)$ and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ realize the same transfer function if $D = \tilde{D}$ and $CA^kB = \tilde{C}\tilde{A}^k\tilde{B}$ for $k = 0, 1, \ldots$

# 11 Equivalent D-T LTI System

LTI system $\begin{cases} x[k+1] = Ax[k] + Bu[k] \\ y[k] \quad = Cx[k] + Du[k] \end{cases}$ after a change of coordinates $x = T\tilde{x}$:

$$\begin{cases} \tilde{x}[k+1] = \tilde{A}\tilde{x}[k] + \tilde{B}u[k] = \left(T^{-1}AT\right)\tilde{x}[k] + \left(T^{-1}B\right)u[k] \\ y[k] \quad = \tilde{C}\tilde{x}[k] + \tilde{D}u[k] = (CT)\tilde{x}[k] + Du[k] \end{cases}$$

- LTI systems $(A, B, C, D)$ and $\left(T^{-1}AT, T^{-1}B, CT, D\right)$ are (algebraically) equivalent
- Two equivalent systems have the same transfer function $H(z) = \tilde{H}(z)$

# 12 Discretization of C-T LTI Systems

Given a continuous-time LTI system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$, suppose

- input, state, and output are sampled every $T$ time:

$$u[k] = u(kT), \quad x[k] = x(kT), \quad y[k] = y(kT), \quad k = 0, 1, \ldots$$

- input $u$ is kept constant during each sampling time interval

# 13 Continuous-Time Controlled LTV Systems

A continuous-time LTV system with input:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

$x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p, A(t), B(t), C(t), D(t)$ of proper dimensions

## 14 Solutions of C-T Controlled LTV Systems

For general $u \neq 0$, the solution of the LTV system is

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t,\tau)B(\tau)u(\tau)d\tau$$

$$y(t) = \underbrace{C(t)\Phi(t)x(0)}_{\text{zero-input resonse}} + \underbrace{C(t)\int_0^t \Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)}_{\text{zero-state response}}$$

## 15 Discrete-Time Controlled LTV Systems

Solution of the discrete-time LTV system

$$\begin{aligned} x[k+1] &= A[k]x[k] + B[k]u[k] \\ y[k] &= C[k]x[k] + D[k]u[k] \end{aligned}$$

under general input $u$ is

$$x[k] = \Phi[k]x[0] + \sum_{i=0}^{k-1} \Phi[k, i+1]B[i]u[i]$$

$$y[k] = C[k]\Phi[k]x[0] + C[k]\sum_{i=0}^{k-1} \Phi[k, i+1]B[i]u[i] + D[k]u[k]$$