

Eigenvalues and Eigenvectors

Yangang Cao

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Consider this equation

$$(A - \lambda I)x = 0$$

It is called **eigenvalue equation** of matrix A , λ and x are called **eigenvalues** and **eigenvectors** respectively, we are interested only in those particular values λ for which there is a nonzero eigenvector x , so $A - \lambda I$ **must be singular**. We emphasize the steps in solving $(A - \lambda I)x = 0$:

- **Compute the determinant of $A - \lambda I$.** With λ subtracted along the diagonal, this determinant is a polynomial of degree n . It starts with $(-\lambda)^n$.
- **Find the roots of this polynomial.** The n roots are the eigenvalues of A .
- **For each eigenvalues solve the equation $(A - \lambda I)x = 0$.** Since the determinant is zero, there are solutions other than $x = 0$. Those are eigenvectors.

There are some properties about λ and x :

- Trace of $A = \lambda_1 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn}$
- $\lambda_1 \cdot \cdots \cdot \lambda_n = |A|$
- If x_1, \cdots, x_k correspond to **different** eigenvalues $\lambda_1, \cdots, \lambda_k$, then those eigenvectors are linearly independent.

Suppose the n by n matrix A has n linearly independent eigenvectors. If these eigenvectors are the columns of a matrix S , then $S^{-1}AS$ is a diagonal matrix Λ . The eigenvalues of A are on the diagonal of Λ :

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

We call S the **eigenvector matrix** and Λ the **eigenvalue matrix**. According to diagonalization, we can calculate powers and products easily:

$$\Lambda^k = (S^{-1}AS)(S^{-1}AS) \cdots (S^{-1}AS) = S^{-1}A^kS$$

Generally, the matrix A and $M^{-1}AM$ (M is any invertible matrix) are **similar**, Going from one to the other is a **similarity transformation**. Λ is a special similar form. Similar matrices share the same eigenvalues.

If columns of Q contain orthonormal eigenvectors of A , we can get:

$$\Lambda = Q^T A Q = Q^{-1} A Q$$

1 Complex Matrices

1.1 Basics

A review of complex numbers is easy to give:

$$\begin{aligned} i^2 &= -1 \\ a + ib &= \overline{a - ib} = r e^{i\theta} \\ r &= |a + ib| = \sqrt{a^2 + b^2} \\ \theta &= \arctan\left(\frac{b}{a}\right) \end{aligned}$$

1.2 Transposes in the Complex Case

By definition, the **complex vector space** C^n contains all vectors x with n complex components:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ with components } x_j = a_j + ib_j$$

For complex matrices, a superscript H (or a star) combines both conjugate and transpose, and it is called **A Hermitian**.

$$\overline{A}^T = A^H = A^*$$

1.3 Hermitian Matrices

Hermitian matrices are defined:

$$A = A^H$$

If $A = A^H$:

- For all complex vector x , the number $x^H A x$ is real.
- Every eigenvalue is real.
- Two eigenvectors of a real symmetric matrix or a Hermitian matrix, if they come from different eigenvalues, are orthogonal to one another.