Lecture 16: Minimality, BIBO Stability, and Canonical Forms

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1 Kalman Decomposition

For any continuous-time n-state m-input p-output LTI system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

Its **Kalman Canonical Form** is ($\tilde{x} = Tx$)

$$\dot{\tilde{x}} = \begin{bmatrix} \tilde{A}_{co} & 0 & \tilde{A}_{13} & 0 \\ \tilde{A}_{21} & \tilde{A}_{c\overline{o}} & \tilde{A}_{23} & \tilde{A}_{24} \\ 0 & 0 & \tilde{A}_{\overline{c}0} & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{\overline{co}} \end{bmatrix} \tilde{x} + \begin{bmatrix} \tilde{B}_{co} \\ \tilde{B}_{c\overline{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \tilde{C}_{co} & 0 & \tilde{C}_{\overline{c}o} & 0 \end{bmatrix} \tilde{x} + Du$$

- Eigenvalues if A is

$$\sigma(A) = \sigma(\tilde{A}) = \sigma\left(\tilde{A}_{co}\right) \cup \sigma\left(\tilde{A}_{c\overline{o}}\right) \cup \sigma\left(\tilde{A}_{c\overline{o}}\right) \cup \sigma\left(\tilde{A}_{\overline{co}}\right)$$

- Transfer function is

$$H(s) = C(sI - A)^{-1}B + D = \tilde{C}_{co}\left(sI - \tilde{A}_{co}\right)^{-1} \tilde{B}_{co} + D,$$

  whose poles (counting multiplicity) are exactly $\sigma(\tilde{A}_{co})$

2 Minimal Realizations

The system (A, B, C, D) is a minimial realization of the transfer function $H(S) = C(sI - A)^{-1}B + D$ iff $\dim(A)$ is the smallest possible

- Equivalnetly, iff (A, B) is controllable and (C, A) is observable
- Or equivalently, poles of $H(s)$ are exactly $\sigma(A)$

If the system is **not** minimal, then

- Either (A, B) is not controllable and/or (C, A) is not observable
- Poles of $H(s)$ are a proper subset of $\sigma(A)$

3 Example

Transfer function $H(s) = \frac{1}{s+1}$

4 BIBO Stability of LTI Systems

Definition (BIBO Stability):
A continuous-time LTI system with transfer function $H(s)$ is called bounded-input bounded-output stable if bounded inputs lead to bounded outputs (with zero initial conditions).

SISO system with impulse response $h(\cdot)$ is BIBO stable if and only if

$$\underbrace{\int_0^\infty |h(\tau)| d\tau < \infty}_{\text{peak gain}}$$

• Equivalently, all poles of $H(s)$ have negative real part

5 BIBO Stability of LTV Systems

Definition (BIBO Stability):
A continuous-time LTV system is called bounded-input bounded-output stable if, under zero initial conditions, there exists $K < \infty$ such that

$$\sup_{t \geq 0} \|y(t)\| \leq K \cdot \sup_{t \geq 0} \|u(t)\|, \quad \forall u$$

A LTV system $(A(t), B(t), C(t), D(t))$ with state transition matrix $\Phi(t, \tau)$ is BIBO stable if and only if $D(t)$ is uniformly bounded and

$$\sup_{t \geq 0} \int_0^t \|h(t, \tau)\| d\tau < \infty$$

Here, $h(t, \tau) = C\Phi(t, \tau)B$

6 Internal vs. External Stability of LTI Systems

A continuous-time (resp. discrete-time) LTI system (A, B, C, D) is called stable if and only if all eigenvalues of A have negative real part (resp. magnitude less than one)

• **Internal** stability, defined in terms of the internal state

With $H(s) = C(sI - A)^{-1}B + D$ (or $H(z) = C(zI - A)^{-1}B + D$ in discrete-time case),the system is called BIBO stable if all the poles of the transfer function H have negative real part (magnitude less than one)

• **External** stability, defined in terms of input-output behavior

7 Controllability Canonical Form of SISO System

Suppose system (A, b, c) is controllable with controllability matrix $\mathcal{C}$

Fact:
There exists a coordinate transform $x = T\tilde{x}$ so that the transformed system $(\tilde{A}, \tilde{b}, \tilde{c})$ has controllability matrix $\tilde{\mathcal{C}} = I$.

Transformed system $(\tilde{A}, \tilde{b}, \tilde{c})$ is of the **controllability canonical form**:

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & \cdots & 0 & -\alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix}$$

$$\tilde{b} = T^{-1}b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{c} = cT$$

where $\chi_A(\lambda) = \chi_{\tilde{A}}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$

8 Properties of Controllability Canonical Form

- Transfer function's poles determined by last column of $\tilde{A}$, and zeros determined by $\tilde{A}$ and $\tilde{c}$ jointly

$$H(s) = \frac{\beta_{n-1}s^{n-1} + \cdots + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0}$$

where $\begin{bmatrix} \beta_{n-1} & \cdots & \beta_0 \end{bmatrix} = \tilde{c} \begin{bmatrix} 1 & \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 \\ & 1 & \alpha_{n-1} & \cdots & \alpha_2 \\ & & \ddots & \ddots & \vdots \\ & & & 1 & \alpha_{n-1} \\ & & & & 1 \end{bmatrix}$

- Minimal realization if and only if $(\tilde{c}, \tilde{A})$ is observable, or equivalently, if and only if there is no zero-pole cancelation

- Extension to controllable multiple-input systems possible

9 Controllability Canonical Form of Example $H(s)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\alpha_0 \\ 1 & 0 & -\alpha_1 \\ 0 & 1 & -\alpha_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \quad H(s) = \frac{\beta_2 s^2 + \beta_1 s + \beta_0}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}$$

$$y = \begin{bmatrix} \tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{bmatrix} 1 & \alpha_2 & \alpha_1 \\ 0 & 1 & \alpha_2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

10 Controller Canonical Form of SISO Systems

Suppose (A, b, c) is controllable with $\chi_A(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$ Using the coordinate transform $x = T\tilde{x}$ where

$$T = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} \begin{bmatrix} 1 & \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 \\ & 1 & \alpha_{n-1} & \cdots & \alpha_2 \\ & & \ddots & \ddots & \vdots \\ & & & 1 & \alpha_{n-1} \\ & & & & 1 \end{bmatrix}$$

Transformed system $(\tilde{A}, \tilde{b}, \tilde{c})$ is of the **controller canonical form**:

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$\tilde{b} = T^{-1}b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{c} = cT = \begin{bmatrix} \beta_{n-1} & \beta_{n-2} & \cdots & \beta_1 & \beta_0 \end{bmatrix}$$

11 Properties of Controller Canonical Form

- Any controllable SISO system has such a form
- Guaranteed to be controllable with centrollability matrix

$$\tilde{\mathcal{C}} = \begin{bmatrix} 1 & \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 \\ & 1 & \alpha_{n-1} & \cdots & \alpha_2 \\ & & \ddots & \ddots & \vdots \\ & & & 1 & \alpha_{n-1} \\ & & & & 1 \end{bmatrix}^{-1}$$

- Transfer function's pole determined by last column of $\tilde{A}$, and zeros determined by $\tilde{c}$ only.

$$H(s) = \frac{\beta_{n-1}s^{n-1} + \cdots + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0}$$

- Minimal realization if and only if $(\tilde{c}, \tilde{A})$ is observable, or equivalently, if and only if there is no zero-pole cancelation
- Very useful for feedback controller design (later)

12 Controller Canonical Form of Example $H(s)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\alpha_2 & -\alpha_1 & -\alpha_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \quad H(s) = \frac{\beta_2 s^2 + \beta_1 s + \beta_0}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}$$

$$y = \begin{bmatrix} \beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

13 Observer Canonical Form of SISO Systems

Suppose (A, b, c) is observable with $\chi_A(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$
Its **observer canonical form** is the dual of its controller canonical form:

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} -\alpha_{n-1} & 1 & 0 & \cdots & 0 \\ -\alpha_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_1 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tilde{b} = T^{-1}b = \begin{bmatrix} \beta_{n-1} \\ \beta_{n-2} \\ \vdots \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

$$\tilde{c} = cT = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

- Has the transfer function $H(s) = \frac{\beta_{n-1}s^{n-1} + \cdots + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0}$
- Minimal if and only if there is no zero-pole cancelation
- Very useful for output feedback state observer design (later)

14 Observer Canonical Form of Example $H(s)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\alpha_2 & 1 & 0 \\ -\alpha_1 & 0 & 1 \\ -\alpha_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$H(s) = \frac{\beta_2 s^2 + \beta_1 s + \beta_0}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}$$

15 Observability Canonical Form of SISO Systems

Suppose (A, b, c) is observable with $\chi_A(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$ Its **observability canonical form** is the dual of controllability c.f.:

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}, \tilde{b} = T^{-1}b$$

$$\tilde{c} = cT = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

- Observability matrix $\tilde{\mathcal{O}} = I$ is the identity
- Has the transfer function $H(s) = \frac{\beta_{n-1}s^{n-1}+\cdots+\beta_0}{s^n+\alpha_{n-1}s^{n-1}+\cdots+\alpha_0}$ where β_i' 's depend on $\tilde{A}$ and $\tilde{b}$ jointly
- Minimal if and only if there is no zero-pole cancelation

16 Observability Canonical Form of Example $H(s)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha_2 & 1 & 0 \\ \alpha_1 & \alpha_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix}$$