Functions of Square Matrices

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1 Functions and Polynomial of Matrix

We can define functions of square matrices $A \in \mathbb{R}^{n \times n}$ like:

$$f(A)=A^{k}, A^{\frac{1}{2}}, e^{A}, \ln(A),\left(I_{n}-A\right)^{-1}$$

Real analytic functions has the properties:

- $f(\lambda)$ is infinitely differentiable
- At any λ_0 where $f(\cdot)$ is defined, its Taylor expansion converges locally:

$$f(\lambda)=f\left(\lambda_{0}\right)+f^{\prime}\left(\lambda_{0}\right)\left(\lambda-\lambda_{0}\right)+\frac{1}{2!} f^{\prime \prime}\left(\lambda_{0}\right)\left(\lambda-\lambda_{0}\right)^{2}+\cdots$$

For polynomials $f(\lambda)=a_k\lambda^k+a_{k-1}\lambda^{k-1}+\cdots+a_0$, the extension is easy:

$$f(A)=a_{k} A^{k}+a_{k-1} A^{k-1}+\cdots+a_{0} I_{n} \in \mathbb{R}^{n \times n}, \quad \forall A \in \mathbb{R}^{n \times n}$$

- replace every occurrence of λ by A
- replace constant term a_0 by $a_0 I_n$

2 Polynomials of Matrices via JCF

Suppose $A \in \mathbb{R}^{n \times n}$ has Jordan Canonical Form (JCF):

$$A=T J T^{-1}=T\left[\begin{array}{ccc}J_{1} & & \\ & \ddots & \\ & & J_{q}\end{array}\right] T^{-1}$$

Clearly, $A^k = TJ^kT^{-1}$, and for a polynomial function $f(\lambda), f(A)$ can be computed via JCF. Each Jordan block J_i of size n_i can be decomposed as $J_i = \lambda_i I + N_i$

- $\lambda_i I$ and N_i commute
- N_i is nilpotent: $(N_i)^{n_i} = 0$

Polynomial function of J_i can be computed using these properties. Let $\chi_A(\lambda) = \det(\lambda l - A)$ be the **characteristic polynomial** of A, for any Jordan block J_i of A,

$$\chi_{A}\left(J_{i}\right)=0$$

Cayley-Hamilton Theorem: for any matrix $A \in \mathbb{R}^{n \times n}$, $\chi_A(A) = 0$, since:

$$\chi_{A}(A)=T\left[\begin{array}{ccc}\chi_{A}\left(J_{1}\right) & & \\ & \ddots & \\ & & \chi_{A}\left(J_{q}\right)\end{array}\right] T^{-1}=0$$

3 Minimal Polynomials(Very Useful)

The minimal polynomial of $A \in \mathbb{R}^{n\times n}$ is the polynomial $\mu_A(\lambda)$ with the **minimum degree** satisfying $\mu_A(A) = 0$. Using the JCF, the minimal polynomial is

$$\mu_A(\lambda) = (\lambda - \lambda_1)^{d_1} \cdots (\lambda - \lambda_\ell)^{d_\ell}$$

- $\lambda_1, \ldots, \lambda_\ell$ are the distinct eigenvalues of A
- d_i is the largest size of Jordan blocks associated with λ_i

4 Implication of C-H Theorem

4.1 Part 1

Given a square matrix $A \in \mathbb{R}^{n\times n}$

- $A^k \in \operatorname{span}\left\{I_n, A, A^2, \ldots, A^{n-1}\right\}$, for $k = 0, 1, 2, \ldots$
- For any polynomial function $f(\lambda)$

  $$f(A) \in \operatorname{span}\left\{I_n, A, A^2, \ldots, A^{n-1}\right\}$$

  which indicate that

  $$f(A) = h(A) \text{ for some polynomial } h(\lambda) \text{ of degree} \leq n-1$$

4.2 Part 2

Given a square matrix $A \in \mathbb{R}^{n\times n}$ and a polynomial function $f(\lambda)$ so that

$$f(A) = h(A)$$

for a polynomial $h(\lambda)$ of degree at most $n-1$ satisfying

$$\left\{\begin{array}{ll} f^{(j)}(\lambda_1) = h^{(j)}(\lambda_1), & j = 0, 1, \ldots, m_1 - 1 \\ f^{(j)}(\lambda_\ell) = h^{(j)}(\lambda_\ell), & j = 0, 1, \ldots, m_\ell - 1 \end{array}\right. \tag{1}$$

- $\lambda_1, \ldots, \lambda_\ell$ are the distinct eigenvalues of A
- $m_1, \ldots, m_\ell$ are their **algebraic** multiplicities
- $f(\lambda)$ and $h(\lambda)$ agree on the spectrum $\sigma(A)$ of matrix A

5 General Functions of Square Matrices

Given an analytical function $f(\lambda)$ that is defined at $\lambda = 0$

$$\begin{aligned} f(\lambda) &= f(0) + f'(0)\lambda + \frac{1}{2!}f''(0)\lambda^2 + \cdots \\ \Rightarrow \quad f(A) &= f(0)I + f'(0)A + \frac{1}{2!}f''(0)A^2 + \cdots \end{aligned}$$

The general function of the square matrix $A \in \mathbb{R}^{n\times n}$ is defined as

$$f(A) = h(A)$$

where $h(\lambda)$ is a polynomial of degree at most $n-1$ that agrees with $f(\lambda)$ on the spectrum $\sigma(A)$ of A.

Example

Find $f(A) = e^A$ for $A_1 = \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix}$, $\quad A_2 = \begin{bmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{bmatrix}$

6 General Functions of Matrices via JCF

Suppose $A \in \mathbb{R}^{n\times n}$ has Jordan Canonical Form (JCF):

$$A = T \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} T^{-1} \Rightarrow f(A) = T \begin{bmatrix} f(J_1) & & \\ & \ddots & \\ & & f(J_q) \end{bmatrix} T^{-1}$$

For each Jordan block J_i:

$$f(J_i) = \begin{bmatrix} f(\lambda_i) & f'(\lambda_i) & \frac{1}{2!}f''(\lambda_i) & \cdots & \frac{1}{(n_i-1)!}f^{(n_i-1)}(\lambda_i) \\ & f(\lambda_i) & f'(\lambda_i) & \ddots & \vdots \\ & & f(\lambda_i) & \ddots & \frac{1}{2!}f''(\lambda_i) \\ & & & \ddots & f'(\lambda_i) \\ & & & & f(\lambda_i) \end{bmatrix}$$

Example

Consider $A = \begin{bmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{bmatrix}$

- $f_1(\lambda) = \lambda^{1000}$, $f_1(A) = A^{1000}$
- $f_2(\lambda) = \frac{1}{s-\lambda}$ for some constant scalar s, $f_2(A) = (sI - A)^{-1}$