Computations with Matrices

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For a positive definite matrix, the solution $x = A^{-1}b$ and the error $\delta x = A^{-1}\delta b$ always satisfy

$$||x|| \geqslant \frac{||b||}{\lambda_{max}} \text{ and } ||\delta x|| \leqslant \frac{\delta b}{\lambda_{min}} \text{ and } \frac{||\delta x||}{||x||} \leqslant \frac{\lambda_{max}}{\lambda_{min}} \frac{||\delta b||}{||b||}$$

Ratio $c = \lambda_{max}/\lambda_{min}$ is the **condition number** of a positive definite matrix A.

The **norm** of A is the number $||A||$ defined by

$$||A|| = \max_{x \neq 0} \frac{||Ax||}{||x||}$$

In other words, $||A||$ bounds the anplifying power of the matrix:

$$||Ax|| \leqslant ||A|| \; ||x|| \quad \text{for all vectors } x$$

In the symmetric case, $||A||$ is the same as λ_{max}, and $||A^{-1}||$ is the same as $1/\lambda_{min}$. The correct replacement for $\lambda_{max}/\lambda_{min}$ is the product $||A|| \; ||A^{-1}||$. The **condition number** of A is $c = ||A|| \; ||A^{-1}||$. The relative error satisfies

$$\delta x \; from \; \delta b \quad \frac{||\delta x||}{x} \leqslant c \frac{\delta b}{b}$$

If we perturb the matrix A instead of the right-hand side b, then

$$\delta x \; from \; \delta A \quad \frac{||\delta x||}{||x + \delta x||} \leqslant c \frac{\delta A}{A}$$

$||A||$ is the square root of the largest eigenvalue of $A^T A$: $||A||^2 = \lambda_{max}(A^T A)$. The vector that A amplifies the most is the corresponding eigenvector of $A^T A$:

$$\frac{x^{\mathrm{T}} A^{\mathrm{T}} A x}{x^{\mathrm{T}} x} = \frac{x^{\mathrm{T}} \left(\lambda_{\max} x\right)}{x^{\mathrm{T}} x} = \lambda_{\max}\left(A^{\mathrm{T}} A\right) = \|A\|^{2}$$