Lecture 17: State Feedback Control

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1 Controller Design

A continuous-time (for discrete-time) LTI system

$$\dot{x} = Ax + Bu \quad \text{or} \quad x[k+1] = Ax[k] + Bu[k]$$

Problem: Design input u so that behavior of system is "better"

- Stabilize the system (if A is unstable)
- Make state trajectories decay faster
- Track a reference state trajectory
- Minimize control cost and tracking error

Strategies:

- Open-loop control: Design $u(t)$ directly
- Closed-loop control: Design $u = f_{sfb}(x, t)$ as a function of state and time (state-feedback control)

2 State Feedback Control

LTI system $\dot{x} = Ax + Bu$, with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$
State feedback control:

$$u = -Kx + v, \quad \text{for some constant (gain) matrix } K \in \mathbb{R}^{m \times n}$$

Problem: Design gain matrix K to achieve desirable closed-loop behaviors

- Many behaviors are characterized by eigenvalues of A_{cl}
- How much can we re-allocate the eigenvalues of A to those of A_{cl}?

3 State Feedback and Controllability

Fact:
For any $K \in \mathbb{R}^{m \times n}$, system $(A - BK, B)$ is controllable if and only if system (A, B) is controllable.

• State feedback cannot make an uncontrollable system controllable

4 Example I

$$\dot{x} = Ax + Bu = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

5 Example II

$$\dot{x} = Ax + Bu = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

6 Pole-Placement of Uncontrollable Systems

Fact:
Eigenvalues of $A - BK$ cannot be arbitrarily re-assigned if (A, B) is not controllable.

7 Example

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

With coordinate transformation $x = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tilde{x}$, we have

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tilde{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

8 Pole-Placement of Controllable Systems

Fact:
Eigenvalues of $A - BK$ can be arbitrarily re-assigned for controllable (A, B)

Proof: Single-input case: (A, b)

Claim: Suppose $\chi_A(\lambda) = \lambda^n + a_1 s^{n-1} + \cdots + a_n$. A coordinate transform $x = Tx_c$ exists that transforms the system to its controller canonical form:

$$A_c = T^{-1}AT = \begin{bmatrix} -a_1 & \cdots & -a_{n-1} & -a_n \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \quad B_c = T^{-1}B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(Proof in Textbook, pp. 234-235)
Choose $K_c = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$. Then

$$\chi_{A_c - B_c K_c}(\lambda) = \lambda^n + (a_1 + k_1)\,\lambda^{n-1} + \cdots + (a_n + k_n)$$

whose roots can be arbitrary (subject to complex conjugate constraint)

9 Example

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

With coordinate transformation $x = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} x_c$, we have

$$\dot{x}_c = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} x_c + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Suppose we want poles at $-1, -1$.

10 Proof of Multi-Input Case

Proof: Let $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix}$. If (A, b_i) is controllable for any i, then previous proof works for single-input system with input u_i. Otherwise, assume without loss of generality that $b_1 \neq 0$.

Define matrix $V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^{n\times n}$ by recursion:

- $v_1 = b_1, v_2 = Av_1, \ldots, v_{n_1} = Ab_{n_1-1}$. Stop when no linearly independent vector is generated
- $v_{n_1+1} = A^{n_1}b_1 + b_2$, where b_2 is any column of B that is linearly independent of previous v_i; $v_{n+2} = Av_{n_1+1}, \ldots, v_{n+n_2} = Av_{n_1+n_2-1}$ Stop when no linearly independent vector is generated
- Continue until n linearly independent v_i' 's are generated

Define $F_0 = \begin{bmatrix} \underbrace{0\cdots0 \quad \mathrm{e}_2}_{n_1} & \underbrace{0\cdots0 \quad \mathrm{e}_3}_{n_2} & \cdots \end{bmatrix} \cdot V^{-1}$

Claim: $(A + BF_0, b_1)$ is controllable

- Check that its controllability matrix is exactly V
- After feedback, closed-loop system controllable from single input b_1

11 Effect on Transfer Function

Fact:
For a contrattable SISO system $\dot{x} = Ax + Bu, y = cx$, state feedback can change the poles. but not the zeros, of its transfer function.

12 Stabilizability

Definition (Stabilizability):
System (A, B) is called stabilizable if there exist a state feedback matrix K such that the closed-loop system $A - BK$ is stable

- If A is stable itself, (A, B) is stabilizable
- If (A, B) is controllable, it is stabilizable as well
- If (A, B) is not controllable, it could still be stabilizable

Theorem:
System (A, B) is stabilizable if all its uncontrolable modes correspond to stable eigenvalues of A

13 Examples