Model Order Reduction

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1 Introduction by A Mechanical System

Suppose a four-mass mechanical system is described by the following dynamics:

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -M^{-1}K & -M^{-1}G \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1}D \end{bmatrix} u \\
y &= \begin{bmatrix} P & Q \end{bmatrix} x
\end{aligned}
$$

where

$$
\begin{aligned}
M &= \operatorname{diag}(m_1, m_2, m_3, m_4) \\
G &= \operatorname{diag}(b_1, 0, 0, b_5) \\
K &= \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4+k_5 \end{bmatrix} \\
D &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T \\
P &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & k_4/m_4 & -(k_4+k_5)/m_4 \end{bmatrix} \\
Q &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_5/m_4 \end{bmatrix} \\
x &= (q^T, \dot{q}^T)^T \in \mathbb{R}^8 \\
y &= (q_1, \ddot{q}_2)^T
\end{aligned}
$$

this is a two-input, **eight**-state, two-output system model. For real case, for example an airplane, a system could be **too complicate** because of **too many state variables**.

2 Model Order Reduction Problem

Given a model (A, B, C), with order(state dimension) n, find a model (A_r, B_r, C_r) with order $r < n$ whose transfer function satisfies:

$$
\begin{aligned}
C_r (sl - A_r)^{-1} B_r &\simeq C(sl - A)^{-1}B \qquad \text{CT-case} \\
C_r (zl - A_r)^{-1} B_r &\simeq C(zl - A)^{-1}B \qquad \text{DT-case}
\end{aligned}
$$

To determine if two transfer functions are **close** to each other, given the same input u, they should produce **similar** outputs y and y_r. Note that:

- similarity between y and y_r is quantified as $||y - y_r||$ using some norm on the vector space of all output signals
- since y and y_r are linear in u, $\|y - y_r\|$ can be very large due to large u
- evaluate system by looking for largest output difference under u with bounded energy:

  $$\sup_{||u||=1} ||y - y_r||$$

  where $||u||$ is **a**(sometimes may not be $\mathcal{L}_2$) norm on the vector space of all input signals

3 Naive Attempt to Achieve Order Reduction

3 intuitive ways of reducing system order are given and later we will focus on how to make this process systematic rather than relying on intuition.

3.1 Kalman Approach

In Kalman form:

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \tilde{B} = T^{-1}B = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}, \tilde{C} = CT = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}$$

- if (A, B) is not controllable, the controllable subsystem $\left(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1\right)$ realizes the same transfer function as (A, B, C) but with a lower order.
- if (C, A) is not observable, then the observable subsystem realizes the same transfer function with a lower order.

The problem is that most systems are both controllable and observable, meaning that this method won't work.

3.2 Drop Method

Given a DT system:

$$A = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.8 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 0.1 \end{bmatrix}, C = \begin{bmatrix} 10 & 0.1 \end{bmatrix}$$

Mode associated with $\lambda_1 = 0.7$ is much more controllable and observable compared to mode associated with $\lambda_2 = 0.8$. Discarding mode 0.8 results in $A_r = 0.7, B_r = 10, C_r = 10$, which lead to:

$$C_r\left(zl - A_r\right)^{-1}B_r = \frac{100}{z-0.7} \quad \simeq \quad C(zl - A)^{-1}B = \frac{100}{z-0.7} + \frac{0.01}{z-0.8}$$

Note that 0.01 is quiet small compare to 100.

3.3 Compromise Approximation

Given a DT system which is slightly different for C:

$$A = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.8 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 0.1 \end{bmatrix}, C = \begin{bmatrix} 0.1 & 10 \end{bmatrix}$$

Mode for $\lambda_1 = 0.7$ is still much more controllable, but now much less observable, than mode for $\lambda_2 = 0.8$. Discarding either mode will result in a large error for approximating

$$H(z) = C(zl - A)^{-1}B = \frac{1}{z-0.7} + \frac{1}{s-0.8}$$

Noting that the two poles of $H(z)$ are very close to each other, we can first approximate $H(z)$ by

$$H_r(z) = \frac{2}{z-0.75} \simeq H(z)$$

and then find a first-order realization of $H_r(z)$, say, $A_r = 0.75, B_r = 2, C_r = 1$, as the reduced-order model.

4 Lower-Rank(by SVD) Approximation of Matrix

A matrix $A \in \mathbb{R}^{m\times n}$ viewed as a map $A : \mathbb{R}^n \to \mathbb{R}^m$ with rank p. The problem is to find a matrix $A_r \in \mathbb{R}^{m\times n}$ with rank $r < p$ to

$$\text{minimize} \sup_{u\in\mathbb{R}^n, \|u\|=1} \|Au - A_r u\|$$

- $\|\cdot\|$ represent Euclidean norm
- Function to be minimized is equal to the $\mathcal{L}_2$-norm of matrix $A - A_r$

Suppose A has singular value decomposition(SVD)

$$A = U \begin{bmatrix} \Sigma+ & 0 \\ 0 & 0 \end{bmatrix} V^T$$

with

$$\Sigma_+ = \operatorname{diag}(\sigma_1, \ldots, \sigma_p), \sigma_1 \geq \cdots \geq \sigma_p > 0$$

then

$$A_r^* = U \begin{bmatrix} \operatorname{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0) & 0 \\ 0 & 0 \end{bmatrix} V^T$$

with the approximation error

$$\|A_r - A_r^*\| = \sigma_{r+1}$$

This can be extended to linear systems.

4.1 LTI System as Linear Operator

Consider the discrete-time LTI system

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] \end{aligned}$$

If $x[0] = 0$(for convenience), the system maps the input sequence to the output sequence:

$$\underbrace{\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ \vdots \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} 0 & & & & \\ CB & 0 & & & \\ CAB & CB & 0 & & \\ CA^2B & CAB & CB & 0 & \\ \vdots & & & & \ddots \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \\ \vdots \end{bmatrix}}_{U}$$

- G is a **Toeplitz matrix** (constant along diagonal direction) because the system is time-invariant
- G is lower triangular matrix because the system is causal
- G is unchanged under coordinate transformation

4.2 ℓ_2-Gain

The ℓ_2-gain of the system is the induced norm of G:

$$\|G\|_2 = \sup_{\mathbf{U}\neq 0} \frac{\|G\mathbf{U}\|}{\|\mathbf{U}\|}$$

- maximal energy magnification from input to output
- useful in robust control ($\mathbf{U}$ is the perturbation)

5 Hankel Operator

The operator G typically has rank infinity, hence difficult to study. Instead, look at the map from past into future output:

$$\underbrace{\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \end{bmatrix}}_{Y_+} = \underbrace{\begin{bmatrix} CB & CAB & CA^2B & \dots \\ CAB & CA^2B & CA^3B & \dots \\ CA^2B & CA^3B & CA^4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\Gamma} \underbrace{\begin{bmatrix} u[-1] \\ u[-2] \\ u[-3] \\ \vdots \end{bmatrix}}_{U_-}$$

- Γ is a **Hankel matrix**(constant along anti-diagonal direction)
- Each column represents the impulse response w.r.t. $u_j[-k]$ at time $k < 0$, hence Γ can be constructed from experimental data
- Γ unchanged under coordinate changes
- Γ has **finite rank**!

5.1 Decomposition of Hankel Operator

Since state $x[0]$ summarizes contributions of past input, we have

$$\mathbf{U}_- \underbrace{\xrightarrow{\Psi_c} x[0] \xrightarrow{\Psi_o}}_{\Gamma} \mathbf{Y}_+$$

In other words(VIP):

$$\Gamma = \Psi_o \Psi_c = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}}_{\Psi_o} \cdot \underbrace{\begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}}_{\Psi_c}$$

After coordinate transformation:

$$\begin{aligned} x &= T\tilde{x} \\ \tilde{\Psi}_c &= T^{-1}\Psi_c \\ \tilde{\Psi}_o &= \Psi_o T \\ \tilde{\Gamma} &= \Gamma \end{aligned}$$

Rank of Γ is **at most** $n = \dim(x)$

5.2 Rank of Hankel Operator

- Controllability operator Ψ_c is full rank if (A, B) is controllable
- Observability operator Ψ_o is full rank if (C, A) is observable

The Hankel operator Γ has rank $n = \dim(x)$ if and only if the system (A, B, C) is minimal.
Proof: Use the two Sylvester rank inequalities for $\Gamma = \Psi_o \Psi_c$:

$$\begin{aligned}
&\operatorname{rank}(\Gamma) \leq \min\{\operatorname{rank}(\Psi_c), \operatorname{rank}(\Psi_o)\} \\
&\operatorname{rank}(\Gamma) \geq \operatorname{rank}(\Psi_c) + \operatorname{rank}(\Psi_o) - n
\end{aligned}$$

5.3 McMillan Degree

- For a transfer function (or matrix) $H(s)$, the state dimension of its minimal realization is called its **Mcmillan degree**
- McMillan degree is the rank of the Hankel operator Γ

5.4 Hankel Singular Values

Suppose A is stable and system (A, B, C) is minimal. Then both controllability and observability gramians exist and are positive definite:

$$W_c = \Psi_c \Psi_c^T = \sum_{k=0}^{\infty} A^k B B^T \left(A^T\right)^k, \quad W_o = \Psi_o^T \Psi_o = \sum_{k=0}^{\infty} \left(A^T\right)^k C^T C A^k$$

$W_o W_c$ is **diagonalizable** with **positive** eigenvalues:

- this is because $W_o W_c$ is similar to $W_c^{1/2} W_o W_c^{1/2} \succeq 0$
- eigenvalues of $W_o W_c$ unchanged under coordinate change $x = T\widetilde{x}$:

$$\widetilde{W}_o = T^T W_o T, \widetilde{W}_c = T^{-1} W_c \left(T^{-1}\right)^T \Rightarrow \widetilde{W}_o \widetilde{W}_c = T^T W_o W_c \left(T^{-1}\right)^T$$

Hankel singular values(VIP) of the system are (nonzero) singular values of Γ or equivalently, the square roots of the eigenvalues of $W_o W_c$.

- The singular values are typically sorted as $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_n > 0$
- Singular values do not depend on state coordinates

5.5 Proof of Properties of Hankel Singular Values

Singular values of Γ are sequare roots of nonzero enginvalues of

$$\Gamma^T \Gamma = (\Psi_o \Psi_c)^T (\Psi_o \Psi_c) = \Psi_c^T W_o \Psi_c = (W_o^{1/2} \Psi_c)^T (W_o^{1/2} \Psi_c)$$

which has the same nonzero eigenvalues as those of

$$(W_o^{1/2} \Psi_c)(W_o^{1/2} \Psi_c)^T = W_0^{1/2} W_c W_o^{1/2}$$

which in turn is similar to $W_o W_c$ (hence have identical eigenvalues).

5.6 Hankel Norm

Suppose A is stable and system (A, B, C) is minimal. Then Γ maps finite energy input $\mathbf{U}_-$ to finite energy output $\mathbf{Y}_+$, **Hankel norm** of system is

$$\|\Gamma\| := \sup_{\mathbf{U}_- \neq 0} \frac{\|\Gamma \mathbf{U}_-\|}{\|\mathbf{U}_-\|} = \sigma_1(\Gamma)$$

It is the maximum energy amplification from past input to future output.

5.7 Lower-Rank Hankel Operator Approximation

Given a Hankel operator Γ with rank n, for $r < n$, find a Hankel operator Γ_r to

$$\begin{array}{rl} \text{minimize} & \|\Gamma - \Gamma_r\| \\ \text{s.t.} & \operatorname{rank}(\Gamma_r) = r \end{array}$$

- Γ corresponds to a minimal system (A, B, C) with state dimension n
- Γ_r corresponds to a minimal system (A_r, B_r, C_r) with state dimension r whose I/O operator is **closest** to that of (A, B, C)
- recall lower-rank approximation of matrices using SVD truncation

The optimal solution Γ_r^* satisfies $\|\Gamma - \Gamma_r^*\| \geq \sigma_{r+1}$, where σ_{r+1} is the $(r+1)$-th largest sigular value of Γ. Lower bound may not be tight due to the Hankel constraint on Γ_r^*.

6 Balanced Realization

Given a minimal system (A, B, C) with A stable, there exists a coordinate transformation $x = T\widetilde{x}$ such that in the new coordinates, the controllability and observability Gramians are equal and diagonal:

$$\widetilde{W}_c = \widetilde{W}_o = \Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$$

where $\sigma_1 \geq \cdots \geq \sigma_n$ are the Hankel singular values of the system.

- state space model with the above properties are called **balanced**
- trongly controllable states are also strongly observable!

Tip:note that $W_c^{1/2} W_o W_c^{1/2} = U \Sigma^2 U^T$ for some orthogonal U and diagonal Σ. By choosing $T = W_c^{1/2} U \Sigma^{-1/2}$ we have

$$\widetilde{W}_o = T^T W_0 T = \Sigma, \quad \widetilde{W}_c = T^{-1} W_c \left(T^{-1}\right)^T = \Sigma$$

7 Balanced Truncation

Given a minimal system (A, B, C) with stable A, first do a coordinate change $x = T\widetilde{x}$ so that $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is a balanced realization:

- $\widetilde{W}_c = \widetilde{W}_o = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$
- states $\tilde{x}_1, \ldots, \tilde{x}_n$ have decreasing controllability and observability

To find a reduced-order model of order $r < n$, keep only the first r states:

$$\left[\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{C}_2 & 0 \end{array}\right] \xrightarrow{(\tilde{x}_1,\ldots,\tilde{x}_n)\to(\tilde{x}_1,\ldots,\tilde{x}_r)} \left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{B}_1 \\ \hline \tilde{C}_1 & 0 \end{array}\right]$$

Reduced-order model $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ is called the r-th order **balanced truncation** of (the transfer function of) the model (A, B, C), and $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ is also balanced and $\tilde{A}_{11}$ is stable

8 Example

Again, given system: $A = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.8 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 0.1 \end{bmatrix}, C = \begin{bmatrix} 0.1 & 10 \end{bmatrix}$, find first-order reduction.

1. controllability and observability gramians are
$$W_c = \begin{bmatrix} 196.0784 & 2.2727 \\ 2.2727 & 0.0278 \end{bmatrix}, \quad W_o = \begin{bmatrix} 0.0196 & 2.2727 \\ 2.2727 & 277.7778 \end{bmatrix}$$
so
$$W_o W_c = \begin{bmatrix} 9.01 & 0.10769 \\ 1076.9 & 12.881 \end{bmatrix}$$
2. $W_o W_c$ has two eigenvalues $\lambda_1 = 21.888, \lambda_2 = 0.0036161$, whose square roots yield the Hankel singular values:
$$\sigma_1 = 4.6784 \quad \sigma_2 = 0.060134$$
alternatively, Matlab command *hsvd* can be used
3. As σ_1 is much larger than σ_2, a first order system can approximate the original system very well.
4. Apply the linear transform $x = T\tilde{x} = \begin{bmatrix} -6.4152 & 7.671 \\ -0.076711 & -0.064152 \end{bmatrix} \tilde{x}$
5. A balanced realization is resulted:
$$\tilde{A} = T^{-1}AT = \begin{bmatrix} 0.75885 & 0.049211 \\ 0.049211 & 0.74115 \end{bmatrix}, \tilde{B} = T^{-1}B = \begin{bmatrix} -1.4086 \\ 0.12559 \end{bmatrix}$$
$$\tilde{C} = CT = \begin{bmatrix} -1.4086 & 0.12559 \end{bmatrix}$$
whose controllability and observability gramians are $\operatorname{diag}(\sigma_1, \sigma_2)$
6. A 1st-order balanced truncation is as follows
$$\tilde{A}_{11} = 0.75885, \quad \tilde{B}_1 = -1.4086, \quad \tilde{C}_1 = -1.4086$$
whose transfer function is $\frac{1.9842}{z-0.75885}$, alternatively, Matlab command *balred* can be used