Lecture 2: State-Space Model vs. Input-Output Model

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Contents

1 Input/Output Models of C-T LTI System

Consider a continuous-time LTI system with zero initial state, e.g.

- Let $h(t)$, $t > 0$, be the system's **impulse response**
- Then, under any $u(t)$, $t \geq 0$, system has the output

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau = h(t) * u(t), t \geq 0$$

- Taking Laplace transform, we obtain the **transfer function model**:

$$\frac{Y(s)}{U(s)} = H(s) = \mathcal{L}[h(t)]$$

2 Input/Output Models of D-T LTI System

For a discrete-time LTI system with zero initial state, e.g.

$$y[k] - 0.5y[k-1] + y[k-2] = u[k] - 0.7u[k-1], k = 0, 1, ...,$$

with $y[-1] = y[-2] = u[-1] = 0$,
Transfer function model obtained by taking z-transform:

$$\frac{Y(z)}{U(z)} = H(z)$$

3 Internal vs. External Models

- Transfer function models are external (or I/O) models
  - Describe only how the input affects the output
  - System viewed as a black box
- State-space models are internal models
  - Describe how the input affects not only the output, but also all the internal state variables
  - More complete models suitable for complicated system
- Many-to-one correspondence between the two representations

4 Obtaining I/O Models from State-Space Models

A continuous LTI system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \quad t \geq 0$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and zero initial condition $x(0) = 0$. Its transfer function (or matrix) is

5 Obtaining I/O Models from State-Space Models

A discrete-time LTI system

$$\begin{cases} x[k+1] = Ax[k] + Bu[k] \\ y[k] = Cx[k] + Du[k] \end{cases}, \quad k = 0, 1, \ldots$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and zero initial condition $x[0] = 0$. Its transfer function (or matrix) is

6 Examples

Example 1:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -2 & 1 \end{bmatrix} x$$

Example 2:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & -1 \end{bmatrix} x$$

7 State-Space Realization

Definition

- A continuous-time state-space model (A, B, C, D) is called a realization of the transfer function $H(s)$ if $C(sI - A)^{-1}B + D = H(s)$
- A discrete-time state-space model (A, B, C, D) is called a realization of the transfer function $H(z)$ if $C(sI - A)^{-1}B + D = H(z)$
- There are infinitely many realizations of a transfer function

8 Obtaining Stare-Space Realizations from I/O Model

I/O model of a single-input single-output (SISO) system:

$$\dddot{y}(t) + a_1\ddot{y}(t) + a_2\dot{y}(t) + a_3 y(t) = b_1\ddot{u}(t) + b_2\dot{u}(t) + b_3 u(t)$$

Transfer function:

$$H(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

- We will list a few **canonical realizations**
- Each realization is derived from a block diagram of $H(s)$

9 Controller Canonical Form

10 Controllability Canonical Form

11 Observer Canonical Form

12 Observability Canonical Form

13 Diagonal Realization

Suppose $H(s)$ has distinct poles:

$$\begin{aligned} H(s) &= \frac{b_1 s^2 + b_2 s + b_3}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} \\ &= \frac{\gamma_1}{s-\lambda_1} + \frac{\gamma_2}{s-\lambda_2} + \frac{\gamma_3}{s-\lambda_3} \end{aligned}$$

Diagonal realization:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

14 Example

Transfer function $H(s) = \frac{1}{s+1}$. State solution:

$$x_1(t) = x_1(0)e^{-t} - 2e^{-t} * u(t)$$

$$x_2(t) = (x_2(0) + \frac{1}{2}x_1(0))e^{t} - \frac{1}{2}x_1(0)e^{-t} + e^{-t} * u(t)$$