Lecture 8: Stability of Linear Systems

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Contents

1 Stability of C-T Autonomous Linear Systems

A continuous-time autononous linear system

$$\dot{x}(t) = A(t)x(t) \tag{1}$$

Definition (Asymptotic Stability):
The linear system (1) is called asymptotically stable at $x_e = 0$ if its solution $x(t)$ starting from any initial condition $x(0) \in \mathbb{R}^n$ satisfies

$$x(t) \to 0 \text{ as } t \to \infty$$

Definition (Exponential Stability):
The linear system (1) is called exponentially stable at $x_e = 0$ if its solution $x(t)$ starting from any initial condition $x(0) \in \mathbb{R}^n$ satisfies

$$\|x(t)\| \leq Ke^{-r}\|x(0)\|, \quad \forall t \geq 0,$$

for some constants $K, r > 0$.

2 Stability of C-T LTI Systems

Theorem:
For a C-T LTI system $\dot{x} = Ax$, the following statements are equivalent

1. System is asymptotically stable
2. System is exponentially stable
3. All eigenvalues of A are in the open left half of the complex plane $\mathbb{C}$

3 Phase Portraits of Stable 2D Systems

4 Unstable Systems

Definition:
The LTI system $\dot{x} = Ax$ is unstable if, starting from some $x(0)$, the solution $x(t)$ will deverge to infinity.

Theorem:
The LTI system is unstable if either of the following is true:

1. A has eigenvalues on the open right half plane of $\mathbb{C}$
2. A has a defective eigenvalues on the $j\omega$-axis

• An eigenvalue is defective if at least one of its Jordan blocks has size greater than one

5 Phase Portraits of Unstable 2D Systems

6 Marginally Stable Systems

Definition:
The LTI system $\dot{x} = Ax$ is marginally stable if, starting from some $x(0)$, the solution $x(t)$ will neither converge to zero nor diverge to infinity.

Theorem:
The LTI system is marginally stable if both of the following hold:

1. All eigen of A are in the closed left half of the complex plane $\mathbb{C}$
2. There are some eigenvalues of A on the $j\omega$-axis, and all the Jordan blocks associated with such eigenvalues have size one

7 Phase Portraits of Marginally Stable 2D Systems

8 Phase Portraits of 3D Systems

9 Stability of C-T LTV Systems

For LTV system $\dot{x}(t) = A(t)x(t)$, its solution is $x(t) = \Phi(t)x(0), t \geq 0$

Theorem:

- LTV system is asymptotically stable if $\Phi(t) \to 0$ as $t \to \infty$
- LTV system is exponentially stable if there exist C, $r > 0$ such that

$$||\Phi(t)|| \leq Ce^{-rt}, \forall t \geq 0$$

10 Stability of D-T Autonomous Linear Systems

A discrete-time LTV system

$$x[k+1] = A[k]x[k], \quad k = 0, 1, \ldots$$

Definition (Asymptotic Stability):
LTV system is **asymptotically stable at time** k_0 if its solution $x[k]$ starting from any initial condition $x[k_0]$ at time k_0 satisfies

$$x[k] \to 0 \text{ as } k \to \infty$$

Definition(Exponential Stability):
LTV system is **exponentially stable at time** k_0 if its solution $x[k]$ starting from any initial condition $x[k_0]$ at time k_0 satisfies

$$\|x[k]\| \leq Kr^{k-k_0} \|x[k_0]\|, \quad \forall k = k_0, k_0+1, \ldots$$

for some constants $K > 0, 0 \leq r < 1$

11 Stability of D-T LTV Systems

For LTV system $x[k+1] = A[k]x[k], \quad k = 0, 1, \ldots$

- LTV system is asymptotically stable at time k_0 if

$$\Phi\left[k, k_0\right] \rightarrow 0 \text{ as } k \rightarrow \infty$$

- LTV system is exponentially stable at time k_0 if there exist $C \geq 0$,

$$\left\|\Phi\left[k, k_0\right]\right\| \leq C r^{k-k_0}, \quad \forall k \geq k_0$$

- Asymptotic stability is **not equivalent** to exponential stability
- The starting time k_0 **does matter**

12 Stability of D-T LTI Systems

Consider the discrete-time LTI system

$$x[k+1] = Ax[k], k = 0, 1, ...$$

Theorem:
The following statements are equivalent

1. The LTI system is asymptetically stable
2. The LTI system is exponentially stable
3. All the eigenvalues of A are inside the open unit disk of the complex plane $\mathbb{C}$

- For LTI systems, the starting time k_0 does not matter

13 Marginal Stability of D-T LTI Systems

Given a LTI system $x[k+1] = Ax[k]$ The LTI system is **marginally stable** if both of the following hold:

1. All the eigenvalues of A are inside the closed unit disk of $\mathbb{C}$
2. There are some non-defective eigenvalues of A on the unit circle

The LTI system is **unstable** if either of the following is true:

1. A has eigenvalues outside the closed unit disk of $\mathbb{C}$
2. A has a defective eigenvalues on the unit circle