Lecture 15: Observability II

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Contents

1 C-T LTI Systems

A continuous-time n-state m-state p-output LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

- Matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$ are known
- Can we determine $x(0)$ from u and y over the time interval $[0, t]$?

Definition (Observability):
The C-T LTI system is observable (at time $t > 0$) if the initial condition $x(0)$ can be uniquely determined based on $u(\tau)$ and $y(\tau), 0 \le \tau \le t$.

2 Characterizing C-T Observability

Consider derivatives of y:

$$\begin{aligned} y &= Cx + Du \\ \dot{y} &= C\dot{x} + D\dot{u} = CAx + CBu + D\dot{u} \\ \ddot{y} &= CA^2x + CABu + CB\dot{u} + D\ddot{u} \\ &\vdots \end{aligned} \quad \Rightarrow \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \mathcal{O}_n x + \mathcal{T}_n \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-1)} \end{bmatrix}$$

Here, the same matrices in the D-T case are encountered:

$$\mathcal{O}_n = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\text{observability matrix } \mathcal{O}} \quad , \quad \mathcal{T}_n = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ CA^{n-2}B & & CB & D \end{bmatrix}$$

3 Characterizing C-T Observability (cont.)

At time $t = 0$, rewrite the above as

$$\mathcal{O}x(0) = \begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} - \mathcal{T} \begin{bmatrix} u(0) \\ \dot{u}(0) \\ \vdots \\ u^{(n-1)}(0) \end{bmatrix}$$

- $x(0)$ can be uniquely determined iff $\mathcal{O}$ is injective, i.e., $\mathcal{N}(\mathcal{O}) = \{0\}$
- **Unobservable subspace** $\mathcal{N}(\mathcal{O})$ gives ambiguity in determining $x(0)$
- Suppose $u \equiv 0$. If $x(0) \in \mathcal{N}(\mathcal{O})$, then $y \equiv 0$

Effect of u can be substract out. Hence we can assume $u \equiv 0$:

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}$$

4 Observability Condition

Theorem:
The C-T LTI system (A, B, C, D), or simply (C, A), is observable (at any time $t > 0$) if the observability matrix $\mathcal{O}$ is injective, or equivalently, full (column) rank n

5 Equivalent Observability Condition

The C-T system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with $\tilde{A} = A^T, \tilde{B} = C^T, \tilde{C} = B^T$, and $\tilde{D} = D^T$ is called the dual of the $C - T$ system (A, B, C, D).

Proposiyion (Controllability-Observability Duality):
C-T system (A, B, C, D) is observable (resp. controllable) if and only if its dual system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is controllable (resp. observable).

Theorem:
Equivalent conditions for the C-T system (A, B, C, D) to be observable:

- The observability matrix $\mathcal{O}$ is full rank
- **PBH Rank Test**: For any $\lambda \in \mathbb{C}, \text{rank} \begin{bmatrix} \lambda I - A \\ c \end{bmatrix} = n$
- **Eigenvector Test**: For any right eigenvector $v \in \mathbb{C}^n$ of A, $Cv \neq 0$
- The matrix $\int_0^t e^{A^t \tau} C^T C e^{A\tau} d\tau$ is nonsingular for some $t > 0$

6 Quantitative Observability

Suppose C-T system (A, B, C, D) is **stable** and observable, and $u \equiv 0$. Starting from $x(0)$, the output energy over time interval $[0, t]$ is

$$\int_0^t \|y(\tau)\|^2 d\tau = x(0)^T \underbrace{\left(\int_0^t e^{A^T \tau} C^T C e^{A\tau} d\tau \right)}_{W_o(t)} x(0)$$

$$\int_0^\infty \|y(\tau)\|^2 d\tau = x(0)^T \underbrace{\left(\int_0^\infty e^{A^T \tau} C^T C e^{A\tau} d\tau \right)}_{C-T \text{ Observability Gramian } W_o} x(0)$$

• The larger the output energy, the “easier” it is to estimate $x(0)$

7 Kalman Observable Form

There exists a coordinate transform T such that

$$\tilde{A} = T^{-1} A T = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$$

$$\tilde{C} = CT = \left[\begin{array}{cc} \tilde{C}_1 & 0 \end{array}\right]$$

$(\tilde{C}, \tilde{A})$ is called the **Kalman Observable Form**

$$\sigma(\tilde{A}) = \sigma(\tilde{A}_{11}) \cup \sigma(\tilde{A}_{22})$$

Unobservable modes: system modes corresponding to eigenvalues of $\tilde{A}_{22}$

8 Minimality

Definition:
A system (A, B, C, D) is called **minimal** if among all the realizations of its transfer function $C(sI - A)^{-1}B + D$, it has the smallest state dimension

• A given transfer function $H(s)$ have infinite many minimal realizations

Theorem:
A system (A, B, C, D) is minimal if and only if it is both controllable and observable.

9 Proof

10 Kalman Decomposition

For a general system (A, B, C, D), a coordinate transform $\tilde{x} = Tx$ exists that can transform the system to its **Kalman Canonical Form**:

$$\dot{\tilde{x}} = \left[\begin{array}{cccc} \tilde{A}_{co} & 0 & \tilde{A}_{13} & 0 \\ \tilde{A}_{21} & \tilde{A}_{c\overline{o}} & \tilde{A}_{23} & \tilde{A}_{24} \\ 0 & 0 & \tilde{A}_{\overline{c}0} & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{\overline{co}} \end{array}\right] \tilde{x} + \left[\begin{array}{c} \tilde{B}_{co} \\ \tilde{B}_{c\overline{o}} \\ 0 \\ 0 \end{array}\right] u$$

$$y = \left[\begin{array}{cccc} \tilde{C}_{co} & 0 & \tilde{C}_{\overline{c}o} & 0 \end{array}\right] \tilde{x} + Du$$

Block diagram:

11 Kalman Decomposition (cont.)

Controllable and observable subsystem:

$$\begin{aligned} \dot{\tilde{x}}_{co} &= \tilde{A}_{cc}\tilde{x}_{co} + \tilde{B}_{co}u \\ y &= \tilde{C}_{cc}\tilde{x}_{co} + Du \end{aligned}$$

Fact:
The original system and above subsystem have the same transfer function:

$$C(sI - A)^{-1}B + D = \tilde{C}(sl - \tilde{A})^{-1}\tilde{B} + D = \tilde{C}_{co}\left(sI - \tilde{A}_{co}\right)^{-1}\tilde{B}_{co} + D$$