Limits and Derivatives

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Contents

A series of problems lead to limits and derivative, here are 2 examples:

- **tangent line**: use **secant line** to approach tangent line
- **instantaneous velocity**: use **average velocity** in a shorter period time to estimate instantaneous velocity

1 Limit of a Function

1.1 Definition of Limit

Limit of a function is written as

$$\lim_{x \to a} f(x) = L$$

it means that the limit of f as x approaches a equals L. Note that the limit has nothing to do with $f(a)$.

1.2 Term Approach

- informally, approach means getting closer to a certain value
- formally definition will be introduced latter

1.3 One-Sided Limits

For left-hand limit:

$$\lim_{x \to a^-} f(x) = L$$

For right-hand limit:

$$\lim_{x \to a^+} f(x) = L$$

Note that

$$\lim_{x \to a} f(x) = L$$

exists only when both left-hand and right-hand side limits exist.

1.4 Infinite Limits

Let $f(x)$ is defined on $(a - \delta, a) \cup (a, a + \delta)$,

$$\lim_{x \to a} f(x) = \infty$$

leads to infinite limits. Note that:

- here ∞ can be either $+\infty$ or $-\infty$
- it often occurs when the function is not defined at $x = a$ or a is pole of ration functions

1.5 Vertical Asymptote

The line $x = a$ is called a vertical asymptote of $y = f(x)$ if

$$\lim_{x \to a, a^+ \text{ or } a^-} = \infty$$

natural log has a vertical asymptote $x = 0$ since

$$\lim_{x \to 0^+} \ln x = -\infty$$

2 Limit Laws and Theorem

2.1 Limit Laws

Suppose f and g are two functions and c is a constant:

1. **sum**: $\lim(f + g) = \lim f + \lim g$
2. **difference**: $\lim(f - g) = \lim f - \lim g$
3. **constant multiplication**: $\lim(cf) = c \lim f$
4. **product**: $\lim(fg) = (\lim f)(\lim g)$
5. **quotient**: $\lim(f/g) = \lim f / \lim g$, if $\lim g \neq 0$

From the laws above, the following laws can be obtained:

- **power**: $\lim f^n = (\lim f)^n$
- **limit of constant**: $\lim c = c$

2.2 Direct Substitution Property

If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

In fact this property is true if the function is **continuous** at $x = a$.

2.3 A Useful Fact

If $f(x) = g(x)$ when $x \neq a$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

provide the limit exists, this is very useful to get the limit of a ration function whose pole that can be canceled by its numerator factor.

2.4 Limit Theorems

- **two sides theorem**: $\lim_{x\to a} f(x) = L$ if and only if

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$$

- **comparison theorem**: if $f(x) \le g(x)$ and both function have limits at $x = a$, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

- **squeeze theorem**: if $f(x) \le g(x) \le h(x)$ and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

3 Precise Definition of Limit

Let f be a function defined on some open interval that contains the number a except possibly at a itself. Then we say the limit of f as x approaches a is L:

$$\lim_{x\to a} f(x) = L$$

if for **every** number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$|x-a| < \delta \Longrightarrow |f(x) - L| < \epsilon$$

Comment: range constrain leads to domain constrain.

3.1 Precise Definition of Left Hand and Right Hand Limit

Similar way can be applied to these 2 definitions:

- **Left-Hand Limit**:

$$\lim_{x\to a^-} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$a - \delta < x < a \Longrightarrow |f(x) - L| < \epsilon$$

- **Right-Hand Limit**:

$$\lim_{x\to a^+} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$a < x < a + \delta \Longrightarrow |f(x) - L| < \epsilon$$

3.2 Precise Definition of Infinite Limit

Let f be a function defined on some open interval that contains the number a except possibly at a itself. Then

$$\lim_{x \to a} f(x) = +\infty$$

if for **every** positive number M there is $\delta > 0$ such that

$$|x - a| < \delta \Longrightarrow f(x) > M$$

Similar definition can be applied to negative infinite limit.

4 Continuity of Functions

4.1 Definition

A function f is continuous at number a if

$$\lim_{x \to a} f(x) = a$$

this indicate 3 conditions:

- $f(a)$ is defined at a
- limit of f at a exists
- limit equals to $f(a)$

A function is continuous on an interval if it is continuous at every number in the interval. Also a function can be either continuous from left or right.

4.2 Types of Discontinuity

There are 3 types:

- **removable**: one point problem
- **infinite**: reach infinity at certain point
- **jump**: left limit doesn't equal to right limit

4.3 Theorems of Continuity

- if f and g are continuous at a and c is a constant, then $f \pm g$, cf, fg, f/g, where $g \neq 0$ are also continuous at a
- polynomials are continuous on $\mathbb{R}$
- any rational function is continuous wherever it is defined(denominator is not zero)
- the following function are continuous everywhere in their domains: polynomials, rational function, root, trig, inverse trig, exponential, logarithmic functions.

- if f is continuous at b and

$$\lim_{x \to a} g(x) = b$$

then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b)$$

- if g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a

- **intermediate value theorem**: suppose f is continuous on interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$, then there exist a number $c \in (a, b)$ such that $f(c) = N$, this is the foundation for finding root of polynomial $P(x)$ in (a, b) if $P(a)pP(b) < 0$

5 Limits at Infinity and Horizontal Asymptote

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} = L \quad \text{or} \quad \lim_{x \to -\infty} = L$$

Example:

$$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2} \quad \text{and} \quad \lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$$

Here is one useful theorem: if $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

And here is the precise definition for **bound limit** as x reach infinity: let f be a function defined on some interval $(a, +\infty)$, then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\epsilon > 0$, there is a corresponding N such that if $x > N$, then

$$|f(x) - L| < \epsilon$$

similar definition works for x approaches $-\infty$. For **unbound** limit as x reach infinity: let f be a function defined on some interval $(a, +\infty)$, then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding number N such that if $x > N$, then $f(x) > M$.

6 Derivatives and Rate of Change

6.1 Tangent Line

The tangent line to the curve $y = f(x)$ at point $P(a, f(a))$ is the line through P with **slope**

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

6.2 Velocity

- velocity can be considered as slope of position function
- acceleration can be considered as slope of velocity function

6.3 Derivative

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

the tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope equals to $f'(a)$. And derivative can also be considered as instantaneous rate of change of a function at certain value.

7 Derivative as Functions

The derivative of function f is defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

here is some common alternative notation:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

A function is differentiable at a if $f'(a)$ exists and it is differentiable on an open interval I if it is differentiable at every number in the interval.

7.1 Differentiable vs Continuous

If f is differentiable at a then it is continuous at a, the reverse may not be true. Meaning differentiability requires a **higher** condition than continuity.

7.2 Function Fail to be Differentiable

- a sharp corner, $y = |x|$
- discontinuity, $y = 1/x$
- vertical tangent, $y = \sqrt[3]{x}$

7.3 Higher Derivative

The derivative of a function is a function, which may have its own derivative.

- second derivative: f''
- third derivative: f'''

Example:

- **velocity** is the (first) derivative of position function
- **acceleration** is the second derivative of position function
- **jerk** is the third derivative of position function

8 Extra

8.1 Greatest Integer Functions

It is defined as:

$$y = f(x) = \lfloor x \rfloor$$

- piece-wise function
- continuous from right side

8.2 Triangle Inequality

It states that:

$$|a + b| \leq |a| + |b|$$