Topic 02 - Limits and Derivatives

Baboo J. Cui

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Contents

A series of problems lead to limits and derivative, here are 2 examples:

- **tangent line**: use **secant line** to approach tangent line
- **instantaneous velocity**: use **average velocity** in a shorter period time to estimate instantaneous velocity

1 Limit of a Function

1.1 Definition of Limit

Limit of a function is written as

$$\lim_{x \to a} f(x) = L$$

it means that the limit of f as x approaches a equals L. Note that the limit has nothing to do with $f(a)$.

1.2 Term Approach

- informally, approach means getting closer to a certain value
- formally definition will be introduced latter

1.3 One-Sided Limits

For left-hand limit:

$$\lim_{x \to a^-} f(x) = L$$

For right-hand limit:

$$\lim_{x \to a^+} f(x) = L$$

Note that

$$\lim_{x \to a} f(x) = L$$

exists only when both left-hand and right-hand side limits exist.

1.4 Infinite Limits

Let $f(x)$ is defined on $(a - \delta, a) \cup (a, a + \delta)$,

$$\lim_{x \to a} f(x) = \infty$$

leads to infinite limits. Note that:

- here ∞ can be either $+\infty$ or $-\infty$
- it often occurs when the function is not defined at $x = a$ or a is pole of ration functions

1.5 Vertical Asymptote

The line $x = a$ is called a vertical asymptote of $y = f(x)$ if

$$\lim_{x \to a, a^+ \text{ or } a^-} = \infty$$

natural log has a vertical asymptote $x = 0$ since

$$\lim_{x \to 0^+} \ln x = -\infty$$

2 Limit Laws and Theorem

2.1 Limit Laws

Suppose f and g are two functions and c is a constant:

1. **sum**: $\lim(f + g) = \lim f + \lim g$
2. **difference**: $\lim(f - g) = \lim f - \lim g$
3. **constant multiplication**: $\lim(cf) = c \lim f$
4. **product**: $\lim(fg) = (\lim f)(\lim g)$
5. **quotient**: $\lim(f/g) = \lim f / \lim g$, if $\lim g \neq 0$

From the laws above, the following laws can be obtained:

- **power**: $\lim f^n = (\lim f)^n$
- **limit of constant**: $\lim c = c$

2.2 Direct Substitution Property

If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

In fact this property is true if the function is **continuous** at $x = a$.

2.3 A Useful Fact

If $f(x) = g(x)$ when $x \neq a$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

provide the limit exists, this is very useful to get the limit of a ration function whose pole that can be canceled by its numerator factor.

2.4 Limit Theorems

- **two sides theorem**: $\lim_{x\to a} f(x) = L$ if and only if

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$$

- **comparison theorem**: if $f(x) \leq g(x)$ and both function have limits at $x = a$, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

- **squeeze theorem**: if $f(x) \leq g(x) \leq h(x)$ and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

  then

$$\lim_{x\to a} g(x) = L$$

3 Precise Definition of Limit

Let f be a function defined on some open interval that contains the number a except possibly at a itself. Then we say the limit of f as x approaches a is L:

$$\lim_{x\to a} f(x) = L$$

if for **every** number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$|x - a| < \delta \Longrightarrow |f(x) - L| < \epsilon$$

Comment: range constrain leads to domain constrain.

3.1 Precise Definition of Left Hand and Right Hand Limit

Similar way can be applied to these 2 definitions:

- **Left-Hand Limit**:

$$\lim_{x\to a^-} f(x) = L$$

  if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$a - \delta < x < a \Longrightarrow |f(x) - L| < \epsilon$$

- **Right-Hand Limit**:

$$\lim_{x\to a^+} f(x) = L$$

  if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$a < x < a + \delta \Longrightarrow |f(x) - L| < \epsilon$$

3.2 Precise Definition of Infinite Limit

Let f be a function defined on some open interval that contains the number a except possibly at a itself. Then

$$\lim_{x \to a} f(x) = +\infty$$

if for **every** positive number M there is $\delta > 0$ such that

$$|x - a| < \delta \implies f(x) > M$$

Similar definition can be applied to negative infinite limit.

4 Continuity of Functions

5 Extra

5.1 Greatest Integer Functions

It is defined as:

$$y = f(x) = \lfloor x \rfloor$$

- piece-wise function
- continuous from right side

5.2 Triangle Inequality

It states that:

$$|a + b| \leq |a| + |b|$$