Lecture 18: Output Feedback Observer Designer

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1 State Observer Problem

A continuous-time (or discrete-time) LTI system

$$\begin{cases} \dot{x} = Ax \quad \text{or} \\ y = c_x \end{cases} \quad \text{or} \quad \begin{cases} x[k+1] = Ax[k] \\ y[k] = Cx[k] \end{cases}$$

Problem:

- A and C are known
- Input u and output y, but not state x, can be measured
- Design an observer to obtain an estimate $\hat{x}$ of the state

Strategy I:

- Run system up to a finite time T
- Use measurements of u and y up to time T to find an estimate $\hat{x}(0)$
- Simulate system from $\hat{x}(0)$ to obtain state estimate $\hat{x}(t), \forall t \geq 0$

2 Feedback Observer

Strategy II (Feedback Observer Design):

- Built a simulator of original system with a guess of initial condition:

$$\begin{cases} \dot{x} = A\hat{x}, \quad \hat{x}(0) = \text{guess} \\ \hat{y} = C\hat{x} \end{cases}$$

- Detect error in guess by comparing $y(t)$ and $\hat{y}(t)$
- Use error $y(t) - \hat{y}(t)$ to improve guess $\hat{x}(t)$ so that $\hat{x}(t) \to x(t)$

3 A Naive Observer Design

Original system with unknown $x(0)$:

$$\dot{x} = Ax, \quad y = Cx$$

Simulator with a guess (say, 0) of initial state:

$$\dot{\hat{x}} = A\hat{x}, \quad \hat{x}(0) = 0, \quad \hat{y} = C\hat{x}$$

Then, **state-observer error** is simply $e = x - \hat{x}$, and satisfies

$$\dot{e} = Ae, \quad e(0) = x(0)$$

If original system is stable, then $e(t) \to 0$; simulator works (trivially).

4 Linear Feedback Observer

Original system with unknown $x(0)$:

$$\dot{x} = Ax$$
$$y = Cx$$

Simulator with linear feedback

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}), \quad \hat{x}(0) = 0$$
$$\hat{y} = C\hat{x}$$

5 Designing Observer Feedback Gain L

Problem:

- For what system (C, A) can we find L such that $A - LC$ is stable?
- For what system (C, A) can we use L to arbitrarily re-assign the eigenvalues of $A - LC$?

Idea: Consider dual system $(\hat{A} = A^T, \hat{B} = C^T)$

6 Re-assigning Eigenvalues of Error Dynamics

Fact:
For an observable system (C, A), all the eigenvalues of $A - LC$ can be arbitrarily re-assigned by proper choices of L.

- For single-output case, transform to observer canonical form, e.g.

$$\dot{x}_0 = \begin{bmatrix} -\alpha_2 & 1 & 0 \\ -\alpha_1 & 0 & 1 \\ -\alpha_0 & 0 & 0 \end{bmatrix} x_0, \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_0$$

- For multi-output case, first find feedback matrix L_0 to make system observable from a single output; then apply single-output result

7 Kalman Observable Form

Use coordinate transform $x = T\tilde{x}$ to obtain Kalman observable form:

$$\dot{\tilde{x}} = \underbrace{\begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}}_{\tilde{A}=T^{-1}AT} \tilde{x}, \quad y = \underbrace{\begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix}}_{\tilde{C}=CT} \tilde{x}$$

In the new coordinate system, observer feedback gain $\tilde{L} = T^{-1}L = \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{bmatrix}$

$$\tilde{A} - \tilde{L}\tilde{C} = \begin{bmatrix} \tilde{A}_{11} - \tilde{L}_1\tilde{C}_1 & 0 \\ \tilde{A}_{21} - \tilde{L}_2\tilde{C}_1 & \tilde{A}_{22} \end{bmatrix}$$

Thus, the eigenvalues

$$\sigma(A - LC) = \sigma(\tilde{A} - \tilde{L}\tilde{C}) = \sigma\left(\tilde{A}_{11} - \tilde{L}_1\tilde{C}_1\right) \cup \underbrace{\sigma\left(\tilde{A}_{22}\right)}_{\text{fixed}}$$

8 Detectability

Definition (Detectability):
System (C, A) is called **detectable** if there exists a cbserver feedback matrix L such that $A = LC$ is stable.

- If A is stable itself, (C, A) is stabilizable
- If (C, A) is observable, it is detectable as well
- If (C, A) is not observable, it could still be detectable

Theorem:
System (C, A) is detectable if all its unobservable medes are stable (i.e. $\tilde{A}_{22}$ is stable in the Kalman observable form).

- If system has some unobservable modes that are unstable, then no feedback gain L can make $A - LC$ stable; thus linear feedback (indeed, any output feedback) observer will not work

9 Example I

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 \end{bmatrix} x(t)$$

10 Example II

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 5 & -4 & 2 \end{bmatrix}}_{A} \times + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} x$$

11 State Feedback Control Using Observer

We have previously studied state feedback control with gain matrix K What if the state can only be estimated?

- Get an estimate $\hat{x}$ of the state by designing a suitable observer (i.e., by linear feedback observer with gain L)
- State feedback control using estimated state $\hat{x}$ with a state-feedback gain matrix K to both original system and simulator
- This is called an **Observer-Based Controller** or **OBC**

Question:

- Will this scheme work?
- How do we design K and L?
- What are the poles of the closed-loop system?

12 Observer-Based Controllers Diagram

13 Observer-Based Controllers Analysis

Closed-loop equations:

$$\begin{cases} \dot{x} = Ax - BK\hat{x} \\ \dot{\hat{x}} = A\hat{x} + L(y - \hat{y}) - BK\hat{x} \end{cases}$$

The aggregated system is:

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

14 Separation Principle of OBC Design

The closed-loop system under observer-based controller has eigenvalues.

$$\sigma(A - BK) \cup \sigma(A - LC)$$

- $\sigma(A - BK)$ characterizes the state dynamics
- $\sigma(A - BK)$ characterizes the state estimation error dynamics
- If the system is both controllable and observable (i.e. minimal), then the eigenvalues of the closed-loop system can be arbitrarily assigned by proper K and I
- If the system is both stabilizable and detectable, then by some proper K and L, the closed-loop system with states x and e will be stable
- Good choices of K can be obtained via optimal control
- Good choices of L can be obtained via Kalman filtering (other courses)