Lecture 12: Controllability I

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Contents

1 Controllability of C-T LTI Systems

A continuous-time n-state m-input LTI system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{1}$$

Given a terminal time $t_f > 0$. Over the time interval $[0, t_f]$, the control input $u(t), 0 \leq t \leq t_f$, steers (or transfers) the state from x_0 to

$$x_f := x\left(t_f\right) = e^{At_f} x_0 + \int_0^{t_f} e^{A(t_f - \tau)} Bu(\tau) d\tau$$

Definition:

- The LTI system is called **controllable at time** $t_f > 0$ if for any initial state $x_0 \in \mathbb{R}^n$ and any target state $x_f \in \mathbb{R}^n$, a control $u(t)$ exists that can steer the system from x_0 to x_f over the time interval $[0, t_f]$.
- It is called **controllable** if it is controllable at a large enough t_f

2 A Example

State equation:

$$\frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

Assume unit inductances, and zero initial current $x(0) = \begin{bmatrix} i_1(0) \\ i_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

3 Question Related to Controllability

- Where can x_0 be transferred to over the time period $[0, t_f]$?
- If doable, how do we find u that transfers x_0 to x_f?
- How quickly can x_0 be transferred to x_f?
- How do we find "efficient" u that transfers x_0 to x_f?

4 Reachable Set

To study the controllability of the LTI system $\dot{x} = Ax + Bu$, we first consider the special case when $x_0 = 0$. Then at a terminal time $t_f > 0$

$$x_f := x\left(t_f\right) = \int_0^{t_f} e^{A(t_f - \tau)} Bu(\tau) d\tau$$

Definition (Reachable Set):
The reachable set at time $t_f > 0$ of the LTI system is the set of states the system can be steered to using arbitrary control inputs over $[0, t_f]$:

$$\mathcal{R}_{t_f} := \left\{ \int_0^{t_f} e^{A(t_f - \tau)} Bu(\tau) d\tau | u(t), 0 \leq t \leq t_f \right\}$$

- $\mathcal{R}_{t_f}$ is a subspace of $\mathbb{R}^n$ since it is the image of the linear map

$$u(t), 0 \leq t \leq t_f \mapsto x_f = \int_0^{t_f} e^{A(t_f - \tau)} Bu(\tau) d\tau$$

- $\mathcal{R}_{t_f} \subset \mathcal{R}_{\tilde{t}_f}$ whenever $t_f < \tilde{t}_f$(can reach more states given more time)

5 Reachability of C-T LTI Systems

Definition (Reachability):
System is called **reachable** at time $t_f > 0$ if $\mathcal{R}_{t_f} = \mathbb{R}^n$, i.e., if it is can be steer from $x_0 = 0$ to any $x_f \in \mathbb{R}^n$ over the time interval $[0, t_f]$.

Proposition (Reachability = Controllability):
At any $t_f > 0$, the LTI system is controllable if and only if is reachable.

6 Controllability of D-T LTI Systems

A discrete-time n-state m-input LTI system

$$x[k+1] = Ax[k] + Bu[k], \quad x[0] = x_0 \tag{2}$$

Definition:

- The LTI system (2) is **controllable at time** $k_f > 0$ if for any $x_0, x_f \in \mathbb{R}^n$, a control $u[k]$, $k = 0, ..., k_f - 1$, exists that can steer the system from x_0 at time 0 to x_f at time k_f
- It is **controllable** if it is controllable at a large enough time k_f

7 Reachability of D-T LTI System

Definition (Reachable Set):
The **reachable set** at time $k_f > 0$ is the set of states the system (2) starting from $x_0 = 0$ can be steered to at time k_f:

$$\mathcal{R}_{k_f} := \left\{ \sum_{i=0}^{k_f - 1} A^{k_f - 1 - i} Bu[i] | u(k), k = 0, 1, \ldots, k_f - 1 \right\}$$

• System is **reachable at time** $k_f > 0$ if $\mathcal{R}_{k_f} = \mathbb{R}^n$

Proposition (Reachability = Controllability):
The D-T LTI system is controllable if and only if it is reachable (at any k_f)

8 Controllability Matrix

For the D-T system $x[k+1] = Ax[k] + Bu[k]$ with $x[0] = 0$,

$$x[k] = \underbrace{\begin{bmatrix} B & AB \cdots A^{k-1}B \end{bmatrix}}_{C_k} \begin{bmatrix} u[k-1] \\ \vdots \\ u[0] \end{bmatrix}$$

- Reachable set at time k is the range of the matrix $\mathcal{C}_k$ i.e., $\mathcal{R}\left(\mathcal{C}_k\right)$
- **Observation**: $\mathcal{R}\left(\mathcal{C}_k\right)=\mathcal{R}\left(\mathcal{C}_n\right)$ for $k \geq n$

Definition (Controllability Matrix):
The controllability matrix of the system is

$$\mathcal{C}:=\mathcal{C}_n=\left[\begin{array}{llll} B & AB & \cdots & A^{n-1}B \end{array}\right]$$

9 Characterizing Controllability

Proposition:
The reachable subspace of the D-T LTI system (A, B) is the range of its controllability matrix $\mathcal{C}=\left[\begin{array}{llll} B & AB & \cdots & A^{n-1}B \end{array}\right]$:

$$\mathcal{R}=\mathcal{R}(\mathcal{C})$$

Theorem:
The D-T LTI system (A, B) is controllable (reachable) if and only if its controllability matrix $\mathcal{C}$ is onto, or equivalently, full (row) rank

10 Example

1. $x[k+1]=\left[\begin{array}{ll} 0 & 0 \\ 0 & 0 \end{array}\right] x[k]+\left[\begin{array}{l} 1 \\ 1 \end{array}\right] u[k]$
2. $x[k+1]=\left[\begin{array}{ll} 2 & 0 \\ 0 & 2 \end{array}\right] x[k]+\left[\begin{array}{l} 1 \\ 1 \end{array}\right] u[k]$
3. $x[k+1]=\left[\begin{array}{ll} 2 & 0 \\ 0 & 1 \end{array}\right] x[k]+\left[\begin{array}{l} 1 \\ 1 \end{array}\right] u[k]$

11 Equivalent Condition for Controllability

Recall that $\mathcal{C}$ is onto if and only if $\mathcal{C}\mathcal{C}^T$ is nonsingular

Theorem:
The D-T LTI system (A, B) is controllable if and only if

$$\mathcal{C}\mathcal{C}^T=\sum_{k=0}^{n-1} A^k B B^T\left(A^T\right)^k$$

is nonsingular

12 PHB Tests of Controllability

Theorem (Popov-Belevitch-Hautus):
The D-T LTI system (A, B) is controllable if and only if

1. **Rank Test**: for any $\lambda \in \mathbb{C}$

$$\mathrm{rank}[\lambda I - A \quad B] = n$$

2. **Eigenvector Test**: for any left eigenvector $w \in \mathbb{C}^n$ of A,

$$w^T B \neq 0$$

13 Proof of PBH Tests

14 Example

1. $x[k+1] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \times [k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]$

2. $x[k+1] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \times [k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]$

3. $A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ -2 & 2 \\ 2 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 3 \end{bmatrix}$

15 Example

Consider the composite system by the negative feedback connection of two matching linear systems (A_1, B_1, C_1) and (A_2, B_2, C_2):

$$\tilde{A} = \begin{bmatrix} A_1 & -B_1 C_2 \\ -B_2 C_1 & A_2 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

Fact:
The composite system $(\tilde{A}, \tilde{B})$ is controllable if and only if both subsystems (A_1, B_1) and (A_2, B_2) are controllable.

16 Minimum Energy Input for Reachability

Suppose system (A, B) is controllable, and $k \geq n$ **Minimum energy input** is the input $u^*[0], \ldots, u^*[k-1]$ that can steer the system from $x[0] = 0$ to x_d at time k with the least energy $\sum_{i=0}^{k-1} \|u[i]\|^2$

- Minimum energy input is

$$u^* = \begin{bmatrix} u^*[k-1] \\ \vdots \\ u^*[0] \end{bmatrix} = \mathcal{C}_k^T \left(\mathcal{C}_k \mathcal{C}_k^T \right)^{-1} x_d$$

- Minimun energy required to reach x_d is

$$\mathcal{E}_{\min} = \|u^*\|^2 = x_d^T \left(\mathcal{C}_k \mathcal{C}_k^T\right)^{-1} x_d = x_d^T \left(\sum_{i=0}^{k-1} A^i B B^T \left(A^T\right)^i \right)^{-1} x_d$$

17 Example

$$x[k+1] = \begin{bmatrix} 1.75 & 0.8 \\ -0.95 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[k], \text{ with } x[0] = 0, x_d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

18 Minimum Energy Over Infinite Horizon

If (A, B) is controllable, the following matrix always exists:

$$P = \lim_{k \to \infty} \left(\sum_{i=0}^{k-1} A^i B B^T \left(A^T\right)^i \right)^{-1}$$

The minimun energy required to reach a point x_d with no limit on k is

$$\mathcal{E}_{\min} = x_d^T P x_d$$