Lecture 7: Autonomous LTV Systems

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1 Continuous-Time Autonomous LTV Systems

Consider an autonomous linear time-varying (LTV) system

$$\dot{x}(t) = A(t)x(t), \quad t \geq 0, \quad \text{with initial condition } x(0) \in \mathbb{R}^n$$

- $A(t) \in \mathbb{R}^{n \times n}$ is a function of $t \geq 0$

Theorem (Existence and Uniqueness of Solutions):
If $A(t)$ is a piecewise continuous function of t, then starting from any $x(0)$, the LTV system has a unique solution $x(t)$ for all $t \geq 0$.

2 Scalar Autonomous LTV Systems

Consider the scalar case with $x(t), a(t) \in \mathbb{R}$:

$$\dot{x}(t) = a(t)x(t), \quad x(0) \in \mathbb{R}$$

3 Solution Space

The solution space $\mathbb{X}$ of the LTV system is the set of all its solutions:

$$\mathbb{X} := \{x(t), t \geq 0 | \dot{x}(t) = A(t)x(t)\}$$

- The solution space $\mathbb{X}$ is a vector space of dimension n

4 Fundamental Matrix

Define the fundamental matrix $\Phi(t)$, $t \geq 0$, as

$$\Phi(t) := [\phi_1(t) \ldots \phi_n(t)] \in \mathbb{R}^{n \times n}, \quad t \geq 0.$$

- $\phi_i(t)$ is the solution with intial condition $x(0) = e_i, i = 1, \ldots, n$
- Solution from any initial condition $x(0)$ can be written as

$$x(t) = \Phi(t)x(0), \quad t \geq 0$$

5 Properties of Fundamental Matrix

Fundamental matrix $\Phi(t) \in \mathbb{R}^{n \times n}$ solves the matrix differential equation:

$$\dot{\Phi}(t) = A(t)\Phi(t), \quad \Phi(0) = I_n$$

- $\Phi(t)$ is nonsingular at all time $t \geq 0$ due to uniqueness of solutions

6 State Transition Matrix

Definition (State Transition Matrix):
The state transition matrix $\Phi(t,s) \in \mathbb{R}^{n\times n}, s.t \geq 0$, for the LTV system is defined from the fundamental matrix $\Phi(t)$ by

$$\Phi(t,s) = \Phi(t)\Phi(s)^{-1}$$

- It relates the state solution at different times: for any solution $x(t)$,

$$x(t_2) = \Phi(t_2, t_1)\, x(t_1), \quad \forall t_1, t_2 \geq 0.$$

- For any $t_1, t_2, t_3 \geq 0$,

$$\Phi(t_3, t_2)\, \Phi(t_2, t_1) = \Phi(t_3, t_1)$$

7 Example I

Consider the LTV system: $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix} x(t)$

8 Example II

Consider the LTV system: $\dot{x}(t) = A(t)x(t) = \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} x(t)$, Note that $A(t), t \geq 0$, can be diagonalized by the same transformation:

$$A(t) = \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} = \begin{bmatrix} j & -j \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -j\omega(t) & \\ & j\omega(t) \end{bmatrix} \begin{bmatrix} j & -j \\ -1 & 1 \end{bmatrix}^{-1}$$

9 A Useful Special Case

Proposition:
If $A(s)$ and $A(t)$ commute for all $s.t. \geq 0$, then

$$\Phi(t) = e^{\int_0^t A(\tau)d\tau}, \quad t \geq 0$$

- State transition matrix becomes

$$\Phi(t,s) = e^{\int_s^t A(\tau)d\tau}, \quad \forall s, t \geq 0$$

- Solution to the LTV system is

$$x(t) = e^{\int_0^t A(\tau)d\tau} x(0), \quad t \geq 0$$

10 Example III

Consider the LTV system: $\dot{x}(t) = A(t)x(t) = \begin{bmatrix} -e^{-t} & \frac{1}{t+1} \\ 0 & -e^{-t} \end{bmatrix} x(t)$

11 Time-Reversed System

Given a LTV system $\dot{x}(t) = A(t)x(t)$, its **time-reversed system** is

$$\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t) = -A(-t)\hat{x}(t)$$

12 Discrete-Time Autonomous LTV Systems

Discrete-Time LTV system

$$x[k+1] = A[k]x[k], \quad k = 0, 1, \ldots$$

with the initial condition $x[0] \in \mathbb{R}^n$
Solution $x[k]$ is

$$x[k] = \underbrace{A[k-1]A[k-2]\cdots A[0]}_{\phi[k]} \cdot x[0], \quad k = 0, 1, \ldots$$

where $\Phi[k] \in \mathbb{R}^{n \times n}$ is the fundamental matrix

- Unlike C-T case, **fundamental matrix** $\Phi[k]$ **may be singular**.
- No time reversal in general, unless every $A[k]$ is nonsingular
- For D-T LTI system, $\Phi[k] = A^k$

13 Discrete-Time State Transition Matrix

Definition (State Transition Matrix):
The state transition matrix $\Phi[k, \ell] \in \mathbb{R}^{n \times n}, k \geq \ell$, for the discrete-time LTV system is defined as

$$\Phi[k, \ell] = A[k-1] \cdots A[\ell] | \quad k \geq \ell$$

- $\Phi[k, \ell]$ is defined only for $k \geq \ell$
- $\Phi[k, k] = I_n$ for all $k = 0, 1, \ldots$
- $\Phi[k_3, k_2] \Phi[k_2, k_1] = \Phi[k_3, k_1]$ for all $k_3 > k_2 \geq k_1$