Topic 01 - Functions and Models

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1 Functions and Representation

1.1 Definition of Function

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $y = f(x)$, in a set R.

- **domain**: D
- **range**: R
- value of function: $f(x)$ is the value of the function at x
- **independent variable**: x
- **dependent variable**: y

1.2 4 Ways to represent a function

There are 4 ways to represent a function:

- verbally by description
- numerically by table
- visually by graph
- algebraically by formula

1.3 Vertical Line Test

It is used to determine if a graph represent a function. A function must be **one-to-one**, so any vertical line can only pass at most one point to function graph.

1.4 Piece-wise Defined Function(Review)

A function is called piece-wise defined function if different formula describe the it in different domain. Here are 2 examples:

- **Absolute value Function**: $f(x) = |x|$, which is defined as:

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

- **Step function**: also known as Heaviside function, which is defined as:

$$f(x) = \begin{cases} 1, & x > 0 \\ 0.5, & x = 0 \\ 0, & x < 0. \end{cases}$$

1.5 Symmetry of Functions

A function is said to be **even** if $\forall x \in D$:

$$f(-x) = f(x)$$

A function is said to be **odd** if $\forall x \in D$:

$$f(-x) = -f(x)$$

A function that doesn't satisfy any condition above is said neither even nor odd. Here are some properties:

- both even and odd functions should have symmetric domain w.r.t y-axis
- if $f(x)$ is odd and defined at $x = 0$, then $f(0) = 0$

1.6 Increasing and Decreasing Function

- A function f is **increasing** on an interval I if $f(x_1) < f(x_2)$, whenever $x_1 < x_2$ in I.
- A function f is **decreasing** on an interval I if $f(x_1) > f(x_2)$, whenever $x_1 < x_2$ in I.

2 Essential Functions

2.1 Mathematical Model

A **mathematical model** is a math description(often by functions or equations) of a real-world phenomenon.

2.2 Linear Model

Linear model has form of:

$$y = f(x) = mx + b$$

- m is the **slope**
- b is the **y-intercept**

2.3 Polynomials

A polynomial has form of:

$$y = P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

- a_i are called **coefficients**
- if $a_n \neq 0$, n is the **degree** or d of the polynomial
- when $d = 2$, it is a **quadratic function**
- when $d = 3$, it is a **cubic function**

2.4 Power Functions

Power function is of form $f(x) = x^a$, where a is constant. It has the following properties:

- when a is positive integer, function is odd if n is odd and even if n is even
- when $a = 1/n$, where n is a positive integer, function is called **root function**
- when n is even, the domain is $(0, +\infty)$, namely not defined for negative value
- when $a = -1$, this is **reciprocal function**

2.5 Rational Function

A rational function is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

the domain $f(x)$ is $\{D(P)\} \cap \{D(Q(x))\} - \{x|Q(x) = 0\}$, where $D(\cdot)$ is domain function

2.6 Algebraic Functions

It is defined as functions that can be constructed using algebraic operations starting from polynomials.

2.7 Trigonometric Functions

There are 6 basic trig functions:

- $\sin(x)$
- $\cos(x)$
- $\tan(x)$
- $\cot(x)$
- $\sec(x)$, reciprocal of cos
- $\csc(x)$, reciprocal of sin

be clear about their domain, range and **fundamental period**!

2.8 Exponential Functions

Exponential functions are of form $f(x) = a^x$ where a is positive constant. It has the following properties:

- domain is $(-\infty, +\infty)$
- range is $(0, +\infty)$
- function is increasing if $a > 1$ and decreasing if $a < 1$

2.9 Logarithmic Functions

Logarithmic functions are of form $f(x) = \log_a x$ where a is positive constant. It has the following properties:

- domain is $(0, +\infty)$
- range is $(-\infty, +\infty)$
- function is increasing if $a > 1$ and decreasing if $a < 1$
- logarithmic and exponential are **inverse function** to each other

3 Build New Functions from Old Ones

3.1 Translation(Shifts)

Given $y = f(x)$ and $c > 0$:

- $y = f(x) + c$ moves $f(x)$ upward c units
- $y = f(x) - c$ moves $f(x)$ downward c units
- $y = f(x + c)$ moves $f(x)$ to left c units
- $y = f(x - c)$ moves $f(x)$ to right c units, also known as **delay**

3.2 Stretching and Reflecting

Given $y = f(x)$ and $c > 1$:

- $y = cf(x)$ stretch $f(x)$ vertically by a factor of c
- $y = (1/c)f(x)$ shrink $f(x)$ vertically by a factor of c
- $y = f(cx)$ shrink $f(x)$ horizontally by a factor of c
- $y = f(x/c)$ stretch $f(x)$ horizontally by a factor of c
- $y = -f(x)$ reflect $f(x)$ about x-axis
- $y = f(-x)$ reflect $f(x)$ about y-axis

3.3 Combination of Functions

Given 2 functions $f(x)$ and $g(x)$, the following functions can be constructed:

- $f + g = f(x) + g(x)$
- $f - g = f(x) - g(x)$
- $fg = f(x)g(x)$
- $f/g = f(x)/g(x)$
- $f \circ g = f(g(x))$

Comments on combination of functions:

- be careful about the domain of combined functions
- usually $f \circ g \neq g \circ f$

4 Exponential Functions

Exponential functions have form$y = a^x$, where a is constant.

4.1 Laws of Exponent

The following laws apply to exponential functions:

- $a^{-n} = \frac{1}{a^n}$
- $a^{p/q} = (\sqrt[q]{a})^p = \sqrt[q]{a^p}$
- $a^{x+y} = a^x a^y$
- $a^{x-y} = \frac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$

4.2 Number e

Essentially, e is a real number

- e is approximately equal to 2.718
- function $y = e^x$ is called **natural exponential** whose slope is 1 at $x = 0$

5 Inverse Functions

5.1 One-to-one Function

A function f is called a one-to-one if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. It has the following properties:

- can be checked by **horizontal line test**, recall that function only need to satisfy vertical line test
- also known as **bijective**(related to linear mapping)

5.2 Definition of Inverse Functions

Let f be a one-to-one function with domain D and range R, then its inverse function f^{-1} has domain R and range D, it is defined as:

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for any y in R. And it has the following properties:

- $D(f)$ is $R(f^{-1})$
- $D(f^{-1})$ is $R(f)$
- $f^{-1}(x) = y \Leftrightarrow f(y) = x$
- **cancellation equation**: $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ (attention to domain!)
- f and f^{-1} are symmetric about $y = x$

5.3 How to Find Inverse Function

3 steps are required:

1. write $y = f(x)$
2. solve the equation for x in terms of y
3. interchange x and y, the resulting equation is $y = f^{-1}(x)$

5.4 Inverse Trigonometric Functions

There are 6 trigonometric functions, so there are 6 corresponding inverses. Here is a list:

- $y = \sin^{-1} x$, $D \in [-1, 1]$, $R \in [-\pi/2, \pi/2]$
- $y = \cos^{-1} x$, $D \in [-1, 1]$, $R \in [0, \pi]$
- $y = \tan^{-1} x$, $D \in \mathbb{R}$, $R \in [-\pi/2, \pi/2]$
- $y = \cot^{-1} x$, $D \in \mathbb{R}$, $R \in [0, \pi]$
- $y = \sec^{-1} x$, $D \in \{x||x| \geq 1\}$, $R \in [0, \pi/2) \cup [\pi, 3\pi/2)$
- $y = \csc^{-1} x$, $D \in \{x||x| \geq 1\}$, $R \in (0, \pi/2] \cup (\pi, 3\pi/2]$

6 Logarithmic Functions

Recall that logarithmic functions are of form $f(x) = \log_a x$ where a is positive constant. It has the following properties:

- Logarithmic function and exponential functions are inverse function to each other
- $\log_a x = y \Leftrightarrow a^y = x$
- $\log_a(a^x) = x, \forall x \in \mathbb{R}$
- $a^{\log_a x} = x, x \in (0, +\infty)$

6.1 Natural Logarithmic Function

When $a = e$, $\log_e x = \ln x$, this is known as natural logarithmic function.

6.2 Laws of Logarithms

- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a(x/y) = \log_a x - \log_a y$
- $\log_a(x^r) = r \log_a x$, where r is real number
- $\ln x = y \Leftrightarrow e^y = x$
- $\ln e = 1$
- $\log_a x = \frac{\ln x}{\ln a}$, this is **change of base formula**

7 Extra(Optional Contents)

7.1 Principle of Mathematical Induction

Let S_n be statement about the positive integer n, suppose that

1. S_1 is true
2. S_{k+1} is true whenever S_k is true

then S_n is true for all possible integers n.

7.2 More About Step Function

Recall that step function is defined as:

$$y = f(x) = \epsilon(t) = \begin{cases} 1, & x > 0 \\ 0.5, & x = 0 \\ 0, & x < 0. \end{cases}$$

Generally, it is defined as:

$$\int_{-\infty}^{\infty} \epsilon(x)\phi(x)dx = \int_{0}^{\infty} \phi(x)dx$$

Its Laplace transformation is

$$\mathcal{L}[\epsilon(t)] = \int_{0}^{+\infty} e^{-st}dt = \frac{1}{s}$$

Unit impulse equals to the derivative of step function:

$$\delta(t) = \frac{d\epsilon(t)}{dt}$$

which indicate that step function is the integration of unit impulse function:

$$\epsilon(t) = \int_{0}^{t} \delta(s)ds$$