State-Space Model vs. Input-Output Model

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Contents

1 Internal vs. External Models

- Transfer function models are external (or I/O) models
  - Describe only how the input affects the output
  - System viewed as a black box
- State-space models are internal models
  - Describe how the input affects not only the output, but also all the internal state variables
  - More complete models suitable for complicated system

Many state-space model may correspond to one transfer function.

2 IO Model

2.1 IO Models of CT LTI System

Consider a continuous-time LTI system with **zero initial state**, let $h(t)$, $t > 0$ be the system's **impulse response**, then, under any $u(t), t \geq 0$, system has the output

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau = h(t) * u(t), t \geq 0$$

Taking Laplace transform, we obtain the **transfer function model**:

$$\frac{Y(s)}{U(s)} = H(s) = \mathcal{L}[h(t)]$$

2.2 IO Models of DT LTI System

For a discrete-time LTI system with zero initial state, **transfer function model** can be obtained by taking z-transform:

$$\frac{Y(z)}{U(z)} = H(z)$$

recall that $ay[k-n]$ has z-transform $az^{-n}Y(z)$

2.3 IO Models from CT State-Space Models

A continuous LTI system with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and zero initial condition $x(0) = 0$. Its transfer function (or matrix) can be deduced by taking Laplace transformation:

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

which lead to

$$Y(s) = \underbrace{[C(sI-A)^{-1}B + D]}_{H(s)} U(s)$$

2.4 IO Models from DT State-Space Models

A discrete-time LTI system with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and zero initial condition $x[0] = 0$. Its transfer function (or matrix) can be deduced by taking z-transformation:

$$\begin{aligned} zX(z) &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z) \end{aligned}$$

which lead to

$$Y(z) = \underbrace{[C(sI - A)^{-1}B + D]}_{H(z)} U(z)$$

Example:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -2 & 1 \end{bmatrix} x \end{aligned}$$

has transfer function

$$H(s) = -\frac{2}{s+1} + \frac{1}{s+2}$$

3 State-Space Realization

- A continuous-time state-space model (A, B, C, D) is called a realization of the transfer function $H(s)$ if $C(sI - A)^{-1}B + D = H(s)$
- A discrete-time state-space model (A, B, C, D) is called a realization of the transfer function $H(z)$ if $C(zI - A)^{-1}B + D = H(z)$

There are **infinitely many** realizations of a transfer function.

3.1 Obtaining State-Space Realizations from IO Model

IO model of a single-input single-output (SISO) system can be written in difference equation form:

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{i=0}^{m} b_i u^{(i)}(t)$$

Transfer function can be directly obtained by Laplace transformation.

3.2 Controller Canonical Form

Should be able to draw block diagram and write system dynamics.

3.3 Controllability Canonical Form

Should be able to draw block diagram and write system dynamics.

3.4 Observer Canonical Form

Should be able to draw block diagram and write system dynamics.

3.5 Observability Canonical Form

Should be able to draw block diagram and write system dynamics.

3.6 Diagonal Realization

Suppose $H(s)$ has distinct poles:

$$\begin{aligned} H(s) &= \frac{b_1 s^2 + b_2 s + b_3}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} \\ &= \frac{\gamma_1}{s-\lambda_1} + \frac{\gamma_2}{s-\lambda_2} + \frac{\gamma_3}{s-\lambda_3} \end{aligned}$$

Diagonal realization will be:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$