Matrix Exponential

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Contents

1 Matrix Exponential

Power series converges for all $\lambda \in \mathbb{R}$ in scalar exponential function:

$$e^{\lambda} = 1 + \lambda + \frac{1}{2!}\lambda^2 + \frac{1}{3!}\lambda^3 + \cdots$$

For any matrix $A \in \mathbb{R}^{n\times n}$, **define** its matrix exponential:

$$e^{A} := I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \in \mathbb{R}^{n\times n}$$

and matrix power series **always converges**.

1.1 Computing Matrix Exponential Directly

- When A is nilpotent:

  $$A^n = \mathbf{0}, n > N \text{ where } N \text{ is certain number}$$

  therefore e^A only depends on the first N terms

- When A is idempotent: $A^2 = A$, there is an analytic solution:

  $$e^A = I + (e-1)A$$

- When A is of rank one, simply use Dyadic expansion:

  $$A = uv^T$$

1.2 Computing Matrix Exponential by Jordan Form

Using the Jordan Canonical Form:

$$A = T\begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}T^{-1} \Rightarrow e^A = T\begin{bmatrix} e^{J_1} & & \\ & \ddots & \\ & & e^{J_q} \end{bmatrix}T^{-1}$$

1.3 Computing Matrix Exponential by Matlab

Use Matlab command:

expm()

note there is an "m".

2 Properties of Matrix Exponential

For any $A \in \mathbb{R}^{n\times n}$

- $e^{\mathbf{0}} = I$
- invariant of eigenvalue: $Av = \lambda v \quad \Rightarrow \quad e^A v = e^{\lambda} v$
- $e^{A^T} = \left(e^A\right)^T$

- $e^{TAT^{-1}} = Te^{A}T^{-1}$ for nonsingular $T \in \mathbb{R}^{n\times n}$
- $\det\left(e^{A}\right) = e^{\operatorname{tr} A} = e^{\lambda_1+\lambda_2+\cdots}$
- If $A, B \in \mathbb{R}^{n\times n}$ commute, i.e., $AB = BA$, then
$$e^{A+B} = e^{A}e^{B} = e^{B}e^{A}$$
Tip: commute indicate that eigenvectors of both A and B are in the same direction
- $\left(e^{A}\right)^{-1} = e^{-A}$
- If A is skew symmetric $\left(A^{T} = -A\right)$, e^{A} is orthogonal: $\left(e^{A}\right)\left(e^{A}\right)^{T} = I$

3 Baker-Campbell-Hausdorff Formula

For $X, Y \in \mathbb{R}^{n\times n}$, we have
$$e^{X+Y} \neq e^{X} \cdot e^{Y}$$
unless X and Y commute. For any $X, Y \in \mathbb{R}^{n\times n}$, we can write
$$e^{X}e^{Y} = e^{Z}$$
for some $Z = \log\left(e^{X}e^{Y}\right) \in \mathbb{R}^{3\times 3}$ given by
$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] - \frac{1}{24}[Y,[X,[X,Y]]] - \cdots$$
where $[X, Y] := XY - YX$ is the **Lie bracket** of X and Y.

4 Matrix Exponential Representation of 3D Rotations

For $\omega = \begin{bmatrix}\omega_1 & \omega_2 & \omega_3\end{bmatrix}^{T} \in \mathbb{R}^{3}$, define a skew-symmetric matrix Ω:
$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$
then $\Omega_V = \omega \times v$ for $v \in \mathbb{R}^{3}$, where $\times$ denotes cross product of vectors. For any nonzero vector $\omega \in \mathbb{R}^{3}, e^{\Omega} \in \mathbb{R}^{3\times 3}$ is an orthogonal matrix that represents the rotation around the axis ω by the angle $||\omega||$. More precisely,
$$e^{\Omega} = I_3 + \frac{\sin(\|\omega\|)}{\|\omega\|}\Omega + \frac{1-\cos(\|\omega\|)}{\|\omega\|^{2}}\left(\omega\omega^{T} - \|\omega\|^{2} I_3\right)$$
Here is an example:
$$A = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} + \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$
therefore:
$$e^{A} = \underbrace{e^{\sigma}}_{\text{magnitude change}} \underbrace{\begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}}_{\text{pure rotation}}$$

5 Time Indexed Case

5.1 Time Indexed Matrix Exponential

The following power series converges for all $\lambda \in \mathbb{R}$ and all $t \in \mathbb{R}$:

$$f(\lambda) := e^{\lambda t} = 1 + t\lambda + \frac{1}{2!}t^2\lambda^2 + \frac{1}{3!}t^3\lambda^3 + \cdots$$

For any square matrix $A \in \mathbb{R}^{n\times n}$, define

$$e^{At} := I_n + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots$$

- The matrix power series converges for all $A \in \mathbb{R}^{n\times n}$ and all $t \in \mathbb{R}$
- For fixed A, e^{At} can be viewed as a matrix-valued function of time t

5.2 Time Derivative of Matrix Exponential

The scalar function $e^{\lambda t}$ as **a function of** $t \in \mathbb{R}$ has the derivative:

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$$

For fixed $A \in \mathbb{R}^{n\times n}, e^{At}$ as a matrix-valued function of $t \in \mathbb{R}$ satisfies

$$\frac{d}{dt}e^{At} = A \cdot e^{At} = e^{At} \cdot A$$

Tip: why this is commute?

5.3 Properties of Matrix Exponential with Time Index

For any $A \in \mathbb{R}^{n\times n}$ and any $t \in \mathbb{R}$:

- $Av = \lambda v \quad \Rightarrow \quad e^{At}v = e^{\lambda t}v$
- $e^{A^T t} = \left(e^{At}\right)^T$
- $\det\left(e^{At}\right) = e^{(\operatorname{tr} A)t}$
- If $A, B \in \mathbb{R}^{n\times n}$ commute, i.e., $AB = BA$, then

$$e^{(A+B)t} = e^{At}e^{Bt} = e^{Bt}e^{At}$$

- $e^{A(t_1+t_2)} = e^{At_1}e^{At_2} = e^{At_2}e^{At_1}, \forall t_1, t_2 \in \mathbb{R}$
- $\left(e^{At}\right)^{-1} = e^{-At}$
- If A is skew symmetric, then e^{At} is orthogonal for all t

5.4 Computing Time-Indexed Matrix Exponential

Three methods can be taken:

- use the **definition**:

$$e^{At} := I_n + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots$$

- use the **Jordan canonical form**:

$$A = T \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} T^{-1} \quad \Rightarrow \quad e^{At} = T \begin{bmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_q t} \end{bmatrix} T^{-1}$$

- the Laplace transform of e^{At} as a function of time t is

$$\mathcal{L}\left[e^{At}\right] = (sI - A)^{-1} \quad \Rightarrow \quad e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$

Here are two examples:

$$A_1 = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \quad e^{A_1 t} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}, (sI - A_2)^{-1} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)(s+2)} & \frac{1}{(s-1)(s+2)^2} \\ 0 & \frac{1}{s+2} & \frac{1}{(s+2)^2} \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix}$$