Systems and State Variables

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Contents

1 System

1.1 Examples of Systems

There are many systems around us, here are some examples:

- electrical system
- mechanical system
- transportation system
- biological system
- ecological system
- stock market

1.2 Structure of Studying System

- **modeling**: difference equation(direct way), transfer functions(undergraduate), state-space models(graduate), etc
- **analysis**(about the properties): solution, stability, controllability, observability, stabilizability, detectability, etc
- **design**: feedback control, optimal control, robust control, etc

1.3 System Models

System essentially is a signal processor:

$$y = \mathcal{T}(u)$$

- u: **input**
- y: **output**
- $\mathcal{T}$: IO **mapping**, could be described by **ODE**(ordinary), **PDE**(partial), **SDE**(stochastic) or difference equations.

1.4 Signals

Signals are talked more in DSP, here is a brief summary. Mathematically, signal is functions over time index $\mathcal{I}$:

- $u : \mathcal{I} \to \mathbb{R}^m$, a m-dimension input signal
- $y : \mathcal{I} \to \mathbb{R}^p$, a p-dimension output signal

Signals are be classified based on characteristics of $\mathcal{I}$:

- $\mathcal{I} = \mathbb{R}$ leads to continuous-time signal, denote as $u(t)$
- $\mathcal{I} = \mathbb{Z}$ leads to discrete-time signal, denoted as $u[k]$
- $\mathcal{I} = [0, +\infty)$ or $\mathcal{I} = \{0, 1, \ldots, \}$ leads to **causal** signal

Mathematically a mapping can be written in 2 ways:

- set mapping: $u : \mathcal{I} \to \mathbb{R}^m$
- set mapping with detailed info: $u : i \in \mathcal{I} \mapsto u(i) \in \mathbb{R}^m$

Admissible signals are allowed to put into systems, the admissible input set $\mathcal{U}$ must have the following properties for space state systems:

- it is a vector space
- closed to time right shift operation(delay)

For transfer function approach, the system must be

- casual
- exponentially bounded

1.5 Classification of Systems

This part is explained in details in DSP class, here is a quick summary:

- **discrete** vs **continuous**-time system:
  - continuous if both IO are continuous signal
  - discrete if both IO are discrete signal
  - **hybrid** if IO have both types of signals
- **linear** vs **nonlinear** system:
  - linear if **superposition principle** is satisfied

    $$\mathcal{T}(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 \mathcal{T}(u_1) + \lambda_2 \mathcal{T}(u_2)$$

  - nonlinear otherwise
- **discrete** vs **continuous**-time system:
  - continuous if both IO are continuous signal
  - discrete if both IO are discrete signal
  - **hybrid** if IO have both types of signals
- **time varying** vs **time invariant** system:
  - invariant: if $\forall u \in \mathcal{U}$ and $\tau \in \mathcal{I}$:

    $$y(\cdot) = \mathcal{T}(u(\cdot)) \Longrightarrow y(\cdot - \tau) \quad = \quad \mathcal{T}(u(\cdot - \tau))$$

  - varying: otherwise
- **causal** vs **non-causal** system:
  - causal: if output only depends on past input
  - non-causal: otherwise

- **lump** vs **distributed** system:
  - lump: if the system has finite number of state variables
  - distributed: otherwise, namely infinite number state variable

  Usually, lumped system are typically modeled by ODE, while distributed systems arise due to PDE or the presence of delay(require infinite memory).

2 State Variables

The **state variables** of a system is a set of internal variables whose values at any moment t_0 together with future input $u(t)$, $t > t_0$, are sufficient to determine the system output $y(t)$, $t > t_0$.

- it summarize the past input history
- also called initial condition in mathematical perspective

2.1 Implication of System with State Variables

A new IO relation can be obtained by introducing state variables:

$$y(t)|_{t\geq t_0} = \mathcal{T}\left(u(t)|_{t\geq t_0}, x(t_0)\right)$$

2.2 Decomposition of Response

The response of a linear system can be decomposed as:

$$\begin{aligned} y(n) &= \mathcal{T}\left(u(t)|_{t\geq t_0}, x(t_0)\right) \\ &= \mathcal{T}\left(u(t)|_{t\geq t_0}, 0\right) + \mathcal{T}\left(0, x(t_0)\right) \end{aligned}$$

- the first term is called **zero-state** response
- the second term is called **zero-input** response

2.3 General State-Space Model of Lumped System

Suppose the system has state variable $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and output $\mathbb{R}^p$:

- a continuous system:

$$\begin{cases} \frac{dx(t)}{dt} = f(x(t), u(t), t) \\ y(t) = g(x(t), u(t), t) \end{cases}$$

- a discrete system:

$$\begin{cases} x[k+1] = f(x[k], u[k], k) \\ y(k) = g(x[k], u[k], k) \end{cases}$$

2.4 State-Space Model of Linear Lumped System

Suppose the system has state variable $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and output $\mathbb{R}^p$:

- a continuous system:

$$\begin{cases} \frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases}$$

- a discrete system:

$$\begin{cases} x[k+1] = A[k]x[k] + B[k]u[k] \\ y(k) = C[k]x[k] + D[k]u[k]) \end{cases}$$

Note that the general f and g now is a function linear to x and u.

2.5 State-Space Model of Lumped Linear Time-Invariant(LTI) System

Suppose the system has state variable $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and output $\mathbb{R}^p$:

- a continuous system:

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

- a discrete system:

$$\begin{cases} x[k+1] = Ax[k] + Bu[k] \\ y(k) = Cx[k] + Du[k]) \end{cases}$$

Note that the (A, B, C, D) are constants. Graphically, the system can be directly realized by the following diagram:

- continuous:
- discrete:

2.6 Two Important Blocks

Integration for CT system and unit delay for DT system are two most important block units:

- integration for CT:
- unit delay for DT:

3 Examples

3.1 Differential Equation

3.2 RLC Circuit

3.3 Mechanical System

4 Extension

4.1 General Linear Mechanical System

A mechanical system with n DOF (degree of freedom)can be written as:

$$M\ddot{q} + D\dot{q} + Kq = F$$

- q: general displacement vector
- M: mass
- D: damping
- K: stiffness
- F: external force

choose x to be:

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

the state equation can be deduced as:

$$\dot{x} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1}???? \end{bmatrix}$$

Proof: ?????