Sequences and Series

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Contents

1 Sequence

A sequence is a list of numbers written in a definite order:

$$a_1, a_2, \cdots, a_n$$

- n could be either **finite** or **infinite**
- a_i is the i-th **term** of the sequence
- can be defined as a function whose domain is $\mathbb{Z}^+$
- can be written as $\{a_n\}$ or $\{a\}_{n=1}^{+\infty}$

1.1 Limit and Convergence of Sequences

A sequence $\{a_n\}$ has the **limit** L, and it is denoted as

$$\lim_{n \to \infty} a_n = L$$

if we can make a_n as close to L as taking n sufficiently large. If L exists, the sequence is said to be **convergent**, otherwise **divergent**.

1.2 Precise Definition of Limit of Sequence

A sequence $\{a_n\}$ has the **limit** L, and it is denoted as

$$\lim_{n \to \infty} a_n = L$$

if for every $\epsilon > 0$, there is a corresponding integer N such that if $n > N$

$$|a_n - L| < \epsilon$$

And

$$\lim_{n \to \infty} a_n = \infty$$

indicates that for every positive number M, there is an integer N such that if $n > N$, then $a_n > M$

1.3 Limit of a Sequence by Function

If

$$\lim_{x \to \infty} f(x) = L$$

and

$$f(n) = a_n, \forall n \in \mathbb{Z}^+$$

then

$$\lim_{n \to \infty} a_n = L$$

1.4 Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent and c is a constant:

- $\lim c = c$

- $\lim(a \pm b) = \lim a \pm \lim b$

- $\lim ca = c \lim a$

- $\lim(ab) = \lim a \lim b$

- $\lim(a/b) = \lim a / \lim b$

- $\lim a^p = (\lim a)^p$

1.5 Squeeze Theorem

If $a_n \le b_n \le c_n$, for $n > n_0$ and

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$$

then

$$\lim_{n \to \infty} b_n = L$$

1.6 Absolute Convergence

If

$$\lim_{n \to \infty} |a_n| = 0$$

then

$$\lim_{n \to \infty} a_n = 0$$

1.7 Convergence of a Sequence Applied to Function

If

$$\lim_{n \to \infty} a_n = L$$

and a function f is continuous at L, then

$$\lim_{n \to \infty} f(a_n) = f(L)$$

1.8 Convergence of Exponential Sequence(DT Signal)

The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent otherwise.

1.9 Monotonic Sequence

A sequence is called **increasing** if $a_n < a_{n+1}$ for all $n \ge 1$, and is called **decreasing** if $a_n > a_{n+1}$ for all $n \ge 1$. A sequence is **monotonic** if it is either increasing or decreasing.

1.10 Bounded Sequence

A sequence $\{a_n\}$ is **bounded above** if there exists a number M such that

$$a_n \le M, \forall n \ge 1$$

It is **bounded below** if there exists a number N such that

$$a_n \ge N, \forall n \ge 1$$

If a sequence is bounded above and below, it is a **bounded sequence**.

1.11 Monotonic Sequence Theorem

Every bounded and monotonic sequence is convergent.

2 Series

A series is the sum(may be partial sum) of a sequence(may be either finite or infinite), it is denoted as:

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$

2.1 Partial Sum and Convergence

Given a series $\sum a_n$ and let s_n denote its n-th **partial sum**:

$$s_n = \sum_{i=1}^{n} a_i$$

If the sequence $\{s_n\}$ is convergent, and

$$\lim_{n \to \infty} s_n = s$$

exists, then the series is called **convergent** and

$$\sum_{n=1}^{\infty} a_n = s$$

the number s is called the **sum** of the series. If sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

2.2 Geometric Series

The geometric series is defined as:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

where r is called the **common ratio**. Partial sum s_n can be directly evaluated:

$$\sum_{n=1}^{\infty} ar^{n-1}$$

The series is convergent if $|r| < 1$ and its value is:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, |r| < 1$$

If $|r| > 1$, the geometric sequence is divergent.

2.3 Harmonic Series

Harmonic series is defined as:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

it is divergent. To prove it, just need to prove that:

$$s_{2^n} > 1 + \frac{n}{2}$$

2.4 Properties of Convergence for Series

- if a series $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$, and $\lim_{n\to\infty} a_n = 0$ cannot lead to the convergence of $\sum a_n$

- if $\lim_{n\to\infty} a_n$ doesn't exist or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ is divergent

- finite number of terms doesn't affect the convergence or divergence of a series, because sum goes way up to infinity number of terms

2.5 Combination of Series

If $\sum a_n$ and $\sum b_n$ are convergent, so are $\sum ca_n(c$ is a constant), and $\sum(a_n \pm b_n)$ and:

- $\sum ca_n = c \sum a_n$
- $\sum(a_n \pm b_n) = \sum a_n + \sum b_n$

2.6 Rearrangement

If $\sum a_n$ is a conditional convergent series, its rearrangement has the same value.(proved by Riemann)

3 Integral Test and Estimations of Sum

Generally, it's difficult to find the exact sum of series, methods will be taken to determine if a series is convergent or divergent without finding the value.

3.1 Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $f(n) = a_n$, then the series $\sum a_n$ is convergent if and only if the improper integration

$$\int_1^\infty f(x)dx$$

is convergent. Since finite number of terms doesn't affect the convergence, therefore the lower bound doesn't have to start from 1.

3.2 p-Series

p-series is defined as:

$$\sum_{n=1}^\infty \frac{1}{n^p}$$

it is:

- convergent if $p > 1$

- divergent if $p \le 1$(Note that $p = 1$ lead to harmonic series)

3.3 Reminder Estimation(VIP for Numerical Analysis)

Reminder R_n is defined as:

$$R_n = s - s_n = \sum_{i=n+1}^\infty a_i$$

Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \ge n$ and $\sum a_n$ is convergent, then:

$$\int_{n+1}^\infty f(x)dx \le R_n \le \int_n^\infty f(x)dx$$

by adding s_n to the inequality, we can estimate s as:

$$s_n + \int_{n+1}^\infty f(x)dx \le s \le s_n \int_n^\infty f(x)dx$$

Although the estimation won't give the accurate result, usually, the approximation is good enough. The proof is quiet straight forward.

4 Comparison

Compare a given series with a series that is known to be convergent or divergent.

4.1 Comparison Tests

Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms and N is some fixed integer:

- if $\sum b_n$ is convergent and $a_n \leq b_n$, for all $n > N$, then $\sum a_n$ is also convergent

- if $\sum b_n$ is divergent and $a_n \geq b_n$, for all $n > N$, then $\sum a_n$ is also divergent

usually, p-series and geometric series are used to be compared.

4.2 Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms, if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

4.3 Estimate Sum by Comparison

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms, and $a_n \leq b_n$, then the reminder of a_n must be less than b_n. Although this won't give the analytic solution, but a more accurate reminder can be estimated if the reminder of b_n is known.

5 Alternating Series

An alternating series is a series whose terms are alternately positive and negative.

5.1 Alternating Series Test

If the alternating series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 + \cdots \qquad b_n > 0$$

satisfy that:

- $b_{n+1} \leq b_n$, for all n

- $\lim_{n \to \infty} b_n = 0$

then the series is convergent. Note that harmonic series is divergent, but alternating harmonic series is convergent(converge to $\ln 2 \approx 0.693$). The proof is in extra part.

5.2 Alternating Series Estimation Theorem

If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that is converge, then:

$$R_n = |s - s_n| \leq b_{n+1}$$

Proof:

$$R_n = |s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}$$

Again, this theorem is for accuracy control.

6 Absolute vs Conditional Convergence

- A series $\sum a_n$ is called **absolute convergent** if the series $\sum |a_n|$ is convergent

- A series $\sum a_n$ is called **conditional convergent** if it is convergent but not absolute convergent

Note that absolute convergent lead to convergent but the reverse is not necessarily true.

7 Ration and Root Test

These are two useful test to determine the convergence.

7.1 Ratio Test

If

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

and if

- $L < 1$, then $\sum a_n$ is absolute convergent, therefore convergent

- $L > 1$, then $\sum a_n$ is divergent

- $L = 1$, ratio test is inconclusive

7.2 Root Test

If

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = L$$

and if

- $L < 1$, then $\sum a_n$ is absolute convergent, therefore convergent

- $L > 1$, then $\sum a_n$ is divergent

- $L = 1$, root test is inconclusive

It's useful for power series.

8   Strategy for Testing Series

1. check if it is p-series

2. check if it is geometric series

3. check if it is p or geometric like series, if so use comparison test

4. check $\lim_{n \to \infty} a_n \neq 0$ for divergence

5. check if it is alternating series

6. check if factorial, product or power is involved, if so, use ratio test

7. check if a_n is in form of $(b_n)^n$, if so, use root test

8. check if integration can be applied or not

9   Power Series

9.1   Definition

A power series is of form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

where x is **variable** and c_i are **coefficients**. In fact it is a function:

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

whose domain is the set of all x for which the series **converges**. More generally:

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots$$

it is called a power series in $(x - a)$ or a **power series centered at** a or **a power series about** a.

9.2   Convergence of Power Series

For given power series:

$$\sum_{n=0}^{\infty} c_n (x - a)^n$$

there are only 3 possibilities:

- converges only when $x = a(R = 0)$, e.g. $\sum n! x^n$

- converges for all $x(R = \infty)$, e.g.

$$\sum \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \qquad \text{known as Bessel function}$$

- converges for if $|x - a| < R$ and diverge if $|x - a| > R$, and case $|x - a| = R$ should be specifically determined(namely, anything could happen), e.g. $\sum x^n$, whose interval of convergence is $(-1, 1)$

R is called the **radius of convergence**. Usually, **ratio test** is used to determine the value of R.

10 Function as Power Series and Taylor Expansion

Functions can be represents as power series.

10.1 An Important Case: $1/(1 - x)$

Function $1/(1 - x)$ can be directly written as the sum of power, namely

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \quad |x| < 1$$

Any function that can be written in this form can directly be transformed into sum of power with **known** radius of convergence $R = 1$(very helpful).

10.2 Differentiation and Integration of Power Series

If power series

$$\sum_{n=0}^{\infty} c_n(x - a)^n$$

has radius of convergence $R > 0$, then its corresponding function

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n$$

has

- derivative:
$$f'(x) = \sum_{n=0}^{\infty} nc_n(x - a)^{n-1}$$
with ROC(radius of convergence) R

- integration:
$$\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1} + C$$
with ROC R

Note that:

- it is term-by-term integration and differentiation, easy for calculation

- though differentiated and integrated function still have ROC to be R, it **doesn't mean the interval of convergence remains the same**

- basis for a powerful method for solving differential equation

Here are some useful facts(quiet tricky):

- $1/(1-x)^2$ should be considered as differentiation of $1/(1-x)$ to use power series

- $\ln(1+x)$ should be considered as integration of $\frac{1}{1-(-x)}$ to use power series

10.3 Taylor and Maclaurin Series

If function f has a power series representation at $x = a$, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then coefficients are given by:

$$c_n = \frac{f^{(n)}(a)}{n!} \quad \text{recall that: } f^{(n)}(a) = \frac{d^n}{dx^n} f(x)|_{x=a}$$

therefore

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

this is called the **Taylor series** of the function f at a. For special case $a = 0$, it becomes:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

this occurs a lot and it has a special name **Maclaurin series**.

- if f can be represented as a power series about a, then f may **not** equal to the sum of its Taylor series for finite number of x. One example is that f doesn't have **derivative of all orders**. More will be discussed in equality constraints

- the ROC is usually found by ratio test

10.4 Decompose Taylor Series to Partial Sum and Reminder

The Taylor series can be decomposed as:

$$f(x) = \sum_{i=0}^{\infty} c_i(x-a)^i = \sum_{i=0}^{n} c_i(x-a)^i + \sum_{i=n}^{\infty} c_i(x-a)^i$$

- the first term is called n-th **degree Taylor polynomial** of f at a, denoted by:

$$T_n(x) = \sum_{i=0}^{n} c_i(x-a)^i$$

clearly, in general:

$$f(x) = \lim_{n \to \infty} T_n(x)$$

- the second term is called **reminder** of Taylor series, clearly:

$$R_n = \sum_{i=n}^{\infty} c_i (x - a)^i$$

10.5 Equality Constraint

If $f(x) = T_n(x) + R_n(x)$ and

$$\lim_{n \to \infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval. Do **not** take this limit for granted.

10.6 Taylor's Inequality

If $|f^{(n)}|(x) \leq M$ for $|x - a| \leq d$, then the reminder $R_n(x)$ of the Taylor series satisfy the inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1}$$

for $|x - a| \leq d$. One useful limit

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0$$

is often applied with this inequality to prove function meets the requirement of equality constraints mentioned above.

10.7 Important Taylor Expansion List

Here is a list of VIP expansions, it can help us to find integration and differentiation easily :

- $1/(1 - x)$:
$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \quad |x| < 1, R = 1$$

- e^x (this can help us to evaluate e by setting $x = 1$):
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad x \in \mathbb{R}, R = \infty$$

- $\sin x$:
$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad x \in \mathbb{R}, R = \infty$$

- $\cos x$:
$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad x \in \mathbb{R}, R = \infty$$

- $(1+x)^k$, known as **binomial series**:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad |x| < 1, R = 1$$

  where

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$

- $\tan^{-1} x$:
$$\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \quad |x| < 1, R = 1$$

- $\ln(1+x)$:
$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n} \quad |x| < 1, R = 1$$

- Two approximation by polynomial multiplication and division:

$$e^x \sin x \approx x + x^2 + \frac{1}{3}x^3 + \cdots$$

  and

$$\tan x \approx x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

Though all equations look quite easy, it requires lots of experience to use all these well. Power series also can be used to approximate function and by incorporating with Taylor's inequality to find error bound.

11 Extra

11.1 Fibonacci Sequence

It is defined recursively by the conditions:

$$a_1 = 1 \qquad a_2 = 1 \qquad a_n = a_{n-1} + a_{n-2}, n \geq 3$$

11.2 Convert a Rational Number into Fraction

Any rational number can be converted into its fraction form by geometric series method, here is one example:

$$2.3171717171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

$$= 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}}$$

$$= \frac{1147}{495}$$

Note that it's geometric series with $r = 1/10^2 < 1$, it is convergent.

11.3 Proof of Alternating Series Test

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11.4 Proof of Taylor's Inequality

11.5 Application in Einstein' s Theory of Special Relativity

11.6 Application in Optics