Lecture 4: Matrix Exponential

Baboo J. Cui, Yangang Cao

July 9, 2019

Contents

1 Matrix Exponential **2**

2 Computing Matrix Exponential Directly **2**

3 Computing Matrix Exponential: Method II **2**

4 Computing Matrix Exponential: Other Methods **2**

5 Properties of Matrix Exponential **2**

6 Baker-Campbell-Hausdorff Formula **3**

7 Matrix Exponential Representation of 3D Rotations **3**

8 Example: **4**

9 Time Indexed Matrix Exponential **4**

10 Time Derivative of Matrix Exponential **4**

11 Other Properties of Matrix Exponential **4**

12 Computing Time-Indexed Matrix Exponential **5**

13 Example **5**

1 Matrix Exponential

Power series converges for all $\lambda \in \mathbb{R}$:

$$e^{\lambda} = 1 + \lambda + \frac{1}{2!}\lambda^2 + \frac{1}{3!}\lambda^3 + \cdots$$

For any matrix $A \in \mathbb{R}^{n\times n}$, define its matrix exponential:

$$e^{A} := I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \in \mathbb{R}^{n\times n}$$

Matrix power series always converges.

2 Computing Matrix Exponential Directly

- When A is nilpotent.
- When A is idempotent, i.e., $A^2 = A$
- When A is of rank one

3 Computing Matrix Exponential: Method II

Using the Jordan Canonical Form:

$$A = T \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} T^{-1} \Rightarrow e^{A} = T \begin{bmatrix} e^{J_1} & & \\ & \ddots & \\ & & e^{J_q} \end{bmatrix} T^{-1}$$

4 Computing Matrix Exponential: Other Methods

- "Nineteen dubious ways to compute the exponential of a matrix," C. Moler and C. F. Van Loan, SIAM Review, 20(4): 801-836,1978.
- "Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later," C. Moler and C. F. Van Loan, SIAM Review, 45(1): 3-49,2003.
  Availab at

$$www.cs.cornell.edu/cv/researchpdf/19ways+.pdf$$

- Matlab command **expm**.

5 Properties of Matrix Exponential

For any $A \in \mathbb{R}^{n\times n}$

- $e^0 = I$

- $Av = \lambda v \Rightarrow e^A v = e^\lambda v$
- $e^{A^T} = \left(e^A\right)^T$
- $e^{TAT^{-1}} = Te^A T^{-1}$ for nonsingular $T \in \mathbb{R}^{n\times n}$
- $\det\left(e^A\right) = e^{\operatorname{tr} A}$
- If $A, B \in \mathbb{R}^{n\times n}$ commute, i.e., $AB = BA$, then

$$e^{A+B} = e^A e^B = e^B e^A$$

- $\left(e^A\right)^{-1} = e^{-A}$
- If A is skew symmetric $\left(A^T = -A\right)$, e^A is orthegonal: $\left(e^A\right)\left(e^A\right)^T = 1$

6 Baker-Campbell-Hausdorff Formula

For $X, Y \in \mathbb{R}^{n\times n}$, we have $e^{X+Y} \neq e^X \cdot e^Y$ unless X and commute

Proposition (Baker-Campbell-Hausdorff Formula):
For any $X, Y \in \mathbb{R}^{n\times n}$, we can write

$$e^X e^Y = e^Z$$

for some $Z = \log\left(e^X e^Y\right) \in \mathbb{R}^{3\times 3}$ given by

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] - \frac{1}{24}[Y,[X,[X,Y]]] - \cdots$$

where $[X,Y] := XY - YX$ is the Lie bracket of X and Y.

7 Matrix Exponential Representation of 3D Rotations

For $\omega = [\omega_1 \quad \omega_2 \quad \omega_3]^T \in \mathbb{R}^3$, define a skew-symmetric matrix Ω:

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

Then $\Omega_V = \omega \times v$ for $v \in \mathbb{R}^3$, where $\times$ denotes crooduct of vectors.

Proposition:
For any nonzero vector $\omega \in \mathbb{R}^3, e^\Omega \in \mathbb{R}^{3\times 3}$ is an orthogonal matrix that represents the rotation around the axis ω by the angle $||\omega||$. More precisely,

$$e^\Omega = I_3 + \frac{\sin(\|\omega\|)}{\|\omega\|}\Omega + \frac{1-\cos(\|\omega\|)}{\|\omega\|^2}\left(\omega\omega^T - \|\omega\|^2 I_3\right)$$

- See “Finite Dimensional Linear Systems” by Roger Brockett

8 Example:

Example: $A = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix}$

9 Time Indexed Matrix Exponential

The following power series converges for all $\lambda \in \mathbb{R}$ and all $t \in \mathbb{R}$:

$$f(\lambda) := e^{\lambda t} = 1 + t\lambda + \frac{1}{2!}t^2\lambda^2 + \frac{1}{3!}t^3\lambda^3 + \cdots$$

For any square matrix $A \in \mathbb{R}^{n\times n}$, define

$$e^{At} := I_n + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots$$

- The matrix power series converges for all $A \in \mathbb{R}^{n\times n}$ and all $t \in \mathbb{R}$
- For fixed A, e^{At} can be viewed as a matrix-valued function of time t

10 Time Derivative of Matrix Exponential

The scalar function $e^{\lambda t}$ as a function of $t \in \mathbb{R}$ has the derivative:

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$$

Proposition:
For fixed $A \in \mathbb{R}^{n\times n}, e^{At}$ as a matrix-valued function of $t \in \mathbb{R}$ satisfies

$$\frac{d}{dt}e^{At} = A \cdot e^{At} = e^{At} \cdot A$$

11 Other Properties of Matrix Exponential

For any $A \in \mathbb{R}^{n\times n}$ and any $t \in \mathbb{R}$:

- $Av = \lambda v \Rightarrow e^{At}v = e^{\lambda t}v$
- $e^{A^Tt} = \left(e^{At}\right)^T$
- $\det\left(e^{At}\right) = e^{(\operatorname{tr} A)t}$
- If $A, B \in \mathbb{R}^{n\times n}$ commute, i.e., $AB = BA$, then

  $$e^{(A+B)t} = e^{At}e^{Bt} = e^{Bt}e^{At}$$

- $e^{A(t_1+t_2)} = e^{At_1}e^{At_2} = e^{At_2}e^{At_1}, \forall t_1, t_2 \in \mathbb{R}$
- $\left(e^{At}\right)^{-1} = e^{-At}$
- If A is skew symmetric, then e^{At} is orthogonal for all t

12 Computing Time-Indexed Matrix Exponential

Method I: use the definition:

$$e^{At} := I_n + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots$$

Method II: use the Jordan canonical form:

$$A = T \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} T^{-1} \Rightarrow e^{At} = T \begin{bmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_q t} \end{bmatrix} T^{-1}$$

Method III: use the following result

Proposition:
The Laplace transform of e^{At} as a function of time t is

$$\mathcal{L}\left[e^{At}\right] = (sI - A)^{-1} \Rightarrow e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$

13 Example

$$A_1 = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \quad e^{A_1 t} =$$

$$A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}, (sI - A_2)^{-1} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)(s+2)} & \frac{1}{(s-1)(s+2)^2} \\ 0 & \frac{1}{s+2} & \frac{1}{(s+2)^2} \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix}$$