Geometry Chern-Weil theory & kähler geometry

2018.09.26 ~ 2018.12.21

$$\nabla : C^\infty(M, E) \to C^\infty(M, E \otimes T^*M)$$

$$\nabla(fs) = f\nabla s + s \otimes df \qquad \text{for } f \in C^\infty(M)$$

$$C^\infty(M, E \otimes T^*M) \otimes C^\infty(M, TM) \xrightarrow{\tau} C^\infty(M, E)$$

covariant derivative of s in the direction of X $\quad \nabla_X s := \tau(\nabla s, X)$

the rank 1 trivial bundle $C^\infty(M, M \times \mathbb{R})$

the connection is the ordinary differential

for $f \in C^\infty(M)$ $\quad X \in \Gamma(TM)$

$$\nabla_X f = (df, X) = \frac{\partial f}{\partial x^i} X_i = Xf$$

base change map $f_{\lambda\mu} : U_\lambda \cap U_\mu \to GL(r, \mathbb{R})$

$$(e_{\mu 1}, \cdots, e_{\mu r}) = (e_{\lambda 1}, \cdots, e_{\lambda r}) f_{\lambda\mu}$$

def: $\nabla e_{\lambda j} = \sum_{i=1}^{r} e_{\lambda i} \otimes \omega_{\lambda j}^{i}$ $\qquad \omega_\lambda = (\omega_{\lambda j}^{i})$ connection form (not a tensor ?)

in $E \otimes T^*M$

$$\omega_\mu = f_{\lambda\mu}^{-1} \omega_\lambda f_{\lambda\mu} + f_{\lambda\mu}^{-1} df_{\lambda\mu}$$

conversely, a set of ω_μ with such relations $\Rightarrow$ a connection local frame $\qquad\qquad$ a cover

$\pi: E \to M$ $\quad \nabla$ a connection $\quad \underline{e_\lambda = (e_{\lambda 1}, \cdots, e_{\lambda r})}$ on $\underline{U_\lambda, \lambda \in \Lambda}$

Def. $\nabla e_\lambda = e_\lambda \otimes \omega_\lambda = (e_{\lambda 1}, \cdots e_{\lambda r}) \otimes (\omega_\lambda{}^i_j)$ $\qquad \omega_\lambda{}^i_j \in \Omega^1(U_\lambda)$

ω_λ : connection form

Prop. on $U_\lambda \cap U_\mu (\neq \phi)$ $\qquad \omega_\mu = f_{\lambda\mu}^{-1} \omega_\lambda f_{\lambda\mu} + f_{\lambda\mu}^{-1} d f_{\lambda\mu}$ $\qquad \underline{e_\mu = e_\lambda f_{\lambda\mu}}$

conversly, given $\{\omega_\lambda\}_{\lambda \in \Lambda}$ satisfying the transform law, we obtain a connection on E

$$e_\mu \omega_\mu = \nabla_{(\omega)} e_\mu = \nabla_{(\omega)} (e_\lambda f_{\lambda\mu}) = \nabla e_\lambda f_{\lambda\mu} + e_\lambda \otimes d f_{\lambda\mu} = e_\lambda \otimes (\omega_\lambda f_{\lambda\mu} + d f_\lambda)$$

$$\overset{\shortparallel}{e_\lambda \otimes f_{\lambda\mu} \omega_\mu} \qquad \Rightarrow f_{\lambda\mu} \omega_\mu = \omega_\lambda f_{\lambda\mu} + d f_{\lambda\mu}$$

Def. on U_α, $\Omega_\alpha = d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha$ curvature 2-form / curvature matrix

Prop. on $U_\alpha \cap U_\mu \neq \phi$ $\Omega_\mu = f_{\alpha\mu}^{-1} \Omega_\alpha f_{\alpha\mu}$ (hence a tensor)

$dP^{-1} = -P^{-1} dP \, P^{-1}$ (on Dong's class!)

$\Omega_\mu = d\omega_\mu + \omega_\mu \wedge \omega_\mu = d(f_{\alpha\mu}^{-1} \omega_\alpha f_{\alpha\mu} + f_{\alpha\mu}^{-1} df_{\alpha\mu}) + (f_{\alpha\mu}^{-1}\omega_\alpha f_{\alpha\mu} + f_{\alpha\mu}^{-1} df_{\alpha\mu}) \wedge$
(- - -)

(steps are omitted, see the photos. Dong had computed this before)

so $\{\Omega_\alpha\}_\alpha$ gives a same endmorphism of E

$\Omega \in C^\infty(M, \mathrm{End}(E) \otimes \Lambda^2 TM)$

$\mathrm{End}(E) = E \otimes E^*$

$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$

Prop. $e\Omega(X,Y) = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})e$

proof. $\Omega(X,Y) = (d\omega + \omega\wedge\omega)(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) + \omega(X)\omega(Y) - \omega(Y)\omega(X)$

right side: $\nabla_X(e\omega(Y)) - \nabla_Y(e\omega(X)) - e\omega([X,Y]) = e\omega(X)\omega(Y) + eX(\omega(Y)) - e\omega(Y)\omega(X)$

$\nabla_Y e = (\nabla e)(Y) = (e\omega)(Y)$ $- eY(\omega(X)) - e\omega([X,Y]) = e\Omega(X,Y)$

if we consider $\nabla e_\alpha = \omega_\alpha e_\alpha$ i.e. connection act on the left

then we get $(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})e$
$\shortparallel$
$(d\omega - \omega\wedge\omega)(X,Y)e$
from similar computation

if we take local coordinate $(x^1, \cdots, x^n)$ on U, $X = \frac{\partial}{\partial x^i}$ $Y = \frac{\partial}{\partial x^j}$ $i \neq j$, then $[X,Y] = 0$

in particular $\Omega(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X$

thus, "the curvature measures the extent in which the covariant derivative commutes"

Exercise. X, Y v.f. on M, $s \in \Gamma(E)$ $f, g, h \in C^\infty(M)$

show: $\Omega(fX, gY)(hs) = fgh\,\Omega(X,Y)s$

fundamental fact in manifold theory

$T: C^\infty(TM) \times \cdots \times C^\infty(TM) \to C^\infty(M)$

$T(f_1 X_1, \cdots, f_n X_n) = f_1 \cdots f_n T(X_1, \cdots, X_n)$ $\Leftrightarrow$ $T \in C^\infty(TM \otimes \cdots \otimes TM)$

Def. $\mathrm{Aut}(E) = \{\varphi \in \mathrm{End}(E) = E\otimes E^* \mid \det\varphi \neq 0\}$ automorphism bundle

$C^\infty(M, \mathrm{Aut}(E))$ gauge grp

$\shortparallel$

$\mathcal{G}(E) \ni \alpha$ gauge transformation

Exercise. (1) $\mathcal{G}(E)$ grp (composition)

(2) $\mathcal{G}(E)$ acts on $C^\infty(M, E)$

(3) $\mathcal{C}(E)$ = space of connections on E

(eg. on TM, a Riemannian metric on M determines a .... connection on TM)

$\mathcal{G}(E)$ acts on $\mathcal{C}(E)$: $(\alpha, \nabla) \mapsto \alpha\nabla\alpha^{-1}$ send section to section make sense

$\left(YM: \mathcal{C}(E) \to \mathbb{R}_{\geq 0} \qquad YM(\nabla) = \int_{(M,g)} \|\Omega\|^2 \, dV_g \right)$

(a) suppose given ∇ on E, then we have a connection on E^*

$\langle \nabla\alpha, s\rangle = d(\langle\alpha, s\rangle) - \langle\alpha, \nabla s\rangle$

$\alpha \in C^\infty(M, E^*)$ show this is a connection

$\nabla\alpha \in C^\infty(M, E^*\otimes T^*M)$

$\langle\nabla\alpha, \cdot\rangle \mapsto \nabla\alpha(\cdot)$

$C^\infty(M, E) \to C^\infty(M, T^*M)$

by def $\Lambda^1 M$!!

(b) E, F two v. bundle with ∇_E, ∇_F resp.

then we have a connection on $E\otimes F$

$\nabla(s\otimes t) = \nabla s\otimes t + s\otimes\nabla t$

(c) applying (a) & (b) to $E\otimes E^* = \mathrm{End}(E)$ $\Rightarrow$ connection on $\mathrm{End}(E)$

Prop. $\alpha \in \mathcal{G}(E)$ $\nabla \in \mathcal{C}(E)$ the connection form of $\alpha(\nabla)$ is given by

$\omega + \alpha(\nabla\alpha^{-1}) = \omega - \nabla\alpha\cdot\alpha^{-1}$? $\to$ (write in this form shows his individual understanding) ω is the same as ∇

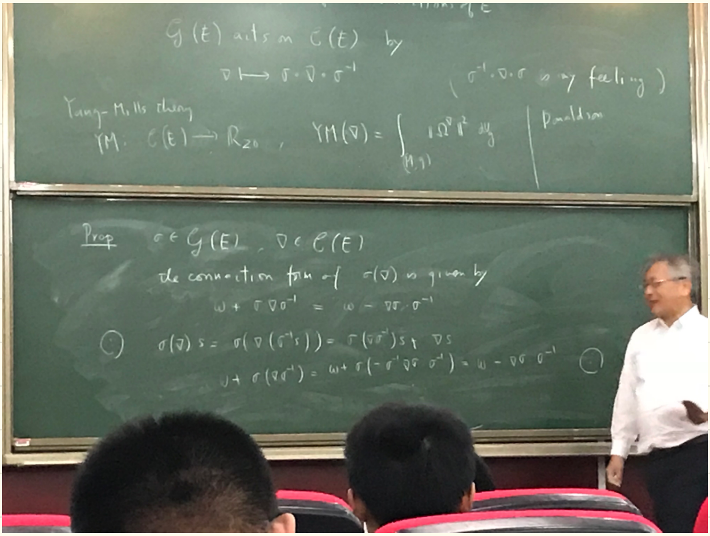
$\alpha(\nabla)s = \alpha(\nabla(\alpha^{-1}s)) = \alpha(\nabla\alpha^{-1})s + \nabla s$

a connection on $\mathrm{End}(E)$

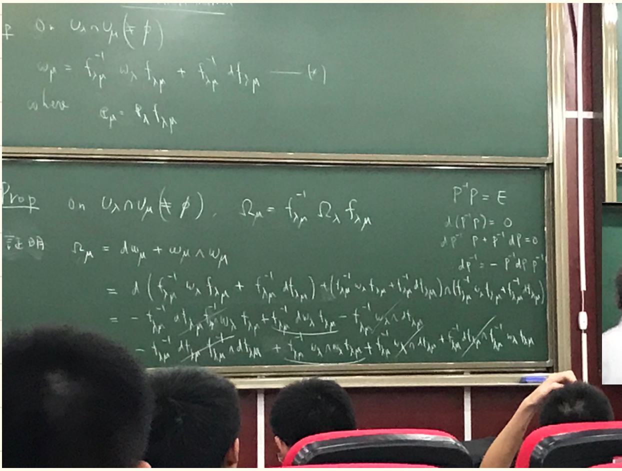
$\omega + \alpha(\nabla\alpha^{-1}) = \omega + \alpha(-\alpha^{-1}\nabla\alpha\,\alpha^{-1}) = \omega - \nabla\alpha\cdot\alpha^{-1}$

section of $E\otimes E^*$

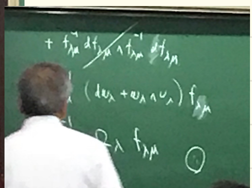
$\rightsquigarrow$ connection on $E\otimes E^*$



G(E) acts on C(E) by
∇ ⟼ σ·∇·σ⁻¹
(σ⁻¹·∇·σ is my feeling)
Yang-Mills theory
YM: C(E) → R≥0 , YM(∇) = ∫(M,g) |Ω∇|² dμg
Donaldson
Prop σ ∈ G(E), ∇ ∈ C(E)
the connection form of σ(∇) is given by
ω + σ∇σ⁻¹ = ω − ∇σ·σ⁻¹
σ(∇)s = σ(∇(σ⁻¹s)) = σ(∇σ⁻¹)s + ∇s
ω + σ(∇σ⁻¹) = ω + σ(−σ⁻¹∇σ σ⁻¹) = ω − ∇σ σ⁻¹



On Uλ ∩ Uμ (≠ φ)
ωμ = fλμ⁻¹ ωλ fλμ + fλμ⁻¹ dfλμ —— (*)
where eμ = eλ fλμ
Prop on Uλ ∩ Uμ (≠ φ), Ωμ = fλμ⁻¹ Ωλ fλμ
P⁻¹P = E
d(P⁻¹P) = 0
dP⁻¹ P + P⁻¹dP = 0
dP⁻¹ = − P⁻¹dP P⁻¹
証明 Ωμ = dωμ + ωμ ∧ ωμ



(dωλ + ωλ ∧ ωλ) fλμ
Ωλ fλμ

Date.

- Exterior Covar. Derivative: $d^\nabla : C^\infty(M, E\otimes\Lambda^k(M)) \to C^\infty(M, E\otimes\Lambda^{k+1}(M))$ s.t.

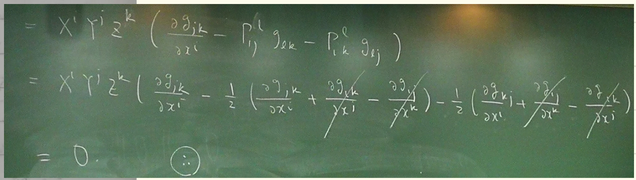
$$d^\nabla(s\otimes\alpha) = \nabla s\wedge\alpha + s\otimes d\alpha$$

↑ skew-symmetrization of $\nabla s\otimes\alpha$.

- **Prop-Exe:**

① $d^\nabla\circ d^\nabla = \Omega$;

② (Bianchi Identity) (2nd): $d^\nabla\Omega = 0$.



2 Riemannian Geometry

- Riemannian Metric: $g\in C^\infty(M, T^*M\otimes T^*M)$ is called a Riemannian metric if locally $g = \sum g_{ij}dx^i\otimes dx^j$ where (g_{ij}) is positive-definite, symmetric at each pt.

Riemannian Mfd: (M,g).

- Thm (Levi-Civita): Given (M,g), $\exists!$ connection ∇ on TM satisfying:

① (Torsion-free): $\nabla_X Y - \nabla_Y X = [X,Y]$, $\forall X, Y\in\mathfrak{X}(M)$;

② (Compatibility): $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, $\forall X, Y, Z\in\mathfrak{X}(M)$,

OR $\nabla g = 0$. "g is parallel"

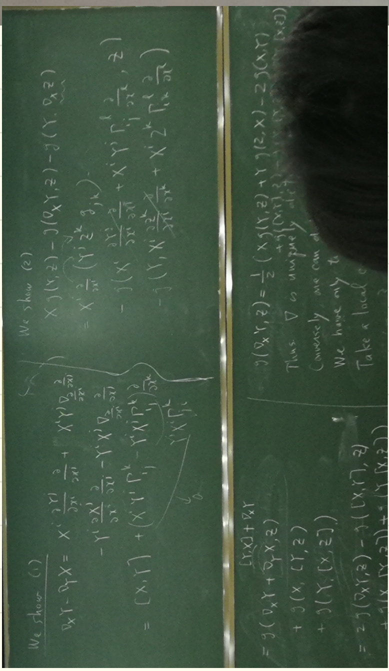
the connection is called the L-C connection / R connection.

(Pf. If there is a ∇ satisfying ①, ②, then

~~$Xg(Y,Z) = g(\nabla_X Y, Z)$~~

$$Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)$$
$$= g(\nabla_X Y + \nabla_Y X, Z) + g(X, [Y,Z]) + g(Y, [X,Z])$$
$$= 2g(\nabla_X Y, Z) - g([X,Y], Z) + g(X, [Y,Z])$$

$\Rightarrow g(\nabla_X Y, Z)$ can be expressed uniquely by g, X, Y, Z $\Rightarrow \nabla$ is unique.



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Conversely, taking local coordinates $(x^1,\cdots,x^n)$,

the expression above implies $g\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$.

Denote $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma^k_{ij}\frac{\partial}{\partial x^k}$ (Christoffel symbols)

and $g^{ij}g_{jk} = \delta^i_k$,

we have $\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)$. ← Memorize!

Then, direct calculation verifies ①, ②. See photos.)

- Corollary. The connection matrix is $\omega = (\Gamma^i_{jk}dx^k)$.

- Curvature: $R(X,Y) = \Omega(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$.

Curvature tensor: The R above is called the curvature tensor.

Define $R(X,Y,Z,W) = g(Z, R(X,Y)W)$, this R is also called the curvature Tensor.

- **Prop-Exc:**

① $R(X,Y,Z,W) = -R(Y,X,Z,W)$;

② $R(X,Y,Z,W) = R(Z,W,X,Y)$;

③ (Jacobi Identity) (1st Bianchi Identity): $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$;

④ (2nd Bianchi Identity): $(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$.

($\Leftrightarrow d^\nabla\Omega = 0$)

- Sectional Curvature: For a 2-dim plane $\Pi\subseteq T_pM$ and an orthonormal basis $X, Y\in\Pi$, $K(\Pi) = R(X,Y,X,Y)$ is called the sectional curvature of Π.

Prop-Exc: $K(\Pi)$ does not depend on the choice of X, Y.

note boo

curvature tensor : $R = R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$ with $R_{ijkl} = -R_{jikl}$ $\boxed{R_{ijkl} = R_{klij}}$

$R(X,Y) \in \mathrm{End}(TM) \doteq C^\infty(M, TM \otimes T^*M)$ $\qquad g(X,Y)$ can be understood like $\underset{T^*M}{{}^tX}\, g\, \underset{TM}{Y}$

two coventions
① if $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ imply $R(X,Y,Z,W) = g(Z, R(X,Y)W) = g(R(X,Y)W, Z)$

the order gets changed

② if $R(X,Y) = -\nabla_X\nabla_Y + \nabla_Y\nabla_X + \nabla_{[X,Y]}$ imply the definition $R(X,Y,Z,W) = g(R(X,Y)Z, W)$

satisfying the same property

both admits the notion of sectional curvature

$R(X,Y,X,Y)$ with $\|X\| = \|Y\| = 1$ & $g(X,Y) = 0$

many Kähler geometer use the opposite convention for $R_{i\bar{j}k\bar{l}}$???

Def (Ricci curvature) on T_pM, let $e_1, \cdots, e_n$ be an orthogonal basis

$$Ric(X,Y) := \sum_{i=1}^n R(X, e_i, Y, e_i)$$ Ric is symmetric in X and Y

imply

$Ric = R_{ij}\, dx^i \otimes dx^j \qquad R_{ij} = R^k_{\ ikj}$

(can also be understood as : $Ric(X,Y) = \mathrm{Tr}(f_{XY})$ where $f_{XY} : T_pM \to T_pM$
$\qquad Z \mapsto R(Z,Y)X$)

(Einstein) if $\exists k \in \mathbb{R}$ constant, s.t. $R_{ij} = k g_{ij}$ i.e. $Ric = kg$, then g is called Ricci-flat or Einstein

note that $\omega \leftrightarrow \Gamma$ can be seen as the 1st derivate of g

and $R = d\omega + \omega\wedge\omega$ the 2-nd

so $Ric = kg$ may be seen as 2-nd partial differential equations in terms of g

it's the " field equation for gravity "

Def (Scalar curvature) $\quad Scal := \sum_{i=1}^n Ric(e_i, e_i) = \underline{\sum_{i,j} R(e_i, e_j, e_i, e_j)} \to$ this is a sectional curvature

eg. (1) $\dim M = 1$ $\quad R = 0$ $\qquad \dim M = 0 \qquad \dim M = 2$ $\quad Scal = R(e_1, e_2, e_1, e_2) + R(e_2, e_1, e_2, e_1)$
$\qquad = 2R(e_1, e_2, e_1, e_2)$

$Ric(e_1, e_1) = R(e_1, e_2, e_1, e_2) = \underset{\overset{\shortparallel}{kg(e_1,e_1)}}{g(e_1, R(e_1,e_2)e_2)}$ $\qquad$ same for (e_2, e_2)
and for (e_1,e_2) or (e_2,e_1)
both are zero

2) $M = \mathbb{R}^n$ $\quad g = \sum_{i=1}^n dx_i^2 \quad \Gamma^k_{ij} = 0$ $\quad$ we define $\nabla_X Y := X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j}$ $\to$ also denoted with XY
claim this's the Levi-Civita con

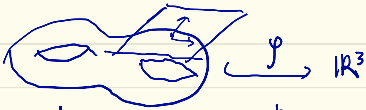
① $\nabla_X Y - \nabla_Y X = XY - YX = [X,Y]$ $\qquad$ both have the term $X^i Y^j \frac{\partial}{\partial x^i \partial x^j}$

② $Xg(Y,Z) = X(\sum Y^i Z^i) = \sum (XY^i) Z^i + \sum Y^i (XZ^i) = g(XY, Z) + g(Y, XZ) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

$\qquad \overset{\shortparallel}{X^j \frac{\partial Y^i}{\partial x^j}} = (XY)^i$ $\qquad\qquad$ flat metric

$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z = (XY - YX - [X,Y])Z = 0$ so $R \equiv 0$ $\quad$ flat connection

3) $M \subseteq \mathbb{R}^n$



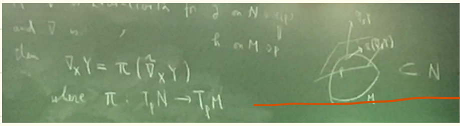
we can strict the flat metric in $\mathbb{R}^n$ to M

choose local coordinate $(U, (x^1, \cdots, x^k))$

let $h_{ij} = g(\varphi_* \frac{\partial}{\partial x^i}, \varphi_* \frac{\partial}{\partial x^j}) = \sum_{k=1}^{n} \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j}$ $\qquad h = (h_{ij}) = \varphi^* g$ induced by φ

more generally, $M^k \overset{\varphi}{\hookrightarrow} (N^n, g)$ we get $h \triangleq \varphi^* g$ $\quad$ what's the Levi-Civita connection?

let $\tilde{\nabla}$ (resp. ∇) denote the L-C of g (resp. h) on N (resp. M)



here, X, Y may extend to be v.f. in N

but $\tilde{\nabla}_X Y$ may not be in the plane T_pM,

so we take the projection π

proof. ① $\nabla_X Y - \nabla_Y X = \pi(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X) = \pi([\tilde{X}, \tilde{Y}]) = [X, Y]$

to be clear, here we use $\tilde{X}$ to denote the extension

fact: $[\tilde{X}, \tilde{Y}] = \widetilde{[X, Y]}$, so we get the equal

② $X h(Y, Z) = X g(Y, Z) = g(\tilde{\nabla}_X Y, Z) + g(Y, \tilde{\nabla}_X Z) = g(\pi(\tilde{\nabla}_X Y), Z) + g(Y, \pi(\tilde{\nabla}_X Z)) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z)$

on where h is defined, the equality holds

belongs to the "plane", since we are going to take inner product with elements in the "plane", it's the same to consider the projection of another variable in the "plane"

Def. for $M^{n-1} \overset{\varphi}{\hookrightarrow} (N^n, g)$ i.e. with codim $= 1$, call $h = \varphi^* g$ the 1-st fundamental form

take a locally orthogonal normal v.f. $e_p \perp T_pM$ for $p \in U$ some small nbhd

the $\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y) e$ for some bilinear form α

$\alpha(fX, gY) = fg\,\alpha(X, Y)$ $\quad$ since $\quad \tilde{\nabla}_{fX} gY = \nabla_{fX} gY + \alpha(fX, gY) e = f(Xg)Y + g\nabla_X Y) + \alpha(fX, gY) e$

$\qquad\qquad\qquad\qquad\qquad\quad \| $

$\qquad\qquad\qquad\quad f(Xg)Y + g\tilde{\nabla}_X Y)$ $\qquad$ so $fg(\tilde{\nabla}_X Y - \nabla_X Y) = \alpha(fX, gY) e$

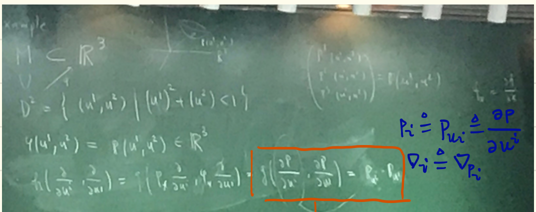
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad fg\alpha(X, Y) e$

$\alpha(X, Y) = \alpha(Y, X)$ $\quad$ since $\nabla_X Y$ & $\nabla_Y X$ differ by $[X, Y]$

$\tilde{\nabla}_X Y$ & $\tilde{\nabla}_Y X$ differ by $[\tilde{X}, \tilde{Y}] = [X, Y]$ on M

eg.



inner product in the standard case

we get that $h = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} P_1 \cdot P_1 & P_1 \cdot P_2 \\ P_2 \cdot P_1 & P_2 \cdot P_2 \end{pmatrix}$ or $h = \sum_{i,j=1}^{2} h_{ij}\, du^i du^j = E\,du^{1\,2} + 2F\,du^1 du^2 + G(du^2)^2$

$$\alpha\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) e = \tilde{\nabla}_{\varphi_* \frac{\partial}{\partial u^i}} \varphi_* \frac{\partial}{\partial u^j} - \nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = \frac{\partial P_j}{\partial u^i} - \pi\left(\underbrace{\frac{\partial P_j}{\partial u^i}}_{\substack{\shortparallel \\ P_{ji}\ (= P_{ij})}}\right) = \left(\frac{\partial P_j}{\partial u^i} \cdot e\right) e$$

so $\alpha = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} P_{11} \cdot e & P_{12} \cdot e \\ P_{21} \cdot e & P_{22} \cdot e \end{pmatrix}$ $\quad \alpha = L(du^1)^2 + 2M\, du^1 du^2 + N(du^2)^2$

Def (Gauss curvature) for $\varphi(D^2) = M \subseteq \mathbb{R}^3$ $\quad \dfrac{\det \alpha}{\det h} = \dfrac{LN - M^2}{EG - F^2}$

Thm (Gauss) $u_0 \in D$ take (u_1, u_2) s.t. $h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ($\frac{\partial}{\partial u^1}\ \frac{\partial}{\partial u^2}$ ortho normal at u_0) $\to$ i.e. $EG - F^2 = 1$

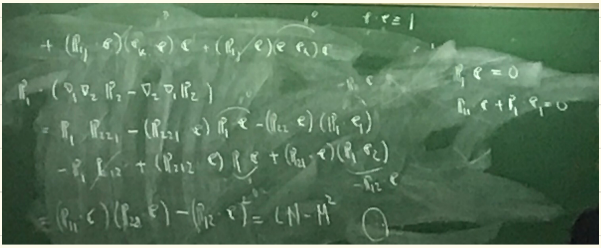
then $\underbrace{R\left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right)}_{\text{sectional curvature, intrinsic}} = \underbrace{LN - M^2}_{\text{related to the outside space}}$

Proof. left $= P_1 \cdot (\nabla_1 \nabla_2 P_2 - \nabla_2 \nabla_1 P_2)$ $\quad$ we have $\left[\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right] = 0$

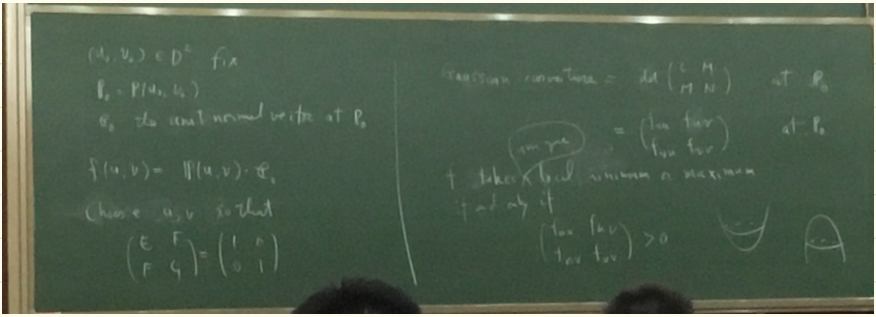
$$\nabla_k \nabla_j P_i = \frac{\partial}{\partial u^k}(\nabla_j P_i) - \left(\frac{\partial}{\partial u^k}(\nabla_j P_i) \cdot e\right) e = \frac{\partial}{\partial u^k}\left(\frac{\partial P_i}{\partial u^j} - \left(\frac{\partial P_i}{\partial u^j} \cdot e\right) e\right) -$$

$$\left(\frac{\partial}{\partial u^k}\left(\frac{\partial P_i}{\partial u^j} - \left(\frac{\partial P_i}{\partial u^j} \cdot e\right) e\right) \cdot e\right) e = P_{ijk} - (P_{ijk} \cdot e)\, e - (P_{ij} \cdot e)\, e_k - (P_{ijk} \cdot e)\, e$$

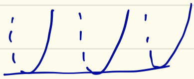
$$+ (P_{ijk} \cdot e)\, e + (P_{ij}\, e_k)\, e + (P_{ij} \cdot e)(e_k \cdot e)\, e + (P_{ij} \cdot e)(e \cdot e_k)\, e$$

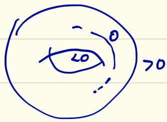


这个实在不想自己算一遍了 ¯_¯||



$\begin{pmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{pmatrix} = 0$





Thm. (Gauss) Gaussian curvature = sectional curvature = described the second derivatives of the 1st fundamental form (Riemannian metric)

Thm (Gauss-Bonnet) M compact surface (in $\mathbb{R}^3$) (Ex. $|P_u \times P_v| \, du \wedge dv$ indep of chart choice)

$$\int_M \text{Gaussian} \, dV = 2\pi \chi(M) \rightsquigarrow \text{Atiyah-Singer}$$

$[(\text{Gaussian})\, dV] \in H^2_{DR}(M)$ is independent of the choice of $M \hookrightarrow \mathbb{R}^3$ (or indep of the metric of M)

Chern $\to$ the Euler class

$\to$ Chern classes for cpt cpx manifold

... vector bundle — Chern connection for holo. v.b.

Kähler mfd

connections, curvature for principal bundle

$\rightsquigarrow$ especially for $U(n)$-bundle

$\downarrow$

Chern class

unitary connections (Hermitian connec) for holo v.b.

M : complex mfd of $\dim_{\mathbb{C}} M = n$ locally open sets of $\mathbb{C}^n$

change of coordinates is holo $w^i(z_1, \dots, z^n)$

$z^i = x^i + \sqrt{-1}\, y^i$ $x^i = \frac{1}{2}(z^i + \overline{z^i})$ $w^i(x^1, y^1, \dots, x^n, y^n) = w^i(z_1, \bar{z}_1, \dots, z_n, \bar{z}_n)$

$$\frac{\partial z^i}{\partial \overline{w^j}} = 0$$

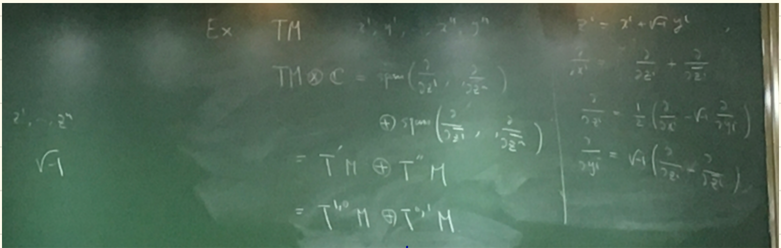
Def. $\pi : E \to M$ ·rank r holo v.b. : i) $\dim_{\mathbb{C}} E = n + r$ cpx mfd

ii) π is holo onto iii) open covering $\{U_\alpha\}$, locally trivialization : $\varphi_\alpha : \pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}^r$ biholo

s.t. $P_\alpha : U_\alpha \times \mathbb{C}^r \to U_\alpha$ the projection $\pi|_{\pi^{-1}(U_\alpha)} = P_\alpha \circ \varphi_\alpha$ holomorphic

(iv) $U_\alpha \cap U_\mu \neq \emptyset$ $\varphi_\alpha \circ \varphi_\mu^{-1}(p, x) = (p, \psi_{\alpha\mu}(p)x)$ $\{\psi_{\alpha\mu} : U_\alpha \cap U_\mu \to GL(r, \mathbb{C})\}$

$(U_\mu \cap U_\alpha) \times \mathbb{C}^r \to (U_\alpha \times U_\mu) \times \mathbb{C}^r$ cpx mfd



T'M : holomorphic v.b. $\psi_{\alpha\mu} = \left(\frac{\partial z^i}{\partial w^j} \right)$ T''M is not $T''M = \overline{T'M}$ conjugate topologically, see "characteristic class"

$T^*M \otimes \mathbb{C} = \mathrm{span}(dz^1, \dots, dz^n) \oplus \mathrm{span}(d\bar{z}^1, \dots, d\bar{z}^n) = T^{*1,0}M \oplus T^{*0,1}M$

p q r s $\otimes$ $\to$ holo v.b.

$\Lambda^n T^{*1,0}M$ K_M canonical

$\Lambda^n T^{1,0}M$ K_M^{-1} anticanonical

D divisor $\quad M = \bigcup U_\alpha$

$U_\alpha \cap D \rightsquigarrow f_\alpha$

$f_{\alpha\mu} := \dfrac{f_\alpha}{f_\mu} \rightsquigarrow$ line bundle

$D + D' \rightsquigarrow L_D \otimes L_{D'}$

for $\mathbb{P}^n_{\mathbb{C}}$ $\quad H = \{[0 : z_1 : \cdots : z_n]\} \subseteq \mathbb{P}^n_{\mathbb{C}}$

$L_H : \mathcal{O}(1)$ $\quad L_H^k := \otimes^k L_H$
$\quad\quad \| $
$\quad \mathcal{O}(k)$

$h \in C^\infty(M, \bar{E}^* \otimes E^*)$ Hermitian metric

h_p Hermitian form for $\forall p \in M$

matrix $\begin{pmatrix} h_{i\bar{j}} \end{pmatrix}$ $\quad H = {}^t\overline{H}$

$h(a, b) = \overline{h(b, a)}$

$h_{i\bar{j}} = \overline{h_{\bar{j} i}}$

maybe an Hermitian metric?

- let D be a non-singular divisor, $M = \bigcup_\alpha U_\alpha$ then $D \cap U_\alpha = \{f_\alpha = 0\}$ where f_α vanishes along D with 1-st order, so there may be local representation $f_\alpha = z f_\alpha^*$ with f_α^* nowhere vanishing holomorphic function
then $f_{\alpha\mu} := \frac{f_\alpha}{f_\mu} = \frac{f_\alpha^*}{f_\mu^*}$ turns to be a nowhere vanishing holomorphic function on $U_\alpha \cap U_\mu$
i.e. $U_\alpha \cap U_\mu \to GL(1, \mathbb{C}) = \mathbb{C}^*$ holomorphic thus gives a line bundle on M

- let M be a cpx mfd with dim $= n$, then if M admits a Riemannian metric g compatible with the almost cpx structure induced from the cpx structure, i.e. $g(Ju, Jv) = g(u, v)$
we can extend J & g to $TM \otimes \mathbb{C} =: T_{\mathbb{C}}M$ $\mathbb{C}$-linearly

- originally we have local coordinates $\frac{\partial}{\partial x^1} \cdots \frac{\partial}{\partial x^n}$ $\frac{\partial}{\partial y^1} \cdots \frac{\partial}{\partial y^n}$ & $J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$ $J(\frac{\partial}{\partial y^i}) = -\frac{\partial}{\partial x^i}$
denote $v \otimes (a+bi)$ as $av + ibv$ in $T_{\mathbb{C}}M$
then define $\frac{\partial}{\partial z^i} = \frac{1}{2}(\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial y^i})$ $\frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}(\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i})$, which also form a basis for $T_{\mathbb{C}}M$
then $J(\frac{\partial}{\partial z^i}) = i\frac{\partial}{\partial z^i}$ $J(\frac{\partial}{\partial \bar{z}^i}) = -i\frac{\partial}{\partial \bar{z}^i}$ thus $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ where $T^{1,0}M$ is the i-eigenvalue part & $T^{0,1}M$ the $-i$ part
(this can be done in general: V is a dim-$2n$ real v.s., with an almost complex structure J
then we can give V a basis $e_1, \cdots, e_n, Je_1, \cdots, Je_n$
then in $V^{\mathbb{C}}$, we have a basis $z_i := \frac{1}{2}(e_i - iJe_i)$ $\bar{z}_i := \frac{1}{2}(e_i + iJe_i)$
extend J $\mathbb{C}$-linearly, then $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ similarly) this is right since both g & J are extended linearly

- then we see that on $T^{1,0}M$ or $T^{0,1}M$, $g = 0$ since $g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}) \overset{!}{=} g(J\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial z^j})$
$= g(i\frac{\partial}{\partial z^i}, i\frac{\partial}{\partial z^j}) = -g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j})$, same for $g(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}) = 0$
define a Hermitian metric on $T^{1,0}M$: $h(u, v) := g(u, \bar{v})$ $\overline{\frac{\partial}{\partial z^i}} = \frac{\partial}{\partial \bar{z}^i}$
note that for u with J-eigenvalue $\pm i$, $\bar{u}$ has J-eigenvalue $\mp i$
$h(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}) = g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}) =: g_{i\bar{j}}$
$\Big(\overset{\shortparallel}{g}(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^i}) = \overline{g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^i})} = \overline{h(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^i})} = \overline{g_{j\bar{i}}} \Big)$
so we have $h = (g_{i\bar{j}})$ a Hermitian metric on $T^{1,0}M$

- we can define a 2-form $\omega_g(u, v) := -g(u, Jv)$ $(= -g(Ju, -v) = g(Ju, v) = g(v, Ju) = -\omega_g(v, u))$
$\longrightarrow$ $\omega_g = i\sum_{i,j} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$ called the Kähler form for (M, J, g) (M is a $2n$-dim real mfd)
where $z_i = x_i + iy_i$ hence $dz_i = dx_i + i\,dy_i$ $d\bar{z}_i = dx_i - i\,dy_i$ $dz_i(\frac{\partial}{\partial z^j}) = \delta_{ij}$
$d\bar{z}_j(\frac{\partial}{\partial \bar{z}^i}) = \delta_{ij}$ $d\bar{z}_j(\frac{\partial}{\partial z^i}) = 0$ $dz_i(\frac{\partial}{\partial \bar{z}^j}) = 0$

not contained in today's course, may be introduced later

- (M, J, g) is called almost Kähler if $d\omega_g = 0$, furthermore if $N(J) = 0$ ($\Leftrightarrow J$ integrable), it's called Kähler, where $N(J): TM \times TM \to TM$, $(v, w) \mapsto [v, w] + J[Jv, w] + J[v, Jw] - [Jv, Jw]$
note that ω_g is always positive definite, so M is symplectic

we have a thm says: if $N(J)=0$, then $d\omega_g=0 \iff \nabla J=0$ where ∇ is the Levi-Civita connection for g, so a kähler mfd is a cpx mfd with the described closed form ω_g, the coefficients of ω_g forms a Hermitian metric, so we may not distinguish kähler form & kähler metric (i.e. Hermitian metric compatible with J)

this topic is also discussed on Chern's book

get back to the course now

- Examples ① $\mathbb{P}^1_{\mathbb{C}}$ $D=\{[0:1]\}$ $U_0=\{z_0\neq 0\}$ with $s=\frac{z_1}{z_0}$ $h_0=\frac{1}{1+|s|^2}$ (for line bundle, a Hermitian metric is just a number at each pt)

$U_1=\{z_1\neq 0\}$ with $t=\frac{z_0}{z_1}$, $h_1=\frac{1}{1+|t|^2}$

$U_0\cap D=\phi$ $U_1\cap D=\{[0:1]\}=\{t=0\}$

$f_0=1$ may define $f_1=t$

on $U_0\cap U_1$, $ts=1$, $f_1\neq 0$, $f_0\neq 0$ $f_{10}=t=\frac{1}{s}$ $f_{01}=\frac{1}{t}=s$ holomorphic

$f_{10}/f_{01}: U_0\cap U_1 \to GL(1,\mathbb{C})=\mathbb{C}^* \rightsquigarrow$ line bundle $\mathcal{O}(1) \rightsquigarrow \mathcal{O}(p):=\otimes^p\mathcal{O}(1)$

(in general, $f_{\lambda\mu}: U_\lambda\cap U_\mu \to GL(r,\mathbb{C})$ holomorphic, $(f_{\lambda\mu}(p))^{-1}=f_{\mu\lambda}(p)$ as matrix, conversely, given $f_{\lambda\mu}$ as above, define an equivalent relation on $\underset{\lambda}{\cup}(U_\lambda\times\mathbb{C}^r)$

$p\in U_\lambda\cap U_\mu$ $\underset{U_\lambda\times\mathbb{C}^r}{(p,v)}\sim\underset{U_\mu\times\mathbb{C}^r}{(p,w)} \overset{\text{def}}{\iff} v=f_{\lambda\mu}(p)w$ ($\iff f_{\mu\lambda}(p)v=w$), this gives a holomorphic v.b.)

claim h_0, h_1 give a well-defined Hermitian metric on $\mathcal{O}(1)$

we need to check the transition condition $h_\lambda={}^t\overline{f_{\mu\lambda}}\,h_\mu f_{\mu\lambda}$

here, it's just $h_1=\frac{1}{1+|t|^2}=\frac{|s|^2}{1+|s|^2}=\frac{\bar{s}s}{1+|s|^2}={}^t\overline{f_{01}}\,h_0 f_{01}$

② first, note that if the transition function for $E\to M$ & $F\to M$ are respectively $f_{\lambda\mu}$, $g_{\lambda\mu}$ then that for $E\otimes F\to M$ is $f_{\lambda\mu}\otimes g_{\lambda\mu}$ (as linear map)

claim $T^{1,0}\mathbb{P}^1\cong\mathcal{O}(2)$ ($=\mathcal{O}(1)\otimes\mathcal{O}(1)$, dim 1, in this case $f_{\lambda\mu}\otimes g_{\lambda\mu}$ is just $f_{\lambda\mu}g_{\lambda\mu}$)

sufficient to show that the transition functions are t^2 & s^2

on U_0, we have local frame $\frac{\partial}{\partial s}$ $\quad -\frac{\partial}{\partial t}=-\frac{\partial s}{\partial t}\frac{\partial}{\partial s}=\frac{1}{t^2}\frac{\partial}{\partial s}=s^2\frac{\partial}{\partial s}$

U_1 $\quad -\frac{\partial}{\partial t}$ $\quad =f_{01}^2\frac{\partial}{\partial s}$

(in his convention, $f_{\mu\lambda}$ is the transition from U_μ to U_λ, I think it's a little bit confusing, and he prefer to use $g_{i\bar{j}}$, not $g_{\bar{i}j}$)

- Def (Fubini-Study metric on $T^{1,0}\mathbb{P}^n$) $U_\lambda=\{z_\lambda\neq 0\}$ local coordinate $U_\lambda\to\mathbb{C}^n$ $t^0_\lambda=\frac{z_0}{z_\lambda}\cdots t^{\hat{\lambda}}_\lambda=1\cdots t^n_\lambda=\frac{z_n}{z_\lambda}$

$g^{(\lambda)}_{i\bar{j}}:=\frac{\partial^2}{\partial t^i_\lambda\partial \bar{t}^j_\lambda}\log(1+|t^0_\lambda|^2+\cdots+1+\cdots+|t^n_\lambda|^2)$ we have

① $g^{(\lambda)}_{i\bar{j}}$ positive definite ② $g^{(\mu)}_{i\bar{j}}={}^t\overline{J_{\lambda\mu}}\,g^{(\lambda)}_{i\bar{j}}\,J_{\lambda\mu}$

eg. $n=1$ $\frac{\partial^2}{\partial\bar{s}\partial s}\log(1+|s|^2)=\frac{\partial}{\partial s}\left(\frac{s}{1+|s|^2}\right)=\frac{1}{1+|s|^2}-\frac{\bar{s}s}{(1+|s|^2)^2}=\frac{1}{(1+|s|^2)^2}$

Lemma. $E \to M$ holo v.b. (transition functions holo)

$$\bar\partial : C^\infty(M, E) \to C^\infty(M, E \otimes T^{*0,1}M)$$

$$s \mapsto \bar\partial s = \bar\partial\left((e_1 \cdots e_n)\begin{pmatrix} s^1 \\ \vdots \\ s^n \end{pmatrix}\right)$$

$$:= (e_1 \cdots e_r)\begin{pmatrix} \bar\partial s^1 \\ \vdots \\ \bar\partial s^n \end{pmatrix}$$

$s^1 \cdots s^n$ smooth functions

$e_1 \cdots e_n$ local frame

$$\bar\partial f = \frac{\partial f}{\partial \bar z^j} d\bar z^j$$

have to show there's no trouble (conflict) when taking another local frame:

$$(e_{\mu 1}, \cdots, e_{\mu r}) = (e_{\lambda 1}, \cdots, e_{\lambda r}) f_{\lambda\mu}$$

$$s = (e_{\mu 1} \cdots e_{\mu r})\begin{pmatrix} t^1 \\ \vdots \\ t^r \end{pmatrix} = (e_{\lambda 1} \cdots e_{\lambda r})\begin{pmatrix} s^1 \\ \vdots \\ s^r \end{pmatrix}$$

$$\shortparallel$$

$$(e_{\lambda 1} \cdots e_{\lambda r}) f_{\lambda\mu}\begin{pmatrix} t^1 \\ \vdots \\ t^r \end{pmatrix}$$

have to show: $\bar\partial s = (e_{\mu 1} \cdots e_{\mu r})\begin{pmatrix} \bar\partial t^1 \\ \vdots \\ \bar\partial t^r \end{pmatrix} \overset{?}{=} (e_{\lambda 1} \cdots e_{\lambda r})\begin{pmatrix} \bar\partial s^1 \\ \vdots \\ \bar\partial s^r \end{pmatrix}$

$$\shortparallel$$

$$(e_{\lambda 1} \cdots e_{\lambda r}) f_{\lambda\mu}\begin{pmatrix} \bar\partial t^1 \\ \vdots \\ \bar\partial t^r \end{pmatrix}$$

since $f_{\lambda\mu}$ is holo (for holo, $\bar\partial = 0$)

$$\shortparallel$$

$$(e_{\lambda 1} \cdots e_{\lambda r})\, \bar\partial\left(f_{\lambda\mu}\begin{pmatrix} t^1 \\ \vdots \\ t^r \end{pmatrix}\right)$$

Rmk. there's no well-defined ∂-operator on $C^\infty(M, E)$

Thm. (Chern connection) $E \to M$ holo-v.b. h: Hermitian metric

then $\exists$ unique connection ∇ satisfying $(\nabla : C^\infty(M, E) \to C^\infty(M, E \otimes (T^*M \otimes \mathbb{C})))$

① if we write $\nabla = \nabla^{1,0} + \nabla^{0,1}$ $\quad C^\infty(M, E) \to C^\infty(M, E \otimes T^{*1,0}M) \oplus C^\infty(M, E \otimes T^{*0,1}M)$

then $\nabla^{0,1} = \bar\partial$

② $d(h(s, t)) = h(\nabla s, t) + h(s, \nabla t)$ i.e. $\nabla h = 0$

ideal: "$d = \partial + \bar\partial$"

proof. Local frame $e_1 \cdots e_r$. If such a ∇ exists

$' \longleftrightarrow 1,0$

$'' \longleftrightarrow 0,1$

$$\underset{\displaystyle e_i \omega^i_j}{\underset{\shortparallel}{\nabla e_j}} = (\nabla' + \bar\partial) e_j = \underset{\displaystyle C^\infty(M, E\otimes T'^* M)}{\underset{\cap}{\nabla' e_j}} \qquad (\bar\partial 1 = 0)$$

So ω^i_j is linear combination of $dz^1 \cdots dz^n$

$$\underset{\displaystyle \partial h_{\bar i j} + \bar\partial h_{\bar i j}}{\underset{\shortparallel}{d h_{\bar i j}}} = h(\overline{\nabla e_i}, e_j) + h(\bar e_i, \nabla e_j)$$

$$= h(\overline{e_k \omega^k_i}, e_j) + h(\bar e_i, e_k \omega^k_j) = \overline{\omega^k_i} h_{\bar k j} + h_{\bar i k} \omega^k_j$$

$$\Rightarrow \partial h_{\bar i j} \underset{\Updownarrow}{=} h_{\bar i k} \omega^k_j \qquad \bar\partial h_{\bar i j} = \overline{\omega^k_i} h_{\bar k j}$$

$$\partial h_{\bar k j} = h_{\bar k i} \omega^i_j$$

taking $(h^{i\bar j}) = (h_{\bar i j})^{-1}$ i.e. $h^{i\bar k} h_{\bar k j} = \delta_{ij}$

then $(\omega^i_j) = (h^{i\bar k} \partial h_{\bar k j})$

may write $\omega = h^{-1}\partial h$ memorize

wish to define ω by this, sufficient to show the transformation law

$$\omega_\mu = \psi_{\lambda\mu}^{-1} \omega_\lambda \psi_{\lambda\mu} + \psi_{\lambda\mu}^{-1} \underline{d\psi_\mu} \; (= \partial + \underset{0}{\bar\partial})$$

$$h_\mu = {}^t\bar\psi_{\lambda\mu} h_\lambda \psi_{\lambda\mu} \qquad \omega_\mu = h_\mu^{-1} \partial h_\mu = \psi_{\lambda\mu}^{-1} h_\lambda^{-1} \, {}^t\bar\psi_{\lambda\mu}^{-1} \left({}^t\bar\psi_{\lambda\mu} \partial h_\lambda \psi_{\lambda\mu} + {}^t\bar\psi_{\lambda\mu} h_\lambda \partial \psi_{\lambda\mu} \right) = \cdots$$

Thm. $\Omega = \bar\partial\omega$ curvature form

pf. $\omega = h^{-1}\partial h$

$$\Omega = d\omega + \omega\wedge\omega = \bar\partial\omega + \partial(h^{-1}\partial h) + h^{-1}\partial h \wedge h^{-1}\partial h$$

$$= \bar\partial\omega + (-h^{-1}\partial h h^{-1} \wedge \partial h + 0) + h^{-1}\partial h \wedge h^{-1}\partial h$$

$$= \bar\partial\omega$$

Cor. $L \to M$ line bundle (holo) h: Hermitian metric locally positive smooth function

then $\Omega = \bar\partial(h^{-1}\partial h) = \bar\partial\partial \log h = -\partial\bar\partial \log h$ memorize

$L \to M$ line bundle $\quad h$: Hermitian metric

$U \subset M$ open $\quad s: U \to L$ nowhere zero section

$e_1 = s \quad h_U := h(\bar{s}, s) > 0$

$\Omega = \bar\partial(h_U^{-1} \partial h_U) = \bar\partial \partial \log h_U$

Lemma 1. $\bar\partial\partial \log h_U$ independent of the choice of local section s

2. $\bar\partial\partial \log h_U = \bar\partial \partial \log h_V$ if $U \cap V \neq \emptyset$

$\rightsquigarrow$ $\bar\partial\partial \log h_U$ is defined on M globally

Def. $c_1(L, h) = \frac{i}{2\pi}\Omega = \frac{i}{2\pi} \bar\partial\partial \log h_U = -\frac{i}{2\pi} \partial\bar\partial \log h_U$

1-st chern form of L w.r.t. h

Thm 1, $c_1(L, h)$ is a real $(1,1)$-form

2, de Rham class $[c_1(L,h)]$ independent of h

Pf. 2, h' another Hermitian metric

$h_V = |\psi|^2 h_U \quad h'_V = |\psi|^2 h'_U \qquad \frac{h_V}{h'_V} = \frac{h_U}{h'_U} = f > 0$

f is a global function on M

$c_1(L,h) = \frac{i}{2\pi} \bar\partial\partial \log h_U = \frac{i}{2\pi} \bar\partial\partial(\log h'_U + \log f) = c_1(L, h') + \frac{i}{2\pi}\bar\partial\partial \log f$

$\cdot \ \frac{i}{2\pi}\bar\partial\partial \log f = d(\bar\partial \log f)$ since $d = \partial + \bar\partial$, $\partial^2 = 0$

exact form

1, $c_1(L,h)$ is closed since $d(\bar\partial\partial) = (\partial + \bar\partial)(\bar\partial\partial)$

Def. $c_1(L) := [c_1(L,h)] \in H^2_{dR}(M,\mathbb{R})$ 1-st Chern class of L

Example. $\mathcal{O}_{\mathbb{P}^n}(k) = H_n^{\otimes k}$ $\quad H_n = \mathcal{O}_{\mathbb{P}^n}(1)$ hyperplane bundle

$n=1 \quad L = \mathcal{O}_{\mathbb{P}^1}(k)$

$\mathbb{P}^1 = S^2 \quad H^2(S^2,\mathbb{Z}) \cong \mathbb{Z}$ $\quad U_0 = \{z^0 \neq 0\} \quad U_1 = \{z^1 \neq 0\}$ $\quad f_{01} = s^k \quad f_{10} = t^k$

$s = \frac{z^1}{z^0} \quad t = \frac{z^0}{z^1}$

$h_{U_0} = \frac{1}{(1+|s|^2)^k}$ $\quad c_1(L, h_{U_0}) = -\frac{i}{2\pi}\partial\bar\partial \log \frac{1}{(1+|s|^2)^k}$

$= k\frac{i}{2\pi}\partial\bar\partial \log(1+|s|^2)$

$\langle c_1(\mathcal{O}_{\mathbb{P}^1}(k)), [\mathbb{P}^1]\rangle = k\int_{U_0} \frac{i}{2\pi}\partial\bar\partial \log(1+|s|^2)$ $\quad \boxed{s \in U_0 \cong \mathbb{C}}$

$\quad\quad \shortparallel$

$\quad k\int_0^\infty \frac{2r\,dr}{(1+r^2)^2}$

$= k$ $\quad (s = re^{i\theta}$ change variable$)$

Cor. $\langle c_1(T'\mathbb{P}^1), [\mathbb{P}^1]\rangle = 2 = \chi(S^2)$

Def. A $\quad M$ n-dim cpt cpx mfd

$$c_1(M) \overset{\text{def}}{\triangleq} -c_1(K_M) = c_1(K_M^{-1})$$

$$K_M = \Lambda^n T^{*\prime}M \quad K_M^{-1} = \Lambda^n T'M$$

Ex. $c_1(\mathbb{P}^n) = (n+1)\,c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ $\quad$ i.e. $\underline{K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-(n+1))}$ $\Rightarrow$ tensor of dual ?

Def. B $\quad M$ cpt cpx mfd $\quad \pi: E \to M$ holo. vector bundle of rank r

h: Hermitian metric $\quad \Omega = \bar\partial(h^{-1}\partial h)$

$$\det(I + t\tfrac{i}{2\pi}\Omega) =: 1 + t\,c_1(h) + \cdots + t^r c_r(h)$$

Fact. $i=1,\cdots,r$ $\quad$ de-Rham class of $[c_i(h)]$ is independent of h

Def $\quad c_i(E) = [c_i(h)] \in H^{r,r}(M,\mathbb{R})$ $\quad \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{S^1}$

i-th Chern class of E $\quad$ (we'll do this in the generalized form on the Chern-Weil theory $U(r)$)

Thm. D divisor on M, L_D the associated line bundle
then $c_1(L_D) \in H^2_{DR}(M,\mathbb{R})$ is the Poincaré dual of $[D] \in H_{2n-2}(M,\mathbb{R})$
i.e. if $Z \in H_2(M,\mathbb{R})$, then $\langle c_1(L_D), Z \rangle = \underbrace{[D]\cdot Z}_{\text{intersection number}}$

Thm (above thm when $n=1$) M Riemann surface
$\underset{\text{cpt}}{\sim}$

$D = \sum_{i=1}^{l} v_i p_i$ $\qquad \langle c_1(L_D), [M] \rangle = \deg D$

pf. $U_i \ni p_i$ $\quad M$

may take a metric on L_D s.t. for $\lambda = 1, \cdots, k$ $\quad h_{U_\lambda} \triangleq h(\bar{e}_\lambda, e_\lambda) = 1$ on U_λ
take

more details for the pf, see the photos

$h(\bar{s}, s)$
$\parallel$
$h(\overline{f_\lambda e_\lambda}, f_\lambda e_\lambda) = \bar{f}_\lambda f_\lambda h_{U_\lambda}$

then $\frac{i}{2\pi} \log h(\bar{s}, s) \in C^\infty(M - \{p_1, \cdots, p_k\})$

$-\frac{i}{2\pi}\partial\bar{\partial} \log h(\bar{s}, s) = -\frac{i}{2\pi}\partial\bar{\partial} \log h_{U_\lambda} = c_1(L_D, h)$

$\langle c_1(L), [M] \rangle = \int_{M - \{p_1, \cdots, p_k\}} c_1(L_D, h) \quad \frac{-i}{2\pi}\int_M \bar{\partial}\partial \log h(\bar{s}, s)$

$c_1(E, h) = \frac{i}{2\pi} \operatorname{tr}\Omega = \frac{i}{2\pi} \operatorname{tr}(\bar{\partial}(h^{-1}\partial h)) = \frac{i}{2\pi}\bar{\partial} \operatorname{tr}(h^{-1}\partial h)$

$= \frac{i}{2\pi}\bar{\partial}\left(\frac{1}{\det h}\partial \det h\right)$

$= \frac{i}{2\pi}\bar{\partial}\partial \log \det h_U$

$$\boxed{\frac{d}{dx}\det(A(x)) = \det A(x) \operatorname{tr}\left(A^{-1}(x)\frac{dA(x)}{dx}\right)}$$

claim when $E = T^{1,0}M$ $\quad c_1(M) \triangleq c_1(T^{1,0}M)$ in the sense of B

it coincides with A $\quad$ since

h: Hermitian metric for $T^{1,0}M$ then $\det h$ is a Hermitian metric for $K_M^{-1} = \Lambda^n T^{1,0}M$

last time: if g is Hermitian for $T^{1,0}M$

write $g = g_{i\bar{j}}\, dz^i \otimes d\bar{z}^j$ then $\mathrm{Re}(g)$ a Riemannian metric

$\mathrm{Im}(g)$ skew-symmetry

Q: we have the chern connection of g on $T^{1,0}M$ (extend $\to T^{1,0}M \oplus T^{0,1}M = T_{\mathbb{C}}M$, $\nabla_X \bar{Y} \triangleq \overline{\nabla_{\bar{X}} Y}$), and also have a Levi-Civita connection of $\mathrm{Re}(g)$ on $T_{\mathbb{C}}M$

when do they coincide?

A: when g satisfying the kähler condition (equivalent)

$\overline{g_{i\bar{j}}} = g_{j\bar{i}}$

Def. $\gamma = -2\mathrm{Im}(g) = i\sum\left(g_{i\bar{j}}\, dz^i \otimes d\bar{z}^j - g_{j\bar{i}}\, d\bar{z}^i \otimes dz^j\right)$

$= i\sum g_{i\bar{j}}\left(dz^i \otimes d\bar{z}^j - d\bar{z}^j \otimes dz^i\right) = i\sum g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$

a real $(1,1)$ form called the fundamental 2-form

Def γ is called kähler if $d\gamma = 0 \iff \nabla\gamma = 0$ where ∇ is the Levi-Civita connection

in this case, g is called kähler

(M, g) a kähler mfd

Thm. $\underset{\nabla}{\text{Chern connection}}$ = Levi-Civita connection

$\Updownarrow$

$d\gamma = 0$

then for kähler mfd they're compatible

$$\frac{\partial g_{i\bar{k}}}{\partial \bar{z}^j} = \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^k}$$

we show $\nabla_X Y - \nabla_Y X = 0 \iff d\gamma = 0$

$$\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = \frac{\partial}{\partial z^p}\, \omega^p_j\left(\frac{\partial}{\partial z^i}\right) = \frac{\partial}{\partial z^p}\left(g^{p\bar{k}} \frac{\partial g_{j\bar{k}}}{\partial z^i}\right)$$

$(\|)$

$$\nabla_{\frac{\partial}{\partial z^j}} \frac{\partial}{\partial z^i} = \left(g^{p\bar{k}} \frac{\partial g_{i\bar{k}}}{\partial z^j}\right)\frac{\partial}{\partial z^p} \iff \frac{\partial g_{i\bar{k}}}{\partial z^j} \text{ symmetric in } i \,\&\, j$$

($\Leftarrow$ symmetric in k & j?)

$$d\gamma = \sqrt{-1}\, \underbrace{\frac{\partial g_{i\bar{k}}}{\partial z^j}\, dz^j \wedge dz^i \wedge d\bar{z}^k}_{(2,1)} + \sqrt{-1}\, \underbrace{\frac{\partial g_{i\bar{k}}}{\partial \bar{z}^j}\, d\bar{z}^j \wedge dz^i \wedge d\bar{z}^k}_{(1,2)}$$

$(=0) \iff \begin{cases} \dfrac{\partial g_{i\bar{k}}}{\partial z^j} = \dfrac{\partial g_{j\bar{k}}}{\partial z^i} \\ \boxed{\dfrac{\partial g_{i\bar{k}}}{\partial \bar{z}^j} = \dfrac{\partial g_{i\bar{j}}}{\partial \bar{z}^k}} \end{cases}$

Example. $\mathbb{P}^n_{\mathbb{C}}$ with Fubini-Study metric is kähler

$$g_{i\bar{j}} = \frac{\partial}{\partial t^i \partial \bar{t}^j} \log\left(1+|t^1|^2+\cdots+|t^n|^2\right)$$ for U_0 $z^0 \neq 0$

$$\gamma = \sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j = \sqrt{-1}\, \partial\bar{\partial} \log\left(1+|t^1|^2+\cdots+|t^n|^2\right)$$

$d\gamma = 0$ obvious

Def. cpt cpx mfd called alg if $\exists N > 0$ s.t. $\exists f : M \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ embedding

gives the reason for the def

Thm (Chow) any cpt cpx submfd in $\mathbb{P}^n_{\mathbb{C}}$ is a zero set of finite number of homogeneous polynomials (i.e. algebraic)

Ex. (M, g) kähler. γ $N \xrightarrow{f} M$ cpx submfd then $d(f^*\gamma) = 0$

$(d f^* = f^* d)$

$N \underset{f}{\hookrightarrow} (\mathbb{P}^N_{\mathbb{C}}, g_{FS})$

$(N, f^* g_{FS})$ kähler alg (mfd) $\subset$ kähler $\subset$ cpx

no proof

(M, g) cpt Riemann mfd $d\mu_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$ (ex. ind of local coord)

volume element consider as a measure

$d : \Lambda^p \to \Lambda^{p+1}$ formal adjoint $d^* : \Lambda^{p+1} \to \Lambda^p$

$\alpha : p$ $\beta : p+1$ $(\alpha, d^*\beta) = (d\alpha, \beta)$ $\Delta_d = dd^* + d^*d$ $\Lambda^p \to \Lambda^p$

what's this pairing? self-adjoint elliptic (Fredholm)

intersection? I didn't understand it well

Hodge theory + de Rham theorem $\quad \ker \Delta_{d,p} \cong H^p_{DR}(M, \mathbb{R}) \cong H^p_{\check{C}ech}(M, \mathbb{R})$ ($\mathbb{C}$ written above $\mathbb{R}$ in both)

or other like singular

on the other hand

if M is a cpt cpx mfd $\quad \bar\partial : \Lambda^{p,q} \to \Lambda^{p,q+1}$

$\bar\partial^* : \Lambda^{p,q+1} \to \Lambda^{p,q}$

$$\Delta_{\bar\partial} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^* \qquad \Lambda^{p,q} \to \Lambda^{p,q}$$

Hodge - Kodaira theory + Dolbeault thm

$$\ker \Delta_{\bar\partial} \cong H^{p,q}_{\bar\partial}(M, \mathbb{C}) \cong H^q(M, \Omega^p)$$

$H^{p,q}_{\bar\partial}$: $\ker\bar\partial / \mathrm{Im}\,\bar\partial$ $\qquad$ Ω^p: sheaf of holo p-forms

Important fact

(M, g) Kähler $\quad$ then $\Delta_d = 2\Delta_{\bar\partial}$

so $\quad \ker \Delta_d = \ker \Delta_{\bar\partial} \cong \bigoplus_{p+q=k} H^q(M, \Omega^p)$

$\ker \Delta_d \cong H^k_{sing/\check{C}ech}(M, \mathbb{C})$

application $\quad H^1(M, \mathbb{C}) = H^{1,0}_{\bar\partial}(M, \mathbb{C}) \oplus \underbrace{H^{0,1}_{\bar\partial}(M, \mathbb{C})}_{= \overline{H^{1,0}_{\bar\partial}(M, \mathbb{C})}}$

if M is Kähler

$b_1 = \dim H^1$ is even $\qquad$ equivalent condition

Thm (Miyaoka - Siu) $\quad M$ cpt cpx mfd, b_1 even, then M is Kähler next:

Chern-Weil theory for principal bundle $\longrightarrow$ to define characteristic class from the geometric aspect

Parallelism & connection of principal bundle

$\updownarrow$

frames

GL_r acts on the right

$E \to M$ $\quad \bigcup_{p\in M} \{e_1, \dots, e_r\}_{E_p}$