unitary connections (Hermitian connec) for holo v.b.

M: complex mfd of $\dim_{\mathbb{C}} M = n$ locally open sets of $\mathbb{C}^n$

change of coordinates is holo $w^i(z_1, \cdots, z^n)$

$z^i = x^i + \sqrt{-1}\, y^i$ $x^i = \frac{1}{2}(z^i + \overline{z^i})$ $w^i(x^1, y^1, \cdots, x^n, y^n) = w^i(z_1, \bar{z}_1, \cdots, z_n, \bar{z}_n)$

$$\frac{\partial z^i}{\partial \overline{w^j}} = 0$$

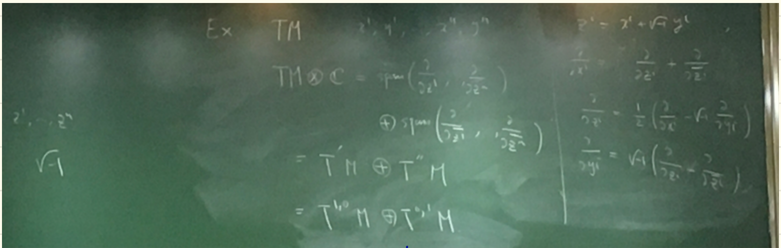
Def. $\pi: E \to M$ · rank r holo v.b. : i) $\dim_{\mathbb{C}} E = n + r$ cpx mfd

ii) π is holo onto iii) open covering $\{U_\alpha\}$, locally trivialization: $\varphi_\alpha : \pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}^r$ biholo

s.t. $P_\alpha : U_\alpha \times \mathbb{C}^r \to U_\alpha$ the projection $\pi|_{\pi^{-1}(U_\alpha)} = P_\alpha \circ \varphi_\alpha$ holomorphic

(iv) $U_\alpha \cap U_\mu \neq \phi$ $\varphi_\alpha \circ \varphi_\mu^{-1}(p, x) = (p, \psi_{\alpha\mu}(p) x)$ $\{\psi_{\alpha\mu} : U_\alpha \cap U_\mu \to \underline{GL(r, \mathbb{C})}\}$ cpx mfd

$: (U_\mu \cap U_\alpha) \times \mathbb{C}^r \to (U_\alpha \times U_\mu) \times \mathbb{C}^r$



$T'M$: holomorphic v.b. $\psi_{\alpha\mu} = \left(\frac{\partial z^i}{\partial w^j}\right)$ $T''M$ is not $T''M = \overline{T'M}$ conjugate topologically, see "characteristic class"

$T^*M \otimes \mathbb{C} = \operatorname{span}(dz^1, \cdots, dz^n) \oplus \operatorname{span}(d\bar{z}^1, \cdots, d\bar{z}^n) = T^{*1,0}M \oplus T^{*0,1}M$

$p\ q\ r\ s$ $\otimes$ $\to$ holo v.b.

$\Lambda^n T^{*1,0}M$ K_M canonical

$\Lambda^n T^{1,0}M$ K_M^{-1} anticanonical

D divisor $\qquad M = \bigcup U_\lambda$

$U_\lambda \cap D \rightsquigarrow f_\lambda$

$f_{\lambda\mu} := \dfrac{f_\lambda}{f_\mu} \rightsquigarrow$ line bundle

$D + D' \rightsquigarrow L_D \otimes L_{D'}$

for $\mathbb{P}^n_{\mathbb{C}}$ $\qquad H = \{[0 : z_1 : \cdots : z_n]\} \subseteq \mathbb{P}^n_{\mathbb{C}}$

$L_H : \mathcal{O}(1) \qquad L_H^k := \bigotimes^k L_H$

$\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad \mathcal{O}(k)$

$h \in C^\infty(M, \overline{E}^* \otimes E^*)$ Hermitian metric

h_p Hermitian form for $\forall p \in M$

matrix $\qquad \begin{pmatrix} h_{\bar{i}j} \end{pmatrix} \qquad H = {}^t\overline{H}$

$h(a, b) = \overline{h(b, a)}$

$h_{\bar{i}j} = \overline{h_{\bar{j}i}}$

maybe an Hermitian metric?

- let D be a non-singular divisor, $M = \bigcup_\alpha U_\alpha$ then $D \cap U_\alpha = \{f_\alpha = 0\}$ where f_α vanishes along D with 1-st order, so there may be local representation $f_\alpha = z f_\alpha^*$ with f_α^* nowhere vanishing holomorphic function
  then $f_{\alpha\mu} := \frac{f_\alpha}{f_\mu} = \frac{f_\alpha^*}{f_\mu^*}$ turns to be a nowhere vanishing holomorphic function on $U_\alpha \cap U_\mu$
  i.e. $U_\alpha \cap U_\mu \to GL(1, \mathbb{C}) = \mathbb{C}^*$ holomorphic thus gives a line bundle on M
- let M be a cpx mfd with $\dim = n$, then if M admits a Riemannian metric g compatible with the almost cpx structure induced from the cpx structure, i.e. $g(Ju, Jv) = g(u, v)$
  we can extend J & g to $TM \otimes \mathbb{C} =: T_{\mathbb{C}}M$ $\mathbb{C}$-linearly
- originally we have local coordinates $\frac{\partial}{\partial x^1} \cdots \frac{\partial}{\partial x^n}$ $\frac{\partial}{\partial y^1} \cdots \frac{\partial}{\partial y^n}$ & $J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$ $J(\frac{\partial}{\partial y^i}) = -\frac{\partial}{\partial x^i}$
  denote $v \otimes (a+bi)$ as $av + ibv$ in $T_{\mathbb{C}}M$
  then define $\frac{\partial}{\partial z^i} = \frac{1}{2}(\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial y^i})$ $\frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}(\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i})$, which also form a basis for $T_{\mathbb{C}}M$
  then $J(\frac{\partial}{\partial z^i}) = i\frac{\partial}{\partial z^i}$ $J(\frac{\partial}{\partial \bar{z}^i}) = -i\frac{\partial}{\partial \bar{z}^i}$ thus $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ where $T^{1,0}M$ is the i-eigenvalue part & $T^{0,1}M$ the $-i$ part
  (this can be done in general: V is a dim-$2n$ real v.s., with an almost complex structure J
  then we can give V a basis $e_1, \cdots, e_n, Je_1, \cdots, Je_n$
  then in $V^{\mathbb{C}}$, we have a basis $z_i := \frac{1}{2}(e_i - iJe_i)$ $\bar{z}_i := \frac{1}{2}(e_i + iJe_i)$
  extend J $\mathbb{C}$-linearly, then $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ similarly)
- then we see that on $T^{1,0}M$ or $T^{0,1}M$, $g = 0$ since $g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}) \overset{!}{=} g(J\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial z^j})$ this is right since both g & J are extended linearly
  $= g(i\frac{\partial}{\partial z^i}, i\frac{\partial}{\partial z^j}) = -g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j})$, same for $g(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}) = 0$
  define a Hermitian metric on $T^{1,0}M$: $h(u, v) := g(u, \bar{v})$ $\overline{\frac{\partial}{\partial z^i}} = \frac{\partial}{\partial \bar{z}^i}$
  note that for u with J-eigenvalue $\pm i$, $\bar{u}$ has J-eigenvalue $\mp i$
  $h(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}) = g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}) =: g_{i\bar{j}}$
  $$\left(\overset{\shortparallel}{g}(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^i}) = \overline{g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^i})} = \overline{h(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^i})} = \overline{g_{j\bar{i}}} \right)$$
  so we have $h = (g_{i\bar{j}})$ a Hermitian metric on $T^{1,0}M$

- we can define a 2-form $\omega_g(u, v) := -g(u, Jv)$ $(= -g(Ju, -v) = g(Ju, v) = g(v, Ju) = -\omega_g(v, u))$
  $\longrightarrow$ $\omega_g = i\sum_{ij} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$ called the Kähler form for (M, J, g) (M is a $2n$-dim real mfd)
  not contained in today's course, may be introduced later
  where $z_i = x_i + iy_i$ hence $dz_i = dx_i + i\,dy_i$ $d\bar{z}_i = dx_i - i\,dy_i$ $dz_i(\frac{\partial}{\partial z^j}) = \delta_{ij}$
  $d\bar{z}_j(\frac{\partial}{\partial \bar{z}^i}) = \delta_{ij}$ $d\bar{z}_j(\frac{\partial}{\partial z^i}) = 0$ $dz_i(\frac{\partial}{\partial \bar{z}^j}) = 0$
- (M, J, g) is called almost Kähler if $d\omega_g = 0$, furthermore if $N(J) = 0$ ($\Leftrightarrow$ J integrable), it's called Kähler, where $N(J): TM \times TM \to TM$, $(v, w) \mapsto [v, w] + J[Jv, w] + J[v, Jw] - [Jv, Jw]$
  note that ω_g is always positive definite, so M is symplectic

we have a thm says: if $N(J)=0$, then $d\omega_g = 0 \iff \nabla J = 0$ where ∇ is the Levi-Civita connection for g, so a kähler mfd is a cpx mfd with the described closed form ω_g, the coefficients of ω_g forms a Hermitian metric, so we may not distinguish Kähler form & kähler metric (i.e. Hermitian metric compatible with J)

this topic is also discussed on Chern's book

get back to the course now

- Examples ① $\mathbb{P}^1_{\mathbb{C}}$ $D=\{[0:1]\}$ $U_0=\{z_0\neq 0\}$ with $s=\frac{z_1}{z_0}$ $h_0=\frac{1}{1+|s|^2}$

  $U_1=\{z_1\neq 0\}$ with $t=\frac{z_0}{z_1}$, $h_1=\frac{1}{1+|t|^2}$

  (for line bundle, a Hermitian metric is just a number at each pt)

  $U_0\cap D=\emptyset$ $U_1\cap D=\{[0:1]\}=\{t=0\}$

  $f_0=1$ may define $f_1=t$

  on $U_0\cap U_1$, $ts=1$, $f_1\neq 0$, $f_0\neq 0$ $f_{10}=t=\frac{1}{s}$ $f_{01}=\frac{1}{t}=s$ holomorphic

  $f_{10}/f_{01}: U_0\cap U_1 \to GL(1,\mathbb{C})=\mathbb{C}^*$ $\rightsquigarrow$ line bundle $\mathcal{O}(1) \rightsquigarrow \mathcal{O}(p):=\otimes^p\mathcal{O}(1)$

  (in general, $f_{\lambda\mu}: U_\lambda\cap U_\mu \to GL(r,\mathbb{C})$ holomorphic, $(f_{\lambda\mu}(p))^{-1}=f_{\mu\lambda}(p)$ as matrix, conversely, given $f_{\lambda\mu}$ as above, define an equivalent relation on $\bigcup_\lambda(U_\lambda\times\mathbb{C}^r)$

  $p\in U_\lambda\cap U_\mu$ $\underset{U_\lambda\times\mathbb{C}^r}{(p,v)}\sim\underset{U_\mu\times\mathbb{C}^r}{(p,u)} \overset{\text{def}}{\iff} v=f_{\lambda\mu}(p)u$ ($\iff f_{\mu\lambda}(p)v=u$), this gives a holomorphic v.b.)

  claim h_0, h_1 give a well-defined Hermitian metric on $\mathcal{O}(1)$

  we need to check the transition condition $h_\lambda={}^t\overline{f_{\mu\lambda}}\,h_\mu\,f_{\mu\lambda}$

  here, it's just $h_1=\frac{1}{1+|t|^2}=\frac{|s|^2}{1+|s|^2}=\frac{\bar{s}s}{1+|s|^2}={}^t\overline{f_{01}}\,h_0\,f_{01}$

  ② first, note that if the transition function for $E\to M$ & $F\to M$ are respectively $f_{\lambda\mu}, g_{\lambda\mu}$ then that for $E\otimes F\to M$ is $f_{\lambda\mu}\otimes g_{\lambda\mu}$ (as linear map)

  claim $T^{1,0}\mathbb{P}^1\cong\mathcal{O}(2)$ $(=\mathcal{O}(1)\otimes\mathcal{O}(1)$, dim 1, in this case $f_{\lambda\mu}\otimes g_{\lambda\mu}$ is just $f_{\lambda\mu}g_{\lambda\mu}$)

  sufficient to show that the transition functions are t^2 & s^2

  on U_0, we have local frame $\frac{\partial}{\partial s}$

  U_1 $-\frac{\partial}{\partial t}$

  $$-\frac{\partial}{\partial t}=-\frac{\partial s}{\partial t}\frac{\partial}{\partial s}=\frac{1}{t^2}\frac{\partial}{\partial s}=s^2\frac{\partial}{\partial s}=f_{01}^2\frac{\partial}{\partial s}$$

  (in his convention, $f_{\mu\lambda}$ is the transition from U_μ to U_λ, I think it's a little bit confusing, and he prefer to use $g_{i\bar{j}}$, not g_{ij})

- Def (Fubini-Study metric on $T^{1,0}\mathbb{P}^n$) $U_\lambda=\{z_\lambda\neq 0\}$ local coordinate $U_\lambda\to\mathbb{C}^n$ $t^0_\lambda=\frac{z_0}{z_\lambda}\cdots t^{\hat\lambda}_\lambda=1\cdots t^n_\lambda=\frac{z_n}{z_\lambda}$

  $g^{(\lambda)}_{i\bar{j}}:=\frac{\partial^2}{\partial t^i_\lambda\partial\bar{t}^j_\lambda}\log(|t^0_\lambda|^2+\cdots+1+\cdots+|t^n_\lambda|^2)$ we have

  ① $g^{(\lambda)}_{i\bar{j}}$ positive definite ② $g^{(\mu)}_{i\bar{j}}={}^tJ_{\lambda\mu}\,g^{(\lambda)}_{i\bar{j}}\,\overline{J_{\lambda\mu}}$

eg. $n=1$ $\frac{\partial^2}{\partial\bar{s}\partial s}\log(1+|s|^2)=\frac{\partial}{\partial s}\left(\frac{\bar{s}}{1+|s|^2}\right)=\frac{1}{1+|s|^2}-\frac{\bar{s}s}{(1+|s|^2)^2}=\frac{1}{(1+|s|^2)^2}$

Lemma. $E \to M$ holo v.b. (transition functions holo)

$$\bar\partial : C^\infty(M,E) \to C^\infty(M, E \otimes T^{*0,1}M)$$

$$s \mapsto \bar\partial s = \bar\partial\left((e_1,\cdots e_n)\begin{pmatrix} s^1 \\ \vdots \\ s^n \end{pmatrix}\right)$$

$$:= (e_1 \cdots e_r)\begin{pmatrix} \bar\partial s^1 \\ \vdots \\ \bar\partial s^n \end{pmatrix}$$

$s^1 \cdots s^n$ smooth functions

$e_1 \cdots e_n$ local frame

$$\bar\partial f = \frac{\partial f}{\partial \bar z^j} d\bar z^j$$

have to show there's no trouble (conflict) when taking another local frame:

$$(e_{\mu 1}, \cdots, e_{\mu r}) = (e_{\lambda 1}, \cdots, e_{\lambda r}) f_{\lambda\mu}$$

$$s = (e_{\mu 1} \cdots e_{\mu r})\begin{pmatrix} t^1 \\ \vdots \\ t^r \end{pmatrix} = (e_{\lambda 1} \cdots e_{\lambda r})\begin{pmatrix} s^1 \\ \vdots \\ s^r \end{pmatrix}$$

$$= (e_{\lambda 1} \cdots e_{\lambda r}) f_{\lambda\mu}\begin{pmatrix} t^1 \\ \vdots \\ t^r \end{pmatrix}$$

have to show: $\bar\partial s = (e_{\mu 1} \cdots e_{\mu r})\begin{pmatrix} \bar\partial t^1 \\ \vdots \\ \bar\partial t^r \end{pmatrix} = (e_{\lambda 1} \cdots e_{\lambda r})\begin{pmatrix} \bar\partial s^1 \\ \vdots \\ \bar\partial s^r \end{pmatrix}$

$$= (e_{\lambda 1} \cdots e_{\lambda r}) f_{\lambda\mu}\begin{pmatrix} \bar\partial t^1 \\ \vdots \\ \bar\partial t^r \end{pmatrix}$$

since $f_{\lambda\mu}$ is holo (for holo, $\bar\partial = 0$)

$$= (e_{\lambda 1} \cdots e_{\lambda r})\, \bar\partial\left(f_{\lambda\mu}\begin{pmatrix} t^1 \\ \vdots \\ t^r \end{pmatrix}\right)$$

Rmk. there's no well-defined ∂-operator on $C^\infty(M,E)$

Thm. (Chern connection) $E \to M$ holo-v.b. h: Hermitian metric

then $\exists$ unique connection ∇ satisfying $(\nabla: C^\infty(M,E) \to C^\infty(M, E \otimes (T^*M \otimes \mathbb{C})))$

① if we write $\nabla = \nabla^{1,0} + \nabla^{0,1}$ $C^\infty(M,E) \to C^\infty(M, E \otimes T^{*1,0}M) \oplus C^\infty(M, E \otimes T^{*0,1}M)$

then $\nabla^{0,1} = \bar\partial$

② $d(h(\bar s, t)) = h(\overline{\nabla s}, t) + h(\bar s, \nabla t)$ i.e. $\nabla h = 0$

ideal: " $d = \partial + \bar\partial$ "

proof. Local frame $e_1 \cdots e_r$. if such a ∇ exists

$' \leftrightarrow 1,0$

$'' \leftrightarrow 0,1$

$$\nabla e_j = (\nabla' + \bar\partial) e_j = \nabla' e_j \qquad (\bar\partial 1 = 0)$$

$$\begin{array}{c} \| \\ e_i w^i_j \end{array} \qquad \begin{array}{c} \mathbin{\rotatebox{90}{$\subset$}} \\ C^\infty(M, E \otimes T'^* M) \end{array}$$

So w^i_j is linear combination of $dz^1 \dots dz^n$

$$d h_{\bar i j} = h(\overline{\nabla e_i}, e_j) + h(\bar e_i, \nabla e_j)$$

$$\begin{array}{c} \| \\ \partial h_{\bar i j} + \bar\partial h_{\bar i j} \end{array} = h(\overline{e_k w^k_i}, e_j) + h(\bar e_i, e_k w^k_j) = \overline{w^k_i}\, h_{\bar k j} + h_{\bar i k} w^k_j$$

$$\Rightarrow \partial h_{\bar i j} = h_{\bar i k} w^k_j \qquad \bar\partial h_{\bar i j} = \overline{w^k_i}\, h_{\bar k j}$$

$$\Updownarrow$$

$$\partial h_{\bar k j} = h_{\bar k i} w^i_j$$

taking $(h^{i\bar j}) = (h_{\bar i j})^{-1}$ i.e. $h^{i\bar k} h_{\bar k j} = \delta_{ij}$

then $(w^i_j) = (h^{i\bar k}\, \partial h_{\bar k j})$

may write $w = h^{-1} \partial h$ memorize

wish to define w by this , sufficient to show the transformation law

$$w_\mu = \psi_{\lambda\mu}^{-1} w_\lambda \psi_{\lambda\mu} + \psi_{\lambda\mu}^{-1} \underline{d\psi_\mu} \; (= \partial + \bar\partial \;)$$

$$\bar\partial \to 0$$

$$h_\mu = {}^t\overline{\psi_{\lambda\mu}}\, h_\lambda \psi_{\lambda\mu} \qquad w_\mu = h_\mu^{-1} \partial h_\mu = \psi_{\lambda\mu}^{-1} h_\lambda^{-1}\, {}^t\overline{\psi_{\lambda\mu}}^{-1} \left({}^t\overline{\psi_{\lambda\mu}}\, \partial h_\lambda \psi_{\lambda\mu} + {}^t\overline{\psi_{\lambda\mu}}\, h_\lambda \partial \psi_{\lambda\mu} \right) = \cdots$$

Thm. $\Omega = \bar\partial w$ curvature form

pf. $w = h^{-1} \partial h$

$$\Omega = dw + w \wedge w = \bar\partial w + \partial(h^{-1} \partial h) + h^{-1}\partial h \wedge h^{-1} \partial h$$
$$= \bar\partial w + (-h^{-1} \partial h\, h^{-1} \wedge \partial h + 0) + h^{-1} \partial h \wedge h^{-1} \partial h$$
$$= \bar\partial w$$

Cor. $L \to M$ line bundle h: Hermitian metric locally positive smooth function
$\;\;\;\;\;\;\;\;\;$holo

then $\Omega = \bar\partial(h^{-1} \partial h) = \bar\partial \partial \log h = -\partial\bar\partial \log h$ memorize

$L \to M$ line bundle $\quad h$: Hermitian metric

$U \subset M$ open $\quad s : U \to L$ nowhere zero section

$e_1 = s \qquad h_U := h(\bar{s}, s) > 0$

$\Omega = \bar\partial (h_U^{-1} \partial h_U) = \bar\partial \partial \log h_U$

Lemma 1. $\bar\partial\partial \log h_U$ independent of the choice of local section s

2. $\bar\partial\partial \log h_U = \bar\partial \partial \log h_V$ if $U \cap V \neq \phi$

$\rightsquigarrow$ $\bar\partial\partial \log h_U$ is defined on M globally

Def. $c_1(L, h) = \frac{i}{2\pi}\Omega = \frac{i}{2\pi} \bar\partial\partial \log h_U = -\frac{i}{2\pi} \partial\bar\partial \log h_U$

1-st chern form of L w.r.t. h

Thm 1, $c_1(L, h)$ is a real $(1,1)$-form

2, de Rham class $[c_1(L,h)]$ independent of h

pf. 2, h' another Hermitian metric

$h_V = |\psi|^2 h_U \qquad h'_V = |\psi|^2 h'_U \qquad \frac{h_V}{h'_V} = \frac{h_U}{h'_U} = f > 0$

f is a global function on M

$c_1(L, h) = \frac{i}{2\pi} \bar\partial\partial \log h_U = \frac{i}{2\pi} \bar\partial\partial(\log h'_U + \log f) = c_1(L, h') + \frac{i}{2\pi} \bar\partial\partial \log f$

$\cdot \frac{i}{2\pi} \bar\partial\partial \log f = d(\bar\partial \log f)$ $\quad$ since $d = \partial + \bar\partial$, $\partial^2 = 0$

exact form

$\otimes$ $c_1(L, h)$ is closed since $d(\bar\partial\partial) = (\partial + \bar\partial)(\bar\partial\partial)$

Def. $c_1(L) := [c_1(L,h)] \in H^2_{dR}(M,\mathbb{R})$ 1-st Chern class of L

Example. $\mathcal{O}_{\mathbb{P}^n}(k) = H_n^{\otimes k}$ $\qquad H_n = \mathcal{O}_{\mathbb{P}^n}(1)$ hyperplane bundle

$n=1 \quad L = \mathcal{O}_{\mathbb{P}^1}(k)$

$\mathbb{P}^1 = S^2 \quad H^2(S^2,\mathbb{Z}) \cong \mathbb{Z}$ $\qquad U_0 = \{z^0 \neq 0\} \quad U_1 = \{z^1 \neq 0\}$ $\qquad f_{01} = s^k \quad f_{10} = t^k$

$s = \frac{z^1}{z^0} \qquad t = \frac{z^0}{z^1}$

$$h_{U_0} = \frac{1}{(1+|s|^2)^k} \qquad c_1(L, h_{U_0}) = -\frac{i}{2\pi}\partial\bar\partial \log \frac{1}{(1+|s|^2)^k}$$

$$= k\frac{i}{2\pi}\partial\bar\partial\log(1+|s|^2)$$

$$\langle c_1(\mathcal{O}_{\mathbb{P}^1}(k)), [\mathbb{P}^1]\rangle = k\int_{U_0}\frac{i}{2\pi}\partial\bar\partial\log(1+|s|^2) \qquad \boxed{s \in U_0 \cong \mathbb{C}}$$

$$\overset{\shortparallel}{k\int_0^\infty \frac{2r\,dr}{(1+r^2)^2}} \quad (s = re^{i\theta} \text{ change variable})$$

$$= k$$

Cor. $\langle c_1(T'\mathbb{P}^1), [\mathbb{P}^1]\rangle = 2 = \chi(S^2)$

Def.A M n-dim cpt cpx mfd

$$c_1(M) \overset{\text{def}}{\triangleq} -c_1(K_M) = c_1(K_M^{-1})$$

$$K_M = \Lambda^n T^{*\prime}M \qquad K_M^{-1} = \Lambda^n T'M$$

Ex. $c_1(\mathbb{P}^n) = (n+1)\,c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ i.e. $\underline{K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-(n+1))}$ $\Rightarrow$ tensor of dual ?

Def. B M cpt cpx mfd $\pi: E \to M$ holo. vector bundle of rank r

h: Hermitian metric $\qquad \Omega = \bar\partial(h^{-1}\partial h)$

$$\det\left(I + t\frac{i}{2\pi}\Omega\right) =: 1 + t\,c_1(h) + \cdots + t^r c_r(h)$$

Fact. $i=1,\cdots,r$ de-Rham class of $[c_i(h)]$ is independent of h

Def $c_i(E) = [c_i(h)] \in H^{r,r}(M,\mathbb{R})$ $\qquad \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, $\overset{\shortparallel}{S^1}$

i-th Chern class of E $\qquad$ (we'll do this in the generalized form on the Chern-Weil theory $U(r)$)

Thm. D divisor on M, L_D the associated line bundle

then $c_1(L_D) \in H^2_{DR}(M, \mathbb{R})$ is the Poincaré dual of $[D] \in H_{2n-2}(M, \mathbb{R})$

i.e. if $Z \in H_2(M, \mathbb{R})$, then $\langle c_1(L_D), Z \rangle = \underbrace{[D] \cdot Z}_{\text{intersection number}}$

Thm (above thm when $n=1$) M Riemann surface
$\underset{\text{cpt}}{\sim}$

$D = \sum_{i=1}^{l} v_i p_i$ $\qquad \langle c_1(L_D), [M] \rangle = \deg D$

pf. $U_i \ni p_i$ $\quad M$

may take a metric on L_D s.t. for $\lambda = 1, \cdots, k$ $\quad h_{U_\lambda} \triangleq h(\bar{e}_\lambda, e_\lambda) = 1$ on U_λ

take

more details for the pf, see the photos

$h(\bar{s}, s)$
$\shortparallel$
$h(\overline{f_\lambda e_\lambda}, f_\lambda e_\lambda) = \bar{f}_\lambda f_\lambda h_{U_\lambda}$

then $\frac{i}{2\pi} \log h(\bar{s}, s) \in C^\infty(M - \{p_1, \cdots, p_k\})$

$-\frac{i}{2\pi} \partial\bar{\partial} \log h(\bar{s}, s) = -\frac{i}{2\pi} \partial\bar{\partial} \log h_{U_\lambda} = c_1(L_D, h)$

$\langle c_1(L), [M] \rangle = \int_{M - \{p_1, \cdots, p_k\}} \quad c_1(L_D, h) \quad \frac{-i}{2\pi} \int_M \bar{\partial}\partial \log h(\bar{s}, s)$

$c_1(E, h) = \frac{i}{2\pi} \operatorname{tr} \Omega = \frac{i}{2\pi} \operatorname{tr}(\bar{\partial}(h^{-1} \partial h)) = \frac{i}{2\pi} \bar{\partial} \operatorname{tr}(h^{-1} \partial h)$

$= \frac{i}{2\pi} \bar{\partial} \left(\frac{1}{\det h} \partial \det h \right)$

$= \frac{i}{2\pi} \bar{\partial} \partial \log \det h_U$

$$\boxed{\frac{d}{dx} \det(A(x)) = \det A(x) \operatorname{tr}\left(A^{-1}(x) \frac{dA(x)}{dx}\right)}$$

claim when $E = T^{1,0}M$ $\quad c_1(M) \triangleq c_1(T^{1,0}M)$ in the sense of B

it coincides with A $\quad$ since

h: Hermitian metric for $T^{1,0}M$ then $\det h$ is a Hermitian metric for $K_M^{-1} = \Lambda^n T^{1,0}M$

last time: if g is Hermitian for $T^{1,0}M$

write $g = g_{i\bar{j}}\, dz^i \otimes d\bar{z}^j$ then $\mathrm{Re}(g)$ a Riemannian metric

$\mathrm{Im}(g)$ skew-symmetry

Q: we have the chern connection of g on $T^{1,0}M$ (extend $\to T^{1,0}M \oplus T^{0,1}M = T_{\mathbb{C}}M$, $\nabla_X \bar{Y} \triangleq \overline{\nabla_{\bar{X}} Y}$), and also have a Levi-Civita connection of $\mathrm{Re}(g)$ on $T_{\mathbb{C}}M$

when do they coincide?

A: when g satisfying the kähler condition (equivalent)

Def. $\overline{g_{i\bar{j}}} = g_{j\bar{i}}$

$$\gamma = -2\,\mathrm{Im}(g) = i\sum \left(g_{i\bar{j}}\, dz^i \otimes d\bar{z}^j - g_{j\bar{i}}\, d\bar{z}^i \otimes dz^j\right)$$
$$= i\sum g_{i\bar{j}}\left(dz^i \otimes d\bar{z}^j - d\bar{z}^j \otimes dz^i\right) = i\sum g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$$

a real $(1,1)$ form called the fundamental 2-form

Def γ is called kähler if $d\gamma = 0 \Leftrightarrow \nabla \gamma = 0$ where ∇ is the Levi-Civita connection

in this case, g is called kähler

(M, g) a kähler mfd

Thm. $\underbrace{\text{Chern connection}}_{\nabla}$ = Levi-Civita connection $\Updownarrow$ $d\gamma = 0$

then for kähler mfd they're compatible

$$\frac{\partial g_{i\bar{k}}}{\partial \bar{z}^j} = \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^k}$$

we show $\nabla_X Y - \nabla_Y X = 0 \Leftrightarrow d\gamma = 0$

$$\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = \frac{\partial}{\partial z^p}\, \omega_j^p\left(\frac{\partial}{\partial z^i}\right) = \frac{\partial}{\partial z^p}\left(g^{p\bar{k}} \frac{\partial g_{j\bar{k}}}{\partial z^i}\right)$$

$$(\|)$$

$$\nabla_{\frac{\partial}{\partial z^j}} \frac{\partial}{\partial z^i} = \left(g^{p\bar{k}} \frac{\partial g_{i\bar{k}}}{\partial z^j}\right) \frac{\partial}{\partial z^p} \Longleftrightarrow \frac{\partial g_{i\bar{k}}}{\partial z^j} \text{ symmetric in } i \,\&\, j$$

($\Leftrightarrow$ symmetric in $k \,\&\, j$?)

$$d\gamma = \sqrt{-1}\, \underbrace{\frac{\partial g_{i\bar{k}}}{\partial z^j}\, dz^j \wedge dz^i \wedge d\bar{z}^k}_{(2,1)} + \sqrt{-1}\, \underbrace{\frac{\partial g_{i\bar{k}}}{\partial \bar{z}^j}\, d\bar{z}^j \wedge dz^i \wedge d\bar{z}^k}_{(1,2)}$$

$(=0) \Leftrightarrow \begin{cases} \dfrac{\partial g_{i\bar{k}}}{\partial z^j} = \dfrac{\partial g_{j\bar{k}}}{\partial z^i} \\ \boxed{\dfrac{\partial g_{i\bar{k}}}{\partial \bar{z}^j} = \dfrac{\partial g_{i\bar{j}}}{\partial \bar{z}^k}} \end{cases}$

Example. $\mathbb{P}^n_{\mathbb{C}}$ with Fubini-Study metric is kähler

$$g_{i\bar{j}} = \frac{\partial}{\partial t^i \partial \bar{t^j}} \log\left(1+|t^1|^2+\cdots+|t^n|^2\right) \quad \text{for } U_0 \; z^0 \neq 0$$

$$\gamma = \sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge d\bar{z^j} = \sqrt{-1}\, \partial\bar{\partial} \log\left(1+|t^1|^2+\cdots+|t^n|^2\right)$$

$d\gamma = 0$ obvious

Def. cpt cpx mfd called alg vrf if $\exists N > 0$ s.t. $\exists f: M \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ embedding

gives the reason for the def

Thm (Chow) any cpt cpx submfd in $\mathbb{P}^n_{\mathbb{C}}$ is a zero set of finite number of homogeneous polynomials (i.e. algebraic)

Ex. (M,g) kähler. γ $\quad N \xrightarrow{f} M$ cpx submfd then $d(f^*\gamma) = 0$

$(d f^* = f^* d)$

$$N \overset{f}{\hookrightarrow} (\mathbb{P}^N_{\mathbb{C}}, g_{FS})$$

$(N, f^* g_{FS})$ kähler $\qquad$ alg (mfd) $\subset$ kähler $\subset$ cpx

no proof

(M,g) cpt Riemann mfd $\quad d\mu_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$ (ex. ind of local coord)

volume element $\quad$ consider as a measure

$d: \Lambda^p \to \Lambda^{p+1}$ formal adjoint $d^*: \Lambda^{p+1} \to \Lambda^p$

$\alpha: p \quad \beta: p+1 \quad (\alpha, d^*\beta) = (d\alpha, \beta)$ $\qquad \Delta_d = dd^* + d^*d \quad \Lambda^p \to \Lambda^p$

what's this pairing? $\qquad$ self-adjoint elliptic (Fredholm)

intersection? I didn't understand it well

Hodge theory + de Rham theorem $\ker \Delta_{d,p} \cong H^p_{DR}(M, \mathbb{R}) \cong H^p_{\check{C}ech}(M, \mathbb{R})$ $\mathbb{C}$ $\mathbb{C}$

or other like singular

on the other hand

if M is a cpt cpx mfd $\bar\partial : \Lambda^{p,q} \to \Lambda^{p,q+1}$

$\bar\partial^* : \Lambda^{p,q+1} \to \Lambda^{p,q}$

$$\Delta_{\bar\partial} = \bar\partial^* \bar\partial + \bar\partial \bar\partial^* \quad \Lambda^{p,q} \to \Lambda^{p,q}$$

Hodge - Kodaira theory + Dolbeault thm

$$\ker \Delta_{\bar\partial} \cong H^{p,q}_{\bar\partial}(M, \mathbb{C}) \cong H^q(M, \Omega^p)$$

$\ker\bar\partial / \operatorname{Im}\bar\partial$

sheaf of holo p-forms

Important fact

(M, g) Kähler then $\Delta_d = 2\Delta_{\bar\partial}$

so $\ker \Delta_d = \ker \Delta_{\bar\partial} \cong \bigoplus_{p+q=k} H^q(M, \Omega^p)$

$\quad \text{\textbackslash\textbackslash}$

$H^k_{sing/\check{C}ech}(M, \mathbb{C})$

application $H^1(M, \mathbb{C}) = H^{1,0}_{\bar\partial}(M, \mathbb{C}) \oplus H^{0,1}_{\bar\partial}(M, \mathbb{C})$

$\quad \text{\textbackslash\textbackslash}$

$\overline{H^{1,0}_{\bar\partial}(M, \mathbb{C})}$

if M is Kähler equivalent condition

$b_1 = \dim H^1$ is even

Thm (Miyaoka - Siu) M cpt cpx mfd, b_1 even, then M is Kähler

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