next:

Chern-Weil theory for principal bundle → to define characteristic class from the geometric aspect

Parallelism & connection of principal bundle

↕

frames

$E \to M$ $\qquad \bigcup_{p\in M} \{e_1, \cdots, e_r\}_{E_p} \leftarrow$ GL_r acts on the right

key: expression of $\nabla_{c_* \frac{d}{dt}} v(t)$

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Now if (M, g) Kähler, then $\Delta_d = 2\Delta_{\bar\partial}$,

hence $H^k(M, \mathbb{C}) \cong \operatorname{Ker}\Delta_d = \operatorname{Ker}\Delta_{\bar\partial} \cong \bigoplus_{p+q=k} H^q(M, \Omega^p)$.

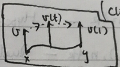
Application: We have $H^1(M, \mathbb{C}) = H^{1,0}_{\bar\partial}(M, \mathbb{C}) \oplus H^{0,1}_{\bar\partial}(M, \mathbb{C}) \cong H^{1,0}_{\bar\partial}(M, \mathbb{C}) \oplus \overline{H^{1,0}_{\bar\partial}(M, \mathbb{C})}$

hence for a Kähler mfd, the first Betti number $b_1 = \dim H^1$ is even.

A theorem of Miyaoka-Siu asserts that if M compact, the inverse holds as well.

· **Parallel Transport**: Given a vector bundle $\pi: E \to M$ and a connection ∇, we want to transport a vector $v(0) \in E_x = \pi^{-1}(x)$ along the curve $c: [0,1] \to M$, ($c(0) = x$, $c(1) = y$.)

It is called parallel, if $\nabla_c v(t) = 0$.



This can be done by solving the following ODE:

Take a frame field $e_1, \cdots, e_r$ around x, let $\nabla e_j = e_i \omega^i_j$, $v(t) = a^i e_i(c(t))$,

then $\nabla_c v(t) = \nabla_{c_* \frac{d}{dt}} v(t) = \frac{da^j}{dt} e_j + a^j \omega^i_j(c(t)) e_i$

$$= \left(\frac{da^i}{dt} + \omega^i_j(c(t)) a^j\right) e_i,$$

i.e. the transport is called parallel if $\frac{da^i(t)}{dt} + \omega^i_j(c(t)) a^j(t) = 0$, $\forall i$.

We call $v(t)$ parallel along $c(t)$, and $v(1)$ be the parallel transport of $v(0)$.

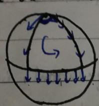
· **Holonomy**: Let $\Gamma_x = \{c: [0,1] \to M \mid c(0) = c(1) = x,\ c \text{ piecewise } C^\infty\}$,

for $\forall c \in \Gamma_x$, define a linear map $P_c: E_x \to E_x$ be the parallel transport along c,

then $\underline{Hol_x} = \{P_c \mid c \in \Gamma_x\} \leq GL(E_x)$ is called the holonomy group of M based at x.

· Eg. For $M = \mathbb{R}^n$, $\omega \equiv 0 \Rightarrow v^i(t) = v^i(0)$ $\forall i, t$. Hence $Hol = 0$.

Eg. For $M = S^2$. $Hol_x = SO(2)$, $\forall x \in S^2$.



note book

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- Restricted holonomy group: $Hol^0_{\nabla,x} = \{P_c \mid c \in \Gamma_x \text{ is a } \underline{\text{contractible curve}}\}$.

- Digression. Marcel Berger found that in each dimension, $\exists$ only finite types of holonomy. (Berger-Simons classification):
  - Holonomy groups of a symmetric space;
  - $SO(n)$;
  - $U(m), n=2m$ — Kähler mfd;
  - $SU(m), n=2m$ — Ricci-flat Kähler mfd; (Calabi-Yau n-fold) (Yau showed that such compact mfd)
  - $Sp(k), n=4k$ — HyperKähler mfd; (Fujiki)
  - $Sp(k)\cdot Sp(1), n=4k$ — Quaternionic mfd;
  - $Spin(n), n=8$;
  - G_2, $n=7$.

  (A good book: Lawson-Michelsohn, Spin Geometry)

- Associate frame bundle: Given a v.b. $E \to M$, define the associate bundle: $P_{GL} \to M$, $P_{GL} = \{(e_1, \cdots, e_r) \mid e_1, \cdots, e_r \text{ is a frame at } x, x \in M\}$.
  Note that $GL(n)$ acts on P_{GL} from the right, and each fiber of P_{GL} can be identified with $GL(n)$.

- Principal bundle: G Lie group, then $\pi: P \to M$ is a principal G-bundle if:
  ① P is a mfd and G acts on P from the right;
  ② Let $M = \bigcup_{\lambda \in \Lambda} U_\lambda$, then for each λ $\exists$ φ_λ:
  $$\begin{tikzcd} \pi^{-1}(U_\lambda) \arrow[r, "\varphi_\lambda"] \arrow[dr, "\pi"'] & U_\lambda \times G \arrow[d, "\text{projection}"] \\ & U_\lambda \end{tikzcd}$$
  ③ φ_λ is G-equivariant, i.e. if $\varphi_\lambda(p) = (\pi(p), \psi_\lambda(p))$ then $\varphi_\lambda(pg) = (\pi(p), \psi_\lambda(p)g)$.

  G is called the structure group.

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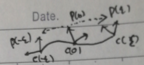


- Eg. Given a curve $c: (-\varepsilon, \varepsilon) \to M$, we define $\tilde{c}: (-\varepsilon, \varepsilon) \to P_{GL}$ by parallel transport (transporting the frame along the curve)
  (For further use, denote elements in P_{GL} as p)
  Choose $p \in P_{GL}$ s.t. $\pi(p) = x$, let $H_p = \{\frac{d\tilde{c}}{dt}(0) \mid c \text{ curve}, c(0) = x, \tilde{c}(0) = p\}$ → n-dim subspace of the $n+r^2$-dim space $T_p P_{GL}$.
  and $H = \coprod_{p \in P_{GL}} H_p$, ⟶ n-dim distribution on P_{GL}
  H is called the horizontal distribution defined by the connection of E.

Lemma 1: $\pi: P_{GL} \to M$ induces an isomorphism $\pi_*: H_p \xrightarrow{\sim} T_{\pi(p)} M$.

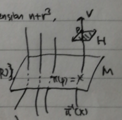
(Pf. As $\pi(\tilde{c}) = c$, $\pi_*(\frac{d}{dt}\tilde{c}) = \frac{dc}{dt}$. It is a surjective linear map.
Since $\dim H_p = \dim T_{\pi(p)} = n$, π_* is an isomorphism.)

Lemma 2: H is right invariant, i.e. $H_{pg} = R_{g*} H_p$ (where $R_g: P_{GL} \to P_{GL}$, $p \mapsto pg$).

(Pf. Let $p = (e_1, \cdots, e_n)$, $pg = (e_1 g, \cdots, e_n g)$.
Note that if $e_i(t)$ is the parallel transport of $e_i(0)$, then $e_i(t)g$ is the parallel transport of $e_i(0)g$.
Hence $H_{pg} = \{\frac{d\tilde{c}'}{dt}|_{t=0} : \tilde{c}'(0) = pg\} = \{\frac{d\tilde{c}}{dt} g|_{t=0} : \tilde{c}(0) = p\} = R_{g*} H_p$.)

- Theorem. Given a connection ∇ on E. Then we have $\underline{TP_{GL} = H \oplus V}$, where H^n is the horizontal distribution, right-invariant, and V^{r^2} is the tangent bundle along the fibers.

(Pf. As H is of dimension n, V is of dimension r^2, TP_{GL} is of dimension $n+r^2$, we need only check that $V \cap H = 0$.



By definition, the fibre passing thru p is $\pi^{-1}(\pi(p)) = \{pg \mid g \in GL(r, \mathbb{R})\}$.
Define $V_p = \{\frac{d}{dt}|_{t=0} p\exp(tX) : X \in \mathfrak{gl}(r, \mathbb{R})\} \subseteq T_p P_{GL}$.
then $\pi_* V_p = 0$. But for $\frac{d\tilde{c}}{dt} \in H_p$, $\pi_*(\frac{d\tilde{c}}{dt}) = \frac{dc}{dt} \not\equiv 0$ unless $c \equiv x$.)

note b

key: the decomposition

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$E \to M$ $\rightsquigarrow$ $P_{GL} \to M$

rank r

locally trivialization given by local frame $e_1 \cdots e_r$

$$\begin{array}{ccc} P_{GL}|_U & & (e_1(x) \cdots e_r(x))\,(a_{ij}) \\ \cong & & \updownarrow \\ U \times GL(r, \mathbb{R}) & & (x, (a_{ij})) \end{array}$$

(local coordinate)

$c : (-\varepsilon, \varepsilon) \to M$ $\rightsquigarrow$ $\tilde{c} : (-\varepsilon, \varepsilon) \to P_{GL}$

$c(0) = x$

$\pi(p) = x$

$\tilde{c}(0) = p$

$\nabla_{\frac{d}{dt}} \tilde{c} = 0$ (how does this make sense: p consists of r vectors $p_1 \cdots p_r$

$\nabla_{\frac{d}{dt}} s_i = 0$

$s_i(x) = p_i$)

$$H_p = \left\{ \frac{d\tilde{c}}{dt}(0) \;\middle|\; \cdots \right\} \cong T_{\pi(p)} M$$

H_p linear : locally elements in P_{GL} expressed as $(e_1, \cdots, e_r)(a^i_j)$

$$\tilde{c}(t) = (e_1(c(t)) \cdots e_r(c(t)))\,(a^i_j(t))$$

$$0 = \nabla_{\frac{d}{dt}} \tilde{c}(t) = (e_1(t) \cdots e_r(t))\,(\omega^i_k(\dot{c}(t)))\,(a^k_j(t)) + (e_1(t) \cdots e_r(t))\left(\frac{da^i_j}{dt}(t)\right) \qquad a^i_j(0) = \delta^i_j$$

$$t = 0 \Rightarrow \frac{da^i_j}{dt}(0) = \omega^i_j(\dot{c}(t))$$

$\frac{d\tilde{c}}{dt}(0)$: $\left(\frac{dc}{dt}(0), \left(\frac{da^i_j}{dt}(0)\right)\right) = \left(\dot{c}(0), -\omega^i_j(\dot{c}(0))\right)$ local rep

from this $\rightsquigarrow$ linear subspace

$E \to M$ $\quad$ $P_{GL} \to M$ $\quad \rightsquigarrow \quad TP_{GL} = H \oplus \bar{V}$ $\quad$ H right-invar
∇

$$\bar{V}_p = \left\{ \frac{d}{dt} \Big| \exp(tX) \Big|_{t=0} \;\Big|\; X \in gl(r, \mathbb{R}) \right\}$$

conversely, right invar distribution H s.t. $TP_{GL} = H \oplus \bar{V}$

$\exists \nabla$ on E

sufficient to give a parrallel transport of a frame along any curve c on M (this is shown in the next page)

since : local section v of E : $\nabla_X v$ for X local v.f.

$$\begin{cases} x \in M \\ c(0) = x \\ \dot{c}(0) = X \end{cases}$$

$p = (e_1, \cdots, e_r)$ frame of E_x $\quad$ parrallel transport of p along c :

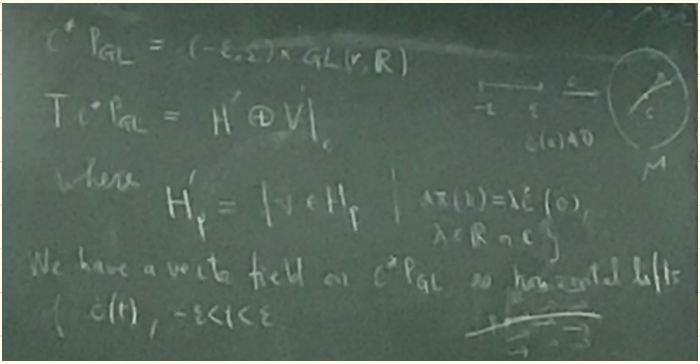
$$p(t) = (e_1(t) \cdots e_r(t))$$

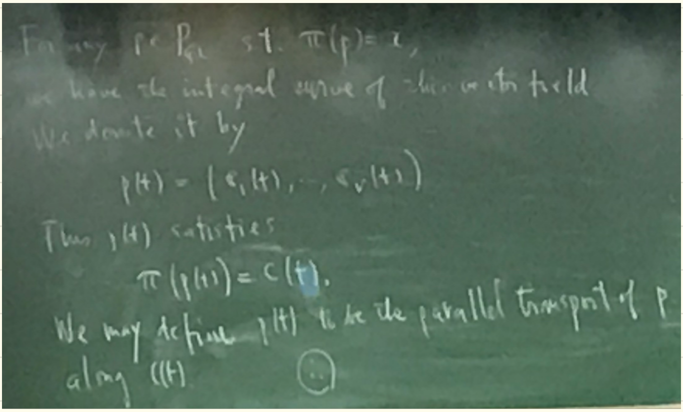
$$v(c(t)) = (e_1(t) \cdots e_r(t)) \begin{pmatrix} a^1(t) \\ \vdots \\ a^r(t) \end{pmatrix}$$

$$\nabla_X v = \nabla_{\frac{d}{dt}} v(c(t)) = (e_1 \cdots e_r) \begin{pmatrix} \frac{da^1}{dt}(0) \\ \vdots \\ \frac{da^r}{dt}(0) \end{pmatrix}$$

$c^* P_{GL} = (-\varepsilon, \varepsilon) \times GL(r, \mathbb{R})$

$T c^* P_{GL}$





P : principal G-bundle ($\Leftarrow\Rightarrow$ free G right-action)

$$p \times G \to P \qquad pg = p \Rightarrow g = id \qquad M = P/G$$
$$(p, g) \mapsto pg$$

for $A \in \mathfrak{g}$ define v.f. A^* on P

$$A^*_p = \frac{d}{dt}\Big|_{t=0} \underline{p\exp(tA)} \quad (\text{curve through } p)$$

A^* : fundamental v.f. of A

$$V_p := \{ A^*_p \mid A \in \mathfrak{g} \}$$
$$V = \coprod_{p \in P} V_p \subset TP \quad \text{vertical distribution} \quad (\text{tangent bundle along the fibre})$$

Lemma. $R_{g*} A^* = (ad(g^{-1})A)^*$

$$Ad(g) : G \to G \quad \rightsquigarrow \quad ad(g) : \mathfrak{g} \to \mathfrak{g} \quad (\text{in matrix grp } X \mapsto gXg^{-1})$$
$$g \in G \qquad x \mapsto gxg^{-1} \qquad \text{since } \mathfrak{g} \simeq T_eG$$

$$(R_{g*}(A^*))_{pg} \underset{\substack{\uparrow \\ \text{right-invar}}}{=} R_{g*}(A^*_p) = \frac{d}{dt}\Big|_{t=0} R_g(p\exp(tA)) = \frac{d}{dt}\Big|_{t=0} p\exp(tA)g$$

$$= \frac{d}{dt}\Big|_{t=0} pg\, Ad(g^{-1})\exp(tA) = (ad(g^{-1})A)^*_{pg}$$

back to P_{GL} for $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ $\qquad G = GL(r, \mathbb{R}) \qquad \mathfrak{g} = gl(r, \mathbb{R})$

Thm. right inv distribution $H \subset TP_{GL}$ is given

$\Leftrightarrow$ $\exists$ $gl(r, \mathbb{R})$-valued 1-form $\tilde{\omega}$ on P_{GL} s.t.

1) $\tilde{\omega}(A^*) = A$

2) $R_g^* \tilde{\omega} = Ad(g^{-1}) \circ \tilde{\omega}$

"$\Rightarrow$"

define $\begin{cases} \tilde{\omega}(A^*) = A \\ \tilde{\omega}(X) = 0 & X \in H \end{cases}$

$(R_g^* \tilde{\omega})(X) = \tilde{\omega}(R_{g*} X) = 0$

$(Ad(g^{-1})\tilde{\omega})(X) = ad(g^{-1})(\tilde{\omega}(X)) = 0$

$(R_g^* \tilde{\omega})(A^*) = ad(g^{-1})A = ad(g^{-1})(\tilde{\omega}(A^*))$

$\Rightarrow$ $R_g^* \tilde{\omega} = ad(g^{-1})\tilde{\omega}$

"$\Leftarrow$" $H := \ker \tilde{\omega}$

Review. $E \to M$ with given ∇

$\iff$ $\exists$ right $GL(r,\mathbb{R})$ - inv decomposition into distribution $H \subset TP_{GL}$

$TP_{GL} = H \oplus V$

$\iff$ $\exists$ $gl(r,\mathbb{R})$-valued 1-form on P_{GL} s.t.

① $\tilde{\omega}(A^*) = A$ ② $R_g^* \tilde{\omega} = ad(g^{-1}) \cdot \tilde{\omega}$

$H_p = \{ \dot{\tilde{c}}(0) \mid c \in C_x ,\ \nabla_{\frac{d}{dt}} \tilde{c} = 0 \}$ & $H = \ker \tilde{\omega}$

$\pi(p) = x$

Def. $\tilde{\omega}$ is called the connection form on P_{GL}

Thm. $e_1 \cdots e_r$ local frame on $U \subset M$, with $P_{GL}|_U \cong U \times GL(r,\mathbb{R})$ $\quad \nabla e = e\omega$ on U

$p = (e_1 \cdots e_r)A \longleftrightarrow (x, (a_j^i))$

then $\tilde{\omega} = A^{-1}dA + A^{-1}\omega A$

Pf. $\hat{c}(t) = e(t)A(t)$ is parrallel

some meanings of notations are omitted

$\iff 0 = e(t)\,\omega(\dot{c}(t))\,A(t) + e(t)\frac{dA(t)}{dt}$

$\iff \frac{dA(t)}{dt} + \omega(\dot{c}(t))\,A(t) = 0$

$\iff A^{-1}\frac{dA}{dt} + A^{-1}\omega A = 0$

$\iff (A^{-1}dA + A^{-1}\omega A)(\dot{A}(0) + \dot{c}(0)) = 0 \iff H = \ker(A^{-1}dA + A^{-1}\omega A) \iff \tilde{\omega} = \cdots$

Thm.7 $P = (e_1 \cdots e_r)$ frame field on U then we have $U \to P_{GL}$, there's $\tilde{\omega}$ on P_{GL}

consider $P^*\tilde{\omega}$ claim this is exactly ω

Pf. $P: U \longrightarrow P_{GL}$

$x \mapsto (x, (\delta_j^i))$

$P^*(A^{-1}dA + A^{-1}\omega A) = \omega$

$A = E$

Def. P_G principal G-bundle on M

a connection on P_G is a right-inv distribution $H \in TP_G$ s.t. $TP_G = H \oplus V$

where V is the tangent bundle along the fibre, i.e. $V = \{ A_p^* \mid A \in \mathfrak{g},\ p \in P_G \}$

H: horizontal V: vertical

$h(v): TP_G \longrightarrow H\ (V)$ projection

Thm. a right-inv distribution H s.t. $TP_G = H \oplus V$

$\Leftrightarrow \exists$ $\mathfrak{g}$-valued 1-form $\tilde{\omega}$ on P_G s.t. the proof is entirely the same

① $\tilde{\omega}(A^*) = A$ ② $R_g^* \tilde{\omega} = ad(g^{-1}) \circ \tilde{\omega}$

Def. (1) $\tilde{\omega}$ is called a connection form on P_G

(2) $\tilde{\Omega} = d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}]$ is called the curvature form

(for $\mathfrak{g}$-valued 1-form α, β $\rightsquigarrow$ $\mathfrak{g}$-valued 2-form $[\alpha,\beta](X,Y) = [\alpha(X)\beta(Y)] - [\alpha(Y)\beta(X)]$

$\rightsquigarrow \frac{1}{2}[\tilde{\omega}, \tilde{\omega}](X,Y) = [\tilde{\omega}(X)\ \tilde{\omega}(Y)]$)

(in the case $G = GL(r, \mathbb{R})$, $[\tilde{\omega}(X)\ \tilde{\omega}(Y)] = \tilde{\omega}(X)\tilde{\omega}(Y) - \tilde{\omega}(Y)\tilde{\omega}(X) = \begin{vmatrix} \tilde{\omega}(X) & \tilde{\omega}(Y) \\ \tilde{\omega}(X) & \tilde{\omega}(Y) \end{vmatrix}$

$\frac{1}{2}[\tilde{\omega}, \tilde{\omega}](X,Y) = \tilde{\omega} \wedge \tilde{\omega}(X,Y)$)

Thm. 7' the same notation in 7, $p^* \tilde{\Omega} = \Omega$

since d & $\wedge$ are linear, we use $p^* \tilde{\omega} = \omega$ freely

Notation: now we directly use ω & Ω

Thm. $\Omega(X,Y) = \Omega(hX, hY)$ $\longrightarrow$ Ω is horizontal

$= d\omega(hX, hY)$

pf. $\Omega(hX, hY) = (d\omega + \frac{1}{2}[\omega,\omega])(hX, hY)$ $\frac{1}{2}[\omega,\omega](hX,hY) = [\omega(hX), \omega(hY)] = 0$

$= d\omega(hX, hY)$ $(\omega \equiv 0$ on $H)$

$= (hX)\underset{0}{\omega(hY)} - (hY)\underset{0}{\omega(hX)} - \omega([hX, hY])$

$= -\omega([hX, hY])$

$\Omega(X,Y) = \Omega(hX+vX, hY+vY) = \Omega(hX,hY) + \Omega(hX, vY) + \Omega(vX, hY) + \Omega(vX, vY)$

assume $vY = A^*$ extend hX to horizontal v.f.

$\Omega(hX, A^*) = d\omega(hX, A^*) = \underbrace{hX\,\omega(A^*)}_{(hX)(A)=0} - A^*\underset{0}{\omega(hX)} - \omega([hX, A^*]) = (hX)A - \omega([hX, A^*])$

$A_p^* = \frac{d}{dt}\Big|_{t=0} p\exp(tA)$ $\qquad$ $\mathrm{Exp}(tA^*) = R_{\exp(tA)}$

$[A^*, hX] = \lim_{t\to 0} \frac{1}{t}\left(hX - R_{\exp(tA)*}\, hX\right)$ $\qquad$ since H is right-inv, $hX - R_{..*}\, hX \in H$ $\Rightarrow [A^*, hX] \in H \Rightarrow \omega([hX, A^*]) = 0$

$\uparrow$ see standard books of mfds

$[X,Y] = \lim_{t\to 0}\frac{1}{t}\left(Y - \mathrm{Exp}(tX)_* Y\right)$

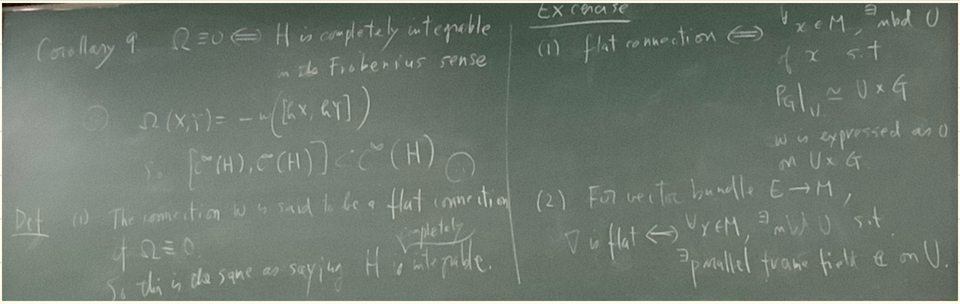
so $\Omega(hX, vY) = 0$

similarly $\Omega(vX, hY) = 0$

$\cdot\ \Omega(A^*, B^*) = d\omega(A^*, B^*) + [\omega(A^*), \omega(B^*)] = \underbrace{A^*B - B^*A}_{=0}\ \underbrace{-\,\omega(\overset{[A,B]^*}{\overset{\|}{[A^*,B^*]}}) + [A,B]}_{=0}$

Cor. $\cdot\ \Omega \equiv 0 \iff H$ is completely integrable in the Frobenius sense

Pf. $\Omega(X,Y) = -\omega([hX, hY]) \equiv 0 \iff [C^\infty(H), C^\infty(H)] \subset C^\infty(H)$



Ex.3 $\quad P_G$ has flat connection $\quad x \in M \quad \Omega\ell_x =$ precewise smooth closed curves based at x

for $c: [0,1] \to M$ $\quad c(0) = c(1) = x$ $\quad \exists\, P_c : [0,1] \to P_G$ s.t. $\pi \circ P_c = c$ & $P_c(0) = P_0 \in P_G|_x$

with $\dot{P}_c \in H$

(this may be done when P_G isn't flat & P_c is called the parallel transport)

a. if $P_c(1)$ is the parallel transport of $P_0(1)$, then

$P_c(1)g$ - - - - - - - - $P_0(1)g$

b. (require flat) $\quad c_1, c_2$ homotopic $\Rightarrow P_{c_1}(1) = P_{c_2}(1)$

c. $\rho: \pi_1(M, x) \longrightarrow G$, $\quad [c] \longmapsto h$ $\qquad \underline{P_c(1) = P_0 h}$ $\qquad \rho$ is a grp-hom

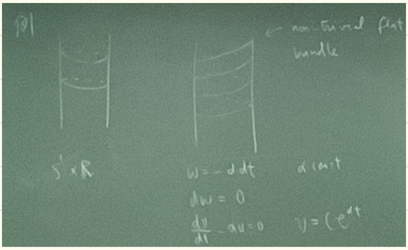
Ex.4 (on flat bundle, holonomy depends on homotopy class)

Ex.4. P_G with flat connection, then $P_G \cong \tilde{M} \times_\rho G$ (ρ: the action in 3)

$\tilde{M}$ is the universal covering $\tilde{M} \to M$ (principal $\pi_1(M)$ bundle)

Ex.5 $\{$ equivalent classes of flat G-bundles over $M \}$ $\xleftrightarrow{1-1}$ $\{ \rho : \pi_1(M) \to G \} / \sim$

$\rho \sim \rho' \iff \exists g \in G$ s.t. $\rho' = g \rho g^{-1}$



α is the "density" different α gives different

$\rho : \underset{\mathbb{Z}}{\underset{\|}{\pi_1(S^1)}} \to \mathbb{R}$ none of two are equivalent

Def. $H \subset G$ subgrp if $\exists$ principal H-bundle P_H with $P_H \overset{i}{\hookrightarrow} P_G$

s.t. $i(ph) = i(p)h$, then P_H is called the reduction of the structure grp to H.

P_H gives a section of $P_{G/H} \to M$

(G/H-bundle not neccessarily principal)

conversly, s a section on $P_{G/H} \to M$ $\rightsquigarrow$ $P_H = \underset{x \in M}{\bigcup} s(x) \hookrightarrow P_G$

$\nwarrow$ H-obrit

Example.1 E v.b. P_{GL} the frame bundle $GL(r,\mathbb{R})/O(r) \cong \{$inner products$\}$

so the reduction to $O(r)$ corresponds to choosing fiber metrics

.2 M oriented Riemannian mfd g, ∇ ∇ is a $so(n)$-connection

$e_1 \cdots e_r$ local $\underset{\text{orthogonal}}{\tilde{\text{frame}}}$ $0 = d\langle e_i, e_j \rangle = \langle \nabla e_i, e_j \rangle + \langle e_i, \nabla e_j \rangle = \omega_i^j + \omega_j^i$ skew-symmetry

i.e. $\omega \in \overset{so(n)}{Lie(SO(n))}$

$\rightsquigarrow$ extend to connection on $P_{SO(n)}$

.3 Hermitian connections are $U(n)$-connections

Lemma.9.5 $\pi : P_G \to M$ if α is a k-form on P_G s.t. $R_g^* \alpha = \alpha$

$\alpha(X_1, \cdots, X_k) = 0$ if $\pi_*(X_i) = 0$ for some i

then $\exists\, \bar{\alpha} \in \Lambda^k(M)$ s.t. $\pi^* \bar{\alpha} = \alpha$

(through this lemma, transform form on P_G to M, related to the cohomology of M,

in order to discuss characteristic class on M)

Pf. v.f. $Z_1 \cdots Z_k$ on M, take R_g^*-inv v.f. $X_1, \cdots, X_k$ on P_G s.t. $\pi_*(X_i) = Z_i$

defined $\bar{\alpha}(Z_1, \cdots, Z_k) := \underline{\alpha(X_1, \cdots, X_k)}$ constant along the fibre due to the property of α and

choice of X_i's

well-defined: suppose Y_i's is another choice s.t. $\pi_*(Y_i) = Z_i$

$\alpha(X_1, \cdots, X_k) - \alpha(Y_1, \cdots, Y_k) = \alpha(X_1, \cdots, X_k) - \alpha(Y_1, \cdots, X_k) + \alpha(Y_1, X_2, \cdots, X_k) - \alpha(Y_1, Y_2, \cdots, X_k) + \cdots$

$+ \alpha(Y_1, \cdots, Y_{k-1}, X_k) - \alpha(Y_1, \cdots, Y_k)$ $\pi_*(X_i - Y_i) = 0$, so this is zero

Characteristic classes (Chern-Weil theory)

G - compact Lie grp

Def. $I^k(G) = \{ f : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k} \to \mathbb{R} \mid f$ symmetric & multi-linear; f is $ad(g)$-inv $\forall g \}$ $\quad I(G) = \bigoplus_{k \geq 0} I^k(G)$ $\mathbb{R}$-algebra

an element of $I^k(G)$ is called a G-inv polynomial on $\mathfrak{g}$ of $\deg = k$

principal G-bundle $\pi: P_G \to M$ with connection ω $\quad (\omega(A^*) = A \quad R_g^* \omega = ad(g^{-1}) \circ \omega) \quad \Omega = d\omega + \frac{1}{2}[\omega, \omega]$

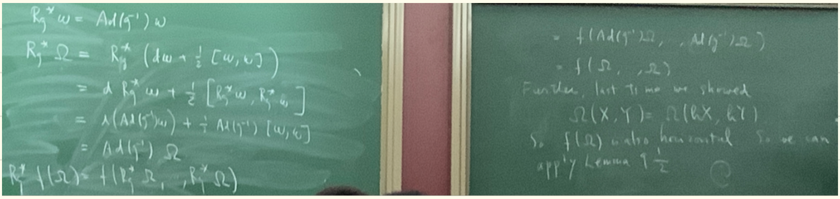
(H is changed when ω changes, V fixed, where $\ker \omega = H$, $TP_G = H \oplus V$)

Lemma.10 $\quad f \in I^k(G)$ define $f(\Omega) := f(\Omega, \cdots, \Omega)$ $\quad 2k$-form on P_G $\quad R_g^* f(\Omega) = f(\Omega)$ since f $ad(g)$-inv and $R_g^* \omega = ad(g^{-1}) \circ \omega$

$f(\Omega)$ is horizontal since Ω is, so $f(\Omega)$ satisfying conditions of Lemma. 9.5 $\Rightarrow \exists$ 2k-form on M $\overline{f(\Omega)}$ s.t. $f(\Omega) = \pi^* \overline{f(\Omega)}$

(then we're going to show that the de-Rham class is independent of the choice of ω

hence get $[\overline{f(\Omega)}]$ related to P_G and f)



(a) $d f(\Omega) = 0$ $\quad (\Rightarrow d\overline{f(\Omega)} = 0)$ $\qquad$ (b) $\underset{\Omega_0}{\omega_0} \ \underset{\Omega_1}{\omega_1}$ $\quad \overline{f(\Omega_0)} - \overline{f(\Omega_1)}$ exact

($\exists$ $Tf(\omega_0, \omega_1) \in \Lambda^{2k-1}(P_G)$ s.t. $f(\Omega_1) - f(\Omega_0) = d\, Tf(\omega_0, \omega_1)$ & $\exists$ $\overline{Tf(\omega_0, \omega_1)}$ s.t. pull back to $Tf(\omega_0, \omega_1)$)

(Lemma: π_* surjective, $\pi^* \alpha = 0 \Rightarrow \alpha = 0$)

Def. $[\overline{f(\Omega)}]$ characteristic class of P_G in terms of f

$$w : I(G) \to H^*(M, \mathbb{R}) \quad \text{alg-hom} \quad \text{Weil-homomorphism}$$
$$f \mapsto [\overline{f(\Omega)}]$$

Def. (extension covariant derivative) $\quad P_G, \underset{H}{\omega}$ $\quad$ for a p-form φ on P_G, define $D\varphi = d\varphi \circ h$

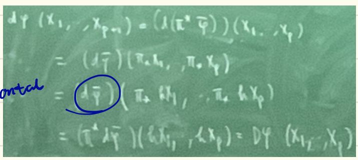
(this is the same as before in the type of formula)

Lemma.11. $\quad D\Omega = 0$ $\qquad D\Omega(X, Y, Z) = d(d\omega + \frac{1}{2}[\omega, \omega])(hX, hY, hZ)$

$= (\frac{1}{2}[d\omega, \omega] - \frac{1}{2}[\omega, d\omega])(hX, hY, hZ) \qquad \omega, d\omega \curvearrowright h = 0$

$= 0$

Lemma.12 $\quad$ if $\underset{\Lambda^p(P_G)}{\varphi} = \pi^* \bar{\varphi}$, $\bar{\varphi} \in \Lambda^p(M)$ $\quad$ then $D\varphi = d\varphi$



horizontal

proof of (a) $\quad d(f(\Omega)) = D(f(\Omega)) = \underbrace{f(D\Omega, \cdots, \Omega)}_{\text{since } \Omega\circ h = \Omega} + \cdots + \underbrace{f(\Omega, \cdots, \Omega, D\Omega)}_{\text{since } D\Omega = 0} = 0$

Lemma.13 φ be a $\mathfrak{g}$-valued 1-form on P_G s.t. $\varphi \circ V = 0$ $\quad R_g^*\varphi = ad(g^{-1})\circ\varphi$

then $D\varphi = d\varphi + \frac{1}{2}[\varphi, \omega] + \frac{1}{2}[\omega, \varphi]$ $\quad (D\varphi := d\varphi\circ h)$

proof. when X, Y are vertical, $hX = hY = 0$ then $D\varphi(X, Y) = 0$

$(d\varphi + \frac{1}{2}[\varphi, \omega] + \frac{1}{2}[\omega, \varphi])(X, Y) = \underbrace{X\underset{0}{\varphi(Y)} - Y\underset{0}{\varphi(X)} - \underset{0}{\varphi([X, Y])} + \underset{0}{\cdots}}_{=0}$ since $\varphi\circ V = 0$, $[X, Y]$ is also vertical

when X, Y are horizontal

$D\varphi(X, Y) = d\varphi(X, Y)$ $\quad (d\varphi + \frac{1}{2}[\varphi, \omega] + \frac{1}{2}[\omega, \varphi])(X, Y) = d\varphi(X, Y)$

$\omega(X) = 0 \quad \omega(Y) = 0$

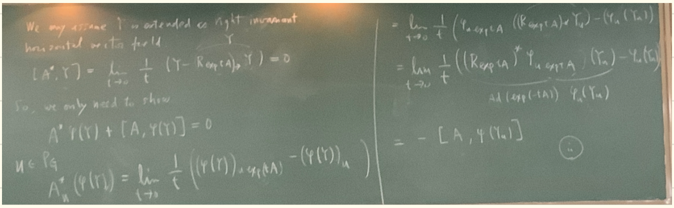
when $X = A^*$ vertical Y horizontal

$D\varphi = 0$ $\quad (d\varphi + \frac{1}{2}[\varphi, \omega] + \frac{1}{2}[\omega, \varphi])(X, Y) = A^*\varphi(Y) - \varphi([A^*, Y]) - \frac{1}{2}[\varphi(Y), \omega(A^*)] + \frac{1}{2}[\omega(A^*), \varphi(Y)]$

??? assume Y extended as R-inv horizontal v.f. $\quad = A^*\varphi(Y) - \varphi([A^*, Y]) + [A, \varphi(Y)]$

$[A^*, Y] = \lim_{t\to 0}\frac{1}{t}(Y - R_{\exp tA *}Y)$

$A^*_u(\varphi(Y)) = \lim_{t\to 0}\frac{1}{t}((\varphi(Y))_{u\exp(tA)} - \varphi(Y)_u)$ $\quad$ i.e. $\varphi_u(Y_u)$



(b) $f(\Omega_1) - f(\Omega_0) = dTf(\omega_0, \omega_1)$ $\quad Tf(\omega_0, \omega_1) = \pi^*\overline{Tf(\omega_0, \omega_1)}$

Proof. $\alpha = \omega_1 - \omega_0$ $\quad \omega_t = \omega_0 + t\alpha$ $\quad = (1-t)\omega_0 + t\omega_1$

$\alpha(A^*) = 0$ $\quad R_g^*\alpha = ad(g^{-1})\alpha$ $\quad$ satisfying conditions of Lemma.13

$\Omega_t = d\omega_t + \frac{1}{2}[\omega_t, \omega_t]$ $\quad$ claim : $\frac{d}{dt}\Omega_t = D_t\alpha$ $\quad$ (D_t the exterior covariant derivative w.r.t. ω_t)

left side $= d\frac{d\omega_t}{dt} + \frac{1}{2}[\frac{d\omega_t}{dt}, \omega_t] + \frac{1}{2}[\omega_t, \frac{d\omega_t}{dt}]$

$= d\alpha + \frac{1}{2}[\alpha, \omega_t] + \frac{1}{2}[\omega_t, \alpha] \underset{\text{lemma 13}}{=} D\alpha$

$1+2(k-1)=(2k-1)$-form

define $Tf(\omega_0,\omega_1) := k\int_0^1 f(\overset{1}{\alpha}, \overset{2}{\Omega_t}, \cdots, \overset{2}{\Omega_t})\,dt$ it's horizontal since Ω_t & α are

it's right-inv since $R_g^* \alpha/\Omega_t = ad(g^{-1})\circ \alpha/\Omega_t$ satisfying conditions of lemma.9.5

$\rightsquigarrow \overline{Tf(\omega_0\omega_1)} \in \Lambda^{2k-1}(M)$ s.t. $\pi^* \overline{Tf(\omega_0,\omega_1)} = Tf(\omega_0,\omega_1)$ then $d(Tf(\omega_0\omega_1)) \underset{\uparrow}{=} D(Tf(\omega_0\omega_1))$ $\forall t$

$= k\int_0^1 Df(\alpha,\Omega_t,\cdots,\Omega_t)\,dt \underset{\substack{\uparrow\\ \text{lemma11}}}{=} k\int_0^1 f(D\alpha,\Omega_t,\cdots,\Omega_t)\,dt = k\int_0^1 f(\frac{d\Omega_t}{dt},\Omega_t,\cdots,\Omega_t)\,dt$ lemma12

$= \int_0^1 \frac{d}{dt} f(\Omega_t)\,dt = f(\Omega_1) - f(\Omega_0)$

so $d\pi^*\overline{Tf(\omega_0,\omega_1)} = \pi^*(\overline{f(\Omega_1)} - \overline{f(\Omega_0)})$ so $\pi^*(d\overline{Tf(\omega_0\omega_1)} - (\overline{f(\Omega_1)} - \overline{f(\Omega_0)})) = 0$

$\overset{\shortparallel}{\pi^* d\,\overline{Tf(\omega_0\omega_1)}}$ $\xRightarrow{\text{surjective}}$ $d\overline{Tf(\omega_0\omega_1)} = \overline{f(\Omega_1)} - \overline{f(\Omega_0)}$ exact

i.e. $[\overline{f(\Omega_1)}] = [\overline{f(\Omega_0)}]$ in $H^{2k}_{DR}(M,\mathbb{R})$

since we don't have lemma.13 for ω_0 but this approach still works due to the simplicity

set $\omega_0 = 0$ (not a connection) $\omega_1 = \omega$ a connection do similar computation formly

$\Rightarrow f(\Omega) = dTf(\omega)$ $Tf(\omega) = k\int_0^1 f(\omega,\Omega_t,\cdots,\Omega_t)\,dt$ $\Omega_t = t\Omega$

i.e. on P_G $f(\Omega)$ is exact, but on M, $[f(\overline{\Omega})]$ isn't trivial

suppose $\overline{f(\Omega)} = 0$ then $f(\Omega) = 0$ then $Tf(\omega)$ is closed, in this situation,

does $[\overline{Tf(\omega)}]$ has any geometric meaning? (by Chern & Simons) Chern-Simons class $[CS]$

Yes, sometimes (foliations, conformal immersions ...)

$[CS] \in H^{2k-1}(M, \underset{\mathbb{R}/\mathbb{Q}}{\mathbb{R}/\mathbb{Z}})$ can be defined sometimes

Cheeges - Simons gave a better explanation later

differential characters

Example. $G = U(r)$ $\mathfrak{g} = \mathfrak{u}(r)$ skew Hermitian matrixes

$\det(E + t\frac{i}{2\pi}A) = 1 + tc_1(A) + \cdots + t^r c_r(A)$ $c_1(A) = \underline{\frac{i}{2\pi}\mathrm{tr}A} \in \mathbb{R}$ since ${}^t\bar{A} = -A$

$c_r(A) = (\frac{i}{2\pi})^r \det A$ $\Rightarrow \mathrm{tr}\bar{A} = -\mathrm{tr}A$

$\overset{\shortparallel}{\overline{\mathrm{tr}A}}$

previously, we work on holomorphic v.b., here, for smooth mfd $P_{U(r)} \to M$

we can define $c_1(E) \cdots c_r(E)$ for smooth cpx v.b. on smooth cpt mfd

Lemma. $I(U(r))$ is generated by $c_1,\cdots,c_r$

$AA^* = A^*A \Leftarrow$ A is diagonalizable by a unitary matrix

$\overset{\uparrow}{A^* = -A}$

$\det(E + \frac{i}{2\pi}PAP^{-1}) = \det(E + \frac{i}{2\pi}A)$ $\Rightarrow$ $\left.\begin{array}{l} c_1 \sim \lambda_1 + \cdots + \lambda_r \\ \vdots \\ c_r \sim \lambda_1\cdots\lambda_r \end{array}\right\}$ fundamental thm for symmetric polynomials

to be continued

in traditional situation $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$

the result should be a function: f. function

hence gives a function

in this case, ω is $\mathfrak{g}$-valued, $d\omega(X,Y)$ should be a function taking value in $\mathfrak{g}$

formly we have the same formula

we can understand $X\omega(Y)$ this way:

X is a v.f. on P_G, $\omega(Y)$ is a $\mathfrak{g}$-valued function

if $\mathfrak{g}$ has a basis, then X can act on the coefficients, hence gives $X\omega(Y)$ a $\mathfrak{g}$-valued function

in that picture, A^* act on B, a constant, hence is zero

I haven't gotten a satisfying understanding yet, maybe I can look it up on some books.

Date.

• **Chern classes**: On a complex v.b. $E \to M$ with an Hermitian metric, the structure grp reduces to $U(r)$. Choose an orthonormal frame e and a connection ω,

define $c_t(E) = [\det(Id + \frac{it}{2\pi}\Omega)]$

$= 1 + tc_1(E) + \cdots + t^r c_r(E)$, $c_i(E) \in H^{2i}_{dR}(M, \mathbb{R})$.

Total chern class: $t=1$, and $c(E) = 1 + c_1(E) + \cdots + c_r(E) \in H^*_{dR}(M, \mathbb{R})$.

• **Alternate Def (Milnor-Stasheff)**: For every complex rk-r v.b. $E \to M$, $\exists!$ $c(E) \in H^*(M, \mathbb{Z} \text{ or } \mathbb{R})$

s.t. ① $c(E) = 1 + c_1(E) + \cdots + c_r(E)$, $c_i(E) \in H^{2i}(M, \mathbb{Z} \text{ or } \mathbb{R})$;

② For $f \in C^\infty(N, M)$, $c(f^*E) = f^*c(E)$; (naturality)

③ For $E \to M$, $F \to M$, $c(E \oplus F) = c(E) \cdot c(F)$; (Whitney sum formula)

The cup product $\cup$, i.e. wedge product for forms.

④ $\langle c_1(\mathcal{O}_{\mathbb{P}^1}(-1)), \mathbb{P}^1 \rangle = -1$. (normalization axiom).

• **Claim**. The Chern classes defined by Chern-Weil theory satisfy ①~④. Hence, by uniqueness, they coincide with the Milnor-Stasheff definitions.

(Pf. ② For a connection ∇ on $E \to M$, $\exists f^*\nabla$ on $f^*E \to M$, $(f^*\nabla)_X s = \nabla_{f_*X}(s \circ f)$.

Then $\Omega(f^*\nabla) = f^*\Omega(\nabla)$, $c(f^*E) = f^*c(E)$.

③ $E \oplus F$ has a curvature form $\begin{pmatrix} \Omega_E & \\ & \Omega_F \end{pmatrix}$,

hence $\left[\det\begin{pmatrix} I+\Omega_E & \\ & I+\Omega_F \end{pmatrix}\right] = [\det(I+\Omega_E)] \cdot [\det(I+\Omega_F)]$

$= c(E) \cdot c(F)$.

④ We have checked this before.)

• **Remark**. For a complex mfd M, define $c_i(M) = c_i(T'M)$.

holomorphic tangent bundle

• **Example**. The Case $G = O(r)$, $\mathfrak{g} = \mathfrak{o}(r)$ (skew-symmetric matrices).

For $A \in \mathfrak{g}$, let $\det(I - \frac{t}{2\pi}A) = 1 + tq_1(A) + \cdots + t^r q_r(A)$

Claim: $q_i = 0$ for i odd.

(Pf. $\det(I - \frac{t}{2\pi}A) = \det({}^t(I - \frac{t}{2\pi}A)) = \det(I + \frac{t}{2\pi}A) = \det(I - \frac{t}{2\pi}(-A))$

$\Rightarrow q_i(A) = q_i(-A) = (-1)^i q_i(A)$.)

• **Pontryagin classes**. Let $E \to M$ be a real v.b. of rk r, take a fibre metric and a compatible connection, we get a connection on $P_{O(r)}$,

define $p_i(\Omega) = q_{2i}(\Omega)$, $p_i(E) = [p_i(\Omega)] \in H^{4i}(M; \mathbb{R})$.

Euler class: Actually $I(SO(2r))$ is generated by $q_2, q_4, \cdots, q_{2r}, e$,

where $e\begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_r \\ & & & -\lambda_r & 0 \end{pmatrix} = \lambda_1 \cdots \lambda_r$.

$[\frac{1}{(2\pi)^r} e(\Omega)]$ is called the Euler class.

• **Alternate Def (Milnor-Stasheff)**: $p_j(E) = (-1)^j c_{2j}(E \otimes \mathbb{C}) \in H^{4j}(M, \mathbb{Z})$.

complexified bundle

It is always of integer value.

• **Claim**. The two definitions of Pontryagin classes coincide.

(Pf. For $A \in \mathfrak{gl}(r, \mathbb{R}) \subseteq \mathfrak{gl}(r, \mathbb{C})$,

$\det(I + t\frac{i}{2\pi}A) = 1 + tc_1(A) + \cdots + t^r c_r(A)$,

$\det(I - t\frac{A}{2\pi}) = 1 + tq_1(A) + \cdots + t^r q_r(A)$

$= \det(I + t\frac{i}{2\pi}(iA))$

$\Rightarrow p_j(A) = q_{2j}(A) = c_{2j}(iA) = i^{2j} c_{2j}(A) = (-1)^j c_{2j}(A)$.)

• **Remark**. The Pontryagin classes were used by Milnor to construct non-diffeomorphic 7-spheres. He presented the 7-sphere as the boundary of two 8-manifolds whose Pontryagin classes did not coincide.

note book

No.

Date.

- **Chern characters**:

$$ch(E) := [\mathrm{tr}\exp(\tfrac{i}{2\pi}\Omega)] = [\mathrm{tr}(I + \tfrac{i}{2\pi}\Omega + \tfrac{1}{2}(\tfrac{i}{2\pi}\Omega)^2 + \cdots + \tfrac{1}{r!}(\tfrac{i}{2\pi}\Omega)^r)] \in H^{*}(M, \mathbb{Q})$$

- **Exercise**. $ch(E \oplus F) = ch(E) + ch(F)$,

  $ch(E \otimes F) = ch(E)\, ch(F)$,

  i.e. ch is a ring homomorphism: $K(M) \to H^{**}(M)$.

Remark. The splitting technique is useful in calculations: Assume formally that $E = L_1 \oplus \cdots \oplus L_r$, L_i are line bundles. See [Bott-Tu].

- Homotopy theoretic definition of char. classes:

Let G be a cpt Lie group, $\exists$ (an infinite dim'l) principal G-bundle (called the **universal bundle**, $E_G \to B_G$ (classifying space)), s.t.

① $\forall$ principal bundle $P_G \to M$ over a C^∞ mfd M, $\exists$ a classifying map $f: M \to B_G$ s.t. $f^* E_G = P_G$;

$$\begin{tikzcd} P_G \arrow[d] & E_G \arrow[l, dashed] \arrow[d] \\ M \arrow[r, "f"] & B_G \end{tikzcd}$$

② The set $[M, B_G]$ of homotopy classes of maps $M \to B_G$ has a one-to-one correspondence with the set of isomorphism classes of principal G-bundles $/M$.

- **Theorem**. $\exists$ an isomorphism $\tilde{w}: I(G) \xrightarrow{\cong} H^*(B_G)$, s.t. given $P_G \to M$, $\exists$ a commutative diagram

$$\begin{tikzcd} I(G) \arrow[r, "\cong", "\tilde{w}"'] \arrow[dr, "w"'] & H^*(B_G) \arrow[d, "f"] \\ & H^*(M) \end{tikzcd}$$

- **Homotopy theoretic definition**. The cohomology classes of $H^*(B_G, \mathbb{Z})$ are called char. classes for principal G-bundles.

Example. When $G = U(1) = S^1$, $B_{S^1} = \mathbb{CP}^\infty = \lim_{n\to\infty} \mathbb{CP}^n$. $H^*(\mathbb{CP}^\infty, \mathbb{Z}) = \mathbb{Z}[c_1]$.

When $G = U(r)$, $E_{U(r)} = \lim_{n\to\infty} U(n+r)/U(n)$

$\downarrow$

$B_{U(r)} = \lim_{n\to\infty} U(n+r)/U(n)\times U(r)$. $H^*(B_{U(r)}, \mathbb{Z}) = \mathbb{Z}[c_1, \cdots, c_r]$.