Chow's theorem and GAGA

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Abstract: This is the lectures for a report on Chow's theorem on a seminar. I'll give some basic definitions and the proof of Chow's theorem by Remmert and Stein first. Then I want to settle enough backgrounds to state the main result of the famous paper GAGA, and sketch the proof. We will use the language of sheaves, I will give some reference.

Contents

1 Introduction

There are deep relations between algeraic geometry and complex geometry. We have already known the definition for an algeraic variety and the fact that smooth complex algeraic variety becomes complex manifold. So in some sense, algebraic structures are stronger than complex structures. As we know, varieties may have singularities, so we also want to extend the concept of complex manifold by the similar way of defining affine varieties, and we get the notion of analytic variety. In some sense we may regard it as complex manifold with singularities. From the defintion later we will see that algeraic implies analytic. In general, the converse is not true, but Chow's theomrem says that in a projective space, analytic implies algeraic. Serre generalized Chow's result to a much more wilder situation in GAGA, which gives a bridge between algebraic geometry and complex analytic geometry. It is of great importance, it is also beautiful in its own way.

2 Basic definitions

Definition 1. ***analytic subvariety in*** $\mathbb{C}^n$

(1) A subset $W \in \mathbb{C}^n$ is called an analytic subspace if it is locally cut out by finitely many holomorphic functions, i.e. $\forall x \in W$, $\exists$open subset $U \subseteq \mathbb{C}^n$ and holomorphic $functions$ f_i, $i = 1, \cdots, r$ on U, s.t. the vanishing $locus$ of all f_i is $just$ $W \bigcap U$.

(2) Assume W is given as above, we give it a structure sheaf $\mathscr{H}_W := i^{-1}(\mathscr{H}_n/\mathscr{A}_W)$, where i is the natural injective map $i : W \to \mathbb{C}^n$, $\mathscr{H}_n$ is the sheaf of holomorphic fuction on $\mathbb{C}^n$, and $\mathscr{A}_U$ is the ideal sheaf of W, i.e. sheaf of holomorphic functions vanishing on W. The name ideal sheaf is due to that $\forall$open subset $U \subseteq \mathbb{C}^n$, $\mathscr{A}_W(U) \subseteq \mathscr{H}_n(U)$ is an ideal, it remains to be checked that $\mathscr{A}_W$ is actually a sheaf (exercise1). Such $(W, \mathscr{A}_W)$ is called an analytic subspace of $\mathbb{C}^n$, if W is assumed to be closed (in general it is locally closed), we call it an analytic subvariety of $\mathbb{C}^n$ (in general, such a structure is a closed immersion). Definition of analytic subsapce of U is clear in this situation, note that the structure sheaf of an analytic subsapce of U is also given by the above method.

Now we have defined the local objects, then we may define morphisms between them, hence the notion of isomorphisms. Using this, we pass to more complicated object analytic variety which is defined to be ringed space locally isomorphic to analytic subspace in $\mathbb{C}^n$, sometimes we assume the underlying set to be Hausdorff (in the analytic case, this is equivalent to seperatedness). Then we define analytic subspace of an analytic variety to be a subset which is locally analytic subspace, and it is called an analytic subvariety if it is closed as topological subspace.

Definition 2. ***analytic variety***

(1) Morphisms: W and X are respectively analytic subspace in $\mathbb{C}^r$ and $\mathbb{C}^s$ with structure sheaves. A morphism consists of a contious map of topological spaces $f : W \to X$ and a morphism of sheaves $f^\sharp : \mathscr{H}_X \to f_*(\mathscr{H}_W)$ which is induced by the pullback through f, i.e. f should pull holomorphic functions in $\mathscr{H}_X(U)$ back to holomorphic functions in $\mathscr{H}_W(f^{-1}(U))$. Since we are working in affine space, this is equivalent to say each coordinate of f is holomorphic, i.e. are sections of $\mathscr{H}_W$. Isomorphisms are invertible morphisms.

(2) Analytic variety and analytic subspace: a ringed space $(X, \mathscr{H}_X)$ is called an analytic variety if X is Hausdorff, $\mathscr{H}_X$ is a subsheaf of $\mathscr{C}_X$(the definition is given in the remark below), and $\forall x \in X$, $\exists$open subset $x \in U \subseteq X$, s.t. $(U, \mathscr{H}_X|_U)$ is isomorphic to an analytic subspace in $\mathbb{C}^n$(the morphism of sheaves should be given by pullback through the topological map). Analytic subspace is just a subset which is analytic(may be defined locally), if the subset is closed, it is called an analytic subvariety or sometimes we directly says closed analytic subspace as in GAGA.

Remark 1. ***about the definition***

(1) Construction of this type is fimilar to us: we first define something locally and then define the object we want to be a ringed space locally isomorphic to the local object. But before this we need to define the morphisms between local objects. It may be a good exercise to rewrite the definition of manifolds and give the definition of general varieties in the language of (locally) ringed sapce (exercis2).

(2) There are various definitions of analytic set, but they all coincides. The difference is for the local pieces, someone may use the notion of analytic subspace of a domain in $\mathbb{C}^n$, or in this case, affine analytic subspace may be directly defined by vanishing locus of finitely many holomorphic functions on a domain globally. The three definitions of local pieces give equivalent definition for analytic variety since if we fixed one definition, then the left two are analytic varieties in that sense (exercise3).

(3) On the structure sheaf: $(\mathbb{C}^n,\ \mathscr{H}_n)$ is actually a locally ringed sapce, i.e. the stalks are local rings, stalk at one point is just the germs of

holomorphic functions at the point. Then by the isomorphism of stalks of a sheaf and its inverse image, order change of taking quotients and taking stalks, and locality is preserved by taking quotients, $(W, \mathscr{A}_W)$ *is also locally ringed. Then since stalks are determined locally, analytic variety is locally ringed, so is its analytic subsapce.*

(4) *Sheaf in another point of view: first, refer to Ex.1.13 of chapter-2 on GTM52 and §.1 of chapter-1 on FAC, in such point of view, we concentrate on the stalks. In this case, firstly, for any topological space we have the sheaf of* $\mathbb{C}$*-valued continious functions. For* W *(resp.* $\mathbb{C}^n$*) we denote the sheaf with* $\mathscr{C}_W$ *(resp.* $\mathscr{C}_{\mathbb{C}^n}$*), since germs in* $\mathscr{C}_{\mathbb{C}^n,x}$ *naturally becomes germs in* $\mathscr{C}_{W,x}$*, we get a restriction map* π_x*. We know that* $\mathscr{H}_{\mathbb{C}^n,x} \subseteq \mathscr{C}_{\mathbb{C}^n,x}$*, so we can restrict* π_x *to* $\mathscr{H}_{\mathbb{C}^n,x}$*, and the image in* $\mathscr{C}_{W,x}$ *is exactly* $\mathscr{H}_{W,x}$*. We have the map between local rings* $\pi_x : \mathscr{H}_{\mathbb{C}^n,x} \to \mathscr{H}_{W,x}$ *with kernel* $\mathscr{A}_{W,x}$*, which consists of locally vanishing holomorphic fuctions near* x*. In this terminology, it is convenient to define morphisms through stalks.*

Until now, we have defined a good category to work with, we also have the category of algeraic varieties over $\mathbb{C}$, the main purpose is to discuss their relations. But things we've talked about are too general. In Chow's theorem, we need not to use so complicated definitions, so let us see the examples in Chow's theorem.

Example 3. ***towards analytic subvariety in*** $\mathbb{P}^n_{\mathbb{C}}$

(1) *Open subset* U *in* $\mathbb{C}^n$ *is analytic subspace of* $\mathbb{C}^n$ *since we can take the holomorphic function on* U *to be the zero funtion.*

(2) *An algeraic variety (locally isomorphic to affine algeraic variety) is analytic since affine algeraic varieties are analytic subvariety in* $\mathbb{C}$*, as polynomial functions are holomorphic.*

(3) $\mathbb{P}^n_{\mathbb{C}}$ *is algeraic hence analytic, since it has* $(n+1)$*-charts of affine space.*

So we can consider analytic subvarieties in $\mathbb{P}^n_{\mathbb{C}}$. Now we're already to state Chow's theorem:

Theorem 4. *(Chow's theorem) Any analytic subvariety in* $\mathbb{P}^n_{\mathbb{C}}$ *is algeraic.*

See Wei's lectures for the 1-st proof (the original proof given by Chow).

3 Proof by Remmert and Stein

3.1 Some notions and the statement

Four years after Chow, Remmert and Stein proved their Theorem stated below, and Chow's theorem becomes a corollary through a frequently used trick: passing to the affine cones, which we have used in the definition of projective varieties. But before the statement, I need to define the notion of dimension for an analytic variety.

Definition 5. ***Dimension***
For an analytic subvariety W in $\mathbb{C}^n$, since it's locally the vanishing locus of finite holomorphic functions, so we can define notions of regular(or smooth) points and singular points. If a neighbourhood of $p \in U \subseteq W$ is given by f_i, $i = 1, \cdots, r$, then p is regular means that the Jacobian has full rank r at p, and the dimension of W at the smooth point p $dim_p W$ is defined to be r. Then the dimension of W is defined to be $sup\{dim_p W\}_{p \in W}$.

Now we can state the theorem:

Theorem 6. *(Remmert-Stein theorem) Let V be an analytic subvariety of domain $D \in \mathbb{C}^n$, W be an irreducible analytic subvariety of $D \backslash V$, if $dimW > dimV$, then the closure $\overline{W}$ is also an irreducible analytic subvariety with $dim\overline{W} = dimW$.*

We can easily extend the category of W in the theorem to analytic subvariety with finitely many branches all with dimension greater than V, since we can consider the problem on each piece.
The basic intuition of Remmert-Stein is that it is conceivable that taking the closure of W could cause pathologies on V, and the theorem states that as long as V is small (in dimension), this does not happen.

Example 7. ***dimW > dimV is necessary***

(1) *In $\mathbb{C}^2$, let $V = \{z_1 = 0\}$, $W = \bigcup\limits_{n=1}^{\infty}\{(\frac{1}{n}, 0)\}$. W is an analytic subvarity in $\mathbb{C}^2 \backslash V$ since it is discrete. But $\overline{W} = W \bigcup \{(0,0)\}$ is not, since near (0,0) there are too many zeros, so it must be identically zero in some neighbourhood(it can be seen as a single-variable holomorphic function on an interval). It also shows that it's important to have finite branches.*

(2) *More interesting one(using Picard's great theorem): V is the same as above, $W = \{z_1 \neq 0, z_2 = e^{\frac{1}{z_1}}\}$, then $\overline{W}$ is not an analytic subvariety:*

if it were, it must have dimension one, hence its intersection with other one dimensional analytic subvariety, $X = \{z_2 = 1\}$ for instance, which isn't contained in it has dimension zero, i.e. is discrete points. But by Picard, (0,1) must be a limit point of $X \bigcap \overline{W}$, hence the intersection is not discrete.

3.2 Induce Chow's theorem

We use the same method as showing vanishing ideal of closed subset in projective space is homogeneous:

Proof. (Chow's theorem)
Consider the natural projection $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{P}^n_{\mathbb{C}}$ with full rank n everywhere. Let V be an analytic subvariety in $\mathbb{P}^n_{\mathbb{C}}$, then $\pi^{-1}(V)$ is an analytic subspace in $\mathbb{C}^{n+1}$ with dimension ≥ 1, it's an affine cone without 0. V is closed in a compact Hausdorff space hence is compact, then it has finitely many branches, it follows that $\pi^{-1}(V)$ is also. So by Remmert and Stein, $\overline{\pi^{-1}(V)} = \pi^{-1}(V) \bigcap \{0\}$ is analytic, it's a cone in $\mathbb{C}^{n+1}$.
We consider the point 0, there is a neighbourhood of $0 \in U \subseteq \overline{\pi^{-1}(V)}$ which is the vanishing locus of $f_i,\ i = 1, \cdots, r$ holomorphic on U, we now consider one of them, say, f. $\forall z \in U, f(z) = 0$, but by definition of the analytic structure on $\pi^{-1}(V)$, $\forall t \in \mathbb{C},\ f(z) = g \circ \pi(z) = g \circ \pi(tz) = f(tz)$ for some holomorphic funtion g on $\mathbb{P}^n_{\mathbb{C}}$, so $f(tz) = 0,\ \forall t \in \mathbb{C}$. Write out the series expansion of f in t^n, we immediately see that each coefficient(homogeneous of degree n) takes zero at z. So U is the vanishing locus of all the homogeneous parts, by Noetherianity, they're finite, which define V in $\mathbb{P}^n_{\mathbb{C}}$ as an algeraic variety. □

Remark 2. *The important observation is that vanishing locally in the affine space can be extended to the whole cone through the projective space. Then this will give homogeneous elements as the vanishing functions, whose vanishing locus can be defined in the projective space, they define the algeraic structure on V.*

3.3 Main idea and the proof

3.4 Proofs of some propositions used

Theorem 8. *Let $(X, \mathscr{H}_X)$ be an analytic space of pure dimension d, $\mathscr{F}$ a Fréchet algebra of holomorphic functions on X such that the injection $i : \mathscr{F} \to \mathscr{H}_X$ is continuous. If X is $\mathscr{F}$-complete, then there exists a $g \in \mathscr{F}^d$ such that X is g-complete.*

Proof. □

Theorem 9. *Let D be a domain in $\mathbb{C}^n$, $X \subseteq D$ a subset, and $g : D \to \mathbb{C}^n$ a holomorphic mapping such that $(X, g, g(D))$ is an analytic cover and X_0 is a complex submanifold of $\mathbb{C}^n$. Then X itself is an analytic subvariety of D.*

Proof. □

4 Towards GAGA

4.1 More on analytic varieties

4.2 The analytic structure of an algebraic variety

4.3 Some notions in algebra and algeraic geometry

5 GAGA

5.1 The statements

5.2 Induce Chow's theorem

5.3 Proofs

5.4 Proofs of some propositions used

6 Applications

6.1 Elementary applications of Chow's theorem

6.2 Some applications given in GAGA