Report for《Introduction to Topology》by V.A.Vassiliev

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Introduction: This book is so thin,but it contains all materials needed for basic algebraic and differentiable topology(including basic general topology,homotopy,covering space,CW-complex,homology,manifolds,fibre,cohomology),and even some higher level contents such as Morse theory and Poincaré duality.It's amazing that a light book like this could offer so much information.And the statement is also charming,so it's quite readable if you take your patience and your hands on.

Contents

1 Topological spaces and some properties

1.1 Topological spaces

Definition 1.1.1. *We call X a **topological space** if it has a structure £ that fits the several conjectures below:*

(1) X *and* $\varnothing \in$ £.

(2) $s_1, s_2 \in$ £ $\Rightarrow s_1 \cap s_2 \in$ £.

(3) $\forall$*family of sets* $S \subseteq$ £ $\Rightarrow (\bigcup_{s \in S} s) \in$ £.

From now on,we use X to denote the topological sapce $(X,$ £$)$ and use topo for a topological space.Objects $\in$ £ are called ***open sets***,and if $s \in$ £, s^c is called a ***closed set***.
Sometimes it's convenient to describle a topo with ***basis***:the basis of X is a family of sets $\mathcal{B} = \{B_i\} \subseteq$ £ such that each open set could be represented as $\bigcup B_i$ for some $B_i s$.
Let's try a special case of topo:

Definition 1.1.2. *A topo X is called a **metric space** if there exists a $\mathbb{R}_+$-function ρ:*

(1) $\rho(x, y) \geq 0$,*and* $\rho(x, y) = 0$ *if and only if* $x = y$.

(2) $\rho(x, y) = \rho(y, x)$.

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

And we define the basis to be all the sets like $\{x \in X | \rho(x, x_0) < r, x_0 \in X, d > 0\}$,then X is a topo.

Exercise 1.1. *Let* $X = \mathbb{R}^n, I^k := \{x = (x_1, \cdots, x_n) \in X | a_i < x_i < b_i\}$,*and let* I^k *be the elements of the basis,does it the same with the topo above?*

Definition 1.1.3. *(without explaination)*

(1) ***Convergence****.*

(2) ***Limitatiom point*** *of a set.*

(3) ***Closure*** *of sets.*

(4) ***Covering*** *of a set.*

(5) ***Restriction*** *of a topo.*

1.2 Homomorphism and some topological properties

Definition 1.2.1. *There is a map $f : X \to Y(X, Y$ are topos):*

(1) f is called continious if for each open set $s \subseteq Y, f^{-1}(s)$ is an open set of X.

(2) f is called a homomorphism when it's bijective and both f and f^{-1} is continious.
(what's the counterexample that f is continious but f^{-1} is not?)

Definition 1.2.2. *Some important topological properties(without explaination):*

*(1) **Connected spaces**.*

*(2) **Path**.*

*(3) **Path − connected spaces**.*

*(4) **Sequence − compact sapces and compact spaces**.*

*(5) **Hausdorff spaces**.*

Theorem 1.2.3. *Something about compact sets.*

(1) All the properties above is fixed under homomorphisms.

*(2) **(Heine − Borel)**For a set $s \in \mathbb{R}^n$:*
s is compact.
$\Leftrightarrow s$ is closed and bounded.
$\Leftrightarrow \forall$ infinite subsets of s has limitation points in s.

(3) A topo with a countable basis[1] is compact if and only if it's sequence-compact.

[1]As the name,a basis is countable if the set $\mathcal{B}$ is countable.

1.3 The operation of topological spaces

Definition 1.3.1. *Some operations(without explaination):*

(1) ***Product****.*

(2) ***Quotient****.*

(3) ***Conglutinant****.(a type of quotient)*

e.g.If X is a topo,use ΣX to denote the topo $X \times [-1,1]/\sim$.
$\sim$:points in $X \times \{-1\}$ or $X \times \{1\}$ is equivalent.ΣX is called suspensioa.[2]

Theorem 1.3.2. *Compactness is fixed under product,conglutinant.*[3]

Exercise 1.2. *Construct the Torus,Klein bottle and Möbius strip through conglutinant.*
(Hint:start from a rectangle.)

[2]纬垂
[3]Pay attention to the conditions

2 Homotopy group and homotopy equivalence

2.1 Homotopy group

Definition 2.1.1. ***Homotopic maps***:
There are two continious maps given: $f : X \to Y, g : X \to Y$*,they're homotopic if there exists a continious map* $F : X \times I \to Y$ *such that* $F(x, 1) = f(x)$ *and* $F(x, 0) = g(x)$*,denote that* $f \simeq g$*.And* F *is called a homotopic map.*

Lemma 2.1.2. $B^n/\partial B^n, i.e. B^n/S^{n-1} \simeq S^{n-1}$.[4]

Proof. Let ∂B^n correspond with the north point of S^n,and from the bound to the center,each surface corresponds with the corresponding circle on S^n from north to south. □

Definition 2.1.3. ***Homotopy group***$(A, X$ *are both path-continious topos.)*:
Choose $a_0 \in A$ *and* $x_0 \in X$*,consider the set consists of homotopy-equivalent classes of maps* $f : A \to X, f(a_0) = x_0$*,denote it with* $\pi[A, X]$*.Specially,we use* $\pi_n(X)$ *for* $\pi[S^n, X]$*.We claim that* $\pi_n(X)$ *has a group structure,and we call it* $\boldsymbol{n - dimension}$ ***homotopy group***.

Exercise 2.1. *Proof that:*

(1) $\pi_n(X)$ *actually compose a group.(Hint:use Lemma 2.1.2.)*

(2) Two homomorphous topos have the same homotopy group.

**(3) The group won't get changed as we change the base point* x_0.[5]

Exercise 2.2. *Proof that while* $n \geq 2, \pi_n(X)$ *is a commutative group.*

Question 2.3. *When will* $\pi[A, X]$ *have a group structure and when will the group be commutative?*
(Direction:if $\exists B$ *such that* $A = \Sigma B$*,*$\pi[A, X]$ *is a group;if* $A = \Sigma\Sigma B$*,*$\pi[A, X]$ *is a commutative group.)*

Definition 2.1.4. *For a given topo,if* $\pi_n(X)(n \leq k)$ *is trivial and* π_{k+1} *is not,then we call* X $\boldsymbol{k - connected}$*.Moreover,we can define "*$\boldsymbol{locally\ k -}$ ***connected****".*

[4] B^n stands for the disc in dimension n,and ∂ is the bound of the space,here,means S^{n-1}.

[5] If it's too difficult,add the condition:$n = 1$.

2.2 Homotopy equivalence

Definition 2.2.1. X, Y *are called* ***homotopy equivalent****:*
If there exist two continious maps:$f : X \to Y, g : Y \to X$*,and two homotopic maps:*$F : X \times [0,1] \to X, G : Y \times [0,1] \to Y$*,such that* $F(x,0) = x, F(x,1) = g \circ f(x), G(y,0) = y, G(y,1) = f \circ g(y)$*,i.e.*$f \circ g \simeq I_Y$ *and* $g \circ f \simeq I_X$*.*

It's easy to see that homomorphism equivalence$\subset$homotopy equivalence,e.g.$I \times S$ *and* S is homotopy equivalent but not homomorphism equivalent.

Exercise 2.4. *Proof that two homotopy equivalent topos have isomorphic homotopy group(for* $\forall n$*).What about the opposite side?*

Oh!Fuck,I don't want to go on with this!