$$\nabla : C^\infty(M, E) \to C^\infty(M, E\otimes T^*M)$$

$$\nabla(fs) = f\nabla s + s\otimes df \quad \text{for } f\in C^\infty(M)$$

$$C^\infty(M, E\otimes T^*M)\otimes C^\infty(M, TM) \xrightarrow{\tau} C^\infty(M, E)$$

covariant derivative of s in the direction of X — $\nabla_X s := \tau(\nabla s, X)$

the rank 1 trivial bundle $C^\infty(M, M\times\mathbb{R})$

the connection is the ordinary differential

for $f\in C^\infty(M)$ $\quad X\in\Gamma(TM)$

$$\nabla_X f = (df, X) = \frac{\partial f}{\partial x^i}X_i = Xf$$

base change map $f_{\lambda\mu} : U_\lambda\cap U_\mu \to GL(r,\mathbb{R})$

$$(e_{\mu 1}, \dots, e_{\mu r}) = (e_{\lambda 1}, \dots, e_{\lambda r})f_{\lambda\mu}$$

def: $\nabla e_{\lambda j} = \sum_{i=1}^{r} e_{\lambda i}\otimes \omega_{\lambda j}^{i}$ in $E\otimes T^*M$ $\quad \omega_\lambda = (\omega_{\lambda j}^{i})$ connection form (not a tensor ?)

$$\omega_\mu = f_{\lambda\mu}^{-1}\,\omega_\lambda\, f_{\lambda\mu} + f_{\lambda\mu}^{-1}\,df_{\lambda\mu}$$

conversely, a set of ω_μ with such relations $\Rightarrow$ a connection local frame / a cover

$\pi: E \to M$ ∇ a connection $\underline{e_\lambda = (e_{\lambda 1}, \cdots, e_{\lambda r})}$ on $\underline{U_\lambda, \lambda \in \Lambda}$

Def. $\nabla e_\lambda = e_\lambda \otimes \omega_\lambda = (e_{\lambda 1}, \cdots e_{\lambda r}) \otimes (\omega_{\lambda\, j}^{\ i})$ $\omega_{\lambda\, j}^{\ i} \in \Omega^1(U_\lambda)$

ω_λ : connection form

Prop. on $U_\lambda \cap U_\mu (\neq \phi)$ $\omega_\mu = f_{\lambda\mu}^{-1} \omega_\lambda f_{\lambda\mu} + f_{\lambda\mu}^{-1} d f_{\lambda\mu}$ $\underline{e_\mu = e_\lambda f_{\lambda\mu}}$

conversly, given $\{\omega_\lambda\}_{\lambda \in \Lambda}$ satisfying the transform law, we obtain a connection on E

$$e_\mu \omega_\mu = \nabla_{(\mu)} e_\mu = \nabla_{(\mu)} (e_\lambda f_{\lambda\mu}) = \nabla e_\lambda f_{\lambda\mu} + e_\lambda \otimes d f_{\lambda\mu} = e_\lambda \otimes (\omega_\lambda f_{\lambda\mu} + d f_{\lambda})$$

$$e_\lambda \otimes f_{\lambda\mu} \omega_\mu \qquad \Rightarrow f_{\lambda\mu} \omega_\mu = \omega_\lambda f_{\lambda\mu} + d f_{\lambda\mu}$$

Def. on U_α, $\Omega_\alpha = d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha$ curvature 2-form / curvature matrix

Prop. on $U_\alpha \cap U_\mu \neq \phi$ $\Omega_\mu = f_{\alpha\mu}^{-1} \Omega_\alpha f_{\alpha\mu}$ (hence a tensor)

$dP^{-1} = -P^{-1} dP P^{-1}$ (on Dong's class!)

$\Omega_\mu = d\omega_\mu + \omega_\mu \wedge \omega_\mu = d(f_{\alpha\mu}^{-1} \omega_\alpha f_{\alpha\mu} + f_{\alpha\mu}^{-1} df_{\alpha\mu}) + (f_{\alpha\mu}^{-1} \omega_\alpha f_{\alpha\mu} + f_{\alpha\mu}^{-1} df_{\alpha\mu}) \wedge (- - -)$

(steps are omitted, see the photos. Dong had computed this before)

so $\{\Omega_\alpha\}_\alpha$ gives a same endomorphism of E

$\Omega \in C^\infty(M, \mathrm{End}(E) \otimes \Lambda^2 TM)$

$\mathrm{End}(E) = E \otimes E^*$

$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$

Prop. $e\Omega(X,Y) = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})e$

proof. $\Omega(X,Y) = (d\omega + \omega\wedge\omega)(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) + \omega(X)\omega(Y) - \omega(Y)\omega(X)$

right side: $\nabla_X(e\omega(Y)) - \nabla_Y(e\omega(X)) - e\omega([X,Y]) = e\omega(X)\omega(Y) + e X(\omega(Y)) - e\omega(Y)\omega(X)$

$\nabla_Y e = (\nabla e)(Y) = (e\omega)(Y)$ $\quad - eY(\omega(X)) - e\omega([X,Y]) = e\Omega(X,Y)$

if we consider $\nabla e_\alpha = \omega_\alpha e_\alpha$ then we get $(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})e$

i.e. connection act on the left $\quad \shortparallel$

$(d\omega - \omega\wedge\omega)(X,Y)e$

from similar computation

if we take local coordinate $(x^1, \cdots, x^n)$ on U, $X = \frac{\partial}{\partial x^i}$ $Y = \frac{\partial}{\partial x^j}$ $i \neq j$, then $[X,Y] = 0$

in particular $\Omega(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X$

thus, "the curvature measures the extent in which the covariant derivative commutes"

Exercise. X, Y v.f. on M, $s \in \Gamma(E)$ $f, g, h \in C^\infty(M)$

show: $\Omega(fX, gY)(hs) = fgh\,\Omega(X,Y)s$

fundamental fact in manifold theory

$T: C^\infty(TM) \times \cdots \times C^\infty(TM) \to C^\infty(M)$

$T(f_1 X_1, \cdots, f_n X_n) = f_1 \cdots f_n T(X_1, \cdots, X_n) \Leftrightarrow T \in C^\infty(TM \otimes \cdots \otimes TM)$

Def. $\mathrm{Aut}(E) = \{\varphi \in \mathrm{End}(E) = E\otimes E^* \mid \det\varphi \neq 0\}$ automorphism bundle

$C^\infty(M, \mathrm{Aut}(E))$ gauge grp

$(\nabla_X J)(Y)$

$\mathcal{G}(E) \ni \alpha$ gauge transformation

$\nabla\left(\sum J(e_i)\otimes e_i^*\right)$
$= \sum\left(\nabla J(e_i)\otimes e_i^* + J(e_i)\otimes \nabla e_i^*\right)$

Exercise.
(1) $\mathcal{G}(E)$ grp (composition)
(2) $\mathcal{G}(E)$ acts on $C^\infty(M,E)$
(3) $\mathcal{C}(E)$ = space of connections on E
(eg. on TM, a Riemannian metric on M determines a .... connection on TM)
$\mathcal{G}(E)$ acts on $\mathcal{C}(E)$: $(\alpha, \nabla) \mapsto \alpha\nabla\alpha^{-1}$ send section to section
make sense

$\left(YM : \mathcal{C}(E) \to \mathbb{R}_{\geq 0} \qquad YM(\nabla) = \int_{(M,g)} \|R^\nabla\|^2 \, dV_g \right)$

(a) suppose given ∇ on E, then we have a connection on E^*
$\langle \nabla\alpha, s\rangle = d(\langle \alpha, s\rangle) - \langle \alpha, \nabla s\rangle$
$\alpha \in C^\infty(M, E^*)$ show this is a connection

$\nabla\alpha \in C^\infty(M, E^*\otimes T^*M)$
$\langle \nabla\alpha, \cdot\rangle \mapsto \nabla\alpha(\cdot)$
$C^\infty(M,E) \to C^\infty(M, T^*M)$
by def $\Lambda^1 M$

(b) E, F two v. bundle with ∇_E, ∇_F resp.
then we have a connection on $E\otimes F$
$\nabla(s\otimes t) = \nabla s\otimes t + s\otimes\nabla t$

(c) applying (a) & (b) to $E\otimes E^* = \mathrm{End}(E)$ $\Rightarrow$ connection on $\mathrm{End}(E)$

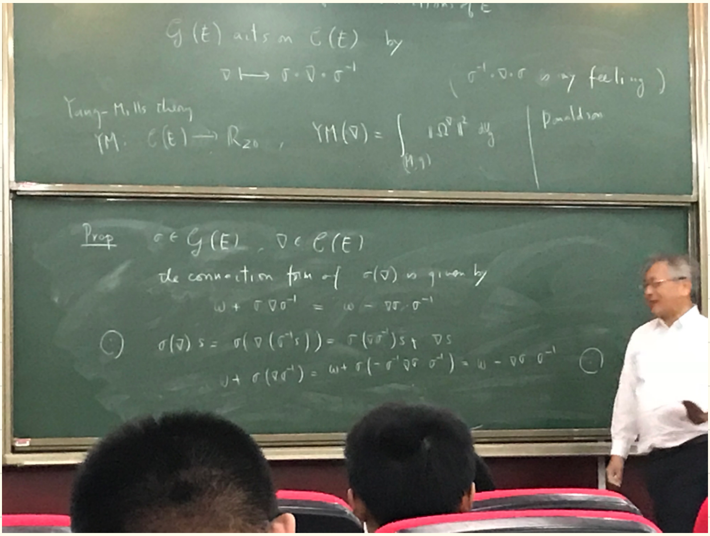
Prop. $\alpha \in \mathcal{G}(E)$ $\nabla \in \mathcal{C}(E)$ the connection form of $\alpha(\nabla)$ is given by
$\omega + \alpha(\nabla\alpha^{-1}) = \omega - \nabla\alpha\cdot\alpha^{-1}$? $\to$ (write in this form shows his individual understanding) ω is the same as ∇

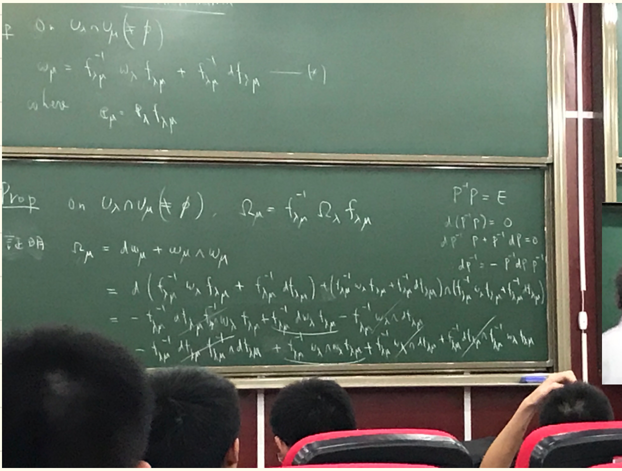
$\alpha(\nabla)s = \alpha(\nabla(\alpha^{-1}s)) = \alpha(\nabla\alpha^{-1})s + \nabla s$

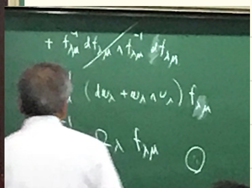
∇: a connection on $\mathrm{End}(E)$

$\omega + \alpha(\nabla\alpha^{-1}) = \omega + \alpha(-\alpha^{-1}\nabla\alpha\,\alpha^{-1}) = \omega - \nabla\alpha\cdot\alpha^{-1}$

∇: $\rightsquigarrow$ connection on $E\otimes E^*$
α^{-1}: section of $E\otimes E^*$


G(E) acts on C(E) by
∇ ↦ σ∘∇∘σ⁻¹
(σ⁻¹∘∇∘σ is my feeling)
Yang-Mills theory
YM: C(E) → R≥0 , YM(∇) = ∫(M,g) |Ω∇|² dμg
Donaldson
Prop σ ∈ G(E), ∇ ∈ C(E)
the connection form of σ(∇) is given by
ω + σ∇σ⁻¹ = ω − ∇σ·σ⁻¹
σ(∇)s = σ(∇(σ⁻¹s)) = σ(∇σ⁻¹)s + ∇s
ω + σ(∇σ⁻¹) = ω + σ(−σ⁻¹∇σ σ⁻¹) = ω − ∇σ σ⁻¹


On Uλ ∩ Uμ (≠ φ)
ωμ = fλμ⁻¹ ωλ fλμ + fλμ⁻¹ dfλμ —— (*)
where eμ = eλ fλμ
Prop On Uλ ∩ Uμ (≠ φ), Ωμ = fλμ⁻¹ Ωλ fλμ
P⁻¹P = E
d(P⁻¹P) = 0
dP⁻¹ P + P⁻¹ dP = 0
dP⁻¹ = − P⁻¹ dP P⁻¹
証明 Ωμ = dωμ + ωμ ∧ ωμ


+ fλμ⁻¹ dfλμ ∧ fλμ⁻¹ dfλμ
(dωλ + ωλ ∧ ωλ) fλμ
Ωλ fλμ

Date.

- ~~Exterior Covar. Derivative~~: $d^\nabla : C^\infty(M, E\otimes\Lambda^k(M)) \to C^\infty(M, E\otimes\Lambda^{k+1}(M))$ s.t.

$$d^\nabla(s\otimes\alpha) = \nabla s\wedge\alpha + s\otimes d\alpha$$

↑ skew-symmetrization of $\nabla s\otimes\alpha$.

- Prop-Exc:

① $d^\nabla\circ d^\nabla = \Omega$;

② (Bianchi Identity) (2nd): $d^\nabla\Omega = 0$.

2 Riemannian Geometry

- Riemannian Metric: $g\in C^\infty(M, T^*M\otimes T^*M)$ is called a Riemannian metric if locally $g=\sum g_{ij}dx^i\otimes dx^j$ where (g_{ij}) is positive-definite, symmetric at each pt.

Riemannian Mfd: (M,g).

- Thm (Levi-Civita): Given (M,g), $\exists!$ connection ∇ on TM satisfying:

① (Torsion-free): $\nabla_X Y - \nabla_Y X = [X,Y]$, $\forall X,Y\in\mathfrak{X}(M)$;

② (Compatibility): $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y,\nabla_X Z)$, $\forall X,Y,Z\in\mathfrak{X}(M)$,

OR $\nabla g = 0$. "g is parallel"

the connection is called the L-C connection / R connection.

Pf. If there is a ∇ satisfying ①,②, then

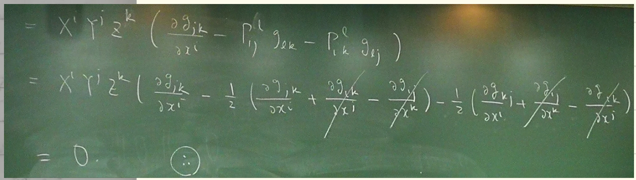
~~$Xg(Y,Z) = g(\nabla_XY, Z)$~~

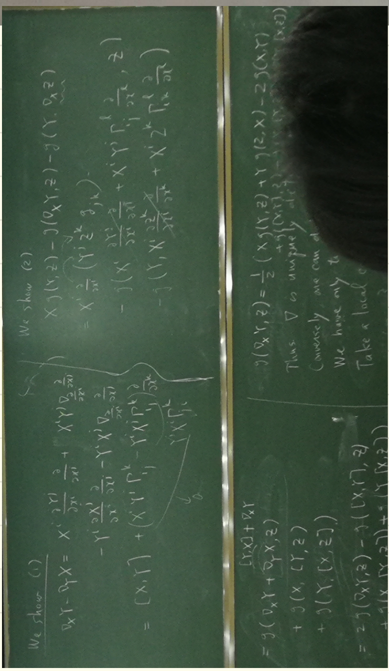
$Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)$

$=g(\nabla_XY+\nabla_YX, Z)+g(X,[Y,Z])+g(Y,[X,Z])$

$=2g(\nabla_XY,Z)-g([X,Y],Z)+g(X,[Y,Z])$

$\Rightarrow g(\nabla_XY,Z)$ can be expressed uniquely by g, X, Y, Z $\Rightarrow \nabla$ is unique.





No.

Date.

Conversely, taking local coordinates $(x^1,\cdots,x^n)$,

the expression above implies $g\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \frac12\left(\frac{\partial g_{ik}}{\partial x^j}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^k}\right)$.

Denote $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma^k_{ij}\frac{\partial}{\partial x^k}$ (Christoffel symbols)

and $g^{ij}g_{jk}=\delta^i_k$,

we have $\Gamma^k_{ij} = \frac12 g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i}+\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^l}\right)$ ← Memorize!

Then, direct calculation verifies ①,②. See photos.

- Corollary. The connection matrix is $\omega = (\Gamma^i_{jk}dx^k)$.

- Curvature: $R(X,Y)=\Omega(X,Y)=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}$.

Curvature tensor: The R above is called the curvature tensor.

Define $R(X,Y,Z,W)=g(Z,R(X,Y)W)$, this R is also called the curvature tensor.

- Prop-Exc:

① $R(X,Y,Z,W)=-R(Y,X,Z,W)$;

② $R(X,Y,Z,W)=R(Z,W,X,Y)$;

③ (Jacobi Identity) (1st Bianchi Identity): $R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W)=0$;

④ (2nd Bianchi Identity): $(\nabla_XR)(Y,Z)+(\nabla_YR)(Z,X)+(\nabla_ZR)(X,Y)=0$.

($\Leftrightarrow d^\nabla\Omega=0$)

- Sectional Curvature: For a 2-dim plane $\Pi\subseteq T_pM$ and an orthonormal basis $X,Y\in\Pi$, $K(\Pi)=R(X,Y,X,Y)$ is called the sectional curvature of Π.

Prop-Exc: $K(\Pi)$ does not depend on the choice of X,Y.

note boo

curvature tensor : $R = R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$ with $R_{ijkl} = -R_{jikl}$ $\boxed{R_{ijkl} = R_{klij}}$

$R(X,Y) \in \mathrm{End}(TM) \doteq C^\infty(M, TM \otimes T^*M)$ $\quad g(X,Y)$ can be understood like ${}^t X\, g\, Y$, with ${}^t X \in T^*M$, $Y \in TM$

two coventions
① if $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ imply $R(X,Y,Z,W) = g(Z, R(X,Y)W) = g(R(X,Y)W, Z)$

the order gets changed

② if $R(X,Y) = -\nabla_X\nabla_Y + \nabla_Y\nabla_X + \nabla_{[X,Y]}$ imply the definition $R(X,Y,Z,W) = g(R(X,Y)Z, W)$

satisfying the same property

both admits the notion of sectional curvature
$R(X,Y,X,Y)$ with $\|X\| = \|Y\| = 1$ & $g(X,Y) = 0$
many Kähler geometer use the opposite convention for $R_{i\bar{j}k\bar{l}}$???

imply

Def (Ricci curvature) on T_pM, let $e_1, \cdots, e_n$ be an orthogonal basis
$$\mathrm{Ric}(X,Y) := \sum_{i=1}^n R(X, e_i, Y, e_i)$$
Ric is symmetric in X and Y
$$\mathrm{Ric} = R_{ij}\, dx^i \otimes dx^j \qquad R_{ij} = R^{k}_{\ ikj}$$
(can also be understood as : $\mathrm{Ric}(X,Y) = \mathrm{Tr}(f_{XY})$ where $f_{XY} : T_pM \to T_pM$, $Z \mapsto R(Z,Y)X$)

(Einstein) if $\exists k \in \mathbb{R}$ constant, s.t. $R_{ij} = k g_{ij}$ i.e. $\mathrm{Ric} = kg$, then g is called Ricci-flat or Einstein

note that $\omega \leftrightarrow \Gamma$ can be seen as the 1st derivate of g
and $R = d\omega + \omega \wedge \omega$ the 2-nd
so $\mathrm{Ric} = kg$ may be seen as 2-nd parital differential equations in terms of g
it's the " field equation for gravity "

Def (Scalar curvature) $\quad \mathrm{Scal} := \sum_{i=1}^n \mathrm{Ric}(e_i, e_i) = \sum_{i,j} \underline{R(e_i, e_j, e_i, e_j)}$ $\to$ this is a sectional curvature

eg. (1) $\dim M = 1$ $\quad R = 0$ $\quad \dim M = 0$ $\quad \dim M = 2$ $\quad \mathrm{Scal} = R(e_1, e_2, e_1, e_2) + R(e_2, e_1, e_2, e_1)$
$= 2R(e_1, e_2, e_1, e_2)$

$\mathrm{Ric}(e_1, e_1) = R(e_1, e_2, e_1, e_2) = g(e_1, R(e_1, e_2) e_2) = k g(e_1, e_1)$
same for (e_2, e_2)
and for (e_1, e_2) or (e_2, e_1)
both are zero

2) $M = \mathbb{R}^n$ $\quad g = \sum_{i=1}^n dx_i^2$ $\quad \Gamma^k_{ij} = 0$ $\quad$ we define $\nabla_X Y := X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j}$ $\to$ also denoted with XY
claim this's the Levi-Civita connection

① $\nabla_X Y - \nabla_Y X = XY - YX = [X,Y]$ $\quad$ both have the term $\underline{X^i Y^j \frac{\partial}{\partial x^i \partial x^j}}$

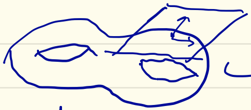
② $Xg(Y,Z) = X(\sum Y^i Z^i) = \sum (XY^i) Z^i + \sum Y^i (XZ^i) = g(XY, Z) + g(Y, XZ) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

$XY^i = X^j \frac{\partial Y^i}{\partial x^j} = (XY)^i$

flat metric

$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = (XY - YX - [X,Y]) Z = 0$ so $R \equiv 0$ flat connection

3) $M \subseteq \mathbb{R}^n$



$\xrightarrow{\varphi} \mathbb{R}^3$

we can stretch the flat metric in $\mathbb{R}^n$ to M

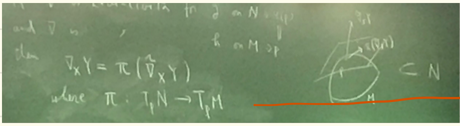
choose local coordinate $(U, (x^1, \cdots, x^k))$

let $h_{ij} = g(\varphi_* \frac{\partial}{\partial x^i}, \varphi_* \frac{\partial}{\partial x^j}) = \sum_{k=1}^{n} \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j}$

$h = (h_{ij}) = \varphi^* g$ induced by φ

more generally, $M^k \xhookrightarrow{\varphi} (N^n, g)$ we get $h \triangleq \varphi^* g$ what's the Levi-Civita connection?

let $\tilde{\nabla}$ (resp. ∇) denote the L-C of g (resp. h) on N (resp. M)



here, X, Y may extend to be v.f. in N but $\tilde{\nabla}_X Y$ may not be in the plane T_pM, so we take the projection π

proof. ① $\nabla_X Y - \nabla_Y X = \pi(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X) = \pi([\tilde{X}, \tilde{Y}]) = [X, Y]$

to be clear, here we use $\tilde{X}$ to denote the extension

fact: $[\tilde{X}, \tilde{Y}] = \widetilde{[X, Y]}$, so we get the equal

② $X h(Y, Z) = X g(Y, Z) = g(\tilde{\nabla}_X Y, Z) + g(Y, \tilde{\nabla}_X Z) = g(\pi(\tilde{\nabla}_X Y), Z) + g(Y, \pi(\tilde{\nabla}_X Z)) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z)$

on where h is defined, the equality holds

belongs to the "plane", since we are going to take inner product with elements in the "plane", it's the same to consider the projection of another variable in the "plane"

Def. for $M^{n-1} \xhookrightarrow{\varphi} (N^n, g)$ i.e. with codim $= 1$, call $h = \varphi^* g$ the 1-st fundamental form

take a locally orthogonal normal v.f. $e_p \perp T_pM$ for $p \in U$ some small nbhd

the $\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y) e$ for some bilinear form α

$\alpha(fX, gY) = fg\,\alpha(X, Y)$ since $\tilde{\nabla}_{fX}\, gY = \nabla_{fX}\, gY + \alpha(fX, gY) e = f(X g) Y + g \nabla_X Y) + \alpha(fX, gY) e$

$\tilde{\nabla}_{fX}\, gY = f((Xg) Y + g \tilde{\nabla}_X Y)$

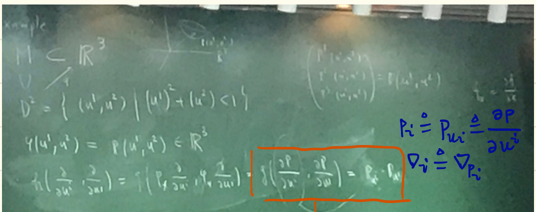
so $fg(\tilde{\nabla}_X Y - \nabla_X Y) = \alpha(fX, gY) e$

$fg(\tilde{\nabla}_X Y - \nabla_X Y) = fg\,\alpha(X, Y) e$

$\alpha(X, Y) = \alpha(Y, X)$ since $\nabla_X Y$ & $\nabla_Y X$ differ by $[X, Y]$

$\tilde{\nabla}_X Y$ & $\tilde{\nabla}_Y X$ differ by $[\tilde{X}, \tilde{Y}] = [X, Y]$ on M

eg.



inner product in the standard case

we get that $h = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} P_1\cdot P_1 & P_1\cdot P_2 \\ P_2\cdot P_1 & P_2\cdot P_2 \end{pmatrix}$ or $h = \sum_{i,j=1}^{2} h_{ij}\, du^i du^j = E\,du^{1\,2} + 2F\,du^1 du^2 + G(du^2)^2$

$\alpha\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) e = \tilde{\nabla}_{\varphi_*\frac{\partial}{\partial u^i}} \varphi_*\frac{\partial}{\partial u^j} - \nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = \frac{\partial P_j}{\partial u^i} - \pi\left(\underset{\substack{\| \\ P_{ji}\ (= P_{ij})}}{\frac{\partial P_j}{\partial u^i}}\right) = \left(\frac{\partial P_j}{\partial u^i}\cdot e\right) e$

so $\alpha = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} P_{11}\cdot e & P_{12}\cdot e \\ P_{21}\cdot e & P_{22}\cdot e \end{pmatrix}$ $\alpha = L(du^1)^2 + 2M\, du^1 du^2 + N(du^2)^2$

Def (Gauss curvature) for $\varphi(D^2) = M \subseteq \mathbb{R}^3$ $\frac{\det\alpha}{\det h} = \frac{LN - M^2}{EG - F^2}$

Thm (Gauss) $u_0 \in D$ take (u_1, u_2) s.t. $h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ($\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}$ ortho normal at u_0) $\to$ i.e. $EG - F^2 = 1$

then $R\left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right) = LN - M^2$

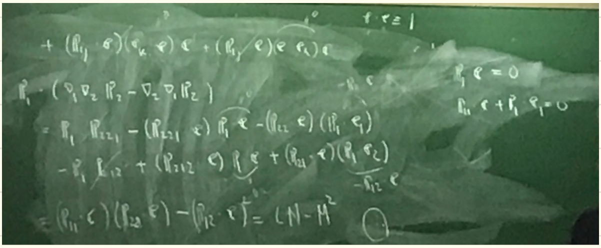
$\downarrow$ sectional curvature, intrinsic

$\downarrow$ related to the outside space

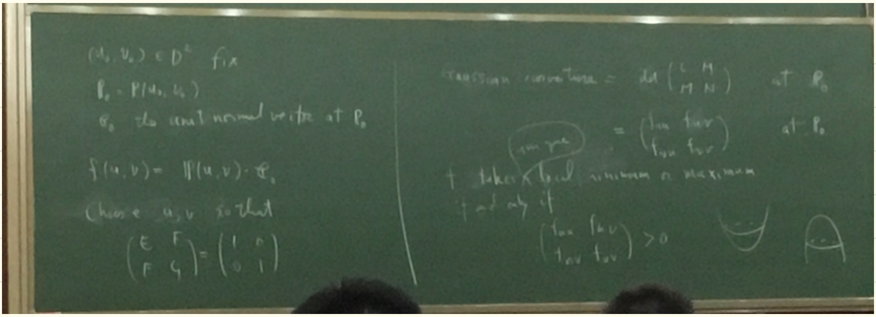
Proof. left $= P_1\cdot(\nabla_1\nabla_2 P_2 - \nabla_2\nabla_1 P_2)$ we have $\left[\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right] = 0$

$\nabla_k\nabla_j P_i = \frac{\partial}{\partial u^k}(\nabla_j P_i) - \left(\frac{\partial}{\partial u^k}(\nabla_j P_i)\cdot e\right)e = \frac{\partial}{\partial u^k}\left(\frac{\partial P_i}{\partial u^j} - \left(\frac{\partial P_i}{\partial u^j}\cdot e\right)e\right) - \left(\frac{\partial}{\partial u^k}\left(\frac{\partial P_i}{\partial u^j} - \left(\frac{\partial P_i}{\partial u^j}\cdot e\right)e\right)\cdot e\right)e = P_{ijk} - (P_{ijk}\cdot e)e - (P_{ij}\cdot e)e_k - (P_{ijk}\cdot e)e$

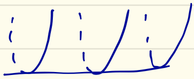
$+ (P_{ijk}\cdot e)e + (P_{ij}\cdot e_k)e + (P_{ij}\cdot e)(e_k\cdot e)e + (P_{ij}\cdot e)(e\cdot e_k)e$

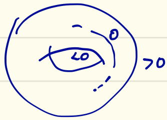


这个实在不想自己算一遍了 ¯_¯||



$$\begin{pmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{pmatrix} = 0$$





Thm. (Gauss) Gaussian curvature = sectional curvature = described the second derivatives of the 1st fundamental form (Riemannian metric)

Thm (Gauss - Bonnet) M compact surface (in $\mathbb{R}^3$) (Ex. $|P_u \times P_v| \, du \wedge dv$ indep of chart choice)

$$\int_M \text{Gaussian } dV = 2\pi \chi(M) \rightsquigarrow \text{Atiyah - Singer}$$

$[(\text{Gaussian})dV] \in H^2_{DR}(M)$ is independent of the choice of $M \hookrightarrow \mathbb{R}^3$ (or indep of the metric of M)

Chern $\to$ the Euler class

$\to$ Chern classes for cpt cpx manifold

... vector bundle

Chern connection for holo. v.b.

Kähler mfd

connections, curvature for principal bundle

$\rightsquigarrow$ especially for $U(n)$ - bundle

$\downarrow$

Chern class