Divisors

curve dim 1

cpt Riemann surface $\sum n_i P_i$

(Noetherian, irr, stalk-integral) $\Rightarrow$ integral

$\Downarrow$ connected

Dedekind domain

affine $\Rightarrow$ separated

$$\begin{tikzcd} & \mathbb{Z} \arrow[r] & A \\ & & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} A \arrow[rr] & & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

the condition suitable for us to discuss interesting theories is somehow narrow:

firstly we assume noetherian, separated & integral & regular in codim 1

(this's true in most geometric settings)

I'd like to discuss focusing on special examples

- Dedekind domain R (seen as curves) here we have decomposition of fraction ideals into prime ideals

since R has dim 1, nonzero prime ideals are just the prime divisor (pt is of codim = 1, closed subscheme, integral $\Leftrightarrow$ the global section of the structure ring is integral in the affine case, i.e. R/p integral, i.e. p prime)

then fraction ideals of R $\xleftrightarrow{1\text{-}1}$ Weil divisors on Spec R

$\updownarrow$ coherent sheaves (locally free of rank 1) $\leftarrow\text{-}\text{-}\rightarrow$ line bundles (invertible sheaves) on Spec R

$\updownarrow$ stalkly free

algebraic fact $\Rightarrow$ fraction ideals = invertible modules on DVR

embedding $\Leftrightarrow$ global section

(ample) sheaves

$\alpha \mathcal{O}_K = \prod p_i^{v_{p_i}(\alpha)}$ in $\mathcal{O}_{K,p}$, $(\alpha) = m^k$, $k = v_p(\alpha)$

principal divisor $\xleftrightarrow{1\text{-}1}$ principal fraction ideal : valuation on R_p is just given by the coefficients of p appeared in decomposition of ideals (local description on DVR: $\underline{\pi^n u} \to \underline{\text{the } n}$)

$\Rightarrow \mathrm{Cl}\operatorname{Spec} R \cong \mathrm{Cl} R$

- cpt Riemann surface : valuation multiplicity of zeros & poles (look at the maximal ideals!)

description of divisors $\longleftrightarrow$ holomorphic line bundles (determined by transform functions or meromorphic sections)

$\Downarrow$ associated sheaves

$\Downarrow$ cohomology

| Picard grp |

(isomorphic classes of invertible sheaves)

when noetherian, normal

$\Leftrightarrow$ Cl X

from divisor to holomorphic line bundle (on M, cpt Riemann surface)

different sides of a line bundle: ① v.b. as a cpx mfd over M, with fibre $\mathbb{C}$ & locally trivialization & holomorphic transform function ($f_\alpha^\beta f_\beta^\alpha = 1$)

② equivalently, we give a cover of $M = \cup M_\alpha$, a set of transform functions $\{f_\alpha^\beta\}$ then we can define an equivalent relation on $\coprod M_\alpha \times \mathbb{C}$ for $x \in M_\alpha \cap M_\beta$ $(x, z_1) \sim (x, z_2)$ ($\in M_\alpha \times \mathbb{C}$, $\in M_\beta \times \mathbb{C}$)

$\Leftrightarrow$ $z_1 = f_\alpha^\beta(x) z_2$ then we have $L = \coprod (M_\alpha \times \mathbb{C}) / \sim$ a quotient topological space,

obviously a cpx mfd of dim=2 & $\pi: L \to M$
locally trivialization: $\varphi_\alpha: \pi^{-1}(M_\alpha) \to M_\alpha \times \mathbb{C}$
} naturally they both satisfy the conditions, and the transform functions to define them is just the transform functions of the bundle

the above two pages is a draft about my talking, but it's not complete

what I talked in the morning (also contained in my talk to the physical student, but I didn't write it down)

Weil divisor : under certain conditions (Noetherian, integral, separated. locally factorial (⇑ integrally closed) & regular in codim 1, since in this case, we only need to show $\dim m/m^2 = 1$, this is due to integrally closed, dim 1, Noetherian ⇔ Dedekind ⇔ DVR ⇔ maximal ideal principal & $\dim \mathrm{krull} \leq \dim m/m^2$), it's locally principal (like the case for Riemann surface)

$\rightsquigarrow$ gives a Cartier divisor (integral $\Rightarrow \Gamma(\mathcal{H}_X^*, V) = K^*$, the function field)

conversely, a Cartier divisor, represented as (U_i, f_i) $\rightsquigarrow$ locally principal Weil divisor (compatible on intersections since $\frac{f_i}{f_j} \in \mathcal{O}_X(U_{ij})$)

they're inverse to each other & sends principal to principal

$\Rightarrow$ $Cl_X \cong CaCl_X$

Picard grp

$$0 \to \mathcal{O}_X^\times \to \mathcal{H}_X^\times \to \mathcal{H}_X^\times / \mathcal{O}_X^\times \to 0$$

$$\Downarrow$$

$$0 \to H^0(\mathcal{O}_X^\times) \to H^0(\mathcal{H}_X^\times) \to H^0(\mathcal{H}_X^\times / \mathcal{O}_X^\times) \to H^1(\mathcal{O}_X^\times) \to H^1(\mathcal{H}_X^\times) \to \cdots$$

when X is integral, $\mathcal{H}_X^\times(U) = K(X)^\times$ for any open $U \subset X$

hence $\mathcal{H}_X^\times$ is flasque, hence $H^1(\mathcal{H}_X^\times) = 0$

$$\Rightarrow \quad H^0(\mathcal{H}_X^\times / \mathcal{O}_X^\times) \big/ \mathrm{im}(H^0(\mathcal{H}_X^\times)) \cong H^1(\mathcal{O}_X^\times) = \mathrm{Pic}\, X$$

$$\parallel$$

$$CaDiv_X / CaP_X = CaCl_X$$

sheaf of modules (particularly quasi-coherent sheaves (modules), coherent sheaves (f.g. modules), locally free sheaves (v.b.) ($\Leftarrow$ "stalks" free))

Def. on a ringed space $(X, \mathcal{O}_X)$, an $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ s.t. $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module &

$$\begin{array}{ccc} \mathcal{O}_X(U)\times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & \circlearrowright & \downarrow \\ \mathcal{O}_X(V)\times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

(in stalk-style language, each stalk $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-mod)

morphism of two $\mathcal{O}_X$-mod $\mathcal{F}\to\mathcal{G}$ should preserve the $\mathcal{O}_X(U)$-mod structure

Rmk. $\mathcal{F}|_U$ of course is an $\mathcal{O}_X|_U$-mod, then we have the presheaf $U\mapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$

it can be checked that this is actually a sheaf

we give it an $\mathcal{O}_X$-mod structure naturally since all rings are commutative

tensor product comes naturally, too.

Def. free, locally free, rank (rank = 1 $\rightsquigarrow$ invertible / line bundle)

sheaves of ideals

Def. (direct / inverse image) $f: X\to Y$ $\quad f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$

$\rightsquigarrow f_*\mathcal{F}$ with $\mathcal{O}_Y$-mod structure $\quad \dfrac{f_*\mathcal{F}}{f_*\mathcal{O}_X\text{-mod}}$ $\quad$ but we have $\mathcal{O}_Y\to f_*\mathcal{O}_X$

$f^{-1}\mathcal{G}(U)$ $\quad \varinjlim_{V\supseteq U}\mathcal{G}(V)$ $\quad f^{-1}\mathcal{O}_Y$-mod

$\quad$?

$\quad \varinjlim_{V\supseteq U}\mathcal{O}_Y(V)$

$$\frac{f^{-1}\mathcal{O}_Y\to\mathcal{O}_X}{\Updownarrow \\ \mathcal{O}_Y\to f_*\mathcal{O}_X}$$

def $f^*\mathcal{F}$ to be the $\mathcal{O}_X$-mod $f^{-1}\mathcal{F}\otimes_{f^{-1}\mathcal{O}_Y}\mathcal{O}_X$

we have the adjoint $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{G})$

Def. $\underset{A\text{-mod}}{M} \rightsquigarrow \widetilde{M}$ $\quad$ A a ring

1. we have M_p, define $\widetilde{M}(U) := \{s: U\to \coprod_{p\in U} M_p \mid s(p)\in M_p$, and $\forall p$, $\exists$ nbhd $p\in V\subset U$, $s|_V = \frac{m}{f}$ for some $m\in M$, $f\notin q\ \forall q\in V\}$ (this means locally comes from section on basis)

then prove: $\widetilde{M}_p = M_p$ $\quad \widetilde{M}(D(f)) = M_f$

2. we define $\widetilde{M}$ by sheaf on a basis $\widetilde{M}(D(f)) := M_f$

then everything is clear

Prop. (some basic properties) 1. the functor $M\longmapsto \widetilde{M}$ is exact (since taking stalks is exact & inj / surj properties of sheaves are determined at stalks)

2. $\widetilde{M\oplus N}\cong \widetilde{M}\oplus\widetilde{N}$ (colimits commutes with colimits)

3. $\widetilde{M\otimes N}\cong\widetilde{M}\otimes\widetilde{N}$ (left adjoint commutes with colimits)

4. $\underset{M}{A} \xrightarrow{f} \underset{N}{B}$ $\quad \mathrm{Spec}\,B\xrightarrow{f^\#}\mathrm{Spec}\,A$ $\quad f^\#_*(\widetilde{N}) = \widetilde{{}_AN}$ $\quad f^{\#*}(\widetilde{M}) = (f^\#)^{-1}(\widetilde{M})\otimes_{\mathcal{O}_{\mathrm{Spec}A}}\mathcal{O}_{\mathrm{Spec}B} = (f^\#)^{-1}(\widetilde{M})\otimes_{\widetilde{A}}\widetilde{B}$

$= \widetilde{M\otimes_A B}$

Def. ((quasi-)coherent) $(X,\mathcal{O}_X)$ a scheme, $\mathcal{F}$ an $\mathcal{O}_X$-mod, $\mathcal{F}$ is called quasi-coherent if $\forall p\in X$, $\exists$ affine nbhd $p\in V=\operatorname{Spec}A\subset X$, s.t. $\mathcal{F}|_V\cong\widetilde{M}$ for some A-module M (this is essentially the same as saying $\exists$ such a cover)

furthermore, if those M's are f.g., $\mathcal{F}$ is called coherent

we will show that in the affine case, quasicoherent is given by M

coherent is given by f.g. M if A is Noetherian

i.e. equivalence of categories

on a nonnoetherian scheme, coherence doesn't perform well

later, we show that it's a property preserved under taking kernel, cokernel & it's a local property

Lemma. $X=\operatorname{Spec}A$, $\mathcal{F}$ quasicoherent

1) $s\in\mathcal{F}(X)$ s.t. $s\equiv 0$ on $D(f)$ then $\exists\, n$ s.t. $f^ns=0$ in $\mathcal{F}(X)$

2) $s\in D(f)$ can be extended by f^ns

Pf. observation: $(X,\mathcal{O}_X)$ a quasicompact scheme $\mathcal{F}$ quasi-coherent on X

then $X=\bigcup_{i=1}^n\operatorname{Spec}A_i$ s.t. $\mathcal{F}|_{\operatorname{Spec}A_i}\cong\widetilde{M_i}$ but there're basis of the type $D(f)$

let $\operatorname{Spec}A_i=\bigcup_{j=1}^{n_i}D(g_{ij})$ $D(g_{ij})\subset\operatorname{Spec}A_i\rightsquigarrow$ morphism $A_i\to A_{i\,g_{ij}}$ $\Rightarrow\mathcal{F}|_{D(g_{ij})}=\underbrace{(M_i\otimes_{A_i}A_{i\,g_{ij}})^{\sim}}_{=\widetilde{M_{ij}}}$

i.e. $X=\bigcup D(g_{ij})$ s.t. $\mathcal{F}|_{D(g_{ij})}\cong\widetilde{M_{ij}}$ for finitely many ij

(if there's no quasi-compactness, the last prop omitted)

(a) $X=\bigcup_{i=1}^m D(g_i)$ $\mathcal{F}|_{D(g_i)}\cong\widetilde{M_i}$ M_i: A_{g_i}-mod $s|_{D(fg_i)}=0\in\widetilde{M_i}(D(f)\cap D(g_i))=M_{i\,f}$

so $\exists\, n_i$ s.t. $f^{n_i}s=0$ in M_i $\rightsquigarrow\exists\, n$ s.t. $f^ns=0$ in M_i for any i

i.e. $f^ns=0$ on $D(g_i),\forall i\Rightarrow f^ns=0$ on X

(b) $s|_{D(fg_i)}\in(M_i)_f$ $\exists\, n$ $t_i\in M_i$ s.t. $f^ns|_{D(fg_i)}=t_i|_{D(fg_i)}$ want to glue t_i's together

$t_i|_{D(fg_ig_j)}=t_j|_{D(fg_ig_j)}$ $\exists\, m$ s.t. $f^m(t_i-t_j)=0$ on $D(g_ig_j)$

$\Rightarrow f^mt_i$'s are compatible $\rightsquigarrow t$ s.t. $t|_{D(f)}=f^{m+n}s$

Prop (affine-local of (quasi-)coherence) scheme $(X,\mathcal{O}_X)$ $\mathcal{F}$: $\mathcal{O}_X$-mod

1) $\mathcal{F}$ quasi-coherent $\Leftrightarrow\forall$ affine open $U=\operatorname{Spec}A\subset X$, $\mathcal{F}|_U\cong\widetilde{M}$ for some A-mod M

2) if X is noetherian, statements in 1) holds for coherence assuming M f.g.

Proof. as the observation in the above lemma, $\mathcal{F}|_{\operatorname{Spec}A}$ is quasi-coherent, so we've reduced to show that if $X=\operatorname{Spec}A$, $\mathcal{F}$ quasi-coherent on X, then $\exists$ A-mod M s.t. $\mathcal{F}=\widetilde{M}$

let $M=\mathcal{F}(X)$, it's an A-mod, my idea is to give a morphism $\widetilde{M}\to\mathcal{F}$ on a basis & show it's an isomorphism

$\widetilde{M}(D(f))=M_f=\mathcal{F}(X)_f$ we have
$$\begin{tikzcd}\mathcal{F}(X)\arrow[r]\arrow[d] & \mathcal{F}(D(f))\\ \mathcal{F}(X)_f\arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$
(if "f" is invertible in $\mathcal{F}(D(f))$) commutes

in particular, $\exists$ a finite cover $\operatorname{Spec}A=\bigcup D(f_i)$ $\mathcal{F}(D(f_i))\cong\widetilde{M_i}$ M_i: A_{f_i}-mod then $\mathcal{F}(D(f_i))=M_i$

on the other hand, the above Lemma told us that $\forall t\in M_i$, $\exists\, n$, $f_i^nt\in M$ exactly

i.e. $M_i=M_f$

hence an isomorphism

furthermore, if A is Noetherian, then M_i is Noetherian module over the Noetherian ring A_{f_i}

$(f_1 \cdots f_r) = A$, we conclude to a purely algebraic problem: showing M is Noetherian over A

Cor. on $X = \operatorname{Spec} A$ quasi-coherent sheaf $\Leftrightarrow$ A-module M; if A is noetherian, coherent sheaf $\Leftrightarrow$ f.g. A-module

Prop. $X = \operatorname{Spec} A$ $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact seq of $\mathcal{O}_X$-mod $\mathcal{F}'$ quasi-coherent $\Rightarrow \Gamma(X, \cdot)$ exact

(it's not flasque however, but the extension property is really like a weaker surjective condition)

claim: for $s \in \mathcal{F}''(X)$ $\forall x \in X$ $\exists D(f) \ni x$ s.t. $s|_{D(f)}$ lifts to a section in $\mathcal{F}(D(f))$, further, $\exists n$, st. $f^n s$ lifts to a global one

(with the claim we can cover X with $D(f_i)$ $i = 1, \cdots, n$ then $(f_1, \cdots, f_n) = (1)$ so $(f_1^{n_1}, \cdots, f_n^{n_n}) = (1)$ $\forall n_i$'s $\in \mathbb{N}$, then we lift $f_i^{n_i}s$ to $t_i \in \mathcal{F}(X)$, we have $1 = \sum a_i f_i^{n_i}$ then $t = \sum a_i t_i$ is the preimage)

Pf. intuition the most basic fact we know is that we can lift sections locally

then there are two ways to find global preimage:

① natural one: cover $\to$ compatible $\to$ glue

② tricky one (rely on ① in fact): cover $D(f_i)$ $\to$ $\sum a_i f_i = 1$ $\xrightarrow[\text{lift of } f_i s : t_i]{\text{find global}}$ $\sum a_i t_i$

② is special for quasi-coherent rely on the extension lemma & quasicompactness

Prop. X a scheme, (1) kernel, image, cokernel of morphism of quasi-coherent sheaves are quasi-coherent

if X Noetherian, same holds for quasi-coherent

Proof (sketch) local problem since (co)kernel $(f|_U) = $ (co) kernel $(f)|_U$ colimit commutes with colimit filtered colimit commutes with finite limit

but in the affine case, everything is given by modules over a ring

(2) $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact where $\mathcal{F}', \mathcal{F}''$ quasi-coherent (noetherian $\to$ coherent) $\Rightarrow$ $\mathcal{F}$ quasi-coherent

this is also local since intersection of affine open can be covered simultaneously by principal open sets for both affines

then consider the module, using five lemma. when X noetherian: $0 \to M' \to M \to M'' \to 0$ M noetherian $\Leftarrow$ M' & M''

Prop (base change) the case of inverse image is a local problem

direct image

on projective scheme $(X, \mathcal{O}_X)$ $X = \mathrm{Proj}\, S$ S graded ring

we use similar construction with $M_{(p)}$ $M_{(f)}$ then $\widetilde{M}$ is quasi-coherent since

$\mathcal{O}_X(D(f)) = S_{(f)}$ $\widetilde{M}|_{D(f)} = \widetilde{M_{(f)}}$ is quasi-coherent on $\mathrm{Spec}\, \underline{S_{(f)}}$ degree 0 part of S_f

if S is noetherian ($\Leftrightarrow$ S_0 noetherian & S f.g. as S_0-algebra), M f.g.

then $\widetilde{M}$ is coherent ($S_{(f)}$ noetherian, $M_{(f)}$ f.g. over $S_{(f)}$)

Def. $\mathcal{O}_X(n) := \widetilde{S(n)}$ $\mathcal{F}$ quasicoherent $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ (twisted)

$\mathcal{O}_X(1)$: twisting sheaf

Prop. assume S is generated by S_1 as S_0-algebra

(1) $\mathcal{O}_X(n)$ invertible (it's previously quasi-coherent)

i.e. $\widetilde{M} \otimes_{\widetilde{S}} \widetilde{N} = \widetilde{M \otimes_S N}$

(2) $\widetilde{M}(n) \cong \widetilde{M(n)}$ $(M(n) \cong M \otimes_S S(n))$ (when $S_1 \ni f$ $M_{(f)} \otimes_{S_{(f)}} N_{(f)} = (M \otimes_S N)_{(f)}$)

(3) $\varphi: S \to T$ preserving degrees, both are generated by deg 1 part over deg 0 part

$\rightsquigarrow$ $f: U \to X = \mathrm{Proj}\, S$ $U \subset Y = \mathrm{Proj}\, T$

then $f_*(\widetilde{N}|_U) \cong \widetilde{{}_S N}$ ($f^{-1}(D(f)) = D(\varphi f)$) $f^*(\widetilde{M}) \cong \widetilde{M \otimes_S T}\,|_U$

$\widetilde{M \otimes_S T}_{(p)} = \widetilde{M}_{\varphi^{-1}(p)} \otimes_{S_{\varphi^{-1}(p)}} T_p$

Def. for an $\mathcal{O}_X$-mod $\mathcal{F}$ def $\Gamma_*(\mathcal{F}) := \bigoplus_{k \geq 0} \Gamma(\mathcal{F}(k))$ as a graded S-module

the action is defined degree-wise through $\mathcal{F}(k) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \mathcal{F}(k+n)$

Facts. 1. for polynomial rings S, $\Gamma_*(\widetilde{S}) = S$

2. similar lemma as in the affine case

3. for S generated by S_1 over S_0, quasi-coherent $\mathcal{F}$ on $\mathrm{Proj}\, S$ $\Leftrightarrow$ $\widetilde{\Gamma_*(\mathcal{F})}$

4. closed subschemes of $\mathbb{P}^r_A$ $\Longleftrightarrow$ homogeneous ideals in $A[x_0, \cdots, x_n]$

5. $f: Y \to \mathrm{Spec}\, A$ projective $\Leftrightarrow$ $Y = \mathrm{Proj}\, S$ with $S_0 = A$, S $\underline{\text{f.g.}}$ by S_1 as S_0-algebra

very ampleness

Def. (Preminary)

$$P^r_A = \mathrm{Proj}\, A[x_0, \cdots, x_r] \quad \text{or} \quad \begin{tikzcd} \mathbb{P}^r_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}A \arrow[r] \arrow[d] & \mathbb{P}^r_{\mathbb{Z}} \arrow[d] \\ \mathrm{Spec}A \arrow[r] & \mathrm{Spec}\,\mathbb{Z} \end{tikzcd}$$

for any scheme Y, $\mathbb{P}^r_Y := \mathbb{P}^r_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} Y$

$$\begin{tikzcd} \mathbb{P}^r_Y \arrow[r, "p"] \arrow[d] & \mathbb{P}^r_{\mathbb{Z}} \arrow[d] \\ Y \arrow[r] & \mathrm{Spec}\,\mathbb{Z} \end{tikzcd}$$

$\mathrm{Hom}_{\text{scheme}}(X, \mathrm{Spec}A) \cong \mathrm{Hom}_{\text{ring}}(A, \mathcal{O}_X(X))$

in particular, when $A = \mathbb{Z}$

there's a unique $\mathbb{Z} \to \mathcal{O}_X(X)$

preserving identity

$\mathbb{Z} \hookrightarrow \mathbb{Z}[x_0, \cdots, x_r]$

$\mathbb{P}^r_{\mathbb{Z}} \to \mathbb{P}^0_{\mathbb{Z}} = \mathrm{Spec}\,\mathbb{Z}$

now we define $\mathcal{O}(1)$ of $\mathbb{P}^r_Y$ to be $p^*(\mathcal{O}(1))$ $\quad$ $\mathcal{O}(1)$ is already defined on $\mathbb{P}^r_{\mathbb{Z}}$

(when $Y = \mathrm{Spec}A$, it's the same as $\mathcal{O}(1)$ already defined on $\mathrm{Proj}\, A[x_0, \cdots, x_r]$ since in this case,

$\mathbb{Z}[x_0, \cdots, x_r] \to A[x_0, \cdots, x_r] \longrightarrow \mathbb{P}^r_A \xrightarrow{f} \mathbb{P}^r_{\mathbb{Z}}$ $\quad$ $f^*(\widetilde{M}) = \widetilde{M \otimes_{\mathbb{Z}[x_0, \cdots, x_r]} A[x_0, \cdots, x_r]}$ $\quad$ we take $M = \mathbb{Z}[x_0, \cdots, x_r](1)$)

then comes the definition of very ample

$X \longrightarrow Y$ $\quad$ an invertible sheaf $\mathcal{F}$ on X is very ample related to Y if:

$\exists\, r$, immersion $f: X \to \mathbb{P}^r_Y$ s.t. $f^*(\mathcal{O}(1)) = \mathcal{F}$

(this def relies on Y, later we will give the definition of ampleness, in an intrinsic way, and we'll know that it's equivalent to that its power to be very ample, which isn't intrinsic)

Rmk. $X \xrightarrow{f} Y$ $\quad$ Y noetherian, then X projective $\Leftrightarrow$ f proper & $\exists$ very ample sheaf on X related to Y

"$\Rightarrow$" is done

"$\Leftarrow$" immersion $i: X \to \mathbb{P}^r_Y$ $\quad$ $\mathcal{L} = i^*(\mathcal{O}(1))$ $\quad$ f is proper, satisfying conditions in Ex. 4.4

$\Rightarrow$ $i(X)$ closed