Riemann Surface

2018.03 ~ 2018.06

0. Review of complex analysis

$f \in \mathcal{O}(D)$ can be expanded as covergent Laurant series on D

$$f(z) = \sum_{-\infty}^{+\infty} a_n (z - z_0)^n$$

孤立奇点

e.g. $D = \Delta(z_0, R) = \{0 < |z - z_0| < R\}$

$f \in \mathcal{O}(D) \Rightarrow f = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$

① z_0 is called removable singularity of f

if: f can be continued holomorphially to

② z_0 is called a pole

$\exists\ n_0 < 0$ $a_{n_0} \neq 0$ & $a_n \neq 0$ finitely

③ essential singularity

$a_n \neq 0$ holds for infinitely many $n < 0$

Ex. prove: ① z_0 is removable $\Leftrightarrow$ f is bounded in a neighbourd of z_0

② z_0 is a pole $\Leftrightarrow$ $\lim_{z \to z_0} f(z) = \infty$

③ ③ $\Leftrightarrow$ 不∃

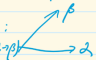
Geometric viewpoint

conformmal map

angle:

(of rotation from $\alpha \to \beta$)

anti-clockwise



$T: \mathbb{R}^2 \to \mathbb{R}^2$ a linear isomorphism

① T is called angle-preserving/isogonal

if: $\angle(T\alpha, T\beta) = \angle(\alpha, \beta)$

② - - - - conformal if $\exists \lambda > 0$ $\|T\alpha\| = \lambda \|\alpha\|$

③ orientation-preserving if $\det T > 0$

What about holomorphic map? Locally

$f: D \subset \mathbb{R}^2 = \mathbb{C} \to \mathbb{R}^2$ differentiable

locally linearly

f isogonal: $df_z = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$ $\mathbb{R}^2 \to \mathbb{R}^2$ $T_z D$ $T_z \mathbb{R}^2$

is isogonal

conformal

orientation

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^2$ a linear isomorphism

T is isogonal $\Leftrightarrow \begin{cases} ① T \text{ is orientation} \\ ② T \text{ is conformal} \end{cases}$

Remark: $f = u + iv \in \mathcal{O}(D)$ $|f'(z)|^2 = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

in particular: $f'(z) \neq 0 \Leftrightarrow \Big| \quad \Big| > 0$

so: isogonal $\Leftrightarrow$ conformal

Thm Assume $f: D \to \mathbb{C} \in C^1(D)$

① if $f \in \mathcal{O}(D)$ and $f' \neq 0$ then f is isogonal

② if f is isogonal, $f \in \mathcal{O}(D)$

Ex. proof it

Recall ① identity thm: zero set of a nonconstant holo f is discrete

② If $f \in \mathcal{O}(\Delta^*)$, f bounded $\Rightarrow f \in \mathcal{O}(\Delta)$

(expand to z_0)

Thm $f: D \to \mathbb{C} \in C^1(D)$ is a nonconstant function

then $f \in \mathcal{O}(D) \Leftrightarrow \begin{cases} \text{(i)} \ |\det df| = 0 \ \text{is discrete} \\ \text{(ii)} \ f|_{D \setminus A} \text{ is isogonal} \end{cases}$

Chapter 1 Recall of basic results about holomorphic / analytic functions

Definition: A function $f: D \to \mathbb{C}$ from a domain (connected open subset of $\mathbb{C}$) D to $\mathbb{C}$ is called holomorphic if
$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists for any $z \in D$

How to understand a holomorphic function?

① analytic viewpoint

② algebraic viewpoint

③ geometric viewpoint

④ topological viewpoint

(1) analytic viewpoint (calculus)

ⓐ viewpoint of differentiation

Notation: $\mathcal{O}(D)$ is the space of all holomorphic functions on D. $\mathcal{O}(D)$ is an algebra over $\mathbb{C}$

Theorem: Let $f \in C^1(D)$, $f = u + iv$. Then $f \in \mathcal{O}(D) \Leftrightarrow f$ satisfies Cauchy-Riemann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

We introduce two operators on $\mathbb{C}$: $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ $\quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

Exercise: Let $f \in C^1(D)$, prove:

① $f \in \mathcal{O}(D) \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0$

② $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$

③ $\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{1}{4}\Delta f$

ⓑ viewpoint of integration

Cauchy's theorem: Assume $f \in \mathcal{O}(D)$, and γ is a closed curve in D whose interior lies in D, then $\int_\gamma f(z)dz = 0$

Morera's theorem: Assume $f: D \to \mathbb{C}$ is continuous, then $f \in \mathcal{O}(D)$ if $\int_\gamma f(z)dz = 0$ holds for every Jordan curve γ in D whose interior lies in D. Define $F(z) = \int_{z_0}^{z} f(z)dz$, then $f(z) = F'(z)$

Viewpoint of differentiation Cauchy – Riemann	$\xleftrightarrow{\text{Green's formula}}$	Viewpoint of integration Cauchy – Morera

Exercise: $f \in C^1(D)$, derive the Cauchy's theorem from Greens formula

Some applications of Cauchy's Morera:

① Cauchy's integral formula: For $f \in \mathcal{O}(D)$, we have $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$ $\quad (*)$

Special case: $z = 0$, $\gamma \sim C(\epsilon)$, $f(0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z} f(z)dz$ $\quad \left(\Leftrightarrow \frac{\partial}{\partial \bar{z}}\frac{1}{z} = \pi\delta_0\right)$

$(*) \Rightarrow f^{(n)}(z)$ exists $\forall n$ and $f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta$

② Cauchy's inequality: Assume $f \in \mathcal{O}(D)$, $|f(z)| \le M$, $\forall z \in D$. Assume $K \subset D$ compact, assume $d(K, \partial D) > \delta > 0$
Then $|f^{(n)}(z)| \le \frac{n!}{\delta^n}M$ $\quad \forall n \ge 1$ $z \in K$

③ Liouville's Theorem: bounded entire functions are constant

$\Rightarrow$ Fundamental theorem of algebra: Any nonconstant polynomial over $\mathbb{C}$ has a root in $\mathbb{C}$.

④ Theorem (Weierstraß) Assume $\{f_n\} \subset \mathcal{O}(D)$ converges uniformly to f, then $f \in \mathcal{O}(D)$; Moreover, $\{f_n^{(j)}\}$ converges to $f^{(j)}$ uniformly on compact sets in D.

Remark (Weierstraß approximation theorem) Continuous functions on a closed interval can be approximated by polynomials.

(2) Algebraic viewpoint (Weierstraß)

Theorem: Let $f \in \mathcal{O}(D)$, $z_0 \in D$, then in some neighborhood of z_0, f can be expanded as convergent power series as
$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

Proof: Assume $z_0 = 0$, assume $\Delta(z_0, r) = \{|z-z_0| < r\} \subset\subset D$. By Cauchy integral formula,
$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta(1-\frac{z}{\zeta})} d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta} \sum_0^\infty (\frac{z}{\zeta})^n d\zeta = \sum_0^\infty a_n z^n$. $\square$

Corollary to the theorem:

① Identity theorem: For a nonconstant $f \in \mathcal{O}(D)$, the zero set of f is a discrete subset of D

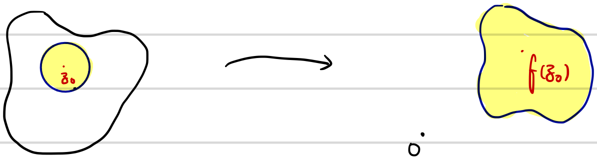
Assume $f(0)=0$, then around $z=0$, $f(z) = z^n g(z)$, $g(0) \neq 0$

② Open mapping theorem: a nonconstant function f maps an open set to an open set.

Assume $f \in \mathcal{O}(D)$, assume $0 \in D$, $f(0) = 0$. May assume 0 is in the interior of $f(D)$. $f(z) = z^n g(z)$, $g(z) \neq 0$.

$\Rightarrow \exists$ a holomorphic function $h(z)$ near 0 s.t. $g(z) = h(z)^n$. Let $w(z) = z h(z)$, then $w'(0) = h(0) \neq 0 \Rightarrow w$ is another coordinate near 0, and $f(w) = w^n$, hence, 0 is in the interior of $f(D)$.

③ Theorem (Maximal principle) Assume $f \in \mathcal{O}(D)$ is nonconstant, then $|f(z)|$ cannot attain its maximal in D.



④ maximal principle $\Rightarrow$ Theorem (Schwarz) Assume $f: \Delta = \{|z|<1\} \to \Delta$ holomorphic and $f(0) = 0$. Then

(i) $|f(z)| \leq |z|$, and "=" holds at some points $\Leftrightarrow f = e^{i\theta} z$ is a rotation

(ii) $|f'(0)| \leq 1$, and "=" holds $\Leftrightarrow f = e^{i\theta} z$ is a rotation

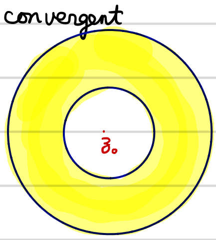
Consider $g(z) = \frac{f(z)}{z} \in \mathcal{O}(\Delta)$ and apply the maximal principle.

⑤ Schwarz lemma $\Rightarrow$ $\mathrm{Aut}(\Delta) = \{f: \Delta \to \Delta: \text{holomorphic bijective}\} = \{z \mapsto e^{i\theta}\frac{z-z_0}{1-\bar{z}_0 z} \mid \theta \in \mathbb{R}, z_0 \in \Delta\} \cong PSL(2,\mathbb{R})$

Remark: In general, by Cauchy's integral formula, we can prove

Theorem: Let $D = \{r < |z-z_0| < R\}$, where $0 \leq r < R \leq \infty$. Assume $f \in \mathcal{O}(D)$, then f can be expanded as a convergent Laurent series as $f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n$

Now let $f \in \mathcal{O}(\Delta^*(z_0, r))$, $\Delta^*(z_0, r) := \{0 < |z-z_0| < r\}$, then $f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n$



Topological View po..

1. multi-valued $f(z) = \mathrm{Ln}\, z = \ln|z| + i\,\mathrm{Arg}\, z$, $z \in \mathbb{C}^*$

$\boxed{\text{Being multi-valued is an essential character}}$ of holo func

suggests that domains in $\mathbb{C}$ aren't the natural definition for some holo func

so: consider domains more general
Riemann surface

2. E.G. $f(z) = \sqrt{z(z-1)(z-\lambda)}$ $\lambda \neq 0, 1$ (由代数曲线诱出对应R-S)

photo: M is the R-S associated ~~with~~ to f

and f is holo (meromorphic) func on M (singular-valued)

评述

Structures on R-S:

① topology Fundamental group
(co)homology group 同调群 子流形??
Intersection theory

② smooth structure (D. mani)
smooth func, field d-forms De Rham cohomo

③ complex/conformal structure
holo func, fields, d-forms, Dolbeault cohomo
mero

viewpoints:

① complex-analytic
A R-S is a 1-dim complex mani

② algebraic: geometry (for compact R-S)
A c-R-S is an algebraic curve

③ algebra
A c-R-S is equivalent to a finite extention of $\mathbb{C}(t)$
the field of rational func over $\mathbb{C}$

Def for Riemann Surface

1. complex projective space ($\mathbb{P}^n$不加说明指$\mathbb{P}^n\mathbb{C}$)

important: $\mathbb{P}^2\mathbb{C} \cong \mathbb{C}^2 \cup L_\infty$ $(L_\infty \cong \mathbb{P}^1)$

L_∞可视为对$\mathbb{C}^2$中 (x, y) 引入齐次坐标 $(\xi, x\xi, y\xi)$ $\xi \neq 0$
对$\xi = 0$ 对应点视为$\frac{\eta}{\zeta}$方向的无穷远点 而ξ差一个$\lambda \in \mathbb{C}$的,
(也即看出,就是在$\mathbb{C}^2$中斜率$\frac{\eta}{\zeta} = \lambda$) L_∞的点 视为等价

$F(\xi, \zeta, \eta) = \sum_{i=0}^{n} f_i(\zeta, \eta)\, \xi^{n_i} = 0$

即可表$\mathbb{P}^2$上一条代数曲线

限到$\mathbb{C}^2$上,就是$F(1, x, y) = 0$

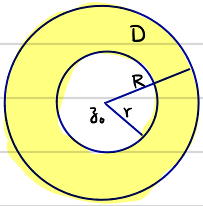
齐次 $\xrightarrow{\text{分解}}$ 因式仍齐次

2. def: R-S
局部同胚 $\mathbb{C}$ (复结构!)
转移映射全纯
双

局部全纯坐标(覆盖)

对于compact 都有有限covering

Lecture 2 7 Mar 2018



$(0 \leq r < R \leq \infty)$ $f \in \mathcal{O}(D)$, then f can be expanded as convergent Laurent series on D

$f = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$

Special case, $r=0$, $D = \Delta(z_0, R) = \{0 < |z - z_0| < R\}$, $z_0 = 0$, $f = \sum_{-\infty}^{\infty} a_n z^n$

Definition: ① z_0 is called a removable singularity of f if f can be continued holomorphically to z_0

② z_0 is called a pole, if (i) $\exists n_0 < 0$ s.t. $a_{n_0} \neq 0$ & (ii) $\exists$ only finitely many terms $a_n \neq 0$, $n < 0$.

③ z_0 is called an essential singularity of f if $a_n \neq 0$ for infinitely many $n < 0$.

Exercise: ① z_0 is removable $\Leftrightarrow$ f is bounded near z_0

② z_0 is a pole $\Leftrightarrow$ $\lim_{z \to z_0} f(z) = \infty$

③ z_0 is essential $\Leftrightarrow$ $\lim_{z \to z_0} f(z)$ does not exist.

(3) Geometric viewpoint

Definition: The angle from α to β is defined to be the angle of rotation from α to β anti-clockwisely, denoted by $\angle(\alpha, \beta)$.

$\angle(\alpha, \beta) = 2\pi - \angle(\beta, \alpha)$

Definition: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear isomorphism. T is called to be

① angle-preserving / isogonal, if $\angle(T\alpha, T\beta) = \angle(\alpha, \beta)$, $\forall \alpha, \beta \in \mathbb{R}^2$, $\alpha, \beta \neq 0$.

② conformal, if $\exists \lambda > 0$ s.t. $\|T\alpha\| = \lambda \|\alpha\|$ $\forall \alpha \in \mathbb{R}^2$

③ Orientation-preserving if $\det T > 0$.

Definition: Let $f: D \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map. Then f is called an isogonal (resp. conformal, resp. orientation-preserving) at $z \in D$, if so is $(f_*)_z$.

Exercise: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear isomorphism. Then T is isogonal if and only if T is both orientation-preserving and conformal.

Remark: Assume $f = u + iv \in \mathcal{O}(D)$, $z \in D$. Then $|f'(z)|^2 = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$. In particular, $f'(z) \neq 0 \Leftrightarrow \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} > 0$

Theorem: Assume $f \in C^1(D; \mathbb{C})$ Then:

① If $f \in \mathcal{O}(D)$, $f' \neq 0$, then f is angle-preserving on D

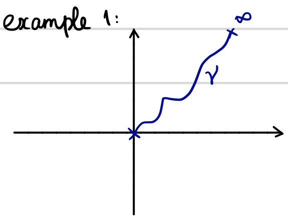
② If f is angle-preserving on D, then $f \in \mathcal{O}(D)$.

Theorem: (geometric characteristic of holomorphic functions) Assume $f: D \to \mathbb{C}$, $f \in C^1(D)$ is a nonconstant function. Then $f \in \mathcal{O}(D)$ if and only if (i) The zero set of $\det(f_*)$ is discrete in D & (ii) $f|_{D-A}: D - A \to \mathbb{R}^2$ is angle-preserving

(4) Topological Viewpoint

multivalued-functions:

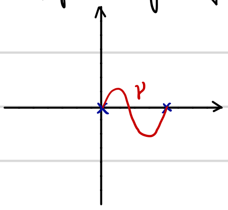
example 1:



On $\mathbb{C} - \gamma$, f has infinitely many single valued branches

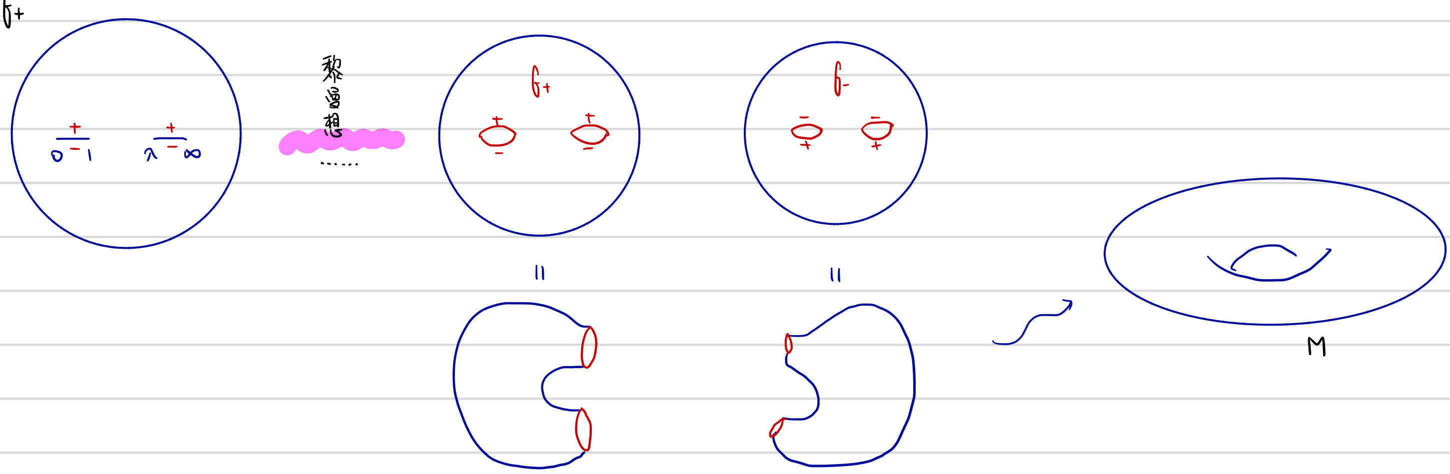
$f(z) = \operatorname{Log} z$

example 2: $f(z)=\sqrt{z(z-1)}$ $\quad z\neq 0,1$ $\quad$ On $\mathbb{C}-\gamma$, $f(z)$ has two single-valued branches.



Riemann's observation: Being multi-valued is an essential character of holomorphic functions, which suggests that domains in $\mathbb{C}$ are not the natural domains for some holomorphic functions, so we need to consider more general "domains", namely, Riemann surfaces.

Example: $f(z)=\sqrt{z(z-1)(z-\lambda)}$, $\lambda\neq 0,1$. f has two single-valued branches on $\hat{\mathbb{C}}\setminus([0,1]\cup[\lambda,\infty])$, $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$, denoted by f_+, f_-; $f_-=-f_+$



Riemann surfaces:

Structures on a Riemann surface:

① topological structure: fundamental group, (co)homology group, intersection theory

② smooth structure: smooth functions, smooth vector fields, differential forms, de Rham cohomology

③ complex/conformal structure: holomorphic/meromorphic functions, vector fields, differential form, Dolbeault cohomology

Viewpoints towards Riemann surfaces

① complex analytic: A Riemann surface is a 1-dimensional complex manifold

② algebro geometric: A compact Riemann surface is an algebraic curve.

③ algebra: A compact Riemann surface is equivalent to a finite extension of $\mathbb{C}(t)$, the field of rational functions of $\mathbb{C}$.

References:

① Weyl "the concept of a Riemann surface"

② Griffiths 《代数曲线》.

③ Forster "Lectures on Riemann surfaces" GTM 81.

④ Farkas & Kra "Riemann surfaces"

Class 3 (3.12).

- Riemann Surface.

Example 1) $\mathbb{C}$ and domains in $\mathbb{C}$ are Riemann Surface.

2) $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$. define topology on $\mathbb{P}^1_{\mathbb{C}}$ as:

- for $z \in \mathbb{C}$, the nbhd basis of z is given in the ordinary sense.
- $z = \infty$. the nbhd basis of z is given by $U_r := \{\tau \in \mathbb{C} \mid |\tau| > r\} \cup \{\infty\}$, $r > 0$.

Under this topology, $\mathbb{P}^1$ is a connected, compact, Hausdorff space

Let $U = \mathbb{C}$, $V = \mathbb{C}^* \cup \{\infty\}$. Then $\{U, V\}$ is an open cover of $\mathbb{P}^1$.

Let $z_1 : U \to \mathbb{C}$, $z \mapsto z$; $\quad z_2 : V \to \mathbb{C}$, $z \mapsto \begin{cases} 1/z & z \neq \infty \\ 0 & z = \infty \end{cases}$. Then z_1, z_2 are homeomorphism.

$U \cap V = \mathbb{C}^*$. $z_1(U \cap V) = \mathbb{C}^*$. $z_2(U \cap V) = \mathbb{C}^*$. $z_2 \circ z_1^{-1} : z_1(U \cap V) \longrightarrow z_2(U \cap V)$, $z \longmapsto \frac{1}{z}$

So $\{(U, z_1), (V, z_2)\}$ gives a Riemann Structure on $\mathbb{P}^1$. hence $\mathbb{P}^1$ is a Riemann Surface.

Example: $X = \mathbb{C}/\Lambda$. let $\omega_1, \omega_2 \in \mathbb{C}$ be two $\mathbb{R}$-independent vectors. let

$\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$. $\Lambda \subset \mathbb{C}$ is a subgroup. $X = \mathbb{C}/\Lambda$.

let $\pi : X \to X/\Lambda$ be the quotient map.

Easy observation shows that π is open, continuous.

$\therefore$ $\mathbb{C}/\Lambda$ is a cpt, connected, Hausdorff space.

Fact: $\exists r > 0$. $\forall z \in \mathbb{C}$, $\pi|_{\Delta(z,r)} : \Delta(z, r) \longrightarrow \mathbb{C}/\pi$ is injective, where $\Delta(z, r) = \{|\tau - z| < r\}$

Since $\mathbb{C}/\Lambda$ is compact, $\exists z_1, \cdots, z_n \in T$. s.t. $\{U_j = \pi(\Delta(z_j, r))\}_{j=1}^n$ is an open cover of $\mathbb{C}/\Lambda$.

$$\begin{tikzcd} \mathbb{C} \supset \Delta(z_j, r) \arrow[d, "\text{holomorphic}"', "\pi|_{\Delta(z_j,r)}"] \\ \mathbb{C}/\Lambda \supset U_j \arrow[u, bend right, "\varphi_j"'] \end{tikzcd}$$

$\varphi_j = (\pi|_{\Delta(z_j,r)})^{-1}$. Since π is an open map, $\pi|_{\Delta(z_j,r)}$ is a homeomorphism

Let $\varphi_j := (\pi|_{\Delta(z_j,r)})^{-1} : U_j \longrightarrow \Delta(z_j, r)$.

One can prove that $\varphi_j \circ \varphi_i^{-1} = z + \gamma$, $\exists \gamma \in \Lambda$. independent of z.

hence $\{(U_j, \varphi_j)\}_{j=1}^n$ defines a Riemann Surface structure on X.

In the viewpoint of differential geometry. We can define angles between two vectors. (Particularly, it is compatible with what is discussed in (Riemann geometry))

X: oriented smooth surface g: a Rie-metric on X

Theorem ($\exists$ of isothermal coordinate) $\to$ 等温坐标的存在性

$\forall p \in X$ $\exists$ a local coordinate $z = x + iy$ s.t.

① z is compatible with the orientation on X.

② g can be represented in term of z as:

$$g = \rho^2(z)(dx^2 + dy^2) = \rho^2(z)|dz|^2$$

ρ is a smooth function

Notion: A coordinate z satisfying ① and ② is called ... of g (等温坐标)

Assume $z = x + iy$, $z' = x' + iy'$ are two ... of g

The the coordinate change $z \to z'$ is holomorphic

So: g induces a Rie-S structure on X, whose holomorphic local coordinate are given by ... of g

E.g. X is an orientaled smooth surface

If $j: X \to \mathbb{R}^3$ is a smooth immersion, then $j^* g_0$ gives a Rie-metric on X

tangent maps are injective

so one can pull it back

(metric is given by inner product on tangent space)

Hence, j induces a conformal structure on X (Rie-S)

Q_1: Any surface can be immersed ... ? (by restriction of g_0)

2: Rie-S ?

A_1: no! torus

2: Theorem: X: a R-S $\Rightarrow$ $\exists$ embedding $j: X \hookrightarrow \mathbb{R}^3$ (locally conformal)

be the conformal structure on X is induced by j

Definition: two Rie-metric g_1, g_2 on surface X is called conformal: $g_2 = \rho g_1$ for some smooth function $\rho > 0$ on X

two conformal Rie-metric on an oriented smooth surface induces the same Rie-S structure on X

单值化定理

Lex X be a compact R-S of genus λ. Then,

① $\lambda = 0$ $\exists$ unique Rie-metric g on X s.t.

- g induces the con-structure on X
- the curvature $\equiv 1$

② $\lambda = 1$ $\exists$ unique Rie-metric ...

- ...
- ... $\equiv 0$
- the volume of $(X, g) = 1$

③ $\lambda > 1$ ...

- ...
- ... $\equiv -1$

Higher dimentional ... : Calabi conjecture

Lecture 3

Chapter 2 Riemann surfaces

Basic Concepts

Definition: A Riemann surface is a connected, Hausdorff topological space X, with an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ and a family of maps $z_\alpha: U_\alpha \longrightarrow \mathbb{C}$ satisfying: ① $z_\alpha: U_\alpha \longrightarrow z_\alpha(U_\alpha)\ (\overset{\text{open}}{\subset} \mathbb{C})$ is a homeomorphism. ② If $\alpha, \beta \in \Lambda$, with $U_\alpha \cap U_\beta \neq \phi$, the transition function $z_\beta \circ z_\alpha^{-1}: z_\alpha(U_\alpha\cap U_\beta) \longrightarrow z_\beta(U_\alpha \cap U_\beta)$ is holomorphic.

Definition: • (U_α, z_α) is called a holomorphic local coordinate on X

• En générale, si $U \subset X$ est ouvert et $z: U \to \mathbb{C}$ une application, alors (U, z) un coordimètre locale holomorphe sur X si : ① $z: U \longrightarrow z(U) \overset{\text{ouvert}}{\subset} \mathbb{C}$ est holomorphe, et ② $\forall \alpha$ avec $U \cap U_\alpha \neq \phi$, l'application $z_\alpha \circ z^{-1}$: $z(U\cap U_\alpha) \longrightarrow z_\alpha(U \cap U_\alpha)$ est holomorphe. Pour exemple, si $U_\alpha' \overset{\text{ouvert}}{\subset} U_\alpha$, alors, $(U_\alpha', z_\alpha|_{U_\alpha'})$ est un coordimètre locale holomorphe.

Examples: $\mathbb{C}$ and domains in $\mathbb{C}$ are Riemann surfaces.

Example: $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$

Define topology on $\mathbb{P}^1$ as:

- For $z \in \mathbb{C}$, the neighborhood basis of z is given by the ordinary topological basis.
- For $z = \infty$, The neighborhood of z is given by $U_r := \{z \in \mathbb{C}: |z| > r\} \cup \{\infty\}$, $r > 0$.

Under this topology, $\mathbb{P}^1$ is a connected, Hausdorff, compact space. Let $U = \mathbb{C}$, $V = \mathbb{C}^\times \cup \{\infty\}$, then $\{U, V\}$ is an open cover of $\mathbb{P}^1$. Let $z_1: U \longrightarrow \mathbb{C}$ $z \mapsto z$ and $z_2: V \to \mathbb{C}$ $z \mapsto \begin{cases} \frac{1}{z} & z \neq \infty \\ 0, & z = \infty. \end{cases}$ Then z_i are homeomorphisms, $U \cap V = \mathbb{C}^\times$ $z_1(U\cap V) = z_2(U \cap V) = \mathbb{C}^\times$, then $z_2 \circ z_1^{-1}: z \mapsto \frac{1}{z}$, $z_1 \circ z_2^{-1}: z \mapsto \frac{1}{z}$, they are holomorphic. Hence, $\{U, V\}$ gives a Riemann surface structure of $\mathbb{P}^1$. $\mathbb{P}^1$ is called the Riemann surface.

Example: $X = \mathbb{C}/\Lambda$. Let $w_1, w_2 \in \mathbb{C}$ be two $\mathbb{R}$-linearly independent vectors. Let $\Lambda = \{m w_1 + n w_2 : m, n \in \mathbb{Z}\}$. $\Lambda \subset \mathbb{C}$ is a subgroup. $X = \mathbb{C}/\Lambda$. Let $\pi: \mathbb{C} \to \mathbb{C}/\Lambda = X$ be the quotient map. The quotient topology on X is defined as follows: a set $U \subset X$ is said to be open, if $\pi^{-1}(U) \subset \mathbb{C}$ is open. Under this topology, $\pi: \mathbb{C} \to X$ is continuous and open (i.e., π maps open subsets of $\mathbb{C}$ onto open sets in $\mathbb{C}/\Lambda$). Let $T =$ , then $\pi(T) = \mathbb{C}/\Lambda$ hence $\mathbb{C}/\Lambda$ is compact, connected, Hausdorff space.

Obvious fact: $\exists r > 0$ s.t. $\forall z \in \mathbb{C}$, $\pi|_{\Delta(z,r)}: \Delta(z,r) \longrightarrow \mathbb{C}/\Lambda$ is injective, where $\Delta(z,r) = \{|\zeta - z| < r\}$; Since $\mathbb{C}/\Lambda$ is compact, $\exists z_1, \dots, z_n \in T$ s.t. $\{U_j, \Delta(z_j, r)\}_{j=1}^n$ is an open cover of $\mathbb{C}/\Lambda$. Easily check that this gives a Riemann surface structure of $\mathbb{C}/\Lambda$.

Lecture 4:

Let X be an oriented smooth surface, let g be a Riemann metric on X.

Theorem (Existence of Isothermal coordinate) For any $p \in X$, there is a local coordinate $z = x+iy$ s.t.
① z is compatible with the orientation of X, and ② g can be represented in term of z as
$g = \rho^2(z)(dx^2+dy^2) = \rho^2(z)|dz|^2$ $(dz = dx+idy)$ where $\rho > 0$ is a smooth function.

Notation: A coordinate satisfying ① and ② is called isothermal coordinate of g.

Exercise: Assume $z = x+iy$, $z' = x'+iy'$ are two isothermal coordinate of g. Then the coordinate change $z \mapsto z'$ is holomorphic.

Therefore, g induces a Riemann surface structure on X, whose holomorphic local coordinates are given by isothermal coordinates of g.

Example. Assume X is oriented smooth surface. If $j: X \to \mathbb{R}^3$ is a smooth immersion, then j^*g_0 gives a Riemann metric on X, where g_0 is the ordinary metric on $\mathbb{R}^3$, here j induces a conformal structure (i.e., Riemann surface structure) on X

Notice that not all metrics come from the immersion to $\mathbb{R}^3$. But we have the following

Theorem (Carsa, 60; Yiedy 70) Let X be a Riemann surface, then $\exists$ embedding $j: X \hookrightarrow \mathbb{R}^3$ s.t. the conformal structure of X is induced by j.

Definition: Two Riemann metrics g_1 and g_2 are called conformal, if $g_2 = \rho g_1$, for some smooth function $\rho > 0$ on X.

It is obvious that two conformal Riemann metric on an oriented smooth surface induces the same Riemann surface structure on X.

Uniformization Theorem. Let X be a Riemann surface with genus λ. Then
① If $\lambda = 0$, then there is a unique Riemann metric g on X s.t. (i) g induces conformal structure on X, and (ii) the curvature $\equiv 1$.
② If $\lambda = 1$, then there is a unique Riemann metric g on X s.t. (i) g induces the conformal structure on X, (ii) The curvature of $g \equiv 0$; (iii) The volume of $(X, g) = 1$
③ If $\lambda \geq 2$, then there is a unique Riemann metric g on X s.t. (i) g induces the conformal structure on X; and (ii) Curvature $g \equiv -1$.

Higher dimensional analogue of the above theorem: Calabi conjecture

Definition: A function $f: X \to \mathbb{C}$ is called holomorphic, if $\exists$ a coordinate cover $\{(U_\alpha, z_\alpha)\}_{\alpha \in A}$ of X s.t. $f \circ z_\alpha^{-1}: z_\alpha(U_\alpha) \to \mathbb{C}$ is holomorphic for any $\alpha \in A$.

Remark: ① The definition of holomorphic functions does not depend on the choice of coordinate neighborhood. In fact, if (U, z) is a local coordinate on X, then $f \circ z^{-1}: z(U) \to \mathbb{C}$ is holomorphic.
② If $D \subset X$ is a domain, then D is a Riemann surface and $f|_D: D \to \mathbb{C}$ is holomorphic.
③ Holomorphicity is a local property: If f is hol. near each point in X, then f is holomorphic on X.

④ A Riemann surface is also smooth manifold, on which we can define smooth functions in the similar way.

Notation: $\mathcal{O}(X)$ = the space of all holomorphic functions on X, $\mathcal{O}(X)$ is an algebra over $\mathbb{C}$.

Basic properties of holomorphic functions:

① Identity Theorem: the zero set of a nonconstant $f \in \mathcal{O}(X)$ is a discrete subset of X.

Proof: Assume by contradiction that $p_0 \in X$ is an accumulation point of the zero set of f. Let (U, z) be a local coordinate near p_0, and $z(p_0) = 0$. Then $f \circ z^{-1}: z(U) \to \mathbb{C}$ is holomorphic. According to identity theorem in domains in $\mathbb{C}$, $f \circ z^{-1} \equiv 0 \Rightarrow f|_U \equiv 0$. By connectedness of X, $f \equiv 0$. $\square$

② Open Mapping Theorem: A nonconstant $f \in \mathcal{O}(X)$ is an open map.

③ Maximal Principle: For a nonconstant $f \in \mathcal{O}(X)$, $|f|$ cannot reach its maximal value on X.

Corollary: If X is a compact, all holomorphic functions on X are constant.

Holomorphic Maps Between Riemann Surfaces.

Definition: Let $f: X \to Y$ is a continuous map, f is called holomorphic if for any local coordinate (V, w) on Y f^*w is a holomorphic function on $f^{-1}(V)$.

Notation: $\mathcal{O}(X;Y)$: the space of all holomorphic maps from X to Y.

Definition: ① If $f \in \mathcal{O}(X,Y)$ is a bijection, then f is called biholomorphic ($\Rightarrow f^{-1} \in \mathcal{O}(X,Y)$)

② If $\exists$ a biholomorphic map $f \in \mathcal{O}(X,Y)$, we say that X and Y are isomorphic. Denoted by $X \cong Y$.

③ A biholomorphic map $f \in \mathcal{O}(X,X)$ is called automorphic of X. Let $\mathrm{Aut}(X)$ be the space of all automorphisms of X, then $\mathrm{Aut}(X)$ is a group, called the automorphism group of X.

Example: $\mathrm{Aut}(\Delta) = \{ z \mapsto e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z} : \theta \in \mathbb{R}, z_0 \in \Delta \} \cong SL(2, \mathbb{R})$

Example: ① $\mathrm{Aut}(\mathbb{P}^1) = \{ z \mapsto \frac{az+b}{cz+d} \} \cong PSL(2, \mathbb{C})$

② For any distinct point $z_1, z_2, z_3 \in \mathbb{P}^1$, then $\exists ! f \in \mathrm{Aut}(\mathbb{P}^1)$ s.t. $f(z_1) = 0$, $f(z_2) = 1$, $f(z_3) = \infty$.

- Holo fuctions

$f: X \to \mathbb{C}$ cover: $\{(U_\alpha, z_\alpha)\}$ of X

$\forall \alpha$ $f \circ z_\alpha^{-1}: z_\alpha(U_\alpha) \to \mathbb{C}$ is holo

then we call f holo

Remark: holo of f is independent of coordinate cover. if f is holo at each neighb... then f is holo on X

R-S is a smooth mani

- $\mathcal{O}(X)$: holo functions on X $\to$ $\mathbb{C}$ algebra

- ① zero set of a nonconstant $f \in \mathcal{O}(X)$, is a discrete subset

② open mapping theorem:

A nonconstant $f \in \mathcal{O}(X)$ is open

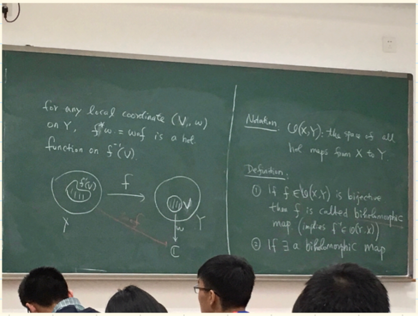
③ maximum principle

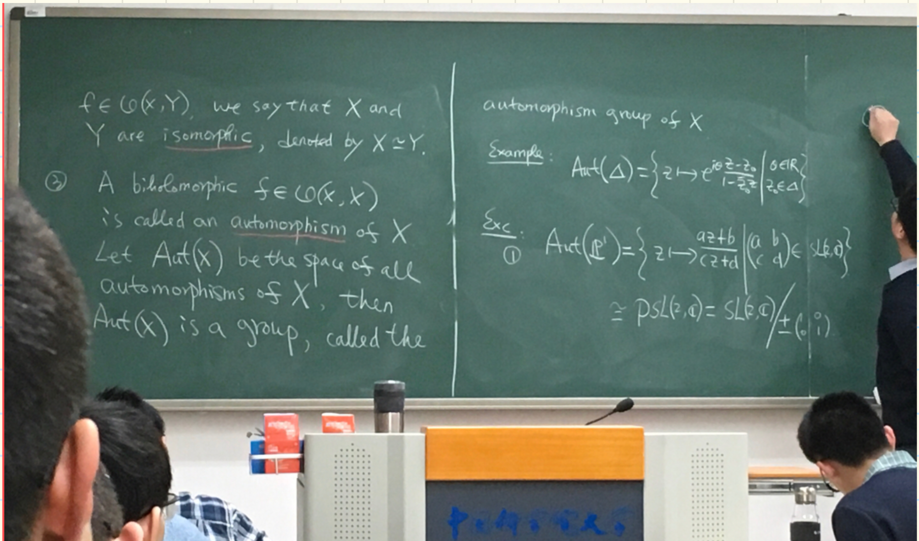
不能取到 maximum

if X is compact $\Rightarrow$ all $f \in \mathcal{O}(X)$ is constant

- mapping

Def: $f: X \to Y$ is continuous, f is called holomorphic, if:





② $\forall$ $z_1, z_2, z_3 \in \mathbb{P}^1$ $\exists!$ $f \in Aut(\mathbb{P}^1)$ s.t. $f(z_1) = 0$

$f(z_2) = 1$ $f(z_3) = \infty$

Lecture 5:

Meromorphic functions:

Definition: A meromorphic function f on X is a holomorphic function f defined on $X-A$ for some discrete $A\subset X$ s.t. $\lim_{z\to a} f(z)=\infty$ for any $a\in A$.

Lemma: A meromorphic function f on X can be viewed as a map $f: X\to \mathbb{P}^1$ as $f(z)=\begin{cases} f(z), & z\in X-A \\ \infty, & z\in A \end{cases}$. It is a holomorphic map. On the other hand, if a map $f: X\to\mathbb{P}^1$ is a nonconstant holomorphic map, then f is a meromorphic function on X.

Notation: $M(X)=\{$all meromorphic functions on $X\}$. $M(X)$ is a field.

Let $f\in \mathcal{O}(X,Y)$ nonconstant

① If $\varphi\in\mathcal{O}(Y)$, then $f^*\varphi\in\mathcal{O}(X)$

② If $\varphi\in M(Y)$, then $f^*\varphi\in M(X)$.

$\Rightarrow$ $f^*: \mathcal{O}(Y)\to\mathcal{O}(X)$ a $\mathbb{C}$-algebra homomorphism; $f^*: M(Y)\to M(X)$ morphism of fields over $\mathbb{C}$.

Theorem: X, Y are Riemann surfaces:

(1) If X and Y are noncompact. Assume $h: \mathcal{O}(Y)\to\mathcal{O}(X)$ is a $\mathbb{C}$-algebra homomorphism, then $\exists$ a biholomorphic map $f: X\to Y$ s.t. $h=f^*$.

(2) If X and Y are compact. Assume $h: M(Y)\to M(X)$ is a $\mathbb{C}$-field isomorphism. Then $\exists$ biholomorphic map s.t. $f: X\to Y$ s.t. $h=f^*$

Example: $M(\mathbb{P}^1)\cong\mathbb{C}(z)$.

Example: $M(\mathbb{C}/\Lambda)=\{f\in M(\mathbb{C}), f$ is Λ-invariant$\}$. An element in $M(\mathbb{C})^\Lambda$ is $\wp: \mathbb{C}\to\mathbb{P}^1$ as
$$\wp(z)=\frac{1}{z}+\sum_{\gamma\in\Lambda}{}' \left(\frac{1}{(z-\gamma)^2}-\frac{1}{\gamma^2}\right)$$

Exercise: ① $\wp\in M(\mathbb{C})^\Lambda$ ② $\wp(z)$ is an even function; $\therefore$ near 0, $\wp(z)$ can be expanded as $\wp(z)=\frac{1}{z^2}+a_0+a_2z^2+a_4z^4+\dots$

③ $\wp'(z)\notin\mathbb{C}(\wp(z))$ ④ $\wp'(z)^2=4\wp(z)^3+g_2\wp+g_3$ for some constants $g_2, g_3\in\mathbb{C}$

Theorem: $M(\mathbb{C}/\Lambda)=\mathbb{C}(\wp,\wp')$

Presentation of Holomorphic Maps Via Local Coordinates

Theorem: Let $f\in\mathcal{O}(X,Y)$ nonconstant. Let $p\in X$, $q\in Y$ with $q=f(p)$. Then $\exists$ a coordinate z near p, and w near q, under which f is locally represented as $z\mapsto w=z^k$ where $k>0$ is an integer that is independent the choice of local coordinates.

Definition: ① k in the above theorem is called the multiplicity of f at p, denote as $m_f(p)$

② The number $b_f(p):=m_f(p)-1$ is called the branching number f of p

③ If $b_f(p)\geq 1$, then p is called a branched point of f; otherwise, p is called an unbranched point of f.

Geometric interpretation of k: $\exists$ a sufficiently small neighborhood U of p in X, and a small V of q in Y, s.t. $\forall y\in Y-q$, $\#(f^{-1}(y)\cap U)=k$.

Definition: Assume $f: X\to\mathbb{P}^1\in M(X)$, let $P\in X$.

If $f(P)=0$ or ∞, we say that p is zero, or pole f of order $m_f(p)$

Definition Let $f \in M(X)$ nonconstant, where X is a compact Riemann surface. Assume $p_1, \dots, p_r$ are zeros of f of order $k_1, \dots, k_r$; $q_1, \dots, q_s$ are poles of f of order $l_1, \dots, l_s$. The divisor associated to f is defined to be the formal sum $(f) = k_1 p_1 + \dots + k_r p_r - l_1 q_1 - \dots - l_s q_s$, which is regarded as an element in the abelian group $Div(X)$ generated by points of X

Remark: Assume $f, g \in M(X)$. If $(f) = (g)$, then $\exists$ constant $c \neq 0$ s.t. $f = cg$. So f is essentially determined by its divisor (f).

Definition: ① $Div(X)$'s elements are called divisor.

② If $\exists f \in M(X)$, s.t., $(f) = D$, D is called a principle divisor.

③ def: $\deg D := m_1 + m_2 + \dots + m_k$ where $D = m_1 p_1 + \dots + m_k p_k$

Exc: $Div(X)^P$ be the space of all principle divisors in $Div(X)$

prove: it's a subgroup of $Div(X)$. $\left(\frac{f}{g}\right) = (f) - (g)$

$V = \#\{vertice\}$ $E = \#\{edges\}$

Riemman–Hurwitz formula:

If X is orientable then $g(X)$ given by $\chi(X) = 2 - 2g(X)$ is called the genus of X.

Def. a continuous map $f: X \to Y$ between two topological spaces is called a covering map, if $\forall y \in Y$, $\exists$ neighborhood V of y in Y, and disjoint opensets $\{U_\alpha\}_{\alpha \in \Lambda}$ s.t. ① $f^{-1}(V) = \bigsqcup_{\alpha \in \Lambda} U_\alpha$ ② $f|_{U_\alpha}: U_\alpha \to V$ is a homeomorphic ($\forall \alpha \in \Lambda$)

Assume $f: X \to Y$ is a covering map, Y is path-connected.

Then for $y \in Y$, $d_f := \#\pi^{-1}(y) \leq +\infty$ is independent of the choice of y, and is called the degree of f. (locally constant (continuous) $\to$ constant)

Lemma: Let X and Y are compact surface, if $\exists$ a covering map $f: X \to Y$ of $\deg = d$, Then $\chi(X) = d\chi(Y)$

Proof. Take a triangulation T of Y, while is fine enough s.t. each triangle σ in T has a neighborhood V s.t. $f^{-1}(V) = U_1 \sqcup \dots \sqcup U_d$ and $f|_{U_j}: U_j \to V$ are homeo...

Then $\tilde{T} := f^{-1}(T)$ is a triangulation of X, and obviously:

$V/E/F(\tilde{T}) = d\, V/E/F(T)$

Let $f: X \to Y$ be a covering map, X, Y are compact and orientable, $\deg f = d$

Branched covering

Definition. A continuous map $f: X \to Y$ between two surfaces is called a branched covering, if:

① f is surjective ② $\exists$ a discrete subset $A \subset Y$ s.t.

$f|_{X \setminus f^{-1}(A)}: X \setminus f^{-1}(A) \to Y \setminus A$ is a covering map and for each $p \in f^{-1}(A)$, there a topological coordinate z near p and w near $f(p)$, s.t. f is locally represented as $z \to w = z^k$, $k \geq 1$

k depends on p

then we can define multiplicity, $b_f(p)$

if $b_f(p) \geq 1$, p is called a branched point.

Def. $f: X \to Y$ branched covering, Y: connected

for $y \in Y$, def: $d_f := \sum_{x \in f^{-1}(y)} m_f(x) \leq +\infty$ called the deg

Exc: $\deg f$ is idependant of the choice of y.

Definition: A continuous map $f: X \to Y$ between two surfaces is called proper if for any compact set $K \subset Y$, $f^{-1}(K)$ is compact

Remark: ① If X is compact, then f must be proper

② Exc. if: $X \to Y$ is proper, then $f(X) \subset Y$ is closed in Y.

③ Exc. Assume X, Y are Riemann Surfaces

$f: X \to Y$ is a holomorphic proper nonconstant map.

Then f is a branched covering

Exc. : X is a compact Riemann surface $\underset{M(X)}{f}: X \to \mathbb{P}^1$ nonconstant

$\Rightarrow$ f is a braded cover

$\Rightarrow$ $\sum_{p \in f^{-1}(0)} m_f(p) = d_f = \sum_{p \in f^{-1}(\infty)} m_f(p)$

i.e. the number of zeros of f and the poles of f is equal, counting multiplicity.

so $\deg(f) = 0$

Theorem: Riemann–Hurwitz formula

Let $f: X \to Y$ be a branched covering between compact, oriented surfaces, $\deg f = d$, Let $b_f = \sum_{p \in X} b_f(p) = \sum_{p \in X} (m_f(p) - 1)$ be the total branched number of f

Proof. Take a triangulation T of Y s.t. $A \subset Y$ is a subset of the vertices of T, where $A \subset Y$ is a finite set s.t. $f|_{X \setminus f^{-1}(A)} : X \setminus f^{-1}(A) \to Y \setminus A$ is a covering map

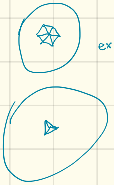
We assume that T is fine enough s.t. $\tilde{T} := f^{-1}(T)$ is a triangulation of X

Then we have $E/F(\tilde{T}) = d\, E/F(T)$

but $V(\tilde{T}) = d\, V(T) - b_f$

$$\chi(X) = d\,\chi(Y) - b_f$$

In particular, b_f is an even number.



example:

$$z \downarrow z^2$$

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Theorem: Assume X and Y are compact orientable surfaces. Let $f: X \to Y$ be a branched covering with total branching number b_f. If the degree of f is d, then we have the following Riemann-Hurwitz formula: $\chi(X) = d\chi(Y) - b_f$. $g(X) = d(g(Y)-1)+1+\frac{b_f}{2}$

① $g(X) \geq g(Y)$

② If $g(X)=g(Y)$

(i) $g(X)=g(Y)=0$, $0=-d+1+\frac{b_f}{2} \Rightarrow d=1+\frac{b_f}{2}$.

(ii) $g(X)=g(Y)=1$, then $b_f=0$, and $f: X \to Y$ is a covering map

(iii) $g(X)=g(Y)>1$, then $d=1$ ($\Rightarrow b_f=0$), $f: X\to Y$ is a homeomorphism.

Easy fact: If X and Y are compact Riemann surfaces, and $f \in \mathcal{O}(X,Y)$. If f is nonconstant, then f is a branched covering.

Chapter 3 Analysis on Riemann surfaces.

§1: Weyl's lemma

$\Omega \subset \mathbb{C}$ domain. Define the topology of $C_c^\infty(\Omega)$ to be $\lim_{n\to\infty} f_n = f \Leftrightarrow \exists K \subset \Omega$ compact s.t. $\operatorname{supp} f_n \subset K$ $\forall n$ and $|f_n - f|_{C^r(K)} \to 0$ $\forall r$

Then $C_c^\infty(\Omega)$ becomes a topological vector space

Definition: A generalized function is a continuous linear functional on $C_c^\infty(\Omega)$

Remark: For a generalized function f on Ω, and $\varphi \in C_c^\infty(\Omega)$, we often denote $f(\varphi)$ by $\int_\Omega f\varphi$

Example: Let $f \in L^1_{loc}(\Omega)$, the space of locally integrable functions on Ω, i.e., for any $\varphi \in C_c^\infty(\Omega)$, $f\varphi$ is integrable on Ω. It is easy to show that $C_c^\infty(\Omega) \to \mathbb{C}$ $\varphi \mapsto \int_\Omega f\varphi$ is a generalized function, which is still denoted by f.

Exercise*: Let $f, g \in L^1_{loc}(\Omega)$, if $\int_\Omega f\varphi = \int_\Omega g\varphi$ $\forall \varphi \in C_c^\infty(\Omega)$, then $f=g$ a.e.

Theorem (Weyl's lemma) Let $u \in L^1_{loc}(\Omega)$ satisfy $\Delta u = 0$, i.e. $\int_\Omega \Delta u.\varphi = \int_\Omega u\Delta\varphi = 0$ $\forall \varphi \in C_c^\infty(\Omega)$. Then $u \in C_c^\infty(\Omega)$ and $\Delta u = 0$ in the ordinary sense.

Lemma (Cauchy's integral formula) Let $\Omega \subset \mathbb{C}$ be a bounded domain, $\partial\Omega$ consists of several smooth curves. Assume $f \in C^1(\overline{\Omega})$, then for $z \in \Omega$, we have $f(z) = \frac{1}{2\pi i}\left[\int_\Omega \frac{1}{\zeta - z}\frac{\partial f}{\partial \bar\zeta}(\zeta) + \int_{\partial\Omega}\frac{f(\zeta)}{\zeta - z}d\zeta\right.$. In particular,

① If $f \in \mathcal{O}(\Omega)$, then $f(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\zeta)}{\zeta - z}d\zeta$

② If $f \in C_c^\infty(\Omega)$, then $f(z) = \frac{1}{2\pi i}\int_\Omega \frac{1}{\zeta - z}\frac{\partial f(\zeta)}{\partial\bar\zeta}d\lambda(\zeta)$, where $d\lambda(\zeta)$ is the Lebesgue measure on $\mathbb{C}$. (Exc)

Lemma: Viewing $f(z) = \frac{1}{4\pi}\log|z|$ as a generalized function on $\mathbb{C}$, then we have $\Delta u = \delta_0$

Theorem: Assume $u \in L^1_{loc}(\mathbb{C})$ satisfies $\int_{\mathbb{C}} u\Delta\varphi = 0$ $\forall \varphi \in C_c^\infty(\mathbb{C})$, then $u \in C^\infty(\mathbb{C})$ and $\Delta u = 0$.

Proof: It suffices to prove that $u \in C^\infty(\mathbb{C})$. Define a function $\omega(z)$ on $\mathbb{C}$ s.t. ① $\omega(z)$ is smooth on $\mathbb{C}^\times$ ② $\omega(z) = \frac{1}{4\pi}\log|z|$ when $|z|<\frac{1}{2}$ ③ $\omega(z)=0$ when $|z|>1$. For an arbitrary $\psi \in C_c^\infty(\mathbb{C})$, define $\varphi(z) = \int_{\mathbb{C}}\omega(z-\zeta)\psi(\zeta)d\lambda(\zeta)$. Changing variables, we get $\varphi(z) = \int_{\mathbb{C}}\omega(\zeta)\psi(\zeta+z)d\lambda(\zeta)$. $\varphi(z) \in C_c^\infty(\mathbb{C})$. $\Delta\omega(z) = \delta_0 + \gamma(z)$, where $\gamma \in C_c^\infty(\mathbb{C}^\times)$. $\Rightarrow \Delta\varphi(z) = \int_{\mathbb{C}}\omega(\zeta)\Delta_z\psi(\zeta+z)d\lambda(\zeta)$
$= \int_{\mathbb{C}}\omega(\zeta)\Delta_\zeta\psi(\zeta+z)d\lambda(\zeta) = \int_{\mathbb{C}}\Delta_\zeta\omega(\zeta).\psi(\zeta+z)d\lambda(\zeta) = \int_{\mathbb{C}}(\delta_0+\gamma(\zeta))\psi(\zeta+z)d\lambda(\zeta) = \psi(z) + \int_{\mathbb{C}}\gamma(\zeta)\psi(\zeta+z)d\lambda(\zeta)$

By assumption $\int_{\mathbb{C}} u\Delta\varphi = 0$, hence $0 = \int_{\mathbb{C}} u\psi + \int_{\mathbb{C}} u(z)\int_{\mathbb{C}}\gamma(\zeta-z)\psi(\zeta)d\lambda(\zeta)\,d\lambda(z)$. The second term $= \int_{\mathbb{C}} d\lambda(\zeta)\psi(\zeta)\overbrace{\int_{\mathbb{C}} d\lambda(z)\,u(z)\gamma(\zeta-z)}^{h(\zeta)}$
$= \int_{\mathbb{C}} d\lambda(\zeta)\,h(\zeta)\psi(\zeta)$ $h \in C^\infty(\mathbb{C})$. Therefore, $0 = \int_{\mathbb{C}}(u+h)\psi$. But $\psi \in C_c^\infty(\mathbb{C})$ is arbitrary, $u = -h \in C^\infty(\mathbb{C})$ □

Lecture 8

Theorem: Assume $u \in L^1_{loc}(\Omega)$ satisfies $\int_\Omega u \Delta\varphi = 0$, $\forall \varphi \in C_c^\infty(\Omega)$. Then $u \in C_c^\infty(\Omega)$ and $\Delta u = 0$.

Proof. $\Delta \frac{i}{4\pi} \ln|z| = \delta_0$ (with cpt-supp) $\xrightarrow{\text{光滑}}$ keep the cpt-supp (idea)

Let $\omega(z)$ be a function on $\mathbb{C}$ s.t. ① $\omega \in C^\infty(\mathbb{C}^*)$ ② $\omega(z) = \frac{i}{4\pi}\ln|z|$, $|z| < \frac{\varepsilon}{2}$ ③ $\omega(z) = 0$ $|z| > \varepsilon$ (where $\varepsilon > 0$ is a constant)

$$\Delta\omega = \Delta\left(\omega(z) - \frac{i}{4\pi}\ln|z|\right) + \Delta\left(\frac{i}{4\pi}\ln|z|\right) = \gamma(z) + \delta_0$$

smooth, $\gamma \in C_c^\infty(\mathbb{C}^*)$ (∞ 的情况，由于 $\ln|z|$ 是 harmonic function)

Let ψ be an arbitrary element in $C_c^\infty(\Omega)$. Define: $\varphi(z) = \int_\Omega \omega(\xi - z)\psi(\xi)\, d\lambda(\xi)$

$\omega(\xi - z)$: vanish if $\xi - z > \varepsilon$; $\psi(\xi)$: cpt-supp

claim: φ is well-defined and cpt-supp on Ω: reason (if ε is sufficiently small)

since support $\psi \subset \Omega$, $\varphi(z) = \int_{\mathbb{C}} \omega(\xi - z)\psi(\xi)\, d\xi = \int_{\mathbb{C}} \omega(\xi)\psi(\xi + z)\, d\xi$

$\Rightarrow \varphi$ is smooth $\Rightarrow \varphi \in C_c^\infty(\Omega)$

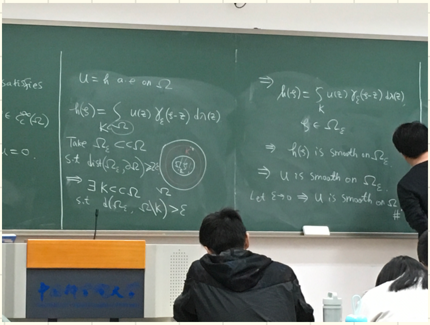
$$\Delta\varphi = \int_{\mathbb{C}} \omega(\xi)\Delta_\xi\psi(\xi+z) \underset{\text{by def}}{=} \int_{\mathbb{C}} \Delta_\xi\omega(\xi)\psi(\xi+z) \underset{\text{代入}\Delta\omega}{=} \cdots \underset{\text{why?}}{=} \psi(z) + \int_{\mathbb{C}} \gamma(\xi)\psi(\xi+z)\, d\lambda(\xi)$$

$$= \psi(z) + \int_{\mathbb{C}} \gamma(\xi - z)\psi(\xi)\, d\lambda(\xi) = \psi(z) + \int_\Omega \gamma(\xi - z)\psi(\xi)\, d\lambda(\xi)$$

$$\Rightarrow 0 = \int_\Omega u\Delta\varphi = \int_\Omega u\psi + \int_\Omega u(z)\left(\int_\Omega \gamma(\xi - z)\psi(\xi)\, d\lambda(\xi)\right) d\lambda(z) = \int_\Omega u\psi + \int_\Omega \Big(\underbrace{\int_\Omega u(z)\gamma(\xi - z)\, d\lambda(z)}_{h(\xi)}\Big)\psi(\xi)\, d\lambda(\xi)$$

$= \int_\Omega u\psi + \int_\Omega h\psi$ $\int_\Omega u\psi = \int_\Omega (-h)\psi$ then by the lemma

$u = -h$ a.e. on Ω want h smooth



Exc. Assume $f \in L^1_{loc}(\Omega)$ $\int_\Omega f \frac{\partial\varphi}{\partial\bar z} = 0$ $\forall\varphi \in C_c^\infty(\Omega)$ (i.e. in the sense of distribution, $\frac{\partial f}{\partial \bar z} = 0$) Then $f \in \mathcal{O}(\Omega)$

($\frac{\partial}{\partial\bar z}$: an elliptic operator) → 基本解 $\frac{1}{z}$: $\Delta\ln|z| = \frac{\partial}{\partial z}\frac{\partial}{\partial\bar z}\ln|z| = \frac{\partial}{\partial z}\frac{\partial}{\partial\bar z}\frac{1}{2}(\ln z + \ln\bar z)$ $= \frac{1}{2}\frac{\partial}{\partial\bar z}\frac{1}{z}$, $\Delta\ln|z| = c\delta_0$

Exc. $f \in L^1_{loc}((0,1))$, $\int_0^1 f(x)\varphi'(x)\, dx = 0$ $(\forall\varphi \in C_c^\infty((0,1))) \Rightarrow f = c$ a.e.

§2 differential forms on Riemann surfaces

Let X be a Riemann surface, U be an open set of X, denote by $\mathcal{E}^\infty(U)$ the space of (complex-valued) smooth functions on $\mathbb{C}$

For $p \in X$, let $\tilde{\mathcal{E}}^\infty_{x,p} := \bigsqcup_{U \ni x} \mathcal{E}^\infty(U)$ we define $f \in C^\infty(U)$ $g \in C^\infty(V)$ to be equivalent iff: $\exists W \subset U \cap V$, s.t. $f|_W = g|_W$ on $\tilde{\mathcal{E}}^\infty_{x,p}$

$\mathcal{E}^\infty_{x,p} := \tilde{\mathcal{E}}^\infty_{x,p}/\sim$ called the space of germs of smooth functions at p

it's a $\mathbb{C}$-algebra which has a unique maximal ideal $\mathfrak{m}_{x,p} := \{f \in \mathcal{E}^\infty_{x,p} \mid f(p) = 0\}$ (then we have $\mathfrak{m}_{x,p}^2$)

Def: A tangent vector v of X at p is a derivate of $\mathcal{E}^\infty_{x,p}$ i.e. $v: \mathcal{E}^\infty_{x,p} \to \mathbb{C}$ linear map

$v(fg) = v(f)g(p) + f(p)v(g)$ Denoted by $T_{x,p}$ ...

Coordinate representation of tangent vectors: Assume $z = x + yi$ is a local (holomorphic) coordinate near p, one can define two tangent vectors as basis: $\frac{\partial}{\partial x}\big|_p$ $\frac{\partial}{\partial y}\big|_p \in T_{x,p}$

$\frac{\partial}{\partial x}\big|_p f = \frac{\partial f}{\partial x}\big|_p$ $\frac{\partial}{\partial y}\big|_p f = \frac{\partial f}{\partial y}\big|_p$ (in local coordinate)

Lemma. $\dim_{\mathbb{C}} T_{x,p} = 2$

Proof. let $f \in \mathcal{E}^\infty_{x,p}$, assume $z(p) = x_0 + iy_0$. Then f can be written as $f(z) = f(p) + a(x - x_0) + b(y - y_0) + h(x,y)$, $h \in \mathfrak{m}_{x,p}^2$

(Exc. $\forall v \in T_{x,p}$ satisfies ① $v(c) = 0$ where c is a constant func ② $v(f) = 0$, $f \in \mathfrak{m}_{x,p}^2$)

$v(f) = 0 + a\,v(x - x_0) + b\,v(y - y_0) + 0$

$= \left(v(x - x_0)\frac{\partial}{\partial x}\big|_p + v(y - y_0)\frac{\partial}{\partial y}\big|_p\right) f$

$= \left(v(x)\frac{\partial}{\partial x}\big|_p + v(y)\frac{\partial}{\partial y}\big|_p\right) f$

$\Rightarrow v = \cdots$ (and v is determined by the value at x and y)

So. $T_{x,p} = \left\{a\frac{\partial}{\partial x}\big|_p + b\frac{\partial}{\partial y}\big|_p \;\middle|\; a, b \in \mathbb{C}\right\}$ (complex 化) ($\dim_{\mathbb{C}} X = 1$ $\dim_{\mathbb{R}} X = 2$, in general, $\dim_{\mathbb{R}} T_p X = 2$ but we get a 复化 to reflect the conformal structure)

Introduce $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial\bar z}$, then it's also a basis

Cotangent space: the dual of $T_{x,p}$, i.e. $T_{x,p}^*$ (forms! make Tensor products!)

Denote $\{dx|_p, dy|_p\}$ the dual basis of $\{\frac{\partial}{\partial x}|_p, \frac{\partial}{\partial y}|_p\}$ and $\{dz|_p, d\bar z|_p\}$ of $\{\frac{\partial}{\partial z}|_p, \frac{\partial}{\partial\bar z}|_p\}$

Let $\omega = u + iv$ be another local coordinate near p

what about the transform matrix?

$$\frac{\partial}{\partial x}\Big|_p = \frac{\partial u}{\partial x}\frac{\partial}{\partial u}\Big|_p + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\Big|_p \quad \cdots \quad \Rightarrow \begin{pmatrix}\frac{\partial}{\partial x}|_p \\ \frac{\partial}{\partial y}|_p\end{pmatrix} = \begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{pmatrix}(p)\begin{pmatrix}\frac{\partial}{\partial u}|_p \\ \frac{\partial}{\partial v}|_p\end{pmatrix}$$

$$\frac{\partial}{\partial z}\Big|_p = \frac{\partial w}{\partial z}\frac{\partial}{\partial w}\Big|_p + \underbrace{\frac{\partial\bar w}{\partial z}}_{0}\frac{\partial}{\partial\bar w}\Big|_p \qquad \frac{\partial}{\partial\bar z}\Big|_p = \underbrace{\frac{\partial w}{\partial\bar z}}_{0}\frac{\partial}{\partial w}\Big|_p + \frac{\partial\bar w}{\partial\bar z}\frac{\partial}{\partial\bar w}\Big|_p = \overline{\left(\frac{\partial w}{\partial z}\right)}\frac{\partial}{\partial\bar w}\Big|_p$$

$$= \frac{\partial w}{\partial z}\frac{\partial}{\partial w}\Big|_p$$

$\Rightarrow T^{1,0}_{x,p} := \{a\frac{\partial}{\partial z}|_p \mid a \in \mathbb{C}\}$ $T^{0,1}_{x,p} := \{a\frac{\partial}{\partial\bar z}|_p \mid a \in \mathbb{C}\}$ is independent of the choice of coordinate.

$T^{1,0}_{x,p} \oplus T^{0,1}_{x,p} = T_{x,p}$ (the break determined the conformal structure!)

(eigenvalue i $-i$)

X. Riemann Surface.

$T^*_{X,p} = (T^*_{X,p})^{1,0} \oplus (T^*_{X,p})^{0,1}$. $\xi \in T^*_{X,p}$. $\xi = a\,dx|_p + b\,dy|_p = \alpha\, dz|_p + \beta\, d\bar{z}|_p$.

$(T^*_{X,p})^{1,0} = \mathbb{C}\, dz|_p$, $(T^*_{X,p})^{0,1} = \mathbb{C}\, d\bar{z}|_p$.

Conjugation on $T_{X,p}$ and $T^*_{X,p}$. $v = a\frac{\partial}{\partial x}|_p + b\frac{\partial}{\partial y}|_p = \alpha\frac{\partial}{\partial z}|_p + \beta\frac{\partial}{\partial \bar{z}}|_p \in T_{X,p}$

$\xi = a\,dx|_p + b\,dy|_p = \alpha\, dz|_p + \beta\, d\bar{z}|_p \in T^*_{X,p}$.

Define: $\bar{v} = \bar{a}\frac{\partial}{\partial x}|_p + \bar{b}\frac{\partial}{\partial y}|_p = \bar{\alpha}\frac{\partial}{\partial \bar{z}}|_p + \bar{\beta}\frac{\partial}{\partial z}|_p$.

Obvious: $\overline{T^{1,0}_{X,p}} = T^{0,1}_{X,p}$; $\overline{(T^*_{X,p})^{1,0}} = (T^*_{X,p})^{0,1}$.

$\tau: V \longrightarrow V$, $v \longmapsto \bar{v}$ $\quad \tau(iv) = -i\bar{v}$, $\tau^2 = Id$

$V_{\mathbb{R}} := \{v \in V \mid \tau(v) = v\}$ $\mathbb{R}$-subspace of V

$iV_{\mathbb{R}} \cap V_{\mathbb{R}} = \{0\}$.

$V = V_{\mathbb{R}} \otimes \mathbb{C} = V_{\mathbb{R}} \oplus iV_{\mathbb{R}}$.

Vector field: Let $TX = \coprod_{p\in X} T_{X,p}$

$TX^{1,0} \triangleq \coprod_{p\in X} T^{1,0}_{X,p}$; $TX^{0,1} \triangleq \coprod_{p\in X} T^{0,1}_{X,p}$

TX is called the tangent bundle of X. ; $TX^{1,0}$ is called the holomorphic tangent bundle of X

Definition: A vector field on X is a map $V: X \to TX$, s.t. $V(p) \in T_{X,p}$. $\forall p \in X$.

If $V(p) \in T^{1,0}_{X,p}$ (or $T^{0,1}_{X,p}$) $\forall p$, we call V a vector field of type $(1,0)$ (or $(0,1)$).

Assume $(U, z = x+iy)$ is a local coordinate of X on U, then on U.

$V(p) = a(p)\frac{\partial}{\partial x}|_p + b(p)\frac{\partial}{\partial y}|_p = \alpha(p)\frac{\partial}{\partial z}|_p + \beta(p)\frac{\partial}{\partial \bar{z}}|_p$ where a, b, α, β are functions on U.

V is called C^∞-vector field if $a, b, \alpha, \beta \in C^\infty(U)$.

V is called a holomorphic vector field if $\beta \equiv 0$. α holomorphic.

Rmk: If V is a holomorphic vector field on X. Let (U_1, z), (U_2, w) be two local coordinates on X with $U_1 \cap U_2 \neq \phi$. $V = \alpha(z)\frac{\partial}{\partial z}$ on U_1. $V = \beta(w)\frac{\partial}{\partial w}$ on U_2.

On $U_1 \cap U_2$, $\alpha(z)\frac{\partial}{\partial z} = \beta(w)\frac{\partial}{\partial w} \Rightarrow \beta(w) = \alpha(z)\frac{\partial w}{\partial z}$.

Example: Holomorphic vector fields on $\mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$. Let $U_1 = \mathbb{C}$, $U_2 = \mathbb{P}^1 \setminus \{0\}$

let z, w be two local coordinates on U_1 and U_2 (resp.) where $w(z) = \begin{cases} \frac{1}{z}, & z \in \mathbb{C}^* \\ 0, & z = \infty. \end{cases}$

On $U_1 \cap U_2 = \mathbb{C}^*$, $w = \frac{1}{z}$.

Let V be a holomorphic vector field on $\mathbb{P}^1$. Assume on $U_1 = \mathbb{C}$: $V = \alpha(z)\frac{\partial}{\partial z}$

on U_2: $V = \beta(w)\frac{\partial}{\partial w} = \beta(w)\frac{\partial z}{\partial w}\cdot\frac{\partial}{\partial z} = \beta(w)(-\frac{1}{w^2})\frac{\partial}{\partial z} \Rightarrow$ on $U_1 \cap U_2$, $\underbrace{\overset{\alpha(\frac{1}{w})}{\overset{\shortparallel}{\alpha(z)}} = -\frac{\beta(w)}{w^2}}$.

α, β are holomorphic. Expand α on $\mathbb{C}$ as: $\alpha(z) = a_0 + a_1 z + \cdots$

Expand β on U_2 as $\beta(w) = b_0 + b_1 w + \cdots$

$\Rightarrow a_0 = -b_2$, $a_1 = -b_1$, $a_2 = -b_0$ $a_j = b_j = 0$ $j \geq 3$.

Conclusion, The space of holomorphic vector fields on $\mathbb{P}^1$ can be identified with $\{a_0 + a_1 z + a_2 z^2 \mid a_0, a_1, a_2 \in \mathbb{C}\}$ whose dimension 3. Rmk: ~~Aut~~ $\mathbb{P}^1 \cong SL(2, \mathbb{C})$. $\dim SL(2,\mathbb{C}) = 3$.

Rmk: The holomorphic vector field on Compact Riemann Surface is always of finite dimension. In particular, except for Riemann sphere and complex ring ($\mathbb{C}/\Lambda$), there is no un-trivial holomorphic vector field.

Exercise: For any holo. vector field V of $\mathbb{P}^1$, compute the numbers of zeros of V, counting multiplicity. (Poincaré - Hopf index theorem) Answer $\chi(X) = 2-2g = 2$.

Exercise: Give all holo. vector fields on $\mathbb{C}/\Lambda$ Answer: constant function

$\underset{\sim}{T^*X} = \coprod_{p\in X} T^*_{X,p}$, $(T^*X)^{1,0} \triangleq \coprod_{p\in X}(T^*_{X,p})^{1,0}$, $(T^*X)^{0,1} \triangleq \coprod_{p\in X}(T^*_{X,p})^{0,1}$.

cotangent bundle of X. $(T^*X)^{1,0}$ is called the holomorphic cotangent bundle of X.

Definition: A 1-form θ on X is a map $\theta: X \to T^*X$ s.t. $\theta(p) \in T^*_{X,p}$ $\forall p \in X$.

θ is called a $(1,0)$ form (or $(0,1)$-form) if $\theta(p) \in (T^*_{X,p})^{1,0}$ (or $(T^*_{X,p})^{0,1}$).

Definition: ① A 1-form θ is smooth if for any coordinate $(U, z = x+iy)$, θ can be represented on U as $\theta = a(z)\,dx + b(z)\,dy = \alpha(z)\,dz + \beta(z)\,d\bar{z}$.

② θ is holomorphic if $\theta = \alpha(z)\,dz$ with α holomorphic.

Notation: · $\mathcal{E}^1(X) :=$ space of smooth 1-forms on X.

· $\mathcal{E}^{1,0}(X)$ or $\mathcal{E}^{0,1}(X)$: space of $(1,0)$ form or $(0,1)$ form on X.

· $\Omega^1(X)$: space of holomorphic 1-forms on X.

Exercise: ① Assume V is a holomorphic vector on $\theta \in \Omega^1(X)$. $\theta(V)$ is a holomorphic functions on X. ② Show that there is no nonzero holomorphic 1-form on $\mathbb{P}^1$.

Definition: A meromorphic 1-form on X is a holomorphic 1-form θ on $X \setminus A$, for some discrete set $A \subset X$. satisfying: $\forall a \in A$, and some local coordinate z near a, θ is represented near a as $\theta = a(z)\,dz$, where $a(z)$ is a meromorphic function near a.

Notation: $\mathcal{M}^1(X) :=$ space of meromorphic 1-forms on X.

· If $f \in \mathcal{M}(X)$, $\theta \in \mathcal{M}^1(X) \Rightarrow f\theta \in \mathcal{M}^1(X)$.

· If $\theta_1, \theta_2 \in \mathcal{M}^1(X)$, $\theta_2 \not\equiv 0$, then $\theta_1/\theta_2 \in \mathcal{M}(X)$.

Rmk: meromorphic function is defined only by its singularities

Let $\theta \in \mathcal{M}^1(X)$, $\theta \not\equiv 0$. X compact, let $p_1, \cdots, p_r$ be zeros of θ of order $k_1, \cdots, k_r$; $q_1, \cdots, q_s$ are poles of θ of order $l_1, \cdots, l_s$. Then the divisor associated to θ is defined to be $(\theta) = k_1p_1 + \cdots k_rp_r - l_1q_1 - \cdots - l_sq_s \in \mathrm{Div}(X)$.

$\deg(\theta) = k_1 + \cdots + k_s - l_1 - \cdots - l_s$.

Obvious. $f \in \mathcal{M}(X)$, $\theta \in \mathcal{M}^1(X)$, $f\theta \in \mathcal{M}^1(X)$, $(f\theta) = (f) + (\theta)$

$\Rightarrow \deg(f\theta) = \deg(f) + \deg(\theta) = \deg(\theta)$.

Let $\theta_1, \theta_2 \in \mathcal{M}^1(X)$, then $\theta_1 = \underbrace{\frac{\theta_1}{\theta_2}}_{\in \mathcal{M}(X)} \cdot \theta_2$, $\deg \theta_1 = \deg(\frac{\theta_1}{\theta_2} \cdot \theta_2) = \deg(\theta_2)$

Corollary: Let X be a compact Riemann surface, then $\exists\, d \in \mathbb{Z}$, s.t. $\deg(\theta) = d$, $\forall \theta \in \mathcal{M}^1(X)$, Fact $d = -\chi(X)$. (to be proven)

Obvious. $f \in \mathcal{M}(X)$, $\theta \in \mathcal{M}^1(X)$, $f\theta \in \mathcal{M}^1(X)$, $(f\theta) = (f) + (\theta)$

$\Rightarrow \deg(f\theta) = \deg(f) + \deg(\theta) = \deg(\theta)$.

Let $\theta_1, \theta_2 \in \mathcal{M}^1(X)$, then $\theta_1 = \frac{\theta_1}{\theta_2} \cdot \theta_2$, $\deg \theta_1 = \deg(\underbrace{\frac{\theta_1}{\theta_2}}_{\in \mathcal{M}(X)} \cdot \theta_2) = \deg(\theta_2)$

Corollary: Let X be a compact Riemann surface, then $\exists\, d \in \mathbb{Z}$, s.t. $\deg(\theta) = d$, $\forall \theta \in \mathcal{M}^1(X)$, Fact $d = -\chi(X)$. (to be proven)

(4.9) Class.

X: Riemann Surface. The differential $df \in \mathcal{E}^1(X)$ is a 1-form, defined in a local coordinate

$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$. $\quad d = \partial + \bar\partial$, where $\partial f = \frac{\partial f}{\partial z} dz \in \mathcal{E}^{1,0}(X)$ $\bar\partial f = \frac{\partial f}{\partial \bar z} d\bar z \in \mathcal{E}^{0,1}(X)$.

Note: If $f \in \mathcal{M}(X)$, df is defined as $df = \frac{\partial f}{\partial z} dz$. $df \in \mathcal{M}^1(X)$.

Recall: Grassman algebra. Let V be a vector space, $\dim V = n$, Can consider $\Lambda^r V$.

$\Lambda^* V = \bigoplus_{r=0}^{n} \Lambda^r V$ is called Grassman algebra of V: $\Lambda^r V \times \Lambda^s V \longrightarrow \Lambda^{r+s} V$ $\quad (u,v) \mapsto u \wedge v$.

$\Lambda^* V$ is a graded algebra.

(2-forms on X)

$p \in X$, tangent space $T_{X,p}$, cotangent space $T^*_{X,p}$. Define: $\Lambda^2_{X,p} = T^*_{X,p} \wedge T^*_{X,p}$ $\quad \dim = 1$.

$dx|_p \wedge dy|_p$ and $dz|_p \wedge d\bar z|_p$ are two basis of $\Lambda^2_{X,p}$.

Assume $w = u + iv$ is another coordinate near p, then $du \wedge dv$, $dw \wedge d\bar w$ are two basis of $\Lambda^2_{X,p}$

The basis change as follows: $du \wedge dv = (\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy) \wedge (\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy) = (\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}) dx \wedge dy$.

$dw \wedge d\bar w = (\frac{\partial w}{\partial z} dz) \wedge (\frac{\partial \bar w}{\partial \bar z} d\bar z) = \left|\frac{\partial w}{\partial z}\right|^2 dz \wedge d\bar z$.

Let $\Lambda^2 X = \coprod_{p \in X} \Lambda^2_{X,p}$. Definition: a 2-form on X is a map $\xi : X \longrightarrow \Lambda^2 X$

s.t. $\xi(p) \in \Lambda^2_{X,p}$, $\forall p \in X$. locally, ξ can be represented as $\xi = a\, dx \wedge dy$ or $\alpha\, dz \wedge d\bar z$

ξ is called smooth if a or α are smooth functions.

Notation: $\mathcal{E}^2(X) :=$ space of smooth 2-form on X.

Differential of 1-form:

Assume $\theta \in \mathcal{E}^1(X)$ be a 1-form, we can define a 2-form, $d\theta \in \mathcal{E}^2(X)$ as an ordinary way. if $\theta = a\,dx + b\,dy$, then $d\theta = (\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y})\, dx \wedge dy$

if $\theta = \alpha\, dz + \beta\, d\bar z$, then $d\theta = (\frac{\partial \beta}{\partial z} - \frac{\partial \alpha}{\partial \bar z})\, dz \wedge d\bar z$

Lemma: $f \in C^\infty(X)$, $\theta \in \mathcal{E}^1(X)$, $d(f\theta) = df \wedge \theta + f\, d\theta$

Lemma: $f \in C^\infty(X)$, $d(df) = 0$.

$$\begin{array}{c} C^\infty(X) \\ \parallel \end{array}$$
$0 \to \mathcal{E}^0(X) \xrightarrow{d} \mathcal{E}^1(X) \xrightarrow{d} \mathcal{E}^2(X) \to 0$ Through this complex, we can define de Rahm cohomology of 0, 1, 2 degree. Let $H^*(X,\mathbb{C}) = H^0(X,\mathbb{C}) \oplus H^1(X,\mathbb{C}) \oplus H^2(X,\mathbb{C})$

The wedge product on differential forms induces product on $H^*(X,\mathbb{C})$. For example:

$$H^1(X,\mathbb{C}) \times H^1(X,\mathbb{C}) \to H^2(X,\mathbb{C}) \quad ([\theta_1],[\theta_2]) \longrightarrow [\theta_1 \wedge \theta_2].$$

Conclusion: $H^*(X,\mathbb{C})$ is a graded ring over $\mathbb{C}$.

Assume X and Y are Riemann Surfaces. $f: X \to Y$ holomorphic map, then f induces linear maps $f^*: \mathcal{E}^0(Y) \to \mathcal{E}(X)$ composition of function. $f^*: \mathcal{E}^1(Y) \to \mathcal{E}^1(X)$. $f^*: \mathcal{E}^2(Y) \to \mathcal{E}^2(X)$.

For example: Let $(U, z = x+iy)$ and $(V, w = u+iv)$ be local coordinates on X and Y, $f(U) \subset V$.

$$\underset{z}{U} \xrightarrow{f} \underset{w}{V}$$

Let $\theta \in \mathcal{E}^1(Y)$, $\theta|_V = a\,du + b\,dv$ or $\theta|_V = \alpha\, dz + \beta\, d\bar{z}$.

$$f^*\theta|_U \triangleq a(f(z))\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + b(f(z))\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) = \left(a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x}\right)dx + \left(a\frac{\partial u}{\partial y} + b\frac{\partial v}{\partial y}\right)dy$$

or $f^*\theta|_V = \alpha(f(z))\dfrac{\partial w}{\partial z}dz + \beta(f(z))\dfrac{\partial \bar{w}}{\partial \bar{z}}d\bar{z}$ (*) From (*). We see

- θ is a holomorphic/meromorphic 1-form + f holo $\Rightarrow$ $f^*\theta$ is holomorphic/meromorphic 1-form

Lemma: $f: X \to Y$ holo (smooth). Then ① $f^*(d\theta) = d(f^*\theta)$ $\theta \in \mathcal{E}^i(X)$ $i=0,1$

② $f^*(\theta_1 \wedge \theta_2) = f^*\theta_1 \wedge f^*\theta_2$ $\forall \theta_1, \theta_2$.

Therefore f^* induces homomorphisms on cohomology groups. $f^*: H^*(Y,\mathbb{C}) \to H^*(X,\mathbb{C})$

f^* is a homomorphism between graded algebras over $\mathbb{C}$.

§ 3. de Rahm isomorphism

Poincaré dual of a closed curve. Assume X is a Riemann Surface. $\gamma: [0,1] \to X$ is a smooth curve, $\theta \in \mathcal{E}^1(X)$. Then $\gamma^*\theta$ is a 1-form on $[0,1]$, which can be expressed as $a(t)\,dt$ for some $a \in C^\infty([0,1])$. Define $\int_\gamma \theta = \int_0^1 \gamma^*\theta = \int_0^1 a(t)\,dt$

§3 de Rham cohomology

X is a Riemann-surface, we have the de Rham complex:

$$0 \to \mathcal{E}^0(X) \xrightarrow{d} \mathcal{E}^1(X) \xrightarrow{d} \mathcal{E}^2(X) \to 0 \qquad d\circ d = 0$$

so we have the cohomology: $H^0(X,\mathbb{C})$ $H^1(X,\mathbb{C})$ $H^2(X,\mathbb{C})$

$H^*(X,\mathbb{C}) = \bigoplus_{i=0}^{2} H^i(X,\mathbb{C})$ is called the de Rham cohomology ring of X,

it's a graded algebra over $\mathbb{C}$ (the structure for 外代数)

- functor property: $f: X \to Y$ is a smooth map, induces a morphism

$\to f^*: H^*(Y,\mathbb{C}) \to H^*(X,\mathbb{C})$ between graded algebras.

§4 Intergration of differential forms

- Intergral of $f \in \mathcal{E}^0(X)$:

$p_1, \cdots, p_r \in X \quad n_1, \cdots, n_r \in \mathbb{Z}, \quad \int f := n_1 f(p_1) + \cdots + n_r f(p_r)$

- Intergral of 1-forms along curves:

$\gamma : [a,b] \to X$ is a smooth curve on X.

$\theta \in \mathcal{E}^1(X) \qquad \int_\gamma \theta := \int_a^b \gamma^* \theta = \int_a^b a(t)\, dt$ where $\gamma^* \theta = a(t)\, dt$

then we may also define $\displaystyle\int_{n_1\gamma_1 + \cdots + n_r\gamma_r} \theta := \sum n_i \int_{\gamma_i} \theta \qquad (n_i \in \mathbb{Z})$

- Intergral of 2-forms on surfaces:

assume X is oriented and $\xi \in \mathcal{E}^2(X)$ with compact-support (not necessary)

if $\operatorname{supp} \xi$ lies in some coordinate neighborhood $(U, z = x+iy)$ which is compatiable with the orientation on X.

$\xi|_U = a(x,y)\, dx \wedge dy$, then $\displaystyle\int_X \xi := \int_U \xi = \int_{z(U) \subset \mathbb{C}} a(x,y)\, dx dy$ which belongs to the general case

$\boxed{\text{why?}}$ the definition is independent of the choice of coordinates.

相差 a Jacobian determine of

in general cases, choose $\{(U_\alpha, z_\alpha)\}_{\alpha \in \Lambda}$ on X compatible with the orientation such that $\{U_\alpha\}_{\alpha\in\Lambda}$ is a locally finite covering of X, then $\exists \{\rho_\alpha\}_{\alpha\in\Lambda}$ be the partition of unity on X subordinate to the covering

$\xi = (\sum_{\alpha\in\Lambda} \rho_\alpha) \xi = \sum_{\alpha\in\Lambda} (\rho_\alpha \xi)$

define $\displaystyle\int_X \xi := \sum_{\alpha\in\Lambda} \int_{U_\alpha} \rho_\alpha \xi$

also: independent of the choice of coordinates and partition of unity.

- Strokes formula: $\Omega \subset X$ is a relatively-compact open set in X with smooth boundary (a curve) ($\bar{\Omega}$ is compact)

$\forall \theta \in \mathcal{E}^1(X) \qquad \int_\Omega d\theta = \int_{\partial\Omega} \theta$

Poincaré duality and de Rham isomorphism.

$-1 \le y \le 1$

E'?

① fundamental group and the first cohomology.

$H_1(X, \mathbb{Z}) := \pi_1(X) / [\pi_1(X), \pi_1(X)]$

Facts. (i) for $\forall [\gamma] \in H_1(X, \mathbb{Z})$, $\exists$ simple closed curves $\gamma_1, \cdots, \gamma_n$ in X s.t. $[\gamma] = [\gamma_1 + \cdots + \gamma_n]$

(ii) X is a compact R-S with genus g

$H_1(X, \mathbb{Z}) = \mathbb{Z}^{\oplus 2g}$

$\dim = 2g$

each "g" has such two curves, all is generated from those.

② Intergral of closed 1-form

Lemma. (monodromy theorem) $\gamma_1, \gamma_2 : [0,1] \to X \qquad \gamma_i(0) = 0 \quad \gamma_i(1) = 1$

assume $\gamma_1 \sim \gamma_2$, then $\displaystyle\int_{\gamma_1} \theta = \int_{\gamma_2} \theta \qquad \forall \theta \in \mathcal{E}^1(X),\ d\theta = 0$

(in complex analysis, $\theta = f(z)\, dz$, $f(z) \in \mathcal{O}(X)$)

Proof. since $\gamma_1 \sim \gamma_2$ $\exists$ continuous (smooth) homotopy map

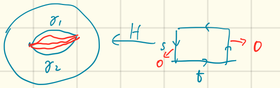
$H : \underset{t}{[0,1]} \times \underset{s}{[0,1]} \to X$

$H^*\theta$ is also a closed 1-form on $[0,1] \times [0,1]$

(stokes) $\Rightarrow$ $\displaystyle\underbrace{\int_{[0,1]\times[0,1]} dH^*\theta}_{\substack{I^2 \\ 0}} = \int_{\partial I^2} H^*\theta = \int_I H(t,0)^*\theta - \int_I H(t,1)^*\theta$

$\| (\text{closed})$

$\displaystyle = \int_I \gamma_1^*\theta - \int_I \gamma_2^*\theta = \int_{\gamma_1}\theta - \int_{\gamma_2}\theta$



(pull-back, determine)

from this lemma, we could see that: $\forall \theta \in \mathcal{E}^1(X)$ with $d\theta = 0$

$\exists$ morphism $\quad l_\theta : \pi_1(X) \hookrightarrow \mathbb{C}$

$\qquad [\gamma] \mapsto \int_\gamma \theta$

if $\gamma_1, \gamma_2 \in \pi_1(X) \qquad \displaystyle\int_{\gamma_1\gamma_2} \theta = \int_{\gamma_1}\theta + \int_{\gamma_2}\theta$

$\Rightarrow$ if $\gamma \in [\pi_1(X), \pi_1(X)] \quad \int_\gamma \theta = 0$

so l_θ can be defined on $H_1(X, \mathbb{Z})$

$l_\theta : H_1(X, \mathbb{Z}) \to \mathbb{C} \qquad$ (duality)

$\qquad [\gamma] \mapsto \int_\gamma \theta$

③ Lemma. (i) $l_\theta = 0 \iff \theta = df$ (exact) i.e. $\theta = 0$ in $H^1(X, \mathbb{C})$

Proof. "$\Rightarrow$" $l_\theta = 0$, for $\forall$ closed curve γ in X, $\int_\gamma \theta = 0$

fix $p_0 \in X$, define a function $f_{p_0} : X \to \mathbb{C}$

$\qquad p \mapsto \int_\gamma \theta \qquad (\gamma(0) = p_0 \ \gamma(1) = p)$

$f_{p_0}(p)$ is independent of γ (Ex. f_{p_0} is smooth and $df_{p_0} = \theta$)

"$\Leftarrow$" $\gamma : [0,1] \to X$ is a closed curve in X

$\displaystyle\int_\gamma \theta = \int_\gamma df = \int_0^1 \frac{df(\gamma(t))}{dt}\, dt = f(\gamma(1)) - f(\gamma(0)) = 0$

from (i) we obtain that: l_θ is defined for cohomology class $[\theta] \in H^1(X, \mathbb{C})$

(ii) $\dim_{\mathbb{C}} H^1(X, \mathbb{C}) \le 2g = \dim H_1(X, \mathbb{Z}) \qquad l_\theta = 0 \Rightarrow \theta = 0$ in $H^1(X, \mathbb{C})$

$\Rightarrow l : H^1(X, \mathbb{C}) \to H_1(X, \mathbb{Z})^*$

$\qquad \theta \mapsto l_\theta \ (\in \operatorname{Hom}(H_1(X,\mathbb{Z}), \mathbb{C}))$ is an injective linear map

(note that $\dim H_1(X, \mathbb{Z})^* = 2g$)

Then we want to prove l is surjective.

Lemma: $l_\theta = 0 \iff \theta$ exact. The above lemma shows that (1) l_θ can be defined for cohomology class $[\theta] \in H^1(X, \mathbb{C})$. (2) $\dim_{\mathbb{C}} \underset{\theta}{\overset{}{H^1(X, \mathbb{C})}} \leq 2g = \dim H_1(X, \mathbb{Z})$

$$l_\theta = 0 \Rightarrow \theta = 0. \text{ in } H^1(X, \mathbb{C}). \Rightarrow l: \begin{array}{ccc} H^1(X, \mathbb{C}) & \hookrightarrow & H_1(X, \mathbb{Z})^* := \mathrm{Hom}(H_1(X, \mathbb{Z}) \to \mathbb{C}) \\ \theta & \longmapsto & l_\theta \end{array}$$

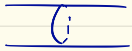
We want to prove l is surjective and $\dim_{\mathbb{Z}} H^1(X, \mathbb{C}) = 2g$.

Method: $\begin{array}{ccc} H^1(X, \mathbb{C}) \times H_1(X, \mathbb{C}) & \longrightarrow & \mathbb{C} \\ (\theta, \gamma) & \longmapsto & \int_\gamma \theta \end{array}$ is nondegenerate.

① $\forall \gamma.\ \int_\gamma \theta = 0 \Rightarrow \theta = 0$ in $H^1(X, \mathbb{C})$

② $\forall \theta$ in $H^1(X, \mathbb{C})$ $\int_\gamma \theta = 0 \Rightarrow \gamma = 0$ in $H^1(X, \mathbb{C})$

Poincaré duality for closed curve in X. Let $\gamma: [0,1] \to X$ be a simple closed curve then $\exists$ n.b.hd T of γ s.t. $T \cong \gamma \times (-1,1)$ diffeomorphic



$\gamma \times (-1,1)$

define $\rho: (-1,1) \to \mathbb{R}$ be a function

① $\rho = 0$, $t < 0$. ② $\rho(t) = 1$. $0 < t < \frac{\varepsilon}{2}$ ③ $\rho \in C^\infty((-1,1) \setminus \{0\})$ $\rho(t) = 0$, $t > \varepsilon$.

ρ can be viewed as a function on $\gamma \times (-1,1)$, define 1-form P_γ on $\gamma \times (-1,1)$ as follows:

$P_\gamma = \begin{cases} 0 & |t| < \frac{\varepsilon}{2} \\ d\rho & \text{o.w.} \end{cases}$ Then ① $dP_\gamma = 0$. ② $\mathrm{supp}\, P_\gamma \subset \{-\varepsilon < t < \varepsilon\}$.

P_γ can be viewed as a 1-form on T. by a diffeomorphism $T \cong \gamma \times (-1,1)$. and also viewed as a 1-form on X. since $\mathrm{supp}\, P_\gamma \subset T$.

Theorem: Assume θ is a closed 1-form on X. Then $\int_X \theta \wedge P_\gamma = \int_\gamma \theta$.

$$\int_X \theta \wedge P_\gamma = \int_T \theta \wedge P_\gamma = \int_{\gamma \times [0,\varepsilon]} \theta \wedge P_\gamma = \int_{\gamma \times [0,\varepsilon]} \theta \wedge d\rho = -\int_{\gamma \times [0,\varepsilon]} d(\rho\theta) = \int_{\gamma \times \{0\} \cup -\gamma \times \{\varepsilon\}} \rho\theta = \int_\gamma \theta$$

Remark: ① $[P_\gamma] \in H^1(X, \mathbb{C})$ is called Poincaré dual of γ in X.

② For any compact submanifold A of a manifold M, $\dim A = r$, $\dim M = n$. we can construct $P_A \in H^{n-r}(M, \mathbb{C})$ s.t. $\forall$ closed r-form θ on M

$\int_M \theta \wedge P_A = \int_A \theta$. P_A is called the Poincaré dual of A in M. The construction of which can be intrinsic. In fact, P_A is given by the Thom class of the normal bundle of A in X. (see Bott & Tu)

- Poincaré duality (special case)

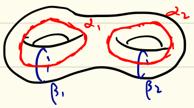
Theorem: Let X be a compact Riemann Surface. We can define a bilinear form:

$$Q: \begin{array}{ccc} H^1(X, \mathbb{C}) \times H^1(X, \mathbb{C}) & \to & \mathbb{C} \\ (\xi, \eta) & \mapsto & \int_X \xi \wedge \eta \end{array}$$

Proof: Assume γ is a simply closed curve in X, let P_γ be the Poincaré dual of γ in X. Then $\int_X \xi \wedge P_\gamma = \int_\gamma \xi = 0$ since γ is arbitrary $\Rightarrow \xi = 0$ in $H^1(X, \mathbb{C})$ $\square$

Remark: Q is nondegenerate and skew-symmetric. and induced an isomorphism $H^1(X, \mathbb{C}) \cong H^1(X, \mathbb{C})^*$. de Rham isomorphism.

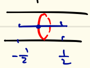
Assume X compact, $g(X) = g$.



$P_{\alpha_1}, P_{\alpha_2}, P_{\beta_1}, P_{\beta_2} \in H^1(X, \mathbb{C})$ Poincaré dual of $\alpha_1, \alpha_2, \beta_1, \beta_2$.

Lemma: $\int_{\alpha_i} P_{\alpha_j} = 0 \quad -\int_{\alpha_i} P_{\beta_j} = \delta_{ij} = \int_{\beta_j} P_{\alpha_i}, \quad \int_{\beta_i} P_{\beta_j} = 0$.

The only nontrivial part, $\int_{\beta_1} P_{\alpha_1} = 1$. We may intergal it on a nice geometry which is diffeomorphic to



Then $\int_{\beta_1} P_{\alpha_1} = \int_{p_0 \times [0, \varepsilon]} P_{\alpha_1} = \int_{p_0 \times [\varepsilon, 0]} d\rho$

$= \rho(0) - \rho(\varepsilon) = 1$ $\square$

Let $H_1(X, \mathbb{C}) = H_1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \{ \sum_{i=1}^{g} (a_i \alpha_i + b_i \beta_i) \mid a_i, b_i \in \mathbb{C} \}$ and $\{\alpha_i, \beta_j\}$ is a basis for $H_1(X, \mathbb{C})$.

Lemma: For $c \in H_1(X, \mathbb{C})$ if $\int_c \xi = 0$. $\forall \xi \in H^1(X, \mathbb{C}) \Rightarrow c = 0$ in $H_1(X, \mathbb{C})$

write $c = \sum_{i=1}^{g} a_i \alpha_i + b_i \beta_i$, then $\forall P_{\alpha_i}$, $\int_c P_{\alpha_i} = b_i = 0$. $\forall P_{\beta_j}$. $\int_c P_{\beta_j} = -a_j = 0$.

Theorem: The following bilinear map $H^1(X, \mathbb{C}) \times H_1(X, \mathbb{C}) \to \mathbb{C}$

$(\theta, \gamma) \mapsto \int_\gamma \theta$ is non-degenerate.

it induces two isomorphisms $H^1(X, \mathbb{C}) \xrightarrow{\cong} H_1(X, \mathbb{C})^*$, $H_1(X, \mathbb{C}) \xrightarrow{\cong} H^1(X, \mathbb{C})^*$.

Therefore $\dim_{\mathbb{Z}} H^1(X, \mathbb{C}) = 2g$.

Theorem θ is a closed 1-form on X, then $\int_X \theta\wedge P_\gamma = \int_\gamma \theta$

(similar to δ-function)

Proof. identify T and $\gamma\times(-1,1)$

then $\int_X \theta\wedge P_\gamma = \int_T \theta\wedge P_\gamma = \int_{\gamma\times[0,\varepsilon]} \theta\wedge P_\gamma = \int_{\gamma\times[0,\varepsilon]} \theta\wedge d\rho$

$= -\int_{\gamma\times[0,\varepsilon]} d(\rho\theta) = -\int_{-\gamma\times\{0\}\cup\gamma\times\{\varepsilon\}} \rho\theta = -\int_{-\gamma\times\{0\}} \theta = \int_\gamma \theta$ $\square$

Remark. ① $[P_\gamma]\in H^1(X,\mathbb{C})$ is called the Poincaré dual of γ in X

② for any compact submanifold A of a manifold M, $\dim A = r$, $\dim M = n$; one can construct a class $P_A \in H^{n-r}(M,\mathbb{C})$ s.t. $\forall$ closed r form θ on M

$\int_M \theta\wedge P_A = \int_A \theta$

P_A is called the Poincaré duality of A in M.

in fact, P_A is given by the Thom class of the normal bundle of A in X.

Poincaré duality (a special case)

Theorem. X is a compact Riemann-surface, define the following bilinear form $Q: H^1(X,\mathbb{C})\times H^1(X,\mathbb{C}) \to \mathbb{C}$ (anti-symmetry)

$(\xi,\eta) \longmapsto \int_X \xi\wedge\eta$

say: Q is nondegenerate.

Proof. γ is arbitrary simple-closed curve on X

let P_γ be the dual of γ in X, then

$\int_X \xi\wedge P_\gamma = \int_\gamma \xi = 0 \Rightarrow \xi = 0$ $\square$

Remark. Q is nondegenerate $\Rightarrow$ induces a linear isomorphism between $H^1(X,\mathbb{C})$ and its duality

de Rham isomorphism.

Lemma. $\int_{\alpha_i} P_{\alpha_j} = 0 \qquad \int_{\beta_i} P_{\alpha_j} = \delta_{ij}$

$\int_{\beta_i} P_{\beta_j} = 0 \qquad \int_{\alpha_i} P_{\beta_j} = -\delta_{ij}$

Proof. we just prove that $\int_{\alpha_1} P_{\alpha_1} = \int_{\alpha_2} P_{\alpha_1} = \int_{\beta_2} P_{\alpha_1} = 0 \quad \int_{\beta_1} P_{\alpha_1} = 1$

- take P_{α_1} s.t. $\operatorname{supp} P_{\alpha_1}$ is sufficiently close to α_1 s.t. $\operatorname{supp} P_{\alpha_1} \cap (\alpha_2\cup\beta_2) = \phi \Rightarrow$
- choose $[\alpha_1'] = [\alpha_1] \in H_1(X,\mathbb{Z})$ $\operatorname{supp} P_{\alpha_1} \cap \alpha_1' = \phi$ $\Rightarrow \int_{\alpha_1} P_{\alpha_1} = \int_{\alpha_1'} P_{\alpha_1} = 0$
- identify a neighbour T of α_1 in X with $\alpha_1\times(-1,1)$ assume $\beta_1\cap T$ is given by $P_0\times(-1,1)$ $P_0\in\alpha_1$ and $\operatorname{supp} P_{\alpha_1} \subset \{0\le t\le\varepsilon\}$ $\varepsilon \ll 1$

then $\int_{\beta_1} P_{\alpha_1} = \int_{P_0\times[0,\varepsilon]} P_{\alpha_1} = \int_{P_0\times[0,\varepsilon]} d\rho = \int_{P_0\times\{0\}\cup P_0\times\{\varepsilon\}} \rho = \rho(0) - \rho(\varepsilon) = \rho(0) = 1$ $\square$

Theorem. $H_1(X,\mathbb{C}) = H_1(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$

also, $\{\alpha_i, \beta_i\}$ is a basis of $H_1(X,\mathbb{C})$

(**Lemma.** $c\in H_1(X,\mathbb{C})$, if $\int_c \xi = 0 \quad \forall \xi\in H^1(X,\mathbb{R}) \Rightarrow c = 0$ in $H_1(X,\mathbb{C})$

Proof. $c = \sum_{i=1}^{g} (a_i\alpha_i + b_i\beta_i) \Rightarrow 0 = \int_c P_{\alpha_j} = \sum_i \left(a_i \int_{\alpha_i} P_{\alpha_j} + b_i \int_{\beta_i} P_{\alpha_j}\right)$

$= b_j \quad \forall j$

for $P_{\beta_j} \Rightarrow a_j = 0 \quad \forall j$ $\square$)

① the following bilinear map $H^1(X,\mathbb{C})\times H_1(X,\mathbb{C}) \to \mathbb{C}$ is nondegenerate

$(\xi, c) \longmapsto \int_c \xi$

② similarly $H^1(X,\mathbb{C}) \cong H_1(X,\mathbb{C})^*$

③ $\dim H^1(X,\mathbb{C}) = 2g$

X: compact Riemann surface, $g(X) = g$

Recall Poincaré duality: $Q: H^1(X,\mathbb{C})\times H^1(X,\mathbb{C}) \to \mathbb{C}$ non-degenerate bilinear form

canonically linear-isomorphism $H^1(X,\mathbb{C}) \cong H^1(X,\mathbb{C})^*$ $(\xi,\eta) \mapsto \int_X \xi\wedge\eta$

de Rham isomorphism: $P: H^1(X,\mathbb{C})\times H_1(X,\mathbb{C}) \to \mathbb{C}$ non-degenerate bilinear form

canonically linear-isomorphism $H^1(X,\mathbb{C}) \cong H_1(X,\mathbb{C})^*$ $(\gamma,\theta) \mapsto \int_X \theta\wedge P_\gamma = \int_\gamma \theta$

Poincaré dual of a closed curve γ: (X need not to be compact)

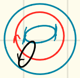
$\exists$ closed 1-form P_γ on X s.t.

① $\operatorname{supp} P_\gamma$ can lies in any given small neighbourhood of γ in X

② $\forall$ 1-form θ where $d\theta = 0$ on X, $\int_X \theta\wedge P_\gamma = \int_\gamma \theta$

particular, if X is compact, we have simple basis.

§1 intersection theory on Riemann surfaces



outside: positive 右手螺旋… 区分+, −

Definition. γ_1、γ_2 two curves on X are called to be transverse if: $\forall p \in \gamma_1 \cap \gamma_2$, $T_p\gamma_1 + T_p\gamma_2 = T_pX$ (γ_1、γ_2 doesn't "tangent" at p)

assume γ_1 and γ_2 are transverse, for $p \in \gamma_1 \cap \gamma_2$, define

$$\mathrm{ind}_p(\gamma_1, \gamma_2) = \begin{cases} 1 & (\gamma_{1*}\frac{\partial}{\partial t}, \gamma_{2*}\frac{\partial}{\partial t}) \text{ gives the orientation of } T_pM \text{ on } X \\ -1 & \text{otherwise} \end{cases}$$

so ind_p is contra-commutative.

Exc. γ_1 and γ_2 are transverse $\Rightarrow$ $\gamma_1 \cap \gamma_2$ finite

Definition. γ_1 and γ_2 are transverse closed curves in X, the intersection number of γ_1 and γ_2: $\gamma_1 \cdot \gamma_2 := \sum_{p \in \gamma_1 \cap \gamma_2} \mathrm{ind}_p(\gamma_1, \gamma_2)$ (also contra-commutative)

unfinished **Example.** X compact, $g(X) = g$, $\alpha_i \cdot \alpha_j =$

Definition. $c \in H_1(X, \mathbb{Z})$, γ be a closed curve, then $\exists$ simple closed curves $\gamma_1 \cdots \gamma_n$ s.t. $c = [\gamma_1 + \cdots + \gamma_n]$ and γ 与 γ_i are transverse. the intersection number of c and $[\gamma]$ is defined as:

$$c \cdot [\gamma] := \sum_{i=1}^{n} \gamma_i \cdot \gamma$$

Exc. the definition is independent of the choice of $\gamma_1 \cdots \gamma_n$ and homo class.

so for $\forall c_1, c_2 \in H_1(X, \mathbb{Z})$, we can define $c_1 \cdot c_2$, and thus get a bilinear form

Lemma. the bilinear form is non-degenerate.

Proof. $\alpha_i \cdot \alpha_j = 0 = \beta_i \cdot \beta_j$ $\alpha_i \cdot \beta_j = \delta_{ij} = -\beta_j \cdot \alpha_i$

since α_i, β_i forms a basis

we have the matrix under this basis

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \quad \text{(symplectic)}$$

$\Rightarrow$ $\exists$ canonical isomorphism

Theorem. for two closed curves γ_1、γ_2 in X, $\gamma_1 \cdot \gamma_2 = \int_X P_{\gamma_1} \wedge P_{\gamma_2}$ dual

(intuition: δ-function)

Lemma. ① $[P_\gamma] \in H^1(X, \mathbb{C})$ depends on γ only

② if $[\gamma_1] = [\gamma_2]$, then $[P_{\gamma_1}] = [P_{\gamma_2}]$ (so we can define Poincaré dual of $c = [\gamma] \in H_1(X, \mathbb{C})$?)

the proof is left for **Exc.**

the following are some directions:

we shall tell that $\forall c \in H_1(X, \mathbb{Z}) \subset H_1(X, \mathbb{C})$, the Poincaré dual P_c of c is defined

$$\begin{tikzcd} H_1(X, \mathbb{C}) \ni c \arrow[r, hook, "\alpha"] \arrow[dr, "P"'] & H^1(X, \mathbb{C})^* \\ & H^1(X, \mathbb{C}) \arrow[u, "\tau"'] \end{tikzcd}$$

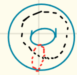
$\langle \alpha(c), \theta \rangle = \int_c \theta$ $\langle \tau(\xi), \theta \rangle = \int_X \theta \wedge \xi$

$P(c)$ satisfies: $\alpha(c) = \tau(P(c))$, as $P(c) = \tau^{-1}(\alpha(c))$

$$\int_c \theta = \int_X \theta \wedge P_c \quad \forall \theta$$

X: compact Riemann surface

Review: the intersection number can be defined for homology classes which spans by



nondegenerate + anti-symmetry $\longrightarrow$ $\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$

the above bilinear form can be extended to $H_1(X, \mathbb{C}) = H_1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$

$H_1(X, \mathbb{C}) \times H_1(X, \mathbb{C}) \longrightarrow \mathbb{C}$ which's also non degenerate

$(c_1, c_2) \longmapsto c_1 \cdot c_2$

$\Rightarrow$ canonical isomorphism $H_1(X, \mathbb{C})^* \cong H_1(X, \mathbb{C})$

Recall. $H^1(X, \mathbb{C}) \times H^1(X, \mathbb{C}) \longrightarrow \mathbb{C}$ (Poincaré duality)

$(\xi, \eta) \longmapsto \int_X \xi \wedge \eta$

$H_1(X, \mathbb{C}) \times H^1(X, \mathbb{C}) \longrightarrow \mathbb{C}$ (de Rham isomorphism)

$(c, \xi) \longmapsto \int_c \xi$

$\Rightarrow$ $H_1(X, \mathbb{C}) \xleftrightarrow{P} H^1(X, \mathbb{C})$ $P(c)$ is the Poincaré dual of c !

$c \in H_1(X, \mathbb{C})$, $P(c) = P_c \in H^1(X, \mathbb{C})$

Fact. for $c_1, c_2 \in H_1(X, \mathbb{C})$ $c_1 \cdot c_2 = \int_X P_{c_1} \wedge P_{c_2}$ (dual)

$(= \int_{c_1} P_{c_2} = -\int_{c_2} P_{c_1})$

Bilinear relation.

let $a_1 \cdots a_g$ $b_1 \cdots b_g$ be a basis of $H_1(X, \mathbb{Z})$ satisfying

$a_i \cdot a_j = 0 = b_i \cdot b_j$ $a_i \cdot b_j = \delta_{ij} = -b_j \cdot a_i$

let $\xi_i = P_{a_i}$ $\eta_i = P_{b_i}$

due to the isomorphism, $\{\xi_i, \eta_i\}$ is a basis of $H^1(X, \mathbb{Z})$

let θ_1、$\theta_2 \in H^1(X, \mathbb{C})$, assume $\theta_1 = \sum^g (u_1^i \xi_i + v_1^i \eta_i)$ $\theta_2 = \sum^g (u_2^i \xi_i + v_2^i \eta_i)$

$$\Rightarrow \int_X \theta_1 \wedge \theta_2 = \int_X (u_1^i \xi_i + v_1^i \eta_i) \wedge (u_2^j \xi_j + v_2^j \eta_j) = \int_X (u_1^i u_2^j \xi_i \wedge \eta_j + v_1^i v_2^j \eta_i \wedge \xi_j)$$

$$= \sum (u_1^i v_2^i - v_1^i u_2^i) \quad \text{(symplectic form)}$$

while $u_1^i = \int_{b_i} \theta_1$ $v_1^i = -\int_{a_i} \theta_1$ $u_2^i = \int_{b_i} \theta_2$ $v_2^i = -\int_{a_i} \theta_2$

$$\Rightarrow \int_X \theta_1 \wedge \theta_2 = \sum_{i=1}^{g} \left(-\int_{b_i} \theta_1 \int_{a_i} \theta_2 + \int_{a_i} \theta_1 \int_{b_i} \theta_2\right)$$

Theorem. (bilinear relation) $\theta_1, \theta_2 \in \Omega^1(X)$ holomorphic 1-form

then $\sum_{i=1}^{g} \left(\int_{a_i} \theta_1 \int_{b_i} \theta_2 - \int_{a_i} \theta_2 \int_{b_i} \theta_1 \right) \left(= \int_X \underbrace{\theta_1 \wedge \theta_2}_{dz \wedge dz = 0}\right) = 0$

§2 Harmonic analysis on Riemann surface

Aim: for a compact Riemann surface X, $H^1(X,\mathbb{C}) \cong \Omega^1(X) \oplus \overline{\Omega^1(X)}$
(to find out $\Omega^1(X)$)

star-operator $*$ on 1-forms (补空间 保角、共形结构) Hodge star

Definition: assume $\theta \in \mathcal{E}^1(X)$ is a smooth 1-form, define a new 1-form $*\theta \in \mathcal{E}^1(X)$ as follows:
$(U, z = x+iy)$ is a local coordinate $\theta|_U = a\,dx + b\,dy$
then $*\theta|_U = -b\,dx + a\,dy$ i.e. $*dx = dy$ $*dy = -dx$
or $*dz = -i\,dz$ $*d\bar{z} = i\,d\bar{z}$ (complex, reduce eigenvalue)
$*^2 = -1$

Note. the definition is independent of the choice of coordinate and $*^2 = -\mathrm{id}$

Example. $f \in \mathcal{O}(X)$, $f = \varphi + i\psi$, $d\varphi, d\psi \in \mathcal{E}^1(X)$
locally, C-R equation is satisfied
$*d\varphi = -\varphi_y dx + \varphi_x dy = \psi_x dx + \psi_y dy = d\psi$
$\Rightarrow df = d\varphi + i * d\varphi$

Remark. θ exact/closed $\not\Rightarrow$ $*\theta$ exact/closed (not bad things!)

Definition. Harmonic function and 1-form

(**Ex.** $\Omega_1, \Omega_2 \in \mathbb{C}$ $f: \Omega_1 \to \Omega_2$ be a holomorphic map
if h is harmonic on Ω_2, then $h\circ f$ is harmonic on Ω_1.)

a function on X is called harmonic if it's harmonic with respect to local coordinates.
an 1-form is called harmonic if $\forall P \in X$, $\exists$ a neighbourhood U and a harmonic function h on U s.t. $\theta|_U = dh$

Note. θ be a holomorphic 1-form, locally $\theta = \alpha(z)\,dz$ (holo)

$d\theta = d\alpha \wedge dz = (\frac{\partial\alpha}{\partial z} dz + \frac{\partial\alpha}{\partial\bar{z}} d\bar{z}) \wedge dz = 0$
$\Rightarrow \theta$ is closed

locally, $\theta = df$ for some holomorphic function f
$\Rightarrow \theta$ is a harmonic form
(if θ anti-holo, $\theta = d\bar{f}$ also ✓)

Lemma. $\theta \in \mathcal{E}^1(X)$ is a harmonic 1-form
then θ can be decomposet uniquely as $\theta = \omega_1 + \overline{\omega_2}$ $(\omega_i \in \Omega^1(X))$
(uniqueness is obvious since dz and $d\bar{z}$ are linear independent)

assume $X = \Delta \subset \mathbb{C}$ and $\theta = d\varphi$, φ is a harmonic function
$\varphi = \varphi_1 + i\varphi_2$ (φ_i real har functions on Δ)
take $f_1, f_2 \in \mathcal{O}(\Delta)$ s.t. $\varphi_1 = f_1 + \bar{f}_1$ $\varphi_2 = f_2 + \bar{f}_2$
$\varphi = f_1 + if_2 + (\bar{f}_1 + i\bar{f}_2) \Rightarrow d\varphi = \underbrace{d(f_1 + if_2)}_{\omega_1} + \underbrace{i\,d(\bar{f}_1 + i\bar{f}_2)}_{\overline{\omega_2}}$

Notation $\mathcal{H}(X)$ = space of harmonic 1-forms on X
then $\mathcal{H}(X) = \Omega^1(X) \oplus \overline{\Omega^1(X)}$

Square intergrable 1-forms

$V/W \simeq W^\perp$ 商掉 exact forms 剩下的 harmonic

X: Riemann surface, θ be a measurable 1-form on X.
(measurable invariant under diffeomorphism)

Definition.
$\|\theta\|^2 = \int_X \theta \wedge *\bar{\theta}$

locally assume $\theta = a\,dx + b\,dy$ $*\theta = -b\,dx + a\,dy$
$\theta \wedge *\bar{\theta} = (a\,dx + b\,dy) \wedge (-\bar{b}\,dx + \bar{a}\,dy) = (|a|^2 + |b|^2)\,dx \wedge dy$

$\|\theta\|^2 = 0 \iff \theta = 0$ a.e.

Review.
- harmonic 1-form $\theta \iff d\theta = d*\theta = 0$
- θ is a harmonic 1-form $\Rightarrow \exists!\ \omega_1, \omega_2 \in \Omega^1(X)$ s.t. $\theta = \omega_1 + \overline{\omega_2}$

Definition. θ square intergrable $:= \|\theta\|^2 < +\infty$
denoted by $L^2(X)$
define a inner product: $(\theta_1|\theta_2) := \int_X \theta_1 \wedge *\bar{\theta}_2$ (Hermite type)
$\Downarrow$ forms a Hilbert space

proof: $\int_X \theta_2 \wedge *\bar{\theta}_1 = \overline{\int_X \theta_1 \wedge *\bar{\theta}_2} = \int_X \bar{\theta}_1 \wedge *\theta_2$

Exc. $(*\theta_1|*\theta_2) = (\theta_1|\theta_2)$

let E = closure of the space $\{df \mid f \in C^\infty(X)$ supp f compact$\}$
E_* = closure of $\{*df \mid f \in C^\infty(X)$ supp f compact$\}$
E, E_* are closed subspaces of $L^2(X)$

Lemma. $E \perp E_*$ i.e. $(df|*dg) = 0$

proof. $(df|*dg) = \int_X df \wedge \overline{*(*dg)} = -\int_X df \wedge d\bar{g} = -\int_X d(f\,d\bar{g}) = -\int_{\partial X} f\,d\bar{g} = 0$

Lemma. if $\theta \in E$ is a smooth 1-form, then θ is exact.

proof: $\forall$ simple closed curve γ in X, $\int_\gamma \theta = 0$ $(\theta|*d\bar{f}) = -\int_X \theta \wedge df \overset{0}{=} -\int_X f\,d\theta$

$\int_\gamma \theta = \int_X \theta \wedge P_\gamma = -(\theta|*P_\gamma)$
it suffices to show: $(df|*P_\gamma) = 0$ $\forall f \in C_c^\infty(X)$
$-\int_X df \wedge P_\gamma = -\int_X d(fP_\gamma)$

$\theta \in E^{*\perp} \iff d\theta = 0$
$\theta \in E^\perp \iff d*\theta = 0$

Lemma $\theta \in L^2(X)$ smooth, then $d\theta = 0 \Leftrightarrow \theta \in E_*^{\perp}$

proof. "$\Rightarrow$" $\forall f \in C_c^\infty(X)$ $(\theta | * df) = -\int_X \theta \wedge d\bar{f} = \int_X d(\bar{f}\theta) = 0$

"$\Leftarrow$" $\forall f \in C^\infty(X)$ $(\theta | * df) = 0$

$$\overset{\parallel}{\int_X \theta \wedge d\bar{f}} = \int_X d(\bar{f}\theta) - \int_X \bar{f}\, d\theta \Rightarrow \int_X \bar{f}\, d\theta = 0 \Rightarrow d\theta = 0$$

let $H = E^{\perp} \cap E_*^{\perp} = (E \oplus E_*)^{\perp} \Rightarrow L^2(X) = E \oplus H \oplus E_*$

Theorem. H is the space of square intergrable harmonic forms on X
in particular, vectors in H are smooth.

proof. $\Leftarrow$ $(\theta | df) = (\theta | * df) = 0$ $(\forall f \in C_c^\infty(X))$

θ harmonic $\Rightarrow d\theta = d*\theta = 0$

$(\theta | df) = \overline{(df|\theta)} = \overline{\int_X df \wedge *\bar{\theta}} = \overline{\int_X d(f*\bar{\theta})} = 0$ i.e. $\theta \in E^{\perp}$

by the lemma above. $d\theta = 0$ & smooth $\Rightarrow \theta \in E_*^{\perp}$

$\Rightarrow \theta \in H$

conversly, if $\theta \in H$, we show that θ is harmonic.

① if θ is smooth, since $\theta \in E_*^{\perp}$, $d\theta = 0$

and $\theta \in E^{\perp}$ i.e. $\forall f \in C_c^\infty(X)$, $0 = (df, \theta) = \int_X df \wedge *\bar{\theta} = -\int_X f\, \overline{d*\theta}$

$\Rightarrow \int_X f\, \overline{d*\theta} = 0 \Rightarrow d*\theta = 0$

$\Rightarrow \theta$ is harmonic

② prove that if $\theta \in H$, θ is smooth (apply Weyl's lemma)

$(U, z = x+iy)$ is a local coordinate on X, let $\varphi \in C_c^\infty(U)$

$\varphi_x, \varphi_y \in C_c^\infty(U)$

$\theta|_U = a\,dx + b\,dy$ where a and b are measurable functions

$0 = (\theta | d\bar{\varphi}_x) = \int_X (a\,dx + b\,dy) \wedge *\varphi_x = \int_X (a\,dx + b\,dy) \wedge *(\varphi_{x_x}dx + \varphi_{xy}dy)$

$= \int_X (a\,dx + b\,dy) \wedge (\varphi_{xx}dy - \varphi_{xy}dx) = \int_X (a\varphi_{xx} + b\varphi_{xy})\, dx \wedge dy$

similarly $\int_X (a\varphi_{yy} - b\varphi_{xy})\, dx \wedge dy = 0$ $((\theta | * d\bar{\varphi}_y) = 0)$

$\Rightarrow \int_U a(\varphi_{xx} + \varphi_{yy})\, dx \wedge dy = 0$ i.e. $\int_U a \Delta\varphi = 0$

by Weyl's lemma, a is smooth $(\Delta a = 0)$

then consider $(\theta | d\bar{\varphi}_y) = (\theta | * d\bar{\varphi}_x) = 0$

then, we assume X is compact

by the theorem above, we have a linear map $h: H \to H^1(X, \mathbb{C})$
$\theta \mapsto [\theta]$

Theorem. assume X is a compact Riemann surface of genus g
then h is a linear isomorphism
in paticular: $\dim H = 2g$

proof. if $\theta \in \ker h$, then $\theta = df$ for some $f \in C^\infty(X) \Rightarrow \theta \in E$
$\Rightarrow \theta = 0$

assume θ is a closed form $\Rightarrow \theta \in E_*^{\perp}$

$\theta = \underset{E}{\theta_1} + \underset{H}{\theta_2}$ $\Rightarrow \theta_2$ smooth $\Rightarrow \theta_1 = \theta - \theta_2$ smooth $\Rightarrow \theta_1$ is exact

$\Rightarrow h(\theta_2) = [\theta]$

in the compact case $H = \Omega_1(X) \oplus \overline{\Omega_1(X)}$
as $H \cong H^1(X, \mathbb{C})$

Theorem. assume X compact, $g(X) = g$
we have a canonical decomposition:
$H^1(X, \mathbb{C}) = \Omega^1(X) \oplus \overline{\Omega^1(X)}$
$\dim \Omega^1(X) = \dim \overline{\Omega^1(X)} = g$

Remark. the decomposition is the spacial case of Hodge decomposition
in complex geometry

research. topology of $H_1(X, \mathbb{C})$? the abelian of $\pi_1(X)$

Lemma. assume $\theta \in \mathcal{E}^1(X)$, then:
① $\theta \in E \Rightarrow \theta = df$ (exact) $\qquad d\theta = 0 \Leftrightarrow \theta \in E_*^{\perp}$
② $\theta \in E_* \Rightarrow \theta = d*f$ $\qquad d*\theta = 0 \Leftrightarrow \theta \in E^{\perp}$

θ smooth $\quad d\theta = 0 \Rightarrow \theta = \alpha + h$ all smooth $\Rightarrow \alpha = df$

$d*\theta = 0 \Rightarrow \theta = \alpha + h$ $\quad \underset{E_*}{\alpha}$ smooth $\Rightarrow \alpha = d*f$

Theorem.
in general, $\forall \theta$ smooth $\in L^2(X)$, $\theta = \alpha + \beta + h$ all smooth

Lemma. assume $\Omega \subset \mathbb{C}$ is a bounded domain with smooth boundary
then $\forall \varphi \in C^\infty(\bar{\Omega})$, $\Delta u = \varphi$ has a solution u on Ω
and u can be taken as $u(z) = \frac{i}{4\pi}\int_\Omega \ln|\zeta - z|\, \varphi(\zeta)\, d\lambda(\zeta)$

Exc. proof

proof. Let (U, z) be a local coordinate on X, $\theta|_U = p\,dx + q\,dy$

$d*\theta|_U = (p_x + q_y)\, dx \wedge dy$

$\Rightarrow \exists u \in C^\infty(U)$ s.t. $\Delta u = p_x + q_y$

$d*du = \Delta u\, dx \wedge dy$
$\Downarrow$
$d*du = d*\theta|_U \qquad d*(\theta|_U - du) = 0$

$\Rightarrow \theta|_U - du \in E(U)^{\perp}$

$\theta|_U - du = \beta' + h' \qquad \Rightarrow \beta'$ smooth

on U $du = \alpha' + h''$

$y^{-1}XVX^{-1}y = y^{-1}V y$

$X^{-1}y$

$(y^{-1}x)V(y^{-1}x)^{-1} = V$

$\underset{E(U)}{\alpha'} + \underset{E_*(U)}{\beta'} + \underset{H(U)}{(h' + h'' - h)} = \alpha + \beta$

$\alpha\vert_U$	not certainly	$\in E(U)$
$\beta\vert_U$		$\in E_*(U)$

$\forall f \in C_c^\infty(U)$ $*df \in \mathcal{E}^*(X)$
$\alpha \perp *df \Rightarrow \alpha \perp E^*(U)$

for X compact $g(X)=g$ $\mathcal{H}(X)\oplus\overline{\mathcal{H}} \cong H^1(X,\mathbb{C}) \Rightarrow \dim\Omega^1(X)=g$

$\Omega^1(X)\oplus\overline{\Omega^1(X)}$ compact: all are in $L^2(X)$

§ some applications

assume θ is a meromorphic 1-form on Δ, $z=0$ is a pole of θ

write $\theta=f(z)dz$ where f meromorphic

$f(z)=\frac{a_{-n}}{z^n}+\cdots+\frac{a_1}{z}+a_0+a_1z+\cdots$

want to prove: a_1 is invariant under coordinate change.

assume ω is another coordinate $\omega=c_1z+\cdots+c_nz^n+\cdots$ $c_1\neq0$

$\theta=g(\omega)d\omega=g(\omega(z))\frac{\partial\omega}{\partial z}\cdot dz$

Review X: Riemann surface $L^2(X)=E\oplus H\oplus E^*$

θ can be decomposed to $\theta=\alpha+\beta+h$ (all are smooth)

$\alpha\in E$, $\beta\in E^*$, $h\in\mathcal{H}$

residue of meromorphic 1-form

theorem: if X is compact, $\theta\in\mathcal{M}^1(X)$, then $\sum_{p \text{ poles of } \theta}\mathrm{Res}_p\theta=0$

construction of harmonic functions and meromorphic 1-forms

X: (needn't to be compact) $p\in X$, $(U,z)\cong\Delta$ is a local coordinate around p with $z(p)=0$

let φ be a function satisfying $\begin{cases}\varphi(z)=\frac{1}{z^n}, & |z|\le\frac12\\ \varphi(z)=0, & |z|\ge1\end{cases}$ $\varphi\in C^\infty(X\setminus\{p\})$

$\Rightarrow d\varphi$ is a smooth 1-form on $X\setminus\{p\}$ which is holomorphic on $0<|z|<\frac12$

$\theta=d\varphi-i{*d\varphi}$ $\Rightarrow\theta$ is a smooth 1-form on $X\setminus\{p\}$

(on $0<|z|<\frac12$ $d\varphi$ holomorphic $\Rightarrow *d\varphi=-i\,d\varphi \Rightarrow i{*d\varphi}=d\varphi$ $\Rightarrow\theta=0$ $0<|z|<\frac12$)

let $\theta=0$ $(z=0)$ we get θ smooth on X

(assume supp θ compact)

$\theta\in L^2(X) \Rightarrow \theta=\alpha+\beta+h$ where α,β smooth

$\Rightarrow \exists f,g\in C^\infty(X)$ s.t. $\theta=df+{*dg}+h$ on $X\setminus\{p\}$

we have $d(\varphi-f)=i{*d\varphi}+{*dg}+h$

$\Rightarrow d{*d}(\varphi-f)=0$ and of course $d\,d(\varphi-f)=0$

$\Rightarrow d(\varphi-f)$ is harmonic on $X\setminus\{p\}$

$\Rightarrow u=\varphi-f$ is harmonic on $X\setminus\{p\}$

note that: $\varphi=\frac{1}{z^n}$ near p and $f\in C^\infty(X)$, we get:

Theorem. (U,z) is a local coordinate near $p\in X$, $z(p)=0$

$\Rightarrow\exists$ harmonic function u on $X\setminus\{p\}$, near p, u be represented as

$u(z)=\frac{1}{z^n}+v(z)$ v is smooth on U

Remark. u is harmonic on X, then $du=\omega_1+\overline{\omega_2}$ $\omega_i\in\Omega^1(X)$

$du=\frac{\partial u}{\partial z}dz+\frac{\partial u}{\partial\bar z}d\bar z\Rightarrow\partial u=\omega_1$ is a holomorphic 1-form

apply to the above u, we get

Corollary. there is a meromorphic 1-form θ on X s.t.

① p is the unique pole of θ

② under the coordinate z near p, $\theta=\frac{dz}{z^{n+1}}+\theta_1$

θ_1 is holomorphic near p

$u(z)=\frac{1}{z^n}+v(z)$ $\theta=\frac{\partial u}{-(n+1)}=\cdots$

Theorem. $p,q\in X$ (U,z), (V,ω) are coordinates near p and q, $z(p)=\omega(q)=0$

then ① $\exists$ harmonic function u on $X\setminus\{p,q\}$ s.t. $u(z)=\ln|z|+v_1(z)$ near p, $u(\omega)=-\ln|\omega|+v_2(z)$ near q

v_1 smooth near p

v_2 - - - q

② $\exists$ meromorphic 1-form θ on X s.t. p,q are the only poles of θ

$\theta=\frac{dz}{z}+$ holo near p

$\theta=-\frac{d\omega}{\omega}+$ holo q

proof. ① special case: assume $\exists$ coordinate $(U,z)\cong\Delta$ s.t. $p,q\in U$

let $z(p)=a$ $z(q)=b$ consider $\ln\left(\frac{z-a}{z-b}\right)$ (it's single-valued except L)

let L be the segment in U connecting p,q

then $\ln\frac{z-a}{z-b}$ has single-valued branch on $U\setminus L$

define: $\varphi(z)=\ln\frac{z-a}{z-b}$ $z\in U_1$ where $L\subset U_1\subset\subset U$

on $X\setminus L$ $=0$ $z\in X\setminus U$

$\varphi\in C^\infty(X\setminus L)$

$d\varphi$ is an 1-form on $X\setminus\{p,q\}$:

on $U_1\setminus\{p,q\}$ $d\varphi=\left(\frac{1}{z-a}-\frac{1}{z-b}\right)dz$

let $\omega=d\varphi-i{*d\varphi}$ then $\omega=0$ on $U_1\setminus\{p,q\}$

ω can be viewed as a smooth 1-form on X

$\omega\in L^2(X) \Rightarrow \omega=df+{*dg}+h$ on $X\setminus L$ $(f,g\in C^\infty(X))$

similarly, $\varphi-f$ is harmonic on $X\setminus L$

$\Rightarrow u=\mathrm{Re}(\varphi-f)$ is harmonic on $X\setminus L$

on $U_1\setminus L$ $u(z)=\mathrm{Re}\ln\frac{z-a}{z-b}-\mathrm{Re}f$

$=\ln|z-a|-\ln|z-b|-\mathrm{Re}f$

$\Rightarrow u$ can be defined and smooth on $X\setminus\{p,q\}$

since $\Delta u=0$ on $U_1\setminus L$ $\Rightarrow$ $\Delta u=0$ on $U_1\setminus\{p,q\}$

general case.

$\partial u\in\mathcal{M}^1(X)$ $\theta=\partial u$ is what we want

Exc. p_i $(1\le i\le n)$ are distinct points on X and $c_1,\cdots,c_n\in\mathbb{C}$ s.t. $\sum c_i=0$

then $\exists$ meromorphic 1-form ω $(\in\mathcal{M}^1(X))$

s.t. ① ω is holomorphic on $X\setminus\{p_1,\cdots,p_n\}$

② p_i are simple poles of ω

③ $\mathrm{Res}_{p_i}\omega=c_i$

(hint: fix $p_0\in X$, construct $\omega\in\mathcal{M}^1(X)$ s.t. under some coordinate z near p_0, $\omega=\frac{dz}{z^2}+$ holo and ω holo on $X\setminus\{p_0\}$

show $\omega=df$ for some $f\in\mathcal{M}(X)$

then: $f:X\to\mathbb{P}^1$ $\cdots$)

Exc. a simply connected compact Riemann surface X must be biholomorphic to Riemann sphere $\mathbb{P}^1$

Recall:

Theorem: X be a Riemann surface, $p \in X$, $n \geq 1$

assume z is a local coordinate near p, $z(p) = 0$

then: $\exists$ harmonic function $\hat{u}$ on $X \setminus \{p\}$, s.t. $u = \frac{1}{z^n}$ + smooth term (near p)

$\exists$ meromorphic 1-form θ on X s.t. p is the only pole of θ and near p θ is represented as $\theta = \frac{dz}{z^{n+1}}$ + holomorphic term (residue $= 0$)

Theorem': for $p, q \in X$ (z, ω)

$\exists$ (real) harmonic u on $X \setminus \{p, q\}$ $u = \ln|z|$ + smooth near p

$= -\ln|\omega|$ + smooth near q

$\exists$ meromorphic 1-form θ ... $\theta = \frac{dz}{z}$ + holo p

$= -\frac{d\omega}{\omega}$ + holo q

X: compact Riemann surface of genus g

compact Riemann surface: main: Riemann-Roch theorem

pre: bilinear relation

$\dim H_1(X,\mathbb{Z}) = 2g$, a basis: $\{a_i, b_i\}$ is called canonical, if:

$a_i \cdot a_j = 0 = b_i \cdot b_j \quad a_i \cdot b_j = \delta_{ij} = -b_j \cdot a_i$

two closed 1-forms θ_1, θ_2 on X

$$\int_X \theta_1 \wedge \theta_2 = \sum \left(\int_{a_i} \theta_1 \int_{b_i} \theta_2 - \int_{a_i} \theta_2 \int_{b_i} \theta_1 \right)$$

Theorem (1-st bilinear form) $\theta_1, \theta_2 \in \Omega^1(X)$

$$\int_X \theta_1 \wedge \theta_2 = \sum \left(\int_{a_i} \theta_1 \int_{b_i} \theta_2 - \int_{a_i} \theta_2 \int_{b_i} \theta_1 \right) = 0$$ □

Lemma $\omega \in \Omega^1(X)$ if $\int_{a_i} \omega = 0 \ \forall i \Rightarrow \omega = 0$ (prove $\|\omega\| = 0$)

proof. $d\omega = 0 = d\bar{\omega}$

$$\|\omega\|^2 = \int_X \omega \wedge * \bar{\omega} = i \int_X \omega \wedge \bar{\omega} = i \sum \Big(- \underbrace{\int_{a_i} \bar{\omega}}_{= \overline{\int_{a_i} \omega} = 0} \int_{b_j} \omega \Big) = 0$$ □

assume $\omega_1 \cdots \omega_g$ is a basis of $\Omega^1(X)$

$A = \left(\int_{a_j} \omega_i \right)_{g \times g}$

Lemma: A is nonsingular (invertible)

proof. $c_1, \cdots, c_g \in \mathbb{C}$ satisfy $0 = \sum c_i \left(\int_{a_1} \omega_i \ \cdots \ \int_{a_g} \omega_i \right)$

i.e. $\forall j \ \int_{a_j} \sum c_i \omega_i = 0$ by the about Lemma

$\sum c_i \omega_i = 0 \Rightarrow c_i = 0 \ \forall i$ □

Theorem $\exists$ a unique basis $\{\omega_i\}$ s.t. $A = I$

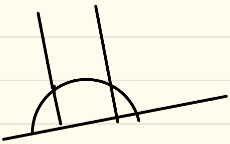
i.e. $\int_{a_j} \omega_i = \delta_{ij}$ (the canonical basis associated to $\{a_i, b_i\}$) □

then define $B = \left(\int_{b_j} \omega_i \right)_{g \times g}$

Theorem: ① B is symmetric ② $\mathrm{Im}(B)$ is positive definite (imaginary part)

proof. ① by the 1-st bilinear relation

② note that $\Omega^1(X) \subset L^2(X)$ the inner product of $L^2(X)$ reduces an inner product on $\Omega^1(X)$, the matrix under the above basis:



保持 intersection ($\wedge$)
transformation

$$H = \left(\int_X \omega_i \wedge * \overline{\omega_j} \right)_{g \times g}$$

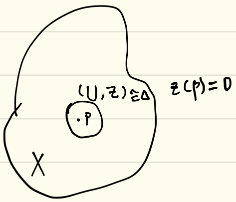
$$\int_X \omega_i \wedge * \overline{\omega_j} = i \int_X \omega_i \wedge \overline{\omega_j} = i \left(\int_{b_i} \overline{\omega_j} - \int_{b_j} \omega_i \right) = i \left(\overline{\int_{b_i} \omega_j} - \int_{b_j} \omega_i \right)$$

$$= 2\,\mathrm{Im}\left(\int_{b_j} \omega_i \right) \qquad H = 2\,\mathrm{Im}(B)$$

as H is positive definite $\Rightarrow$ ② □

Theorem (no proof): if $B_X \sim B_Y$ (up to a transformation)

then $X \sim Y$ conformal equivalent



assume $\omega \in \Omega^1(X)$ $\eta \in \mathcal{M}^1(X)$, $p \in X$ is the only pole of η ($\mathrm{res}_p \eta = 0$) assume $p \notin a_i, b_i$

define $\omega \# \eta = \sum \left(\int_{a_i} \omega \int_{b_i} \eta - \int_{a_i} \eta \int_{b_i} \omega \right)$ aim: compute $\omega \# \eta$

near p, assume $\omega = (a_0 + a_1 z + \cdots + a_i z^i + \cdots)\, dz$

$$\eta = \left(\frac{b_{-n}}{z^n} + \cdots + \frac{b_{-2}}{z^2} + b_0 + b_1 z + \cdots \right) dz$$

let f be a meromorphic function on U s.t. $\eta|_U = df$

take $\rho \in C^\infty(X)$ s.t. $\begin{cases} \rho(z) = 0 & \text{near } p \\ \rho(z) = 1 & |z| \geq \frac{1}{2} \end{cases}$

let $\eta' = d(\rho f)$ (on U) $\qquad \eta' = \eta$ on $X \setminus \{|z| \leq \frac{1}{2}\}$

then η' is closed on X $\Rightarrow$ for U sufficiently small

$$\omega \# \eta' = \omega \# \eta$$

$$\overset{\shortparallel}{\int_X \omega \wedge \eta'} = \int_{|z| \leq \frac{1}{2}} \omega \wedge \eta' \qquad F(z) = a_0 z + \tfrac{1}{2} a_1 z^2 + \cdots \qquad dF = \omega|_U$$

$$= \int_{|z| \leq \frac{1}{2}} dF \wedge \eta' = \int_{|z| \leq \frac{1}{2}} d(F \eta') = \int_{|z| = \frac{1}{2}} F \eta' = \int_{|z| = \frac{1}{2}} F \eta$$

$$= 2\pi i\, \mathrm{Res}_p (F\eta) = 2\pi i \sum_{j=2}^{n} \frac{b_{-j} a_{j-2}}{j-2}$$

Exc. $p_1 \cdots p_n \in X$ $\notin a_i, b_i$ $\qquad c_1 \cdots c_n \in \mathbb{C}^*$ $\qquad \sum c_i = 0$

then ① $\exists!$ $\omega \in \mathcal{M}^1(X) \cap \Omega^1(X \setminus \{p_1, \cdots, p_n\})$ and p_i are simple poles of ω

s.t. $\mathrm{Res}_{p_i} \omega = c_i$ and $\int_{a_i} \omega = 0$

② $\exists!$ $\omega \in \mathcal{M}^1(X) \cap \Omega^1(X \setminus \{p_1, \cdots, p_n\})$ and p_i are simple poles of ω

$\mathrm{Res}_{p_i} \omega = c_i$ and $\int_{a_i} \omega$ $\int_{b_i} \omega$ are purely imaginary

X : compact Riemann surface of genus g

Review :

bilinear relation for meromorphic forms :

$\omega \# \eta \qquad 2\pi i \sum_{j=2}^{n} \frac{1}{j-1} a_{j-2} b_{-j}$

Riemann–Roch theorem

Definition. $\mathrm{Div}(X) = \{ \sum_{i=1}^{r} n_i P_i \mid r \geq 0 \quad P_i \in X \quad n_i \in \mathbb{Z} \}$

for $D \in \mathrm{Div}(X)$ $\deg D = \sum n_i$

we have the group morphism : $\deg : \mathrm{Div}(X) \longrightarrow \mathbb{Z} \quad D \mapsto \deg D$

denote $\mathrm{Div}_0(X) = \{ D \in \mathrm{Div}(X) \mid \deg D = 0 \}$

$D = \sum n_i P_i$ is called effective iff $n_i \geq 0 \ \forall i$ write $D \geq 0$ / $D > 0$
(intergral)

say $D_1 \geq D_2 / D_1 > D_2$

let $f \in \mathcal{M}(X) \setminus \{0\}$, define $\mathrm{ord}_p(f) = \begin{cases} 0 & f(p) \neq 0/\infty \\ r & f(p) = 0 \text{ of order } r \\ -r & f(p) = \infty \text{ of order } r \end{cases}$

question: $\deg D = 0$ then $\Rightarrow \exists f$ $D = (f)$?

for $\omega \in \mathcal{M}'(X)$ we have a similar definition

then we have principle divisors as follows

$(f) := \sum_{p \in X} \mathrm{ord}_p(f) \in \mathrm{Div}(X)$ $\quad \deg (f) = 0$ (theorem)

$= \# f^{-1}(0) - \# f^{-1}(\infty)$

$(f\omega) = (f) + (\omega)$ (morphism)

equivalent : $D_1 - D_2 = (f)$ denote $D_1 \sim D_2$

note : $D_1 \sim D_2 \Rightarrow \deg D_1 = \deg D_2$

let $\mathcal{P}$ be the space of principle divisors

then $\mathcal{P} \subset \mathrm{Div}_0(X)$ is a subgroup as it's abelian ($\mathrm{Div}(X)$)

we have $T := \mathrm{Div}_0(X) / \mathcal{P}$ called the abel variety use divisors to describle $\mathcal{M}'(X)$ $\mathcal{M}(X)$

$\omega_1, \omega_2 \in \mathcal{M}'(X) \Rightarrow \frac{\omega_2}{\omega_1} = f \in \mathcal{M}(X)$

$\omega_2 = f\omega_1 \qquad (\omega_2) = (f) + (\omega_1) \Rightarrow \omega_1 \sim \omega_2$

$\Rightarrow \deg \omega_1 = \deg \omega_2$

Lemma.: the above facts

Definition. $\omega \in \mathcal{M}'(X)$ (ω) is called a canonical divisor

the canonical divisor is unique up to equivalence

let $D \in \mathrm{Div}(X)$, we can define two vector spaces:

actually one kind of $\rightarrow$ cohomology group

$H^0(X, D) = \{ f \in \mathcal{M}(X) \mid (f) + D \geq 0 \}$ Euler number

$H^1(X, D) = \{ \omega \in \mathcal{M}'(X) \mid (\omega) \geq D \}$

$h^0(X, D) := \dim H^0(X, D)$ resp $h^1(X, D)$

Example 1 $D = 0$ $H^0(X, D)$ is holomorphic $= \mathcal{O}(X) \cong \mathbb{C}$

$H^1(X, D) = \Omega^1(X)$ $h^0 = 1$ $h^1 = g$

2 $p \in X$, $D = np$ $(n > 0)$ $H^0(X, D)$: the only pole of f is p and the multiple is less than $n+1$

$H^1(X, D)$: $\omega \in \Omega^1(X)$ $\mathrm{ord}_p(\omega) \geq n$

a rough estimate of h^0 & h^1

assume z is a local coordinate near p with $z(p) = 0$

define a linear map $\alpha: H^0(X, np) \longrightarrow \mathbb{C}^n$

$\ker \alpha = \mathcal{O}(X) \subseteq \mathbb{C}$ $\qquad f \mapsto (a_{-n}, \cdots, a_{-1})$

$1 < h^0 \leq 1 + n$

$\tau: \Omega^1(X) \longrightarrow \mathbb{C}^n$ $\qquad \omega \mapsto (a_0, \cdots, a_{n-1})$

$\ker \tau = H^1(X, np)$ $\qquad g = h^1 + \dim \mathrm{Im}\,\tau \leq h^1 + n$

$\Rightarrow h^1 \geq g - n$ $\qquad$ and $\dim \mathrm{Im}\,\tau \leq g$

$(f)+D_1=(f)+(f_0)+D_2$

$\quad \geq 0 \qquad \overset{''}{(ff_0)}+D_2$

Lemma: $D_1\sim D_2 \Rightarrow h^0/_1(X,D_1)=h^0/_1(X,D_2)$

proof. $D_1=D_2+(f)$ for some $f\in\mathfrak{M}(X)$

$H^0(X,D_1)\to H^0(X,D_2)$ $\quad f\mapsto f_0 f$ $\quad$ linear isomorphism

$H^1(X,D_1)\to H^1(X,D_2)$ $\quad \omega\to\frac{1}{f_0}\omega$

$(\omega)\geq D_1 \Leftrightarrow (\omega)\geq (f_0)+D_2$

$\Leftrightarrow (\frac{1}{f_0}\omega)\geq D_2$

duality

a space be achieved as the dual space

Lemma. D_1+D_2 canonical $\Rightarrow h^0(D_1)=h^1(D_2)$ $\quad h^1(D_1)=h^0(D_2)$

$D_1+D_2=(\omega_0)$

$(f)+D_1\geq 0 \Leftrightarrow (f)+(\omega_0)-D_2\geq 0 \Leftrightarrow (f\omega_0)\geq D_2$

$H^0(X,D_1)\to H^1(X,D_2)$ $\quad f\mapsto f\omega_0$

Exc. h^0 h^1 $<\infty$ $\forall D$

$(\omega)\geq D_1 \Leftrightarrow (\omega)-(\omega_0)+D_1\geq 0 \Leftrightarrow (\frac{\omega}{\omega_0})+D_1\geq 0$

as we say above

H^0, H^1 some sense are "homology group" then

h^0-h^1 some sense is the "Euler number"

a topology invariant (related to D this case)

Theorem (R-R) $h^0-h^1=\deg D-g+1$ $\quad \forall D$

special cases: $D=0 \Rightarrow 1-\dim\Omega^1=\deg D-g+1 \Rightarrow \dim\Omega^1=g$

$D=(\omega_0)$ $\quad H^1(X,D)=\{\omega\in\mathfrak{M}^1(X)\mid (\frac{\omega}{\omega_0})\geq 0\}$ i.e. $\frac{\omega}{\omega_0}$ is holo

$=\{c\omega_0\}\cong\mathbb{C}$

$h^0(X,D)=h^1(X,(\omega_0)-D)=h^1(X,0)=\dim\Omega^1=g$

$g-1=\deg(\omega_0)-g+1 \Rightarrow \deg(\omega_0)=2g-2=-\chi(X)$

(Poincaré-Hopf index)

cotangent negative $\qquad$ tangent

so powerful!

X compact, genus g

Review $\mathrm{Div}(X)$ (f) (ω) $(\deg\omega = 2g-2 = -\chi(X))$

$h^0 - h^1 = \deg D - g + 1 \longrightarrow$ top-inv

Betti, algebra/analysis -inv

top-inv of X

corresponding to a linear bundle and $\deg D$ is its Chern class

proof of R-R theorem

Proposition 1. R-R holds for effective divisors.

proof: ① $D=0$ i.e. prove $1-\dim\Omega^1(X) = 1-g$ but we already know that $\dim\Omega^1(X) = g$)

② $0 \leq D \in \mathrm{Div}(X)$ $D = n_1p_1 + \cdots + n_rp_r$ $n_i \geq 0$ ($\exists\, n_i \neq 0$)

$H^0(X,D) = \{ f \in \mathcal{M}(X) \cap \mathcal{O}(X \setminus \{p_1,\cdots,p_r\}) \mid \mathrm{ord}_{p_i} f \geq -n_i \}$

$D' := \sum (n_i+1)p_i$ $H^1(X,-D') = \{ \omega \in \mathcal{M}^1(X) \mid (\omega) + D' \geq 0 \}$

fix canonical basis $\{\alpha_i\} \cup \{\beta_i\}$ $p_i \notin \alpha_j, \beta_j$ $\forall i,j$

$V := \{ \omega \in H^1(X,-D') \mid \mathrm{res}_{p_i}\omega = 0 \ \int_{\alpha_i}\omega = 0 \}$

claim: $\dim V = \deg D$

reason: z_i be a coordinate near p_i $(z_i(p_i) = 0)$

for $k \geq 0$ $\exists\, \tau_{p_i}^k \in \mathcal{M}^1(X)$ p_i is the only pole of $\tau_{p_i}^k$

$\tau_{p_i}^k = \frac{dz}{z^k} +$ holomorphic terms near p_i

if we assume $\int_{\alpha_j}\tau_{p_i}^k = 0$ then it exists and it's unique

then $\tau_{p_i}^k \in V$ $(\forall\, 2 \leq k \leq n_i+1 \ \ i)$

for $\omega \in V$ $\omega = \sum_{k=2}^{n_i+1} \frac{a_i^k}{z^k}\, dz +$ holomorphic term near p_i

$\omega' := \omega - \sum_i \sum_{k=2}^{n_i+1} a_i^k \tau_{p_i}^k \in \Omega^1(X)$

$\int_{\alpha_l}\omega' = 0 - 0 = 0$ $\forall l$ $\Rightarrow \omega' = 0$

$\Rightarrow \{\tau_{p_i}^k\}$ is a basis of V $\Rightarrow \dim V = \deg D$

$d: H^0(X,D) \longrightarrow V$ $f \mapsto df$ $h^0 = 1 + \dim \mathrm{Im}\, d$

$\ker d \cong \mathbb{C}$

estimate $\dim \operatorname{Im} d$: $V \supset \operatorname{Im} d$

claim $\operatorname{Im} d = \{\omega \in V \mid \int_{b_\ell} \omega = 0 \ \forall \ell\}$

$\Downarrow$

g equations $\to \dim \operatorname{Im} d \geq \deg D - g$

$(\phi : \omega \mapsto (\int_{b_1} \omega, \cdots, \int_{b_g} \omega)$ $\quad \dim \ker \phi = \deg D - \dim \operatorname{Im} \phi \geq \deg D - g$

$V \to \mathbb{C}^g$ $\quad \dim \ker\phi = \dim \operatorname{Im} d \quad (\operatorname{Im} d = \ker \phi)$

$\Rightarrow h^0 \geq 1 + \deg D - g$ (Riemann - inequality)

compute $\dim \operatorname{Im} \phi$: assume $\omega_1, \cdots, \omega_g$ the corresponded basis

$\omega_\ell \# \omega = \sum_i \left(\int_{a_i} \omega_\ell \int_{b_i} \omega - \int_{a_i} \omega \int_{b_i} \omega_\ell \right)$ $\quad (\int_{a_i} \omega_j = \delta_{ij}, \int_{a_i} \omega = 0)$

$= \int_{b_i} \omega$

expand ω_j at P_i : $\omega_j = \left(\sum_{k=0}^{\infty} \lambda^k_{P_i, j} z_i^k \right) dz$

assume $\omega = \sum_{k=2}^{n_i+1} a_i^k \tau_{P_i}^k$ near P_i

by bilinear relation for meromorphic forms

$\int_{b_\ell} \omega = \omega_\ell \# \omega = 2\pi i \left(\sum_{k=2}^{n_i+1} \frac{1}{k-1} \lambda^{k-2}_{P_i, \ell} a_i^k \right)$

then we get a linear equation system

the coefficients form the matrix C $\quad \operatorname{rank} C =: s$

$\dim \operatorname{Im} d = \dim \ker \phi = -\dim \operatorname{Im} \phi + \dim V = \deg D - s$

want to show that $\quad s = g - h' \Leftrightarrow s + h' = g$

$H'(X, D) = \{\omega \in \Omega'(X) \mid \operatorname{ord}_{P_i} \omega \geq n_i\}$

$\omega = c_1 \omega_1 + \cdots + c_g \omega_g \quad (\omega \in \Omega'(X))$

near P_i: $\omega = \sum_{k=0}^{\infty} \left(\sum_{j=1}^{g} c_j \lambda^k_{P_i, j} \right) z_i^k \, dz_i$

then $\omega \in H'(X, D) \Leftrightarrow \sum_{j=1}^{g} c_j \lambda^k_{P_i, j} = 0$ for $k \leq n_i - 1 \quad i = 1, \cdots, r$

$\Rightarrow E = {}^tC \quad\quad t = \operatorname{rank} E = s$

$g - \ker E = g - h'$

finally $\quad \deg D - \dim \operatorname{Im} d = \operatorname{rank} D = \operatorname{rank} E = g - h'$ $\quad$ nontrivial

$\operatorname{Im} d = \ker \phi \quad \phi : V \to \mathbb{C}^g$ $\quad$ $\deg D = \dim V$

$h^0 - 1$

Recall
R-R theorem

Proposition1. R-R holds for effective divisors

Corollary. $g(X)=0 \Rightarrow X \cong \mathbb{P}^1$

proof. $\forall p \in X$ let $D = 1\cdot p \Rightarrow h^0 = 2$ since

$H^0(X,D) = \{f \in \mathcal{M}(X) \cap \mathcal{O}(X \setminus \{p\}) \mid ord_p f \geq -1\}$ $H^1(X,D) = \{\omega \in \mathcal{M}^1(X) \mid (\omega) \geq p\} \cap \Omega^1(X) = \{0\}$

$\Rightarrow \exists f \in H^0(X,D)$ s.t. $ord_p f = -1$

$\Rightarrow f: X \to \mathbb{P}^1$ is a biholomorphic map i.e. $X \cong \mathbb{P}^1$

Corollary. $\omega \in \mathcal{M}^1(X) \setminus \{0\}$ $\deg(\omega) = -\chi(X) = 2g-2$

proof. ① $g=0$ consider $\mathbb{P}^1$

dz is a meromorphic form, for $\omega = \frac{1}{z}$ (the coordinate on $\mathbb{C}^* \cup \{\infty\}$)

$dz = -\frac{1}{\omega^2} d\omega = \theta$ $\deg(\theta) = -2$

② $g \geq 1$ $\dim \Omega^1(X) \geq 1$ $\exists \omega \in \Omega^1(X), \omega \neq 0$

$D = (\omega)$ effective $h^0(X,D) - h^1(X,D) = \deg D - g + 1$

$0 + D = (\omega)$ canonical $h^0(X,D) = h^1(X,0) = \dim \Omega^1(X) = g$

$h^1(X,D) = h^0(X,0) = \dim \mathcal{O}(X) = 1$

$\Rightarrow \deg D = \deg(\omega) = 2g-2$

Proposition2. R-R holds if ① $D \sim$ an effective divisor or

② $K - D \sim$ an effective divisor, where K canonical

proof. ① $h^0, h^1, \deg D$ are all invariant up to equivalence.

② by ①, R-R holds for $K-D$

$$h^0(X,D) - h^1(X,D) = h^1(X,K-D) - h^0(X,K-D)$$
$$= g-1-\deg(K-D) = \deg D + g - 1 - (2g-2)$$
$$= \deg D - g + 1$$

Proposition 3. R–R theorem

proof. consider D that neither D or $K-D$ is equivalent to an effective divisor

then $H^0(X,D)=\phi$ since $\exists f\in \mathcal{M}(X)$ s.t. $(f)+D\geq 0$ means $(f)+D$ is effective $\Leftrightarrow D\sim$ an effective divisor

$H^1(X,D)=\phi$ since $\exists\ \omega\in\mathcal{M}^1(X)$ s.t. $(\omega)\geq D$ means $(\omega)-D\geq 0$ effective where (ω) canonical

i.e. prove that $\deg D = g-1$

assume $D = D^+ - D^- = (n_1p_1+\cdots+n_rp_r)-(m_1q_1+\cdots m_sq_s)$

$\deg D = \deg D^+ - \deg D^-$

$= h^0(X,D^+) - h^1(X,D^+) + g - 1 - \deg D^-$

$h^0(X,D^+)\geq \deg D + \deg D^- + 1 - g$

claim: $\deg D < g$ if $\deg D\geq g \Rightarrow h^0(X,D^+)\geq \deg D^- + 1$

$H^0(X,D) = \{f\in\mathcal{M}(X) \mid (f)+D^+-D^-\geq 0\}$

$= \{f\in H^0(X,D^+) \mid \mathrm{ord}_{q_i} f\geq m_i\}$

$h^0(X,D)\geq h^0(X,D^+) - \deg D^- \geq 1 > 0$ contradiction

$q_i \times m_i$ linear equations

$i=1,\cdots,s$ $f(q_i)\cdots f^{(k)}(q_i)\cdots f^{(m_i)}(q_i)=0$

similarly $\underset{\deg K - \deg D}{\underset{\parallel}{\deg(K-D)}} < g$ i.e. $\deg D > g-2$

$\Rightarrow \deg D = g-1$

Remark. vanishing theorem as $h^0 - h^1 = \deg D - g + 1$

$h^0 \geq \deg D - g + 1 \Rightarrow$ say: when $\deg D$ is large enough, $h^1 = 0$

(and Riemann inequality $\Rightarrow$ equality)

- $\deg D < 0 \Rightarrow h^0 = 0$

since if $\exists f\in H^0(X,D)$ $\underset{0}{\underset{\parallel}{\deg(f)}} + \deg D\geq 0$

- $\deg D = 0$ if $\exists f\in H^0(X,D)$ $(f)+D\geq 0$ and $\deg((f)+D)=0$

$\Rightarrow (f)+D=0$ $D = -(f) = (\frac{1}{f})$?

if $(f)+D\geq 0$ $(g)+D\geq 0 \Rightarrow (f)=(g) \Rightarrow f = cg$

$h^0 = 1/0$ $h^0 = 1 \Leftrightarrow D$ is principle

Corollary: • $\deg D > 2g-2$ (then $\deg(K-D) < 0$)

$h^0(X, K-D) = 0$

$h^1(X, D)$ ''

• $\deg D = 2g-2$ (then $\deg(K-D) = 0$)

$h^1(X,D) = 0/1$ $\quad h^1(X,D) = 1 \iff K-D$ principle i.e. $K \sim D$

(D canonical(?))

$\Rightarrow$ ($\deg D > 2g-2 \Rightarrow h^0 = \deg D - g + 1$)

Remark: above propositions are special cases of Kodaira vanishing theorem. (say when a complex manifold $M \hookrightarrow \mathbb{P}^n$ or M is algebraic see: G&H)

Exercise. X: compact Riemann surface, $p_1, \cdots, p_n$ distinct on X

any $c_1 \cdots c_n \in \mathbb{C}$ $\exists f \in \mathcal{M}(X)$ s.t. $f(p_i) = c_i$

Holomorphic differential forms of higher dgree

X: Riemann surface

$p \in X$ define $K_{X,p}^n := (T_{X,p}^*)^{1,0} \otimes \cdots \otimes (T_{X,p}^*)^{1,0}$ $\dim K_{X,p}^n = 1$

let z be a local coordinate near p. then dz is a basis of $(T_{X,p}^*)^{1,0}$ then $dz^n := dz \otimes \cdots \otimes dz$ is a basis of $K_{X,p}^n$

ω be another coordinate $d\omega = \frac{d\omega}{dz} dz$

then $d\omega^n = \left(\frac{d\omega}{dz}\right)^n dz^n$

let $K_X^n = \bigsqcup_{p \in X} K_{X,p}^n$

Definition. a holomorphic differential on X of dgree n is a map $\omega : X \to K_X^n$

where $\omega(p) \in K_{X,p}^n$ satisfying if (U, z) is a local coordinate $\omega = f dz^n$ for some $f \in \mathcal{O}(U)$ on U denoted with $\Omega^n(X)$

(similarly $\mathcal{M}^n(X)$)

Remark. $\omega_1 \in \Omega^n(X)$ $\omega_2 \in \Omega^m(X)$ $\omega_1 \otimes \omega_2 \in \Omega^{m+n}(X)$

if locally $\omega_1 = f_1 dz^n$ $\omega_2 = f_2 dz^n$, then $\omega_1 \otimes \omega_2 = f_1 f_2 dz^{n+m}$

let $\Omega^*(X) = \bigoplus_{n \geq 0} \Omega^n(X)$ $(\Omega^0(X) = \mathcal{O}(X))$

which is graded algebra called the canonical ring of X

the Spec $\leftarrow$

$\downarrow$

projective variety

(same to X itself)

Fact. X : compact $\Rightarrow$ $\Omega^*(X)$ is finitely generated

(for higher dim, Y.T. Siu has proved)

for $\omega \in \mathcal{M}^n(X)$, we can define zeros and poles and orders

$$\operatorname{ord}_p \omega = \begin{cases} 0 & \text{neither pole nor zero} \\ r & \text{zero of order } r \\ -r & \text{pole of order } r \end{cases}$$

then X: compact genus: g

define the divisor (ω):

$$(\omega) = \sum_{p \in X} (\operatorname{ord}_p \omega) \cdot p$$

assume $\omega_1, \omega_2 \in \Omega^n(X)$

$(\omega_1 \otimes \omega_2) = (\omega_1) + (\omega_2)$ $\quad \deg(\omega_1 \otimes \omega_2) = \deg(\omega_1) + \deg(\omega_2)$

Exc.

$$\dim \Omega^n(X) = \begin{cases} (2n-1)(g-1) & n \geq 2,\ g \geq 2 \\ 1 & g = 1 \\ 0 & g = 0 \end{cases}$$

Remark. ① degree 2: quadratic differential

② $\dim \Omega^2(X) = 3(g-1)$

③ $\Omega^2(X)$ can be identified with the cotangent space of moduli space of Riemann surfaces of genus g

Weierstrass points

X: compact from now on

Question: $\forall\, p \in X$, existence of $f \in \mathcal{M}(X)$ s.t. p is the unique pole of order n?

Definition. n is called a gap of X at p if no

$$H^0(X, np) = \{f \in \mathcal{M}(X) \mid (f) + np \geq 0\} = \{f \in \mathcal{M}(X) \cap \mathcal{O}(X \setminus \{p\}) \mid \mathrm{ord}_p f \geq -n\}$$

respectively $H^0(X, (n-1)p) \subset H^0(X, np)$

n is a gap $\Longleftrightarrow$ $h^0(X, np) = h^0(X, (n-1)p)$

not $\Longleftrightarrow$ $h^0(X, np) = h^0(X, (n-1)p) + 1$

$$\left(h^0(X, np) - h^0(X, (n-1)p) \leq 1\right)$$

$$\shortparallel$$

$$1 + h^1(X, np) - h^1(X, (n-1)p)$$

by vanishing theorem $n-1 \geq 2g-1$ i.e. $n \geq 2g$

then $h^1(X, (n-1)p) = 0 = h^1(X, np)$

then n isn't a gap

for $n = 2g$ $\quad h^0(X, 2gp) = 2g - g + 1 = g + 1$

$$\shortparallel$$

$$h^0(X, 2gp) - h^0(X, (2g-1)p) + \cdots + h^0(X, p) - h^0(X, 0) + 1$$

$\Rightarrow$ g gaps

for $g \geq 1$ $\quad n_1 = 1$ is a gap otherwise the meromorphic function is a biholomorphic map from X to $\mathbb{P}^1$

$$1 = n_1 < n_2 < \cdots < n_g \leq 2g - 1$$

7:30 ~ 8:30
8:35 ~ 9:35
9:40 ~ 10:30
12:15 ~ 13:15
13:30 ~ 15:10
15:20 ; 30 ~

Definition. Weierstrass point p : if $n_j > j$ for some $1 \le j \le g$

Remark ① if p isn't a Weierstrass point
gaps are $1, 2, \cdots, g$
② if p is a Weierstrass point
$\exists\, 1 \le j \le g \quad f \in \mathfrak{M}(X) \cap \mathcal{O}(X \setminus \{p\})$ where $\mathrm{ord}_p f = -j$

Definition. weight of p $\quad wt(p) := \sum_{j=1}^{g} (n_j - j)$

(Theorem. Weierstrass points are finite)

or wronskian
Definition. (Wronski determinant) let $D \subseteq \mathbb{C}$ $\quad \varphi_1, \cdots, \varphi_n \in \mathcal{O}(D)$

$$\omega(\varphi_1, \cdots, \varphi_n) := \begin{vmatrix} \varphi_1 & \cdots & \varphi_n \\ \vdots & & \\ \varphi_1^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix} \in \mathcal{O}(D)$$

$(\psi_1, \cdots, \psi_n) = (\varphi_1, \cdots, \varphi_n) A \qquad \omega(\psi_1, \cdots, \psi_n) = |A|\, W(\varphi_1, \cdots, \varphi_n)$

Lemma. $f \in \mathcal{O}(D) \quad \omega(f\varphi_1, \cdots, f\varphi_n) = f^n \omega(\varphi_1, \cdots, \varphi_n)$

assume $a \in D$ & $\mathrm{ord}_a \varphi_1 < \cdots < \mathrm{ord}_a \varphi_n$
then $\mathrm{ord}_a W(\varphi_1, \cdots, \varphi_n) = \sum_{j=1}^{n} (\mathrm{ord}_a \varphi_j + 1 - j)$
proof. induction on n $\qquad n=1$ case is obvious
assume n-case holds $\qquad$ near a $\frac{\varphi_v}{\varphi_1}$ is holomorphic
then $W(\varphi_1, \cdots, \varphi_{n+1}) = \varphi_1^{n+1} W(1, \frac{\varphi_2}{\varphi_1}, \cdots, \frac{\varphi_{n+1}}{\varphi_1})$
$= \varphi_1^{n+1} W((\frac{\varphi_2}{\varphi_1})', \cdots, (\frac{\varphi_{n+1}}{\varphi_1})')$
$\mathrm{ord}_a W((\frac{\varphi_2}{\varphi_1})', \cdots, (\frac{\varphi_{n+1}}{\varphi_1})') = \sum_{j=1}^{n} (\mathrm{ord}_a (\frac{\varphi_{j+1}}{\varphi_1})' + 1 - j)$
$= \sum_{j=1}^{n} (\mathrm{ord}_a \varphi_{j+1} - \mathrm{ord}\,\varphi_1 - 1 + 1 - j) = \sum_{j=1}^{n} (\mathrm{ord}_a \varphi_{j+1} - \mathrm{ord}\,\varphi_1 - j)$
$\mathrm{ord}_a W(\varphi_1, \cdots, \varphi_{n+1}) = \mathrm{ord}_a \varphi_1 + \sum_{j=2}^{n+1} (\mathrm{ord}_a \varphi_j + 1 - j) = \sum_{j=1}^{n} (\mathrm{ord}_a \varphi_j + 1 - j)$

wonskian of holomorphic 1-forms on X: compact R-S $g \geq 2$

let $\omega_1, \cdots, \omega_g$ be a basis of $\Omega'(X)$

assume (U, z) is a local coordinate

on U $\quad \omega_i = \varphi_i \, dz \quad$ consider $W(\varphi_1, \cdots, \varphi_g)$

if (V, z) is another coordinate on V $\omega_i = \psi_i \, dw$

Exercise. show that: $W(\varphi_1, \cdots, \varphi_g) \, dz^{g(g+1)/2} = W(\psi_1, \cdots, \psi_g) \, dw^{g(g+1)/2}$

(namely, $W(\varphi_1, \cdots, \varphi_g) = W(\psi_1, \cdots, \psi_g) \left(\frac{dw}{dz}\right)^{g(g+1)/2}$

therefore $W(\varphi_1, \cdots, \varphi_g) \, dz^{g(g+1)/2}$ gives a global holomorphic form of

$\deg = g(g+1)/2$ denoted by $W(\omega_1, \cdots, \omega_g) =: W_X$

W_X is independent of the choice of basis up to a nonzero factor

Theorem. for $p \in X$

① p is a Weiertrass point iff $W_X(p) = 0$
and $\operatorname{ord}_p W_X = wt(p)$

② $\sum_{p \in X} wt(p) = (g-1) g (g+1)$

proof. ① $\iff h'(X, (n-1)p) - h'(X, np) = 1 \iff \exists \omega \in \Omega'(X) \ \operatorname{ord}_p \omega = n-1$

assume $n_1 < \cdots < n_g$ are all gaps

than $\exists \omega_1, \cdots, \omega_g \in \Omega'(X)$ s.t. $\operatorname{ord}_p \omega_j = n_j - 1$

they're linearly independent and hence form a basis

$W_X = W(\omega_1, \cdots, \omega_g) \qquad (U, z)$ near p

on U $\quad \omega_i = \varphi_i(z) dz \qquad \operatorname{ord}_a \varphi_i = n_i - 1$

$W_X|_U = W(\varphi_1, \cdots, \varphi_n) \, dz^{g(g+1)/2} \qquad \operatorname{ord}_p W_X = \sum_{j=1}^{g} (n_j - j) = wt(p)$

$\operatorname{ord}_p W_X > 0 \iff wt(p) \geq 0 \iff p$ is Weiertrass

$\Updownarrow$

$W_X(p) = 0$

② by ① $\sum_{p \in X} wt(p) = \sum_{p \in X} \operatorname{ord}_p W_X = \deg W_X = g(g+1)/2 \cdot 2(g-1) = \cdots$

Exercise. $g \geq 2$ N be the number of Weiertrass points

then $2g+2 \leq N \leq (g-1) g (g+1)$

if $N = 2g+2$ then gaps at p are $1, 3, 5, \cdots, 2g-1$

Weierstrass points on hyperelliptic R-S

Definition. a compact R-S X is called hyperelliptic if there $\exists$ a holomorphic map $f: X \to \mathbb{P}^1$ of degree 2

($y^2 = (x-a_1)\cdots(x-a_n)$: hyperelliptic curve $\{(x,y) \in \mathbb{C}^2 \mid \quad\}$

Exercise. assume X is a hyperelliptic surface of genus $g \geq 2$ and $f: X \to \mathbb{P}^1$ is the nonconstant holomorphic map of degree 2 then ① the set of branched points of f coincides with the set of Weierstrass points of X

② (by ①) the number is $2g+2$

chapter 6

Abel - Jacobi map

§1 compact R-S of genus 1

assume $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$

$\Gamma := \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$ $X_\Gamma := \mathbb{C}/\Gamma$ is a compact R-S of genus 1

(i.e. a discrete subgroup of $\mathbb{C}$ of rank 2)

assume $\omega_1', \omega_2' \in \mathbb{C}$ are another such group, forms Γ' and we get $X_{\Gamma'}$

Question: when will X_Γ and $X_{\Gamma'}$ be conformal equivalent?

assume $f: X_\Gamma \to X_{\Gamma'}$ be a biholomorphic map

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{f}} & \mathbb{C} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ X_\Gamma & \xrightarrow{f} & X_{\Gamma'} \end{array}$$

$\tilde{f}$: the lifting of f

$\forall \gamma \in \Gamma$ $z \in \mathbb{C}$ $\exists \gamma' \in \Gamma'$ $\tilde{f}(z+\gamma) = \tilde{f}(z) + \gamma'$

when z changes, as $\tilde{f}$ is continuous on z

& γ' is discrete $\Rightarrow$ γ' is independent of z

$\Rightarrow$ $\tilde{f}'(z+\gamma) = \tilde{f}'(z)$ for $\forall \gamma \in \Gamma$

the $\tilde{f}'(z)$ is bounded since it's determined on the set $\{m\omega_1 + n\omega_2 \mid 0 \le m, n \le 1\}$

$\Rightarrow \tilde{f}'(z) = \alpha \in \mathbb{C}^*$ $\Rightarrow$ $\tilde{f}(z) = \alpha z + \beta$ ($\tilde{f}$ is invertible)

$\Rightarrow \alpha\gamma = \gamma'$ $\alpha\Gamma \subset \Gamma'$

conversely, consider f^{-1} as $\pi' \circ \tilde{f} = f \circ \pi \Leftrightarrow f^{-1} \circ \pi' = \pi \circ \tilde{f}^{-1}$

then $\tilde{f}^{-1} \underset{\substack{\uparrow \\ \text{one of}}}{=} \widetilde{f^{-1}}$ $\tilde{f}^{-1}(z) = \dfrac{z}{\alpha} - \dfrac{\beta}{\alpha}$

$\Rightarrow \frac{1}{\alpha}\Gamma' \subset \Gamma$ $\Rightarrow$ $\alpha\Gamma = \Gamma'$

Theorem. $X_\Gamma \simeq X_{\Gamma'} \Leftrightarrow \exists \alpha \in \mathbb{C}^*$ s.t. $\alpha\Gamma = \Gamma'$

Exercise

Definition. a compact Riemann surface X is called an elliptic curve if $X \cong X_\Gamma$ for some lattice $\Gamma \subseteq \mathbb{C}$

Notation. $\mathcal{E}$:= the isomorphism class of elliptic curves

$\widetilde{\omega} := \{(\omega_1, \omega_2) \in \mathbb{C}^2 \mid \omega_1, \omega_2$ linearly independent over $\mathbb{R}\}$

for $(\omega_1, \omega_2) \in \widetilde{\omega}$, $\Gamma_{(\omega_1,\omega_2)} := \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$

define "$\sim$" on $\widetilde{\omega}$: $(\omega_1, \omega_2) \sim (\omega_1', \omega_2') \iff X_{\Gamma_{(\omega_1,\omega_2)}} \cong X_{\Gamma_{(\omega_1',\omega_2')}}$

$\iff \exists \alpha \in \mathbb{C}^*$ s.t. $\alpha \Gamma_{(\omega_1,\omega_2)} = \Gamma_{(\omega_1',\omega_2')}$

$\omega := \widetilde{\omega}/\sim$ then $\omega \sim \mathcal{E}$ canonically by definition

$$\omega \longleftrightarrow \mathcal{E}$$
$$[\omega_1,\omega_2] \longleftrightarrow [X_{\Gamma_{(\omega_1,\omega_2)}}]$$

for $\forall (\omega_1, \omega_2) \in \widetilde{\omega}$ $\quad (\omega_1, \omega_2) \sim (1, \frac{\omega_2}{\omega_1}) \sim (1, -\frac{\omega_2}{\omega_1})$

so we only need to consider $(1, c)$ where $c \in \mathbb{H} = \{x+iy \mid y > 0\}$

$(1,\tau) \sim (1, \tau') \iff \Gamma_{(\alpha, \alpha\tau)} = \Gamma_{(1,\tau')}$ $\exists \alpha \in \mathbb{C}^*$

then $\exists a, b, c, d \in \mathbb{Z}$ s.t. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ \tau' \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha\tau \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$

$\Rightarrow \frac{b + d\tau'}{a + c\tau'} = \tau$

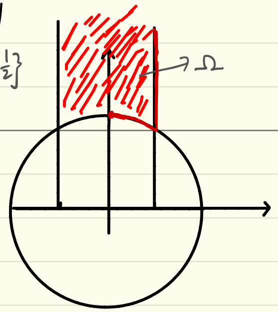
...

Theorem. $(1,\tau) \sim (1, \tau') \iff \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ s.t. $\tau' = \frac{a\tau + b}{c\tau + d}$

identify : $\omega \quad \mathcal{E} \quad \mathbb{H}/SL_2(\mathbb{Z})$ (or $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ as this's a left-action) fundamental domain of $SL_2(\mathbb{Z})$ in $\mathbb{H}$

$\Omega = \{\tau \in \mathbb{H} \mid |\tau| > 1, -\frac{1}{2} < \mathrm{Re}\,\tau \le \frac{1}{2}\} \cup \{\tau \in \mathbb{H} \mid |\tau| = 1, 0 \le \mathrm{Re}\,\tau \le \frac{1}{2}\}$

$G = PSL(2,\mathbb{Z}) := SL(2,\mathbb{Z})/\pm \mathrm{Id}$



G has 2 elements T & S

$T\tau = \tau + 1 \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$S\tau = -\frac{1}{\tau} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$G' = \langle T, S \rangle_{\mathbb{Z}} \subseteq G$

isotropic group?!

important

Theorem. (1) $\forall \tau \in \mathbb{H}$, $\exists g \in G'$ s.t. $g\tau \in \Omega$

(2) assume $\tau, \tau' \in \Omega$ if $\exists\, g \in G$ s.t. $g\tau = \tau'$ the $\tau = \tau'$

(3) $\tau \in \Omega$, if $\exists\, g \in G$, $g \ne \mathrm{Id}$. s.t. $g\tau = \tau$ then $\tau = i$ or $e^{\frac{\pi}{3}i}$

proof. (1) it's sufficient to prove that $\{g\tau \mid g \in G'\} \cap \Omega \ne \emptyset$ (the obrit)

$g\tau = \frac{a\tau+b}{c\tau+d} = \frac{1}{|c\tau+d|^2}(a\tau+b)(c\bar{\tau}+d) = \frac{1}{|c\tau+d|^2}(ac|\tau|^2 + bd + ad\tau + bc\bar{\tau})$

$\mathrm{Im}\, g\tau = \frac{ad-bc}{|c\tau+d|^2}\mathrm{Im}\,\tau = \frac{1}{|c\tau+d|^2}\mathrm{Im}\,\tau$

take $g_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in G'$ s.t. $|c_0\tau + d_0|^2 = \min\{|c\tau+d|^2 \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G'\}$

$\tau' = g_0\tau$ then $\mathrm{Im}\,\tau' = \sup\{\mathrm{Im}\, g\tau \mid g \in G'\}$

by T, one can assume $-\frac{1}{2} < \mathrm{Re}\,\tau' \le \frac{1}{2}$

claim $|\tau'| \ge 1$: if $|\tau'| < 1$ let $\tau'' = -\frac{1}{\tau'}$ $\quad \mathrm{Im}\,\tau'' = \frac{\mathrm{Im}\,\tau'}{|\tau'|^2} > \mathrm{Im}\,\tau'$ ✗

① $|\tau'| > 1 \Rightarrow \tau' \in \Omega$ ② $|\tau'| = 1$ if $\tau' \notin \Omega$ then $\mathrm{Im}(-\frac{1}{\tau'}) = \mathrm{Im}\,\tau'$

and $-\frac{1}{\tau'} \in \Omega$

action: free

↓

the quotient space is a manifold

otherwise, there're some singularities

(2) assume $\tau, \tau' \in \Omega$ s.t. $\tau' = g\tau$ for some $g \in G$ and $\mathrm{Im}\,\tau' \ge \mathrm{Im}\,\tau$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$

$|c\tau + d| \le 1$

① $c = 0 \Rightarrow d = \pm 1$ may assume $d = 1$ then $a = 1$ $\qquad g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

$\tau' = \tau + b \Rightarrow b = 0 \quad \tau' = \tau$

② $\underset{\frac{\sqrt{3}}{2}}{\underset{\text{IV}}{\left|\tau + \frac{d}{c}\right|}} \le \frac{1}{|c|} \quad \Rightarrow c = \pm 1 \qquad$ assume $c = 1$

$|\tau + d| \le 1$ then $\mathrm{Im}(\tau + d) \ge \frac{\sqrt{3}}{2}$ if $d \ne 0$

$\mathrm{Re}(\tau + d) \ge \frac{1}{2}$

(i) $d = 0$ $|\tau| = 1$ $\quad g = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ $\quad \tau' = -\frac{1}{\tau} + a$

then $\tau = i$ $a = 0$ or $\tau = e^{\frac{\pi}{3}i}$ $a = 1$

(ii) $d \ne 0$ then $\tau = e^{\frac{\pi}{3}i}$ $\quad d = -1$ $\quad \mathrm{Im}\,\tau' = \frac{\mathrm{Im}\,\tau}{|\tau - 1|^2} = \mathrm{Im}\,\tau$

then $\tau' = e^{\frac{\pi}{3}i}$ (or $g\tau = a - \frac{1}{\tau - 1}$ then $a = 0$)

§2 Abelian variety

(g-dim complex manifold)

X: compact Riemann surface

$\dim \Omega^1(X) = g$

natural map: $\phi: H_1(X,\mathbb{Z}) \longrightarrow \Omega^1(X)^*$

$\gamma \longmapsto \phi(\gamma)$

$\phi(\gamma)(\omega) := \int_\gamma \omega$ (obviously linearly)

$\phi(\gamma)=0$ means, for all $\omega \in \Omega^1(X)$ $\int_\gamma \omega = 0$ then $\int_\gamma \overline{\omega} = 0$

$\Rightarrow \int_\gamma \theta = 0 \quad \forall\ \theta \in H^1(X,\mathbb{C}) = \Omega^1(X) \oplus \overline{\Omega^1(X)}$

$\Rightarrow \gamma = 0$ by de Rham isomorphism

$\Rightarrow \Lambda := \phi(H_1(X,\mathbb{Z}))$ is a discrete subgroup in $\underset{\dim_{\mathbb{R}} = 2g}{\underbrace{\Omega^1(X)^*}}$ of rank $2g$ (real sense)

compact complex Lie group so important

then $J(X) := \Omega^1(X)^*/\Lambda$ is a complex torus of dimension g called abelian/Jacobi variety of X

$\{a_i, b_i \mid i=1,\cdots,g\}$ basis of $H_1(X,\mathbb{Z})$ $\omega_1, \cdots, \omega_g$ basis of $\Omega^1(X)$

dual basis $*$

Linear isomorphism $\Omega^1(X)^* \longrightarrow \mathbb{C}^g$ (the coordinate in row)

(this is not canonical)

$\phi(a_i) = \sum \left(\int_{a_i} \omega_j\right) \omega_j^*$

we get $H_1(X,\mathbb{Z}) \xrightarrow{\psi} \mathbb{C}^g$ $\quad a_i$ (or b_i) $\longmapsto \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \\ \int_{a_i} \omega_g \end{pmatrix}$

$(\psi(a_1) \cdots \psi(a_g)\ \psi(b_1) \cdots \psi(b_g))$

called a period matrix of X

$\Lambda = \{\sum m_i \psi(a_i) + n_i \psi(b_i) \mid m_i, n_i \in \mathbb{Z}\}$ is a lattice in $\mathbb{C}^g$

$\mathbb{C}^g/\Lambda \cong J(X)$

we may choose a canonical basis of $H_1(X,\mathbb{Z})$

$$\begin{pmatrix} a_1 \\ \vdots \\ b_1 \\ \vdots \end{pmatrix} \cdot (a_1 \cdots b_1 \cdots) = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

intersection

(say, associated to (α, β))

then $\exists!$ canonical basis of $\Omega^1(X)$ for the fixed basis of $H_1(X,\mathbb{Z})$

$z_{ij} = \int_{b_j} \omega_i \qquad Z = (z_{ij}) \qquad$ the period matrix is $(I_g \ Z)$

$\Lambda = \{ \sum m_i e_i + n_i z_i \}$

note that $\begin{cases} Z = Z^T \\ \operatorname{Im} Z \text{ is positive definite} \end{cases}$

$\mathcal{H}_g := \{ B \in GL(g, \mathbb{C}) \mid B^T = B$ and $\operatorname{Im} B$: positive definite$\}$

called the Siegel upper half space

(a, b) isn't unique

assume (a', b') be another canonical basis

$\exists P \in GL(2g, \mathbb{Z}) \qquad (a', b') = (a, b) P$

$$\begin{pmatrix} a' \\ b' \end{pmatrix} \cdot (a', b') = P^T \begin{pmatrix} a \\ b \end{pmatrix} (a \ b) P \qquad P \in Sp(g, \mathbb{Z})$$

$$\Rightarrow \quad P^T \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} P = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

$(I_g \ Z') \qquad (I_g \ Z)$

Exercise. $\exists P \in Sp(g, \mathbb{Z}) \quad P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad Z' = (AZ + B)(CZ + D)^{-1}$

$$M \in \mathcal{H}_g \Rightarrow P \cdot M \overset{\triangle}{=} (AM + B)(CM + D)^{-1} \in \mathcal{H}_g$$

Summary. under two basis $PZ = Z' \quad \exists P \in Sp(g, \mathbb{Z})$

then $\mathbb{C}^g/\Lambda \cong \mathbb{C}^g/\Lambda' \cong J(X)$

Note. ① $\mathcal{H}_g / Sp(g,\mathbb{Z})$ moduli space of g-abelian varieties

(can be defined as g-dim complex torus which can be embedded into projective space)

$$\dim \mathcal{H}_g / Sp(g,\mathbb{Z}) = \dim \mathcal{H}_g = \frac{g(g+1)}{2} \ ?$$

② dimension of Riemann surface of genus g $(g \geq 2)$ is $3g-3$

→ compact

③ $X \sim X' \iff J(X) \cong J(X')$ (Torelli theorem)

so there're $M \in \mathcal{H}_g$ which isn't a period matrix of any Riemann surface

unsolved : what's the neccessary & sufficient condition for M to be the period matrix of a R-S

§3 Abel-Jacobi map

X: compact Riemann surface $g(X) \geq 1$ a canonical basis $(a,b) = \{a_i, b_i \mid 1 \leq i \leq g\}$ of $H_1(X, \mathbb{Z})$

$\omega_1, \cdots, \omega_g$: canonical basis of $\Omega'(X)$ associated to (a,b)

$$\Pi_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th term} \in \mathbb{C}^g \qquad \Pi_{g+i} = \begin{pmatrix} \int_{b_i} \omega_1 \\ \vdots \\ \int_{b_i} \omega_g \end{pmatrix}$$

$$\Lambda := \left\{ \sum_{i=1}^{g} m_i \Pi_i + n_i \Pi_{g+i} \;\middle|\; m_i, n_i \in \mathbb{Z} \right\}$$

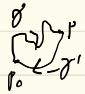
$A(X) \cong \mathbb{C}^g / \Lambda$ fix $p_0 \in X$

the Abel-Jacobi map is defined to be

$$I: X \longrightarrow A(X) = \mathbb{C}^g/\Lambda$$
$$p \longmapsto I(p) = \begin{bmatrix} \int_{p_0}^{p} \omega_1 \\ \vdots \\ \int_{p_0}^{p} \omega_g \end{bmatrix}$$

($[\ ]$ denote the equivalent class)

where $\int_{p_0}^{p} \omega := \int_{\gamma} \omega, \ \forall \gamma$



since $\int_{\gamma + (-\gamma')} \omega \in \Lambda$ this is well-defined

another way:

$\int_{p_0}^{p} \omega$ multiple function module the 单值分支 (branch?)

in general, I can be defined on $Div(X)$:

for $D = \sum_{i=1}^{r} m_i p_i$ $\quad I(D) := \sum_{i=1}^{r} m_i I(p_i) = \begin{bmatrix} \sum m_i \int_{p_0}^{p_i} \omega_1 \\ \vdots \\ \sum m_i \int_{p_0}^{p_i} \omega_g \end{bmatrix}$

it's a group morphism

$Div_p(X) \subset Div_0(X)$

$\forall D \in Div(X) \quad D = D_+ - D_- \qquad \mathcal{Y}(D) \qquad I(D^+) - I(D^-) = I(D)$

$$\begin{bmatrix} \sum_{i=1}^{r} \int_{q_i}^{p_i} \omega_1 \\ \vdots \\ \sum_{i=1}^{r} \int_{q_i}^{p_i} \omega_g \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{r} \int_{p_0}^{p_i} \omega_1 \\ \vdots \\ \sum_{i=1}^{r} \int_{p_0}^{p_i} \omega_g \end{bmatrix} - \begin{bmatrix} \sum_{i=1}^{r} \int_{p_0}^{q_i} \omega_1 \\ \vdots \\ \sum_{i=1}^{r} \int_{p_0}^{q_i} \omega_g \end{bmatrix}$$

$I|_{Div_0(X)} : Div_0(X) \longrightarrow A(X)$ $\qquad$ new I is defined to be $I|_{Div_0(X)}$

Theorem (Abel) $\ker I = Div_p(X)$

(Jacobi inversion) I is surjective

$$0 \to Div_p(X) \hookrightarrow Div_0(X) \twoheadrightarrow A(X) \to 0 \qquad A(X) \cong Div_0(X)/Div_p(X)$$

§4 introduction to complex mfd

Definition. $D \subset \mathbb{C}^n$ $f: D \to \mathbb{C}$ is called holomorphic if:
$\forall a \in D$, f can be expanded as power series
$f(z_1, \cdots, z_n) = \sum_{r_1, \cdots, r_n \geq 0} C_{r_1 \cdots r_n} (z_1 - a_1)^{r_1} \cdots (z_n - a_n)^{r_n}$ converges uniformly on some neighbourd of a in D

Notation. $\mathcal{O}(D) = \{$ holomorphic functions on $D\}$ $D_1 \subset \mathbb{C}^m$ $D_2 \subset \mathbb{C}^n$
$f = (f_1, \cdots, f_n) : D_1 \to D_2$ is called a holomorphic map if $f_1, \cdots, f_n \in \mathcal{O}(D_1)$

Fact. f, g holomorphic $\Rightarrow$ $g \circ f$ holomorphic $f: D_1 \to D_2$ $g: D_2 \to D_3$

Definition, $D_1, D_2 \subset \mathbb{C}^n$ are called biholomorphically equivalent (isomorphic) if: $\exists$ biholomorphic map $f: D_1 \to D_2$ (f is bijective and f^{-1} & f are both holomorphic)

Definition. X: Hausdorff topological space with countable base
complex atlas, holomorphic transform
...
thus, holomorphic functions, maps, isomorphisms ... all can be defined

Examples. 1. domain in $\mathbb{C}^n$; Riemann surface

2. $\mathbb{P}^n_{\mathbb{C}}$ as a set: 1-dim linear-subspace of $\mathbb{C}^{n+1}$
$(\mathbb{C}^{n+1})^* := \mathbb{C}^{n+1} \setminus \{0\}$ $\forall z = (z_0, \cdots, z_n) \in (\mathbb{C}^{n+1})^*$
determines a unique 1-dim subspace $\underset{\{\lambda z \mid \lambda \in \mathbb{C}\}}{}$ $\to$ a point in $\mathbb{P}^n$
w, z determines the same point $\Leftrightarrow$ $\exists \lambda \in \mathbb{C}^*$ $w = \lambda z$ (natural $\mathbb{C}$-v.c.)
identify $\mathbb{P}^n$ with $(\mathbb{C}^{n+1})^* / \mathbb{C}^*$ $\pi: (\mathbb{C}^{n+1})^* \to \mathbb{P}^n$
$\pi(z)$ denoted by $[z_0 : z_1 : \cdots : z_n]$ (homogeneous coordinate on $\mathbb{P}^n$)
as a quotient space, $\mathbb{P}^n$ is endowed with natural topology
open covering: $U_i = \{[z_0, \cdots, z_n] \mid z_i \neq 0\}$ $i = 0, \cdots, n$
$\varphi_i: U_i \to \mathbb{C}^n$ $[z_0, \cdots, z_n] \mapsto \left(\frac{z_0}{z_i}, \cdots, \widehat{\frac{z_i}{z_i}}, \cdots, \frac{z_n}{z_i}\right)$
$\varphi_i^{-1}: (t_1, \cdots, t_n) \mapsto [t_1, \cdots, t_i, 1, t_{i+1}, \cdots, t_n]$

let $S^{2n+1} = \{ z \in \mathbb{C}^{n+1} \mid |z| = 1 \}$

$\pi : S^{2n+1} \longrightarrow \mathbb{P}^n$ surjective $\pi(z) = \pi(w) \Longleftrightarrow \exists \lambda \in S^1$ $\lambda z = w$

$\mathbb{P}^n \cong S^{2n+1}/S^1$ compact

the quotient map is called a Hopf fibration
(principle bundle) structure group S^1

§5 submanifolds

let X be a complex manifold

Definition. a subset $A \subset X$ is called an analytic subset if: $\forall p \in X$, $\exists$ neighbour U of p and $f_1, \dots, f_r \in \mathcal{O}(U)$ s.t. $A \cap U = \{x \in U \mid f_1(x) = \dots = f_r(x) = 0\}$

[for several variable analytic function, the set of zeros isn't discrete, and always have dim $n-1$ if $\neq \emptyset$]

Definition. a complex submanifold of X is a connected set $A \subset X$ analytic and $df_1(p) \wedge \dots \wedge df_r(p) \not\equiv 0$

$\dim A = n - r$

(i.e. $\exists$ local coordinate $(z_1, \dots, z_n)$ near p s.t. rank $\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & & \vdots \end{pmatrix}(p) = r$

Definition. holomorphic embedding

let X, Y be complex manifolds of dimension $m \le n$

a holomorphic map $f: X \to Y$ is called a holomorphic embedding if: 1) f is injective 2) $f(X) \subset Y$ is closed & $f: X \to f(X)$ is homeomorphic 3) $\forall x \in X$, $\forall$ local coordinate $\underset{\text{near } x}{(z_1, \dots, z_m)}$ & $(w_1, \dots, w_n)$ near $f(x)$, rank $df_x = m$ i.e. injective

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Theorem (implicit function theorem) $D \subseteq \mathbb{C}^n$ a domain $a \in D$

(1) $f = (f_1, \dots, f_n): D \to \mathbb{C}^n$ holomorphic s.t. $\mathrm{rank}(J_a) = n$ then $\exists$ neighbour U of a s.t. $f(U)$ open & $f|_U: U \to f(U)$ is biholomorphic

(2) $f_1, \dots, f_r \in \mathcal{O}(D)$ $(r \le n)$ s.t. rank $\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & & \vdots \end{pmatrix}(a) = r$ then $\exists$ neighbour U of a & $f_{r+1}, \dots, f_n \in \mathcal{O}(U)$ s.t. $f \doteq (f_1, \dots, f_n): U \longrightarrow f(U)$ is biholomorphic

$f(U) \overset{\text{open}}{\subset} \mathbb{C}^n$

Exercise. (1) $A \subset X$ a complex submanifold, then A is a complex manifold s.t. $j: A \hookrightarrow X$ is holomorphic

(2) $f: Y \to X$ is a holomorphic embedding, the $f(Y)$ is a complex submanifold of X

non-reduced space

Definition. (1) X is called Stein if X can be holomorphically embedded in some $\mathbb{C}^n$
(2) X is called projective if ------ in $\mathbb{CP}^n$ for some n

Remark. (1) (Grauert) X is stein iff it has an exhaustive strictly plurisubharmonic function
(2) (Kodaira) assume X is compact, then X is projective iff X has a positive line bundle.

embedding of compact Riemann surface into projective spaces

canonical map: if $g(X) \geq 1$, $\{\omega_1, \cdots, \omega_g\}$ is a basis of $\Omega^1(X)$

$$\Phi : X \longrightarrow \mathbb{P}^{g-1}$$
$$x \longmapsto [\omega_1(x) : \cdots : \omega_g(x)]$$

more precisely: (U, z) is a local coordinate on X, and assume $\omega_1 = a_1(z)dz \cdots \omega_g = a_g(z)dz$
then $\Phi|_U$ be represented as $\Phi(z) = [a_1(z) : \cdots : a_g(z)]$
(by a coordinate transform, all divided by $\frac{dw}{dz}$, which is the same in $\mathbb{P}^{g-1}$)

by Lemma. $\forall p \in X$ $\exists\, \omega \in \Omega^1(X)$ s.t. $\omega(p) \neq 0$, $a_i(z)$ won't vanish simultaneously

proof. $H^1(X, p) = \{\omega \in \Omega(X) \mid (\omega) \geq p\} = \{\omega \in \Omega^1(X) \mid \omega(p) = 0\}$
since $H^0(X, p) = \mathbb{C}$ $\quad 1 - h^1 = 1 - g + 1 \Rightarrow h^1 = g - 1$

then prove ~~that~~ Φ is injective
why?

Lemma 2. assume X isn't hyperelliptic (non exist meromorphic function with 2 poles i.e. double-cover of $\mathbb{P}^1$)

then $\forall p, q \in X$ $p \neq q$ $\exists \omega \in \Omega^1(X)$ $\omega(p) = 0$ $\omega(q) \neq 0$

proof. suffices to show $h^1(X, p) > h^1(X, p+q)$
$\overset{\shortparallel}{g-1}$

since X isn't hyperelliptic

$h^1(X, p+q) = h^1(X, p) = 1$

Lemma 3 not hyperelliptic then $\forall p$ $\exists \omega \in \Omega^1(X)$ $\mathrm{ord}_p \omega = 1$

similarly

$\Phi : X \to \Phi(X)$ compact to compact bijection

Φ continuous

$\Downarrow$

Φ^{-1} continuous

i.e. homeomorphic

i.e. a compact Riemann surface with $g(X) = g$ which is not hyperelliptic can be holomorphically embedding into $\mathbb{P}^{g-1}$

Review. X : compact with $g(X)=g\geq 1$

$\omega_1, \cdots, \omega_g$ be a basis of $\Omega^1(X)$

canonical map $x \mapsto [\omega_1(x) : \cdots : \omega_g(x)]$

(in fact, similar to $\frac{\omega_2}{\omega_1}, \cdots, \frac{\omega_g}{\omega_1}$ $(g-1)$ number of meromorphic function

let $f_0, f_1, \cdots, f_n \in \mathcal{M}(X)\setminus\{0\}$

we can define a map $\Phi: X \to \mathbb{P}^n$ as follows

$x \longmapsto [f_0(x) : f_1(x) : \cdots : f_n(x)]$

more precisely: assume z is a local coordinate near x.

with $z(x)=0$, $r = \min_i \{\mathrm{ord}_x f_i\}$

then $f_i = z^r g_i(z)$ where g_i is holomorphic near 0 & they won't vanish simultanously.

then near x $\Phi(z) = [g_0(z) : g_1(z) : \cdots : g_n(z)] \in \mathbb{P}^n$

which is holomorphic

Example. $X = \mathbb{P}^1$ $f_0 = 1$, $f_1 = z, \cdots, f_n = z^n \in \mathcal{M}(\mathbb{P}^1)$

$\min\{\mathrm{ord}_\infty f_i\} = -n$ near ∞, let the coordinate be $\omega = \frac{1}{z}$ $f_0 = \frac{1}{\omega^n}\omega^n \cdots$ $f_n = \frac{1}{\omega^n}\cdot 1$

near ∞, the map $\omega \mapsto [\omega^n : \omega^{n-1} : \cdots : 1]$

so $\infty \mapsto [0:0:\cdots:0:1]$

Let $D \in \mathrm{Div}(X)$ $H^0(X, D) = \{ f \in \mathcal{M}(X) \mid (f) + D \geq 0 \}$

assume $h^0 \geq 2$, $f_0, f_1, \cdots, f_n$ is a basis of $H^0(X, D)$

$$\Phi_D : X \longrightarrow \mathbb{P}^n \qquad \underline{\text{holomorphic}}$$
$$x \longmapsto [f_0(x) : \cdots : f_n(x)]$$

Theorem. $\deg D \geq 2g+1 \Rightarrow \Phi_D$ is a holomorphic embedding

proof. exercise

Example. $g=1$ i.e. $2g+1=3$, let $p \in X$, $D = 3p$

$$H^1(X, D) = \{ \omega \in \Omega^1(X) \mid \mathrm{ord}_p\, \omega \geq 3 \}$$

$\dim \Omega^1(X) = 1$ & Lemma 1 $\Rightarrow \exists\, \omega \in \Omega^1(X)$. $\omega(p) \neq 0$

so $H^1(X, D) = \{0\} \Rightarrow h^0 = 3$

$\Phi_D : X \longrightarrow \mathbb{P}^2$

theorem $\Rightarrow$ (1) compact Riemann surface of genus 1 can be embedding into $\mathbb{P}^2$

(2) compact Riemann surface is projective

Remark. ① all compact Riemann surface can be embedding into $\mathbb{P}^3$ ($\mathbb{P}^2$ ×)

② all non-compact Riemann surface $\xrightarrow{\text{embedding}} \mathbb{C}^3$

$\mathbb{C}^2$: unknown

$\mathbb{P}^n = (\mathbb{C}^{n+1})^* / \mathbb{C}^*$

Definition. $P(z_0, z_1, \cdots, z_n) \in \mathbb{C}[z_0, z_1, \cdots, z_n]$ homogeneous ...

we can define the zero set

$\downarrow$

algebraic set

an algebraic set of $\mathbb{P}^n$ is an analytic subset (obvious, by the $n+1$ charts U_i)

Theorem (Chow) analytic set in $\mathbb{P}^n$ is algebraic

a non-singular plane algebraic curve is a complex submanifold $\Rightarrow$ a compact Riemann Surface

proof

$\text{genus}(A) = \frac{(d-1)(d-2)}{2}$

two curves

d_1 d_2

intersection: $d_1 \cdot d_2 = A_1 \cdot A_2$

$\Downarrow$

compact R-S with some genus can't be embedded into $\mathbb{P}^2$

$F(x, y, z)$ homogeneous of deg d

if $\frac{\partial F}{\partial x}$ $\frac{\partial F}{\partial y}$ $\frac{\partial F}{\partial z}$ have no common zero

then $A = V_F \subseteq \mathbb{P}^2$ is non-singular

X : compact Riemann-surface
$g(X) \geq 1$
$\{a_i, b_i\}$ canonical basis of $H_1(X, \mathbb{Z})$ with corresponding basis $\omega_1, \dots, \omega_g$ of $\Omega^1(X)$

$$\Pi_i = \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \\ \int_{a_i} \omega_g \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow i\text{-th term} \\ \\ \\ \end{array}$$

$$\Pi_{g+i} = \begin{pmatrix} \int_{b_i} \omega_1 \\ \vdots \\ \int_{b_i} \omega_g \end{pmatrix}$$

$A(X) := \mathbb{C}^g / \Lambda$ where $\Lambda = \left\{ \sum_{i=1}^{g} (m_i \Pi_i + n_i \Pi_{g+i}) \mid m_i, n_i \in \mathbb{Z} \right\}$
Abel variety (algebraic group)

$I_{p_0}: \mathrm{Div}(X) \longrightarrow A(X)$ fixed $p_0 \in X$

$$D = \sum_{i=1}^{r} n_i p_i \longmapsto \begin{pmatrix} \sum_{i=1}^{r} n_i \int_{p_0}^{p_i} \omega_1 \\ \vdots \\ \sum_{i=1}^{r} n_i \int_{p_0}^{p_i} \omega_g \end{pmatrix} \mod (\Lambda)$$

$I_{p_0}(D)$ depends on p_0. (相差一个 vector in $\mathbb{C}^g$)

but if $\deg D = 0$ $D = (p_1 + \cdots + p_r) - (q_1 + \cdots + q_r)$

then $$I_{p_0}(D) = \begin{pmatrix} \sum_{i=1}^{r} \int_{q_i}^{p_i} \omega_1 \\ \vdots \\ \sum_{i=1}^{r} \int_{q_i}^{p_i} \omega_g \end{pmatrix}$$ which is independent of the choice of p_0

so we get $I: \mathrm{Div}_0(X) \longrightarrow A(X)$ group ~~topology~~ (variety $\Rightarrow$ cplx)
(group structures)

- Abel: $\mathrm{Ker}\, I = \mathrm{Div}_p(X)$
- Jacobi inversion: I is surjective

$\Rightarrow$ $0 \to \mathrm{Div}_p(X) \to \mathrm{Div}_0(X) \xrightarrow{I} A(X) \to 0$ exact sequence of Abelian groups

Exercise. show that: $\forall$ fixed $p_0 \in X$, $f: X \to A(X)$ is a holomorphic embedding

$$p \longmapsto \begin{pmatrix} \int_{p_0}^{p} \omega_1 \\ \vdots \\ \int_{p_0}^{p} \omega_g \end{pmatrix} \mod (\Lambda)$$

Example. $g = 1$
period matrix $(1, \tau)$
$A(X) = \mathbb{C}/\Lambda_\tau$ $f: X \to A(X)$ biholomorphic

so X is of the form $\mathbb{C}/\Lambda$

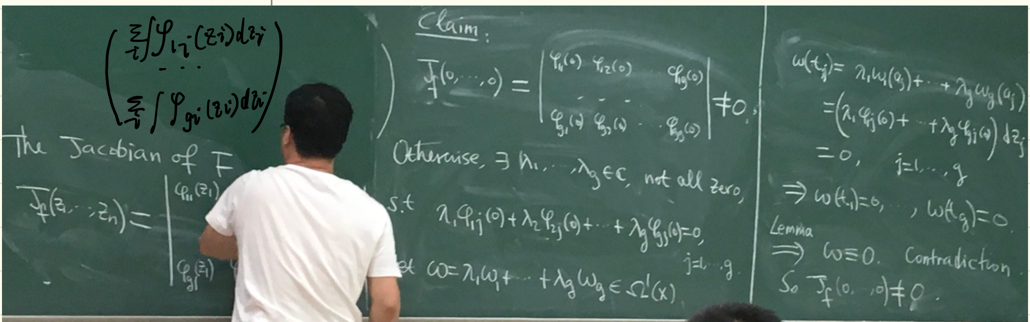
Lemma. $\exists$ distinct points $t_1, \dots, t_g \in X$ s.t. if $\omega \in \Omega^1(X)$ satisfying $\omega(t_i) = 0$, then $\omega = 0$

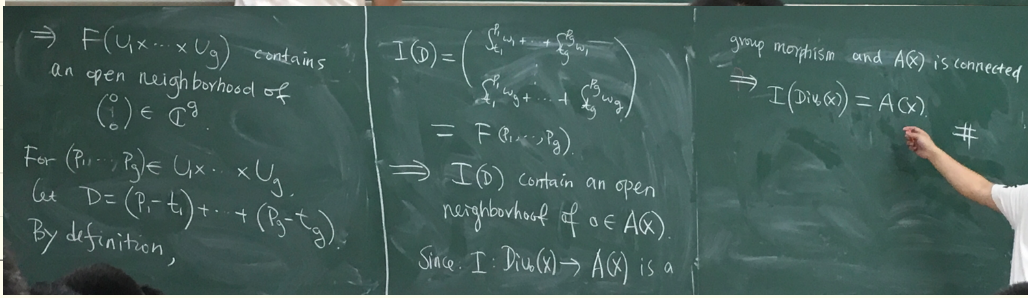
let (U_i, z_i) be local coordinates near t_i s.t. $z_i(t_i) = 0$

consider
$$F: U_1 \times \cdots \times U_g \to \mathbb{C}^g$$
$$(P_1, \dots, P_g) \mapsto \begin{pmatrix} \sum_i \int_{t_i}^{P_i} \omega_1 \\ \vdots \\ \sum_i \int_{t_i}^{P_i} \omega_g \end{pmatrix}$$

if near t_j, $\omega_i = \varphi_{ij}(z_j)\, dz_j$

then F is represented as





$$\omega \# \omega_{PQ} = 2\pi i \int_Q^P \omega$$