CSCI-4961 - RPI, 05/20 - 08/16/2024

Lecture $|0\rangle$: Syllabus for Quantum Computing @ RPI

Tue./Fri. 6:30pm –8:35pm, Location: DCC 337

Contributor: *Lecturer: Yanglet Xiao-Yang Liu*

Figure 1: Photo of the utility-scale IBM Quantum System One, which was unveiled on April 5, 2024 in RPI's Voorhees Computing Center, a former cathedral. It is the first IBM Quantum System One to be installed on a university campus. (Credit: IBM)

Class Kickoff

Video [1] (1 min): Quantum Ribbon Cutting Celebration at RPI
Video [2] (1 min): First of its kind quantum computers unveiled at RPI
Video [3] (5 min): What if we had working quantum computers today?

[1] `https://www.youtube.com/watch?v=-1FEnRAW-A0`
[2] `https://www.youtube.com/watch?v=hLlwxqRwUrc`
[3] `https://www.youtube.com/watch?v=fNOJjR1Znds`

Course Summary

Enabling a transition from an academic research interest to a nascent industry.

No assumption of background knowledge! We only require your interest and commitment.

Building on a foundation of computing (CSCI 2200) and algorithms (CSCI 2300), this course delves into fundamental principles such as qubits, superposition, entanglement, quantum gates, and algorithms. Students will engage in hands-on lab sessions using Qiskit/OpenQASM and/or AWS/Azure cloud services, exploring both quantum circuits and algorithms. Students will also explore applications in optimization and machine learning, with discussions on the latest in quantum hardware, including IBM Quantum System One and Google's Sycamore circuits.

We take a mathematical-and-algorithmic approach and classical-and-quantum approach, accessible to all STEM students. Two professional levels are expected: 1). Quantum-aware engineers via lectures/reports/poster, and 2). quantum-proficient engineers via labs/projects/presentations.

At the *undergraduate* level, we aim to address the educational gap between

- quantum-related graduate programs focused mainly on PhDs, with a few MS programs. E.g., the M.S. in Quantum Science and Technology program at Columbia University will start on September 2024.
- excitement generated both by the popular media and the increased interest in introducing QIST in before-colleague schools;

In particular, students are encouraged to play with the IBM Quantum System One at RPI campus. Please request access via link [4].

Catalog Description: Data structures and algorithms, and the mathematical techniques necessary to design and analyze them. Basic data structures: vector, matrix, associative structures, graphs. Mathematical techniques for designing algorithms, analyzing worst-case and expected-case algorithm efficiency, and characterizing speedups. Algorithm design techniques: Shor's factoring algorithm, Grover's search algorithm, simulated annealing algorithm, etc.. We will also try to cover: QAOA algorithms, quantum supremacy, Qiskit, etc.

Learning objectives include a solid grasp of quantum computing principles, an understanding of quantum advantages (including the quantum supremacy in the NISQ era), proficiency in reading and analyzing simple quantum algorithms, and insights into practical applications.

[4] https://itssc.rpi.edu/hc/en-us/articles/17609648152845-Quantum-computing

Grading and Logistics

There are 12 weeks, 60 students.

Scores	90%	86%	82%	78%	74%	70%	66%	62%	58%	54%	$< 54\%$
Grades	A	A -	B+	B	B-	C+	C	C-	D+	D	fail

Class participation	quizzes (12%) and classroom sharing (6%)	18%
Homework (3)	$3 \times 5\%$	15%
Exam	15%	15%
Labs (4)	$2 \times 2\% + 2 \times 3\%$	10%
Project I, II	$2 \times 3\%$	6%
Proposal/Update	$2 \times 5\%$	10%
Final Report	6%	6%
Presentation	20%	20%
Total		100%

Week	Monday	Tuesday (math/algo)	Wed.	Friday (hands-on)
1 (05/20 - 05/26)		L1 (Ch2.2)		L2 (Ch2.4, 2.6.3, **Lab1**)
2 (05/27 - 06/02)		L3 (Ch2.3)	HW1	L4 (Ch3)
3 (06/03 - 06/09)		L5 (Ch3)		L6 (Ch4.4, **Lab2**)
4 (06/10 - 06/16)	HW2	L7 (Ch4.3, 4.6, **Lab3**)		L8 (Ch4.7)
5 (06/17 - 06/23)		L9 (Ch6)		L10 (Ch6)
6 (06/24 - 06/30)		L11 (Ch6)		L12 (Ch7) **HW3**
7 (07/01 - 07/07)	(**Break**)			
8 (07/08 - 07/14)		L13 (QEC)		L14
9 (07/15 - 07/21)	Proposal	L15 (Review; QEC)		L16 (Review; **Lab4**)
10 (07/22 - 07/28)		L17 (Review)		Exam
11 (07/29 - 08/04)	Update	L18 (**Proj I**)		L19
12 (08/05 - 08/11)		L20 (**Proj II**)		L21
13 (08/11 - 08/16)		Presentations (<15 min)		Presentations (<15 min)
08/23 (ddl)	Report			

Logistics:

Course Learning Management System (LMS)

All course materials will be available via Submitty: `https://submitty.cs.rpi.edu/courses/u24/csci4961`

Log in using your RCS ID (e.g., "liux33")

We will use Submitty's Discussion Forum for course announcements and for asking questions. Please post questions there; also answer questions...

The course schedule will be posted in Submitty (but will likely change)

Check your RPI email at least once per day. Turn on all email notifications in Submitty — you are responsible for staying current with course announcements, assignment changes, etc.

Suggestions: Wednesday OHs are QA sessions for labs/projects. For students with a quantum RCOS project, RCOS sessions could also be your regular group meetings

- Solving problems in group. Each student team has $1 \sim 3$ students (can switch groups for different tasks): HWs, Labs, Projects and Presentations allow collaborations. **But, NO collaboration in exam**!
- **Students can use ChatGPT in this class, including reports and presentations**! Student should state that they use ChatGPT if they do use it. Please avoid fake references/sources (or other misinformation) generated by ChatGPT.

Notes: 1). Friday's lab/project sessions are for attendance purpose; gradings will be done through Proposal/Update and Final Report. Final Report and Presentation should be be on the same topic and within the same student team.

We will provide templates for Proposal/Update and Final Report, and Presentations.

Bonus points: $1 \sim 5\%$

Project I, II; Final Report, Presentation, and other projects with professor's approval, or creating course materials. Course webpage [5]

Each lecture (2 hours):

$10 \sim 15$ minutes for videos of physical experiments;

$70 \sim 80$ minutes for lecture;

$10 \sim 15$ minutes for quizzes;

$15 \sim 30$ minutes for students' sharing.

TA and Undergraduate mentors and Summer URP students:

- Dong Hu (OH for labs: Wednesday 4pm - 6pm at DCC 324): Design. Tutor and Check off Labs, Projects, and Grading Presentations; and join Discussion Forum on Submitty.

[5] `https://yangletliu.github.io/Intro-to-Quantum-Computing-Website/index.html`

- Yue Han (OHs: Monday & Thursday 1pm - 3pm at AE 110): Design and Grade HWs, Reports, and Presentations; and join Discussion Forum on Submitty.
- Undergraduate mentors (Benjamin Rodriguez, Sam Garnett, Garrison Trahant, Yolanda Zhao): Tutor Labs, Projects; and join Discussion Forum on Submitty.
- Yanglet Xiao-Yang Liu (OH at AE 208: Tuesday 1pm - 2pm): Lectures; and Grading Presentations.

Do NOT email our TAs and mentors...instead, attend OHs and post questions on the Discussion Forum. OH will primarily be in-person with a few online-only sessions. Always check the posted schedule and watch for announcements in case of OH change.

Summer URP Students (Ethan McCartney; Deron Maurice Diaz; Joshua Pile; Dannong Wang): we have several URP students on quantum computing in Summer 2024, who will provide help with lab tasks and projects.

NO late days! However, school allows Excused Absence. Use Emails for record.

Textbooks and Materials

Preparation Knowledge

Prerequisites: CSCI 2200 FOCS and CSCI 2300 Algos.
Complex numbers; Trigonometric functions; Sets and functions; Probability
(Complex) linear algebra, matrix operations; tensor product and tensor networks.

- Book 1: Introduction to classical and quantum computing, By Thomas G Wong
- Book 2: Introduction to quantum algorithms via Linear Algebra, Second Edition. Richard J. Lipton and Kenneth W. Regan
- Book 3 (advanced): Quantum computation and quantum information. Cambridge University Press, 2001. Michael A. Nielsen and Isaac L. Chuang
- Lectures [6], Understanding quantum information & computation, by John Watrous (IBM Quantum Education); YouTube [7]
- Lectures, Computing and Quantum Computing, by Malik.
- Lectures [8], Introduction to quantum computing, by Henry Yuen, Columbia University
- Lectures [9], Intro to quantum information science, by Scott Aaronson [10]

[6] https://learning.quantum.ibm.com/course/basics-of-quantum-information
[7] https://www.youtube.com/watch?v=tI1SfFuuX-o&list=PLOFEBzvs-VvqoeIypXYLLf0PY-WOQMLR3&index=1
[8] https://www.henryyuen.net/classes/spring2021/
[9] https://scottaaronson.blog/?p=3943
[10] https://www.scottaaronson.com/democritus/

- Lectures, The mathematics of quantum mechanics, by Martin Laforest, U. Waterloo, Institute for Quantum Computing.
- Book 4, A Course in Quantum Computing, by Michael Loceff. YouTube [11]
- OptimaLab, Introduction to quantum computing [12]

[11] https://www.youtube.com/@michaelloceff2873
[12] https://akyrillidis.github.io/notes/

Statements on the Use of AI

Within this class, you are encouraged to use foundation models (ChatGPT, GPT, DALL-E, Stable Diffusion, Midjourney, GitHub Copilot, and anything after) in a **totally unrestricted fashion, for any purpose, at no penalty**.

However, you should note that all large language models still have a tendency to make up incorrect facts and fake citations, code generation models have a tendency. You will be responsible for any inaccurate, biased, offensive, or otherwise unethical content you submit regardless of whether it originally comes from you or a foundation model. If you use a foundation model, **its contribution must be acknowledged in the hand in; you will be penalized for using a foundation model without acknowledgement**.

Having said all these disclaimers, the use of foundation models is highly encouraged. I am happy to observe how those advanced AI tools are used to help the young generation to understand quantum.

The Prospects of Quantum Computing

Quantum computing promises several disruptive changes:

- Feymman's argument that the exact simulation of a quantum system is only possible with a quantum computer.
- (End of) Moore's Law. Moore's Law states that the density of transistors on a chip roughly doubles every eighteen months. Current estimates say that in about a decade this should be down to single electron transistors. This is the end of the road for further miniaturization of classical computers based on electronics. Long before that chip designers will have to contend with quantum phenomena. Quantum computation provides a method of bypassing the end of Moore's Law, and also provides a way of utilizing the inevitable appearance of quantum phenomena.
- Cryptography. Quantum computation allows us to do cryptography in a way that doesn't require assumptions about factoring primes, etc. It also allows us to break classical cryptography schemas. Obviously, if we are interested in cryptography, we'll also have to be interested in quantum computation.
- Quantum machine learning.... any speedups are summarized in the Nature 2017 paper: Biamonte, Jacob, Peter Wittek, Nicola Pancotti, Patrick Rebentrost, Nathan Wiebe, and Seth Lloyd. "Quantum machine learning." Nature 549, no. 7671 (2017): 195-202.

Table 1: Classical Information Science.

Computing	Communication	Sensing
Turing Theory	Information Theory	Probability
Digital logic	Digital Communication	Detection & Estimation
DSP	Data Compression	a.k.a., statistical inference
Algorithms	Computer Networks	
	Cryptography	

Table 2: Quantum Information Science.

Quantum Computing	Quantum Communication	Quantum Sensing
Quantum Mechanics	Q. Information	Q. Detection (modulation, radar, illumination)
Q. Circuits	Q. Communication	Q. Estimation (metrology, homography)
Q. Algorithms	Q. Cryptography	

Becoming A Quantum Professional

Becoming a quantum computer scientist/engineer?

Educating quantum computing: "*to expand the number of researchers, educators, and students with training in quantum information science and technology to develop a workforce pipeline*" (the National Quantum Initiative (NQI) Act). In this class, we train undergraduate and graduate students to enter the workforce.

Before developing a curriculum, we need to answer the first question:

What are the skills that are needed?

Skill set valued by employers in the quantum industry: coding, statistical methods for data analysis, laboratory experiences, electronics knowledge, problem solving, materials properties, and quantum algorithms.

Fox, M.F., Zwickl, B.M. and Lewandowski, H.J. Preparing for the quantum revolution: What is the role of higher education? Physical Review Physics Education Research, 2020.

Big picture

Table 1 lists classes in the field of Classical Information Science. It includes computing, communication, and sensing.

Table 2 lists classes in the field of Quantum Information Science. Correspondingly, it includes quantum computing, quantum communication, and quantum sensing.

Topics

Overview: Computing and Computational Thinking.

- Computing as a discipline: from classical to quantum
- Overview of quantum computing
- Background on classical and quantum physics (quantum mechanics)

Part $|0\rangle$: One Qubit
The math.

- One Qubit Representation: Arithmetic-Algebra; Trigonometry-Complex Numbers; Cartesian Coordinates
- One Qubit: Superposition, Qubit, Bloch Sphere, Spherical Coordinates
- One-Qubit Gates: Quantum Gates, (Complex) Linear Algebra
- Two-Qubit Gates: Tensor Product
- Multiple Qubits: Density Matrices, Block Ball
- Languages and Turing Machine

Part $|1\rangle$: Quantum Circuits. The operations.

- Circuits
- Languages and Turing Machine

Part $|2\rangle$: Quantum Algorithms
The advantages.

- Complexity: P, NP, and Quantum Speedups
- Quantum Supremacy and Empirical Quantum Advantages
- Factoring: Shor's Algorithm
- Search: Grover's Algorithm
- Quantum Approximate Optimization Algorithm (QAOA)
- Quantum Error Correction. NISQ.

Part $|3\rangle$: Quantum Applications
The potentials.

- Optimization + Quantum Computing
- Quantum Machine Learning
- Quantum Simulation for Physical, Chemical, and Biological Systems
- Finance + Quantum Computing
- Material Science + Quantum Computing

Part $|4\rangle$: Lab Sessions and Projects
The hands-on learning.

- Lab 1: IBM Quantum Learning Suite (the Composer)
- Lab 2: Examples of GHZ Gate and Google's Sycamore Circuits
- Lab 3: Image Classification (quantum machine learning)
- Lab 4: Quantum approximate optimization algorithm (QAOA) [13]
- Project 1: Finding the Ground State of Ising model Using Learning-to-Anneal Algorithm
- Project 2: Challenging the Claim of Quantum Supremacy
- Project 3: Quantum Basic Linear Algebra Subroutines (qBLAS)

Note: Your final presentations and reports can be based on one of Lab 3-4, Project 1-3, or implementation of a quantum algorithm in Chapter 7, or other research ideas (need approval from a professor). Alternatively, the presentation can be about a paper from professor's list.

[13] https://learning.quantum.ibm.com/tutorial/quantum-approximate-optimization-algorithm

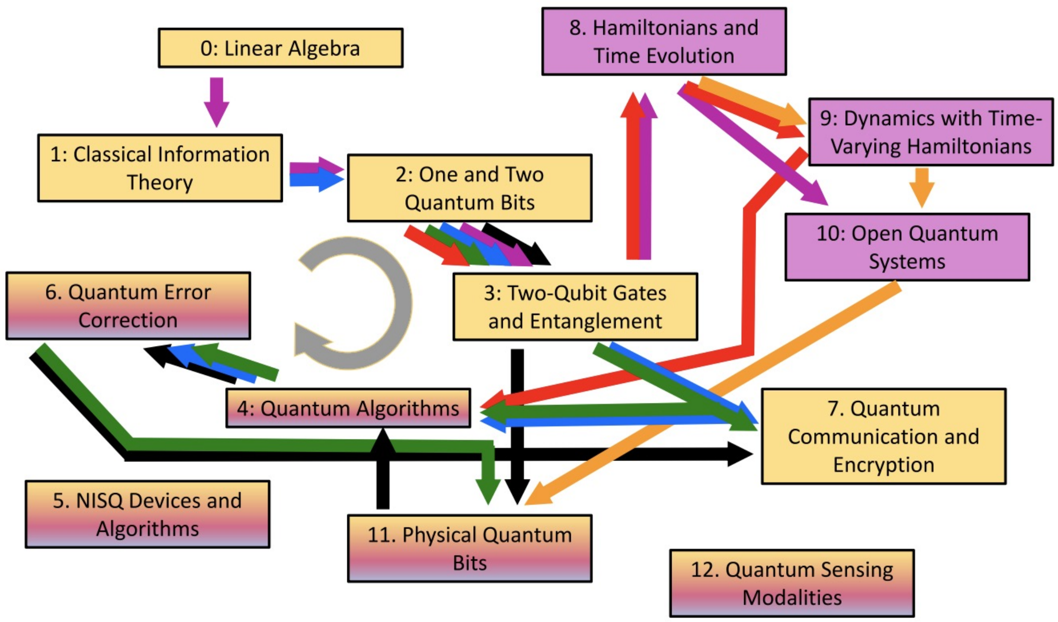

Figure 2: Course modules for STEM students at any level (orange shaded box), all levels plus advanced students (orange faded to purple), and advanced student only (purple).

Course Modules for Quantum Computing

Here, I will describe it based on my survey and resources. The goal is to have a set of modules that can be reorganized for a series of classes at RPI.

Building a Quantum Engineering Undergraduate Program. IEEE TRANSACTIONS ON EDUCATION, 2022.

Module 0 Introductory

To stimulate students' interest through concepts and applications, here we will take a quantum information perspective.

Linear algebra is a strong prerequisite. Either requiring students took it in a separate course (with a refresher lecture), or teaching it as a focused unit (span 2 - 3 lectures).

Concepts-Focused QISE (E)

By learning concepts and application first, (without much math), students may gain appreciation of QIST and intuitions. This would allow the connection between end-use applications and discussion of potential career pathways.

(Complex) Linear Algebra for QISE (E)

Topics: vector spaces (superposition, concept of a basis), linear transformations, matrix multiplication, noncommutativity, diagonalization, inversion, Hermitian and unitary operators, trace and partial traces, outer and tensor products, scaling up to larger matrices numerically.

We will teach those math concepts in the context of single qubit system, i.e., 2×2 matrices, and their tensor products.

Module 1: Classical Information Theory (E)

Topics: basics of bits, gates, communication, randomness and statistics, error correction, parity and data compression.

For the transition from classical bits to qubits, classical bits are represented as vectors and gates as matrices.

Module 2: One and Two Quantum Bits (E)

Topics:

- Qubits, superpositions states, measurements and the Born rule.
- Single-qubit Hilbert space: linear operators, Dirac notation, orthonormal bases and basis changes, qubit rotations, and the Bloch sphere.

Module 3: Two-Qubit Gates and Entanglement (E)

Topics: The CNOT gate and the circuit model of computation. Bell states and nonclassical correlations.

Module 4: Quantum Algorithms (E/A)

Topics:

- Early examples of quantum advantage in computation: the Deutsch, Deutsch-Jozsa, Bernstein-Vazirani, Simon's algorithms.
- Phase kick back from controlled unitaries.
- Oracle algorithms
- Grover's search algorithm, phase estimation.
- Shor's factoring algorithm: QFT and period-finding algorithm.

Module 5: NISQ Devices and Algorithms (E/A)

Topics: NISQ. VQE, QAOA, quantum machine learning. Error mitigation

Module 6: Quantum Error Correction (E/A)

Topics: Error models; Shor code.

Module 7: Quantum Communication and Encryption (E/A)

Module 8: Hamiltonian and Time Evolution (E/A)

Module 9: Dynamics with Time-Varying Hamiltonian (A)

Module 10: Open Quantum Systems (A)

Module 11: Physical Quantum Bits (E/A)

Module 12: Quantum Sensing Modalities (E/A)

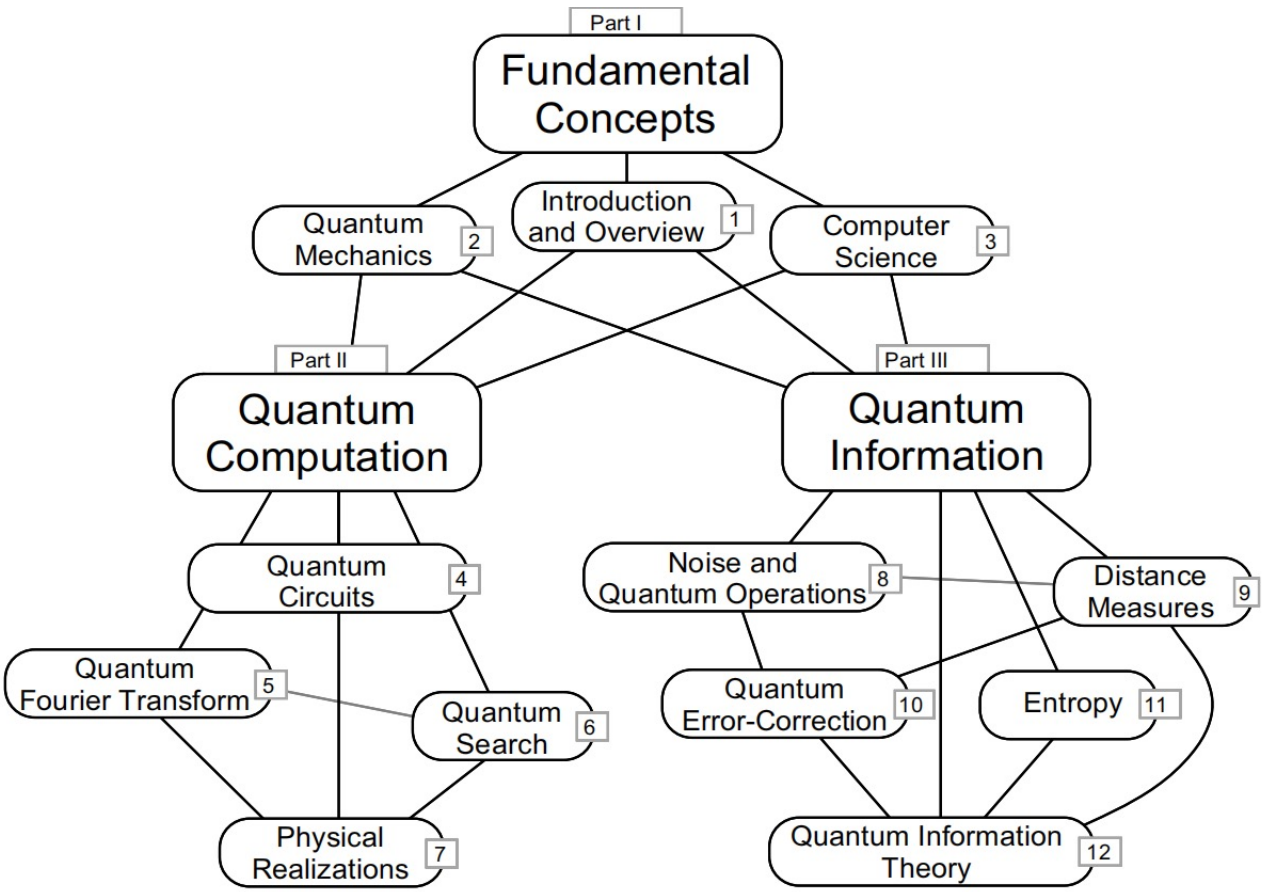

Figure 3: Structure of the Mik-Ike book

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Lecture $|1\rangle$: One Qubit (1/3)

May 21, Tue. 6:30pm –8:35pm

Contributor: ______________________________ *Lecturer: Yanglet Liu*

Richard Feynman: To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature ... If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in.

Topics: Arithmetic and Algebra; System and State; Trigonometry and Complex numbers $\mathbb{C}$

$$\boxed{|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle}$$

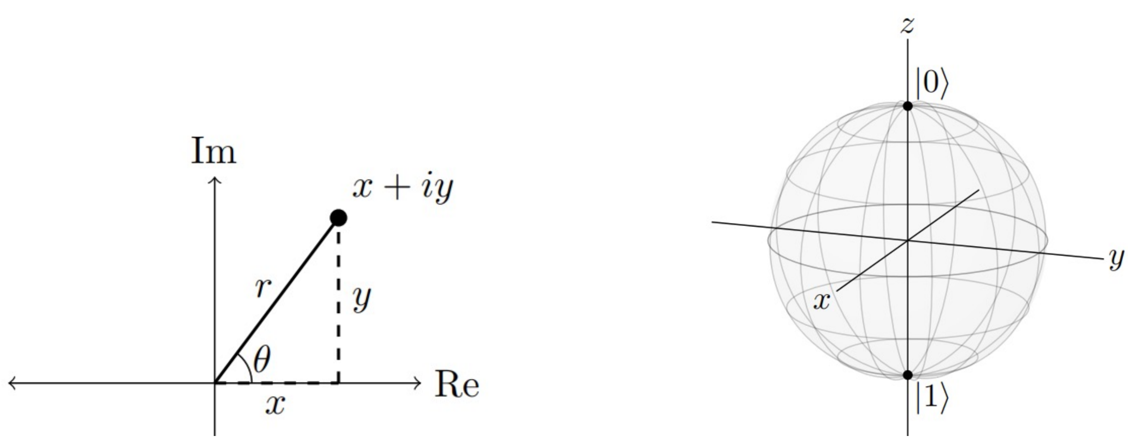


Figure 4: Complex plane (left) and the Bloch sphere (right).

Pop Quizzes and Exercises

1). Powers of i, where $\boxed{i = \sqrt{-1}}$ (imaginary unit number).
$i^0 =$ _____ $i^1 =$ _____ $i^2 =$ _____ $i^3 =$ _____
$i^4 =$ _____ $i^5 =$ _____ $i^6 =$ _____ $i^7 =$ _____

2). $i^0 =$ _____ $i^{-1} =$ _____ $i^{-2} =$ _____ $i^{-3} =$ _____
$i^{-4} =$ _____ $i^{-5} =$ _____ $i^{-6} =$ _____ $i^{-7} =$ _____

3). Trigonometric functions: SOH-CAH-TOA
A complex number $z = x + iy$, with $r = \sqrt{x^2 + y^2}$, as shown in Fig. 4 (left).
Sine is opposite over hypotenuse (SOH), $\sin\theta =$ _____

Cosine is adjacent over hypotenuse (CAH), $\cos\theta \ = $ _____
Tangent is opposite over adjacent (TOA), $\tan\theta \ = $ _____

4). On the Bloch sphere in Fig. 4 (right), mark the four frequent *pure states*: "plus", "minus", "i", and "minus i".

5). **Exercise 2.3** (pp. 79) Identify where a qubit locates on the Bloch Sphere, shown in Fig. 5.

a). $|0\rangle$;

b). $|1\rangle$;

c). half $|0\rangle$ and half $|1\rangle$;

d). more $|0\rangle$ than $|1\rangle$;

e). more $|1\rangle$ than $|0\rangle$.

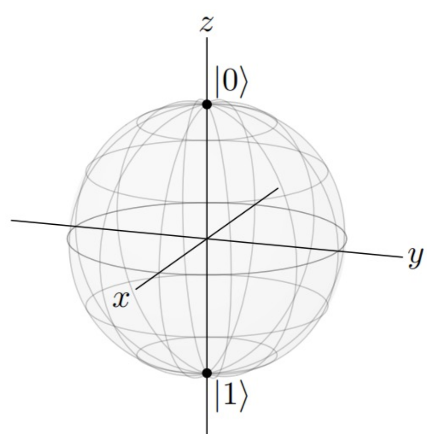


Figure 5: The Bloch sphere.

6). **Exercise 2.4** (pp. 83) Consider complex number $z = 1 + 2i$.

a). Find $\Re(z)$.

b). Find $\Im(z)$.

c). Plot z as a point in a complex plane.

d). Write z in a polar form $re^{i\theta}$.

e). Find z^*.

f). Find $|z|$.

g). Find $|z|^2$.

7). **Exercise 2.5** (pp. 83) Consider complex number $z = -3 - i$.

a). Find $\Re(z)$.

b). Find $\Im(z)$.

c). Plot z as a point in a complex plane.

d). Write z in a polar form $re^{i\theta}$. Hint: The angle should be between π and $3\pi/2$ (i.e., 180° and 270°).

e). Find z^*.

f). Find $|z|$.

g). Find $|z|^2$.

Outline

- Arithmetic and Algebra
- Trigonometry and Complex Numbers
- System and State

Notations

- Imaginary unit $\boxed{i = \sqrt{-1}}$
- $x, y, a, b \in \mathbb{R}$
- Euler's number $e \approx 2.71828...$
- Complex number $z \in \mathbb{C}$: $z = x + iy$, $z = a + ib$, or $z = re^{i\theta}$
- Conjugate $z^* = x - iy = re^{-i\theta}$, or $\bar{z}$,

Videos

Video [14] (3 min): Big Bang Theory: "Schrodinger's relationship" [15]

Short survey of our class:

- Remember trigonometric functions? Complex numbers?
- Familiar with Cartesian coordinates? Spherical coordinates?
- Euler's formula? Euler's identity
- Taken linear algebra? matrix operations? tensor product?
- Anyone did physical experiments?
- Know quantum physics?

[14] `https://www.youtube.com/watch?v=oerZnryFxX0`
[15] `https://www.youtube.com/watch?v=2c66_cha8kg`

Warm-Ups

- Find a solution to polynomial equations

$$x^2 - 1 = 0, \qquad\qquad x^2 + 1 = 0.$$

  Does $i = \sqrt{-1}$ bother you?

- **Euler's formula**. Do you see the beauty of $e^{i\pi} + 1 = 0$?
  How about $e^{ix} = \cos x + i \sin x$ for $x \in \mathbb{R}$? How to prove it?

- Are you comfortable with $|0\rangle$? How about $|10\rangle = |1\rangle \otimes |0\rangle$?
  How about $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ with $\alpha, \beta \in \mathbb{C}$, and $|\alpha|^2 + |\beta|^2 = 1$.

1 Arithmetic and Algebra

Let us start with our early childhood experiences of computing.
We denote x is an element of a set S as $x \in S$.

1.1 Arithmetic

Arithmetics we learned in primary school:

- $+$: natural numbers $\mathbb{N} = \{0, 1, 2, 3, ...\}$, $\mathbb{N}^+ = \{1, 2, 3, ...\}$
- $-$: integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$
- $\times$: e.g., $3 \times 7 = 21$
- $/$: rational numbers $\mathbb{Q} = \{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}^+\}$

Order of operations: we multiply (and divide) before adding (and substracting).

Recall the recursive definition of $\mathbb{N}^+$ in CSCI 2200 FOCS:

1). $1 \in \mathbb{N}$ [Base case]

2). $x \in \mathbb{N} \to x + 1 \in \mathbb{N}$ [Constructor]

1.2 Algebra

Algebra is the abstract encapsulation of our intuition for composition.

Examples

- The concept of *Unity*. The number 1. As a little baby, I learned to count by starting with 1.
- $\mathbb{N} = \{1, 2, 3, \cdots\}$, equipped with two natural operations $+$ and $\times$.
- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$. Additive inverses exist.
- $\mathbb{Q} = \{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}\}$,

Group and Abelian Group. Let G denote a set and $a \in G$ denote a is an element of G. A *group* $(G, \star)$ is a set that satisfies:

- Closure: a binary operation $\star : G \times G \to G$ such that $\forall a, b \in G$, $a \star b \in G$.
- Associativity: $(a \star b) \star c = a \star (b \star c)$, $\forall a, b, c \in G$.
- Existence of identity: $\exists\, e \in G$ such that $e \star a = a \star e = a$, $\forall a \in G$.
- Existence of inverses: Given $a \in G$, $\exists\, a^{-1} \in G$, such that $a \star a^{-1} = a^{-1} \star a = e$.

Example: $(\boldsymbol{N}, +)$ is not a group.

A group $(G, \star)$ is *Abelian* iff $\star$ is a commutative operation, i.e.,

$$a \star b = b \star a, \quad \forall a, b \in G. \tag{1}$$

Difference between Arithmetic and Algebra.

1.3 Boolean Algebra

{NOT, AND, OR} is a universal gate set. Bits: A, B, C

Order of operations: AND is done first, then OR.

Example: $A + BC = A + (BC)$

1.3.1 Association, Commutativity, and Distribution

Associative:

- ABC = (AB)C = A(BC)
- A + B + C = (A + B) + C = A + (B + C)

Commutative:

- AB = BA

S No.	Arithmetic	Algebra
1	It is the branch of mathematics that deals with numbers, their writing systems, and their properties.	It is the branch of mathematics that deals with variables and constants.
2	The operations are carried out with the help of the information provided.	The operations are carried out with the help of standard formulae and expressions.
3	It is generally applicable in real life and associated with elementary education.	Its direct application is not often observed in daily life and is associated with high school education.
4	It has four basic methods of operation (addition, subtraction, multiplication, and division).	It uses numbers, variables, and general rules or formulae to solve problems.
5	It is related to the numbers and number systems.	It is related to equations and formulae.

- A + B = B + A

Distributive:

- A(B+C) = AB + AC
- A + (BC) = (A + B)(A + C) (special for Boolean, check the truth table)

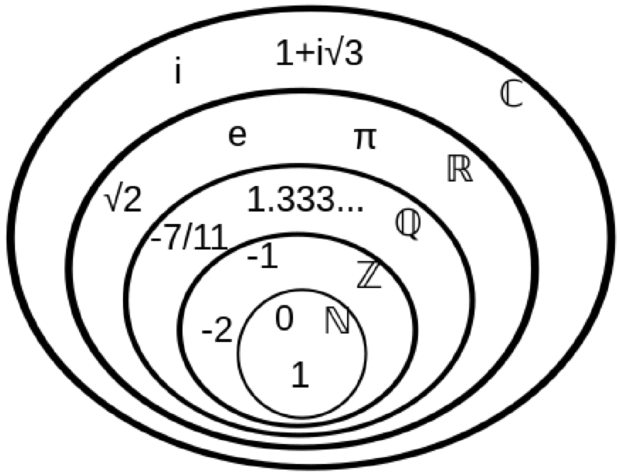
1.3.2 Identities Involving Zero and One

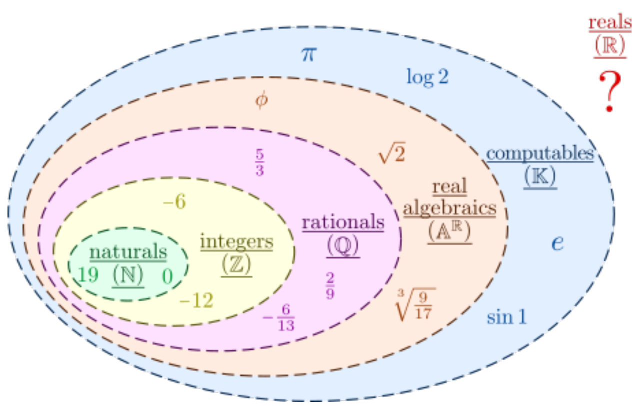
- A0 = 0
- A1 = A
- A + 0 = A
- A + 1 = 1

2 Real Numbers and Complex Numbers

A number is a mathematical object used to count, measure, and label.
Terms: number, numerals, digits

2.1 Real Numbers $\mathbb{R}$





$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$
Natural numbers: $\mathbb{N}$ (with 0), $\mathbb{N}^+$ (without 0)
Integers: $\mathbb{Z}$
Rational numbers: $\mathbb{Q}$
(real) Algebraics: $\mathbb{A}^{\mathbb{R}}$, such as $\sqrt{2}$.
transcendental numbers such as π and e
Computable numbers: $\mathbb{K}$
Real numbers: $\mathbb{R}$
Complex number: $\mathbb{C}$

2.2 Complex Numbers $\mathbb{C}$

Complex numbers arise from solving equation. Find a solution to polynomial equations

$$x^2 - 1 = 0, \qquad x^2 + 1 = 0. \tag{2}$$

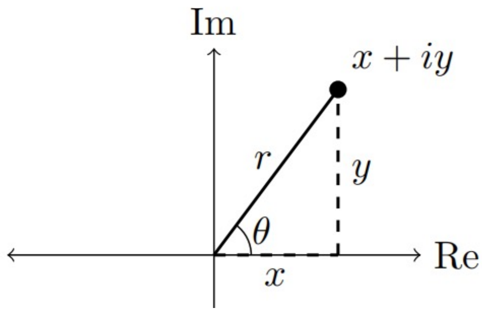


Figure 6: Complex plane.

We invented an *imaginary number* $\boxed{i = \sqrt{-1}}$, such that $i^2 = -1$.

Complex numbers are everywhere in quantum mechanics. Nowadays, they also have many applications in physics, chemistry, biology, electrical engineering, statistics, and even finance and economics.

Three forms for a complex number $z \in \mathbb{C}$:

$$z \;=\; \underbrace{x + iy}_{\text{Cartesian form}} \;=\; \underbrace{\cos\theta \;+\; i\sin\theta}_{\text{Polar Form}} \;=\; \underbrace{r\;\; e^{i\theta}}_{\text{Exponential Form}} \tag{3}$$

where $x, y, r, \theta \in \mathbb{R}$.

2.2.1 Cartesian Form

Every complex number can be uniquely represented in the Cartesian form

$$\boxed{z = x + iy, \quad \text{where } x = \text{Re}(z),\; y = \text{Im}(z)}, \tag{4}$$

where $i = \sqrt{-1}$ or $i^2 = -1$.

Examples: $z_1 = 1 + 3i$ and $z_2 = \frac{1}{\sqrt{2}}(1 + i) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

The *modulus* (a.k.a., length) of z is $|z| = \sqrt{x^2 + y^2}$. The modulus squared, $|z|^2 = x^2 + y^2$, is always a real, positive number.

The *complex conjugate* of $z = x + iy$ is

$$z^* = x - iy, \quad \text{replace } i \text{ with } -i. \tag{5}$$

Examples: $z_1^* = 1 - 3i$ and $z_2^* = \frac{1}{\sqrt{2}}(1 - i) = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

2.2.2 Polar Form (and Exponential Form)

The polar form

$$\boxed{z = re^{i\theta}, \quad \text{where } r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right)} \tag{6}$$

The conjugate is $z^* = re^{-i\theta}$.

Summary
Complex numbers $z_1 = a + bi$ and $z_2 = c + di$, or $z = a + bi$

- Imaginary unit: $\boxed{i = \sqrt{-1}}$
- Addition: $\boxed{z_1 + z_2 = (a + c) + (b + d)i}$
- Multiplication: $\boxed{z_1 z_2 = (ac - bd) + (ad + bc)i}$
- Conjugate: $\boxed{z^* = a - bi = re^{-i\theta}}$
- Modulus: $\boxed{|z| = r = \sqrt{zz^*} = \sqrt{z^* z} = \sqrt{a^2 + b^2}}$
- Euler's formula: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$
- Polar form: $\boxed{z = re^{i\theta}}$
- Periodicity: $\boxed{e^{i(\theta \pm 2\pi)} = e^{i\theta}}$

3 System and State

Classical States

A *system*: a collection of objects (or smaller systems) that can be identified. A *state* is a possible configuration of a system.

- in physics: a set of physical objects that are interrelated in complex ways. Ex: solar system; cities; machines. Three types of physical systems: isolated, closed, open.
  - Isolated: a system in which no matter or energy is being exchanged with surroundings.
  - Closed: a system in which only energy is being exchanged with surroundings.
  - Open: a system in which both matter and energy is being exchanged with surroundings.

- in chemistry: A specific portion of matter under study or observation, typically isolated from its surroundings for the purpose of analysis. Ex. Salt Solution, Closed Container Reaction, Biological System (e.g., enzymes in a cell).

Quantum State

- $|0\rangle$, $|1\rangle$ - Alignment with/Opposition to external field in Z-direction

A quantum state is a mathematical entity that embodies the knowledge of a quantum system. Quantum mechanics specifies the construction, evolution, and measurement of a quantum state. The result is a quantum-mechanical prediction for the system represented by the state. Knowledge of the quantum state, and the quantum mechanical rules for the system's evolution in time, exhausts all that can be known about a quantum system.

Quantum states may be defined differently for different kinds of systems or problems. Two broad categories are wave functions describing quantum systems using position or momentum variables and the more abstract vector quantum states (we focus on the later one in this class).

Quantum computer: takes a current state $|\psi\rangle$ to a new state; in the new state, we measure some observable that corresponds to the result.