HW1

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Question 1

(a)

$$\begin{aligned}
\|x-y\|^2 = \langle x-y, x-y\rangle &= \int_{-\infty}^{+\infty}(x-y)^2dt \\
&= \int_{-\infty}^{0}x^2(t)dt + \int_{T}^{+\infty}x^2(t)dt + \int_0^T[x(t)-c]^2dt \\
&= \int_{-\infty}^{+\infty}x^2(t)dt + \int_0^T(-2cx(t))dt + \int_0^T c^2dt \\
&= \int_{-\infty}^{+\infty}x^2(t)dt + \int_0^T(-2cx(t))dt + c^2T
\end{aligned}$$

$$\min_c \|x-y\|^2 = \min_c\left(\int_0^r -2cx(t)dt + c^2T\right)$$

It's a convex quadratic function of C.
When $c = \frac{\int_0^T x(t)dt}{T}$, $\|x-y\|^2$ get the minima.

(b)

$$e(t) = \begin{cases} c & 0<t<T \\ 0 & o.w. \end{cases}$$

$$\langle x(t), e(t)\rangle = \int_0^T x(t)\cdot cdt \quad x = u\cdot\cos\alpha.$$

$$\|e(t)\| = c^2T,$$

$$y(t) = \frac{\langle x(t), e(t)\rangle}{\|e(t)\|} = \frac{c\int_0^T x(t)dt}{c^2T}\cdot c = \frac{\int_0^T x(t)dt}{T}$$

(c)

Obviously.And I prefer the first one because it starts from the original definition of projection rather than the Kindergarten formula.

Question2

(a)

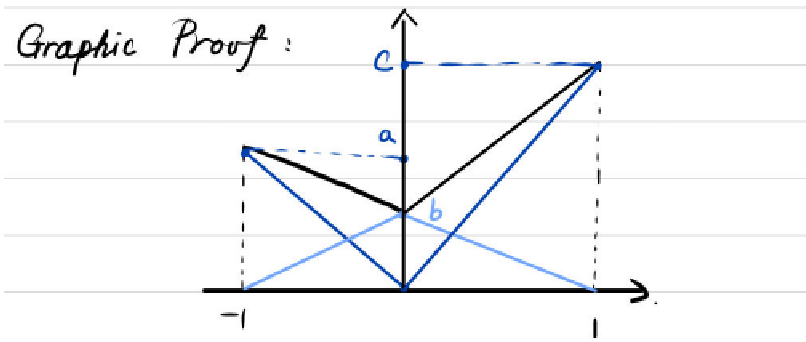

Figure 1: Graphic Proof

$\forall f \in V, f = f(-1) \cdot \varphi_{-1} + f(0) \cdot \varphi_0 + f(1) \cdot \varphi_1$

(b)

$\varphi_{-1}(t)$ and $\varphi_1(t)$

Because $\langle \varphi_{-1}, \varphi_1 \rangle = \int_{-1}^{1} \varphi_{-1}(t) \cdot \varphi_1(t) dt = 0$

(c)

Kindergarten formula:$P_{v_1}(\varphi_0) = \langle \varphi_0, \varphi_{-1} \rangle \varphi_{-1} + \langle \varphi_0, \varphi_1 \rangle \varphi_1$

$$\langle \varphi_0, \varphi_{-1} \rangle = \int_{-1}^{0} -t \cdot (t+1) dt = \frac{1}{6}$$

$$\langle \varphi_0, \varphi_1 \rangle = \int_{0}^{1} t(1-t) dt = \frac{1}{6}$$

$$P_{v_1}(\varphi_0) = \frac{1}{6} \cdot \varphi_{-1}(t) + \frac{1}{6} \varphi_1(t)$$

$$\hat{\varphi}_0 = \varphi_0 - P_{V_1}(\varphi_0)$$

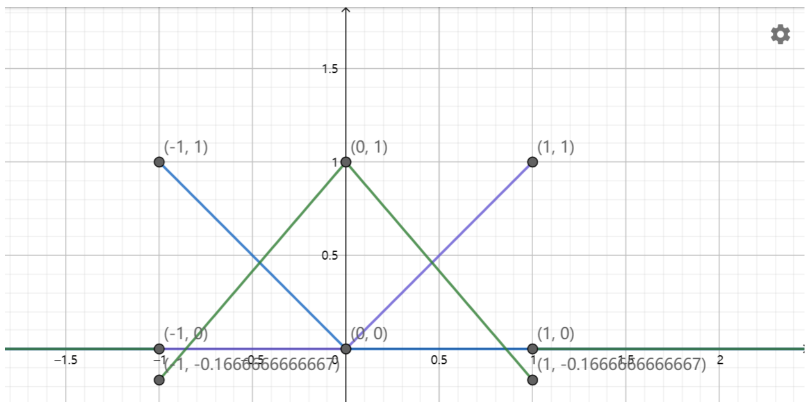

Figure 2: 2(c)

(d)

Suppose $f(t) = a\varphi_{-1}(t) + b\varphi_0(t) + c\varphi_1$

Best approximation: min $\|g(t) - f(t)\|^2$ $\|g(t) - f(t)\|^2 = \langle g - f, g - f \rangle$

$$= \int_{-1}^{0} (1 + at)^2 dt + \int_{0}^{\frac{2}{3}} (1 - bt)^2 dt + \int_{\frac{2}{3}}^{1} (bt)^2 dt + \cdots$$

$$= -\frac{5}{3} + \left(\frac{a^2}{3} - a\right) + \left(\frac{2}{3}b^2 - \frac{17}{9}b\right) + \left(\frac{1}{3}c^2 - \frac{4}{9}c\right) + \frac{ab}{3} + \frac{bc}{3}$$

When $a = \frac{11}{12}, b = \frac{7}{6}, c = \frac{1}{12}$, $\|g - f\|^2$ get global minima.

$$\hat{g}(t) = \frac{11}{12}\varphi_{-1} + \frac{7}{6}\hat{\varphi}_0 + \frac{1}{12}\varphi_1$$

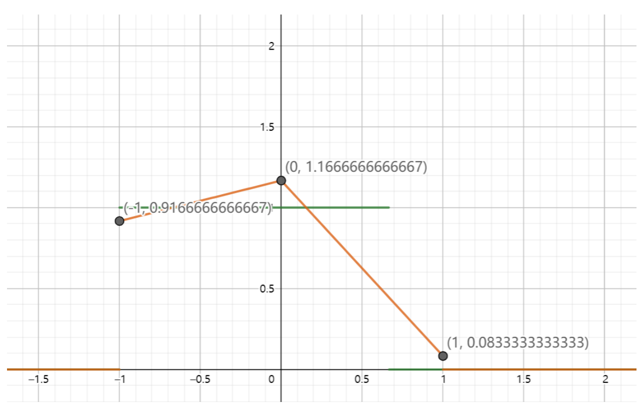

Figure 3: 2(d)