Problem Set 1

February 22, 2022

Deadline: 13:00 March 8, 2022

Question 1

In the real signal space $L^2(\mathbb{R})$ with the inner product $\langle \mathrm{f}, \mathrm{g}\rangle = \int_{-\infty}^{\infty} f(t)g(t)\mathrm{d}t$, consider the subspace V of functions $\mathrm{f} = (f(t))_{t\in\mathbb{R}}$ such that $f(t)$ is constant in $[0, T)$ and zero outside. For any $\mathrm{x} \in L^2(\mathbb{R})$, the goal is to determine the orthogonal projection $P_V(\mathrm{x})$ of x onto the space V.

(a) Determine $y = P_V(x)$ according to the following method. Since $y \in V, y(t)$ must be equal to a constant c in $[0, T)$ and zero outside. Express $\|\mathrm{x} - \mathrm{y}\|^2$ in terms of $x(t)$ and c. Then, express the value c that minimizes the distance between x and y, in terms of $x(t)$.

(b) Determine $\mathrm{y} = P_V(\mathrm{x})$ according to the following second method. Show that V can be presented as the span of a single function $\mathrm{e} = (e(t))_{t\in\mathbb{R}}$ where you have to describe $e(t)$. Then find y by pure inner-product manipulation, in a way similar to an example covered in class in $\mathbb{R}^2$.

(c) Show that the two methods give the same result. Which method do you prefer?

Question 2

In this problem, all numbers must be *exact*, no approximate decimal number will be accepted. Consider the Euclidean space $E = L^2([-1, 1])$ of real signals with the inner product

$$\langle \mathrm{f}, \mathrm{g}\rangle=\int_{-1}^{1} f(t)g(t)\mathrm{d}t.$$

We call V the subspace of the functions $\mathrm{f} = (f(t))_{t\in[-1,1]}$ that are linear on the two intervals $[-1,0]$ and $[0,1]$ while being continuous at 0.

(a) Consider any $\mathrm{f} \in V$. Show that f can be expressed as a linear combination of the three functions in Figure 1. Express the coefficients of the linear combination in terms of f.
Hint: Work graphically.

(b) With question **(a)**, you have proved that $V = Span\{\varphi_{-1}, \varphi_0, \varphi_1\}$. Among the functions φ_{-1}, φ_0 and φ_1, which pairs of two functions are orthogonal ?

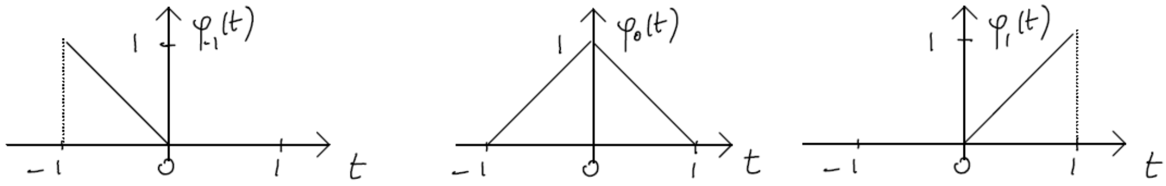

Figure 1

(c) Define $V_1 = Span\{\varphi_{-1}, \varphi_1\}$. Find the funtion

$$\hat{\varphi}_0 = \varphi_0 - P_{V_1}(\varphi_0).$$

What can you say about the family of functions $\{\varphi_{-1}, \hat{\varphi}_0, \varphi_1\}$. Plot these three functions on $[-1,1]$ on the same graph. Make sure to write their values at -1,0,1.

(d) Consider the function

$$g(t) = \begin{cases} 1, & t \in [-1, \frac{2}{3}) \\ 0, & t \in [\frac{2}{3}, 1]. \end{cases}$$

Find the best approximation of $g(t)$ by a function $f(t)$ of V with respect to the norm of E. Plot $g(t)$ and $f(t)$ on the same graph with exact scale. Make sure you show the values of $f(-1), f(0)$ and $f(1)$ on the graph.
Hint 1: what is another interpretation for "best approximation by a vector in a subspace".
Hint 2: most of the integrals can be calculated graphically.