Minkowski Functionals

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Overview

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Background

- Goal:bring new constraints on inflation theory and primordial non Gaussianities.
- Non-Gaussianity parameters:f_{NL}, g_{NL}(non-linear 'local' coupling parameters)
- bi-spectrum measurements—$f_{NL} = 32 \pm 21$(68% confidence level)
- one of the alternative statistics to the bi-spectrum:Minkowski functionals(pros&cons)
- Minkowski functionals describe the morphological features of random fields over excursion sets

Two basic theorems

- d-dimensional Euclidean space
  Hadwiger's Theorem:
  under a few simple requirements,any morphological descriptor is a linear combination of only d+1 functionals;these are the so-called Minkowski functionals V_j,with j ranging from 0 to d.

$$V_0(Q) = \int_Q dv$$

$$V_j(Q) = \frac{1}{\omega_{j-1} C_d^j} \int_{\partial Q} ds E_j(\kappa_1 ... \kappa_{d-1})$$

$E_1 = 1, E_2 = \kappa_1 + ... + \kappa_{d-1}$,and $E_d = \kappa_1 ... \kappa_{d-1}$

$$\omega_j = \frac{2\pi^{(j+1)/2}}{\Gamma((j+1)/2)}$$

Two basic theorems

- Spaces of constant curvature

  Gauss-Bonnet Theorem:the Euler characteristic is a linear combination of all Minkowski functionals:

$$\chi(Q) = \sum_{j=0}^{d} c_j V_j(Q)$$

If d-j even:$c_j = C_d^j \frac{2(kK)^{(d-j)/2}}{\omega_{d-j}}$

If d-j odd:$c_j = 0$

Two-dimensional unit sphere

- Two-dimensional spherical surface(which we care about):consider a smooth scalar field $u(x)$,for example the temperature anisotropies of the Microwave sky.
- Any morphological property can be expanded as a linear conbination of three MFs.
- For a given threshold ν,define the excursion set Q_ν and its boundary ∂Q_ν of a smooth scalar field u as follows:

$$Q_\nu = \{x \in S^2 | u(x) > \nu\}$$

$$\partial Q_\nu = \{x \in S^2 | u(x) = \nu\}$$

- some formulas(how to obtain them?)

Three MFs can be written as follows:

First MF represents the area of Q_ν

- $v_0(\nu) = \int_{Q_\nu} \frac{da}{4\pi}$
- In a pixelized map: $v_0(\nu) = \frac{1}{Npix} \sum_{k=1}^{Npix} \Theta(u-\nu)$ ($\Theta(x)$ is Heaviside step function)

Second MF represents the circumference of Q_ν

- $v_1(\nu) = \int_{\partial Q_\nu} \frac{dl}{16\pi}$
- In a pixelized map: $v_1(\nu) = \frac{1}{Npix} \sum_{k=1}^{Npix} I_1(\nu, x_k)$
- $I_1(\nu, x_k) = \frac{\delta(u-\nu)}{4} \sqrt{u_{;\theta}^2 + u_{;\phi}^2}$

Three MFs can be written as follows:

Third MF represents the integrated geodetic curvature of Q_ν

- $v_2(\nu) = \int_{\partial Q_\nu} \frac{kdl}{8\pi^2}$
- In a pixelized map: $v_2(\nu) = \frac{1}{Npix} \sum_{k=1}^{Npix} I_2(\nu, x_k)$
- $I_2(\nu, x_k) = \frac{\delta(u-\nu)}{2\pi} \frac{2u_{;\theta} u_{;\phi} u_{;\theta\phi} - u_{;\theta}^2 u_{;\phi\phi} - u_{;\phi}^2 u_{;\theta\theta}}{u_{;\theta}^2 + u_{;\phi}^2}$

Algorithm used to compute MFs

STEP 1:Use synfast

- Use synfast(a subroutine in Healpix) to generate first and second derivatives of B-map.
- Define a double array named Mink with three columns
- First column saves the value of B in every pixel
- Second column saves the results of $\sqrt{u_{;\theta}^2 + u_{;\phi}^2}$ in every pixel.
- Third column saves the results of $\frac{2u_{;\theta}u_{;\phi}u_{;\theta\phi} - u_{;\theta}^2 u_{;\phi\phi} - u_{;\phi}^2 u_{;\theta\theta}}{u_{;\theta}^2 + u_{;\phi}^2}$ in every pixel.

Algorithm used to compute MFs

STEP 2:Calculate three statistics of B-map

- Expectation value of B : $\mu_B = \sum_{i=0}^{Npix-1} \frac{Mink[i,0]}{Npix}$
- Variance of B : $\sigma_B = \frac{\sum_{i=0}^{Npix-1} Mink^2[i,0]}{Npix} - \mu_B^2$
- Variance of the gradient of B : $\tau_B = \frac{1}{2}\frac{\sum_{i=0}^{Npix-1} Mink^2[i,1]}{Npix}$

STEP 3

Give the array of discrete threshold $\nu[i] = 0.25i - 3$,width $\Delta\nu = 0.25$.

Algorithm used to compute MFs

STEP 4:Calculate three MFs

Make a new variable derivated by B which we use it as a smoothed scalar field : $u_B[i] = \frac{Mink[i,0]-\mu_B}{\sqrt{\sigma_B}}$ in every pixel.
(We use this variable not just B to calculate MFs because the given thresholds are symmetric and the center is zero.The expectation value of new variable equals zero and its variance equals one.It's convenient to compare with the standard MFs)

Algorithm used to compute MFs

First MF

Algorithm

Outer loop: for i=0 to Npix-1

if $-3.0 < u_B[i] < 3.0$

$k1 = [\frac{u_B[i]+3.0}{\Delta\nu}]$(k1 = the maximal serial number of which the value of B in this pixel greater than threshold)

Inner loop: for j=0 to k1

$v_0[j] += \frac{1}{Npix}$

Algorithm used to compute MFs

Second and Third MFs

Algorithm

Outer loop: for i=0 to Npix-1

if $-3.125 < u_B[i] < 3.125$

(here we set an offset $\chi = \frac{1}{8} = 0.125$ to offset the numerical errors in calculating derivatives)

$k2 = [\frac{u_B[i]+3.125}{\Delta\nu}]$(find the boundary)

$v_1[k2]+ = \frac{Mink[i,1]}{4\Delta\nu\sqrt{\sigma_B}Npix}$

(In numerical computations,the delta function can be numerically approximated through Heaviside step function:

$\delta(u-\nu) = \frac{\Theta(u-\nu+\frac{\Delta\nu}{2})-\Theta(u-\nu-\frac{\Delta\nu}{2})}{\Delta\nu} = \frac{1}{\sqrt{\sigma_B}\Delta\nu}$)

$v_2[k2]+ = \frac{Mink[i,2]}{2\pi\Delta\nu\sqrt{\sigma_B}Npix}$

Standard Minkowski Functionals

- Use figures generated by standard MFs to compare with the results of simulation,and then make sure that our codes are correct.
- MFs calculated by standard Gaussian field is regarded as Standard Minkowski Functionals
- Standard Gaussian field(one-dimension):

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \mu = 0, \sigma = 1$$

- In the no lensing and full sky simulation,B-mode should be Gaussian field just caused by primordial GW.So the results of the simulation should be same as Standard MFs.

Standard Minkowski Functionals

Analytical Algorithm

For MFs,the expectation values for a Gaussian random field is given by

- $< v_0^g(\nu) >= \frac{1}{2}(1 - erf(\frac{\nu-\mu}{\sqrt{2}\sigma}))$
- $< v_1^g(\nu) >= \frac{1}{8}\frac{\sqrt{\tau}}{\Delta\nu}\sqrt{\frac{\pi}{2}}[erf(\frac{\nu-\mu+\Delta\nu/2}{\sqrt{2}\sigma}) - erf(\frac{\nu-\mu-\Delta\nu/2}{\sqrt{2}\sigma})]$
- $< v_2^g(\nu) >= \frac{1}{(2\pi)^{3/2}}\frac{\tau}{\sigma^2}\frac{\sigma}{\Delta\nu}[\exp(-\frac{(\nu-\mu-\Delta\nu/2)^2}{2\sigma^2}) - \exp(-\frac{(\nu-\mu+\Delta\nu/2)^2}{2\sigma^2})]$
- Normal Gaussian Random Field:
  $\mu = 0, \sigma = 1, \tau = \frac{1}{2} < |\nabla u|^2 >$
  (HOW TO CALCULATE τ?)

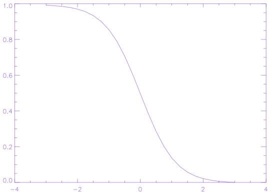
Standard Minkowski Functionals

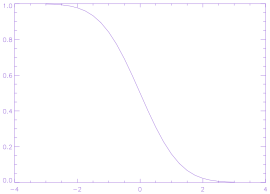
Numerical Algorithm

- Step1: C_l^{BB} generated by CAMB
- Step2: use gaussbeam function to smooth the power spectrum(parameter is 11.2)
  "beam1=gaussbeam(11.2,1024,4)
  cl[l]=cl[l]*beam1[l,2]*beam1[l,2]"
- Step3:calculate $\sigma \& \tau$
  $\sigma = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)C_l$
  $\tau = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)C_l\frac{l(l+1)}{2}$
- Step4:the expectation values for MFs
  $< v_0^g(\nu) >= \frac{1}{2}(1 - erf(\frac{\nu-\mu}{\sqrt{2\sigma}}))$
  $< v_1^g(\nu) >= \frac{1}{8}\sqrt{\frac{\tau}{\sigma}}\exp(-\frac{(\nu-\mu)^2}{2\sigma})$
  $< v_2^g(\nu) >= \frac{\tau(\nu-\mu)}{(2\pi\sigma)^{3/2}}\exp(-\frac{(\nu-\mu)^2}{2\sigma})$

Results of the simulation(left)& Standard MFs(right)

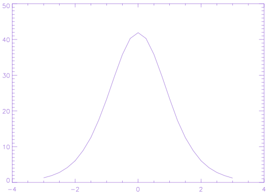
First Minkowski Functional of B-mode

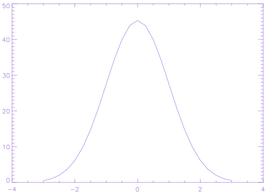




Results of the simulation(left)& Standard MFs(right)

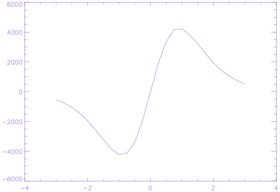
Second Minkowski Functional of B-mode

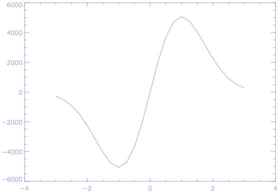




Results of the simulation(left)& Standard MFs(right)

Third Minkowski Functional of B-mode





Thank You