Notes on ESPRIT me21321ods for large array processing

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1 Introduction

In this pa123123per, we would like to evaluate the performance of the estimation of Direction of Arrival (DoA) method ESPRIT [**?**], the idea of which is described in the following section.

2 System Model

For a Unitary Linear Array (ULA) of size N, we consider the following model for the signal received at time $t = 1, \ldots, T$

$$\mathbf{x}(t) = \sum_{\ell=1}^{K} \mathbf{a}(\theta_\ell) s_\ell(t) + \mathbf{z}(t) \in \mathbb{C}^N \tag{1}$$

with $\mathbf{a}(\theta_\ell) \in \mathbb{C}^N$ the steering vector of source s_ℓ at angle of arrival θ_ℓ, its j-th entry given by[1]

$$[\mathbf{a}(\theta_\ell)]_j = \frac{1}{\sqrt{N}} e^{\imath \frac{2\pi d}{\lambda_0}(j-1)\sin(\theta_\ell)} \equiv \frac{1}{\sqrt{N}} e^{\imath \omega (j-1)\sin(\theta_\ell)}, \quad \omega \equiv \frac{2\pi d}{\lambda_0} \tag{2}$$

where there is in total k signal sources $\{s_\ell\}_{\ell=1}^K$, at angle $\{\theta_\ell\}_{\ell=1}^K$ for some $K \ll \min(N, T)$, as well as some independent Gaussian noise $\mathbf{z}(t) \overset{i.i.d.}{\sim} \mathcal{CN}(\mathbf{0}, \mathbf{I}_N)$ for all t.

The above signal model can be rewritten in matrix model by cascading the total T observations as

$$\mathbf{X} = \mathbf{AS} + \mathbf{Z} \tag{3}$$

with $\mathbf{X} = [\mathbf{x}(1), \ldots, \mathbf{x}(T)] \in \mathbb{C}^{N \times T}$, $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_k)] \in \mathbb{C}^{N \times K}$, $\mathbf{S} = [\mathbf{s}(1), \ldots, \mathbf{s}(T)] \in \mathbb{C}^{K \times T}$, column vector $\mathbf{s}(t) = [s_1(t), \ldots, s_k(t)]^\mathsf{T} \in \mathbb{C}^K$ and $\mathbf{Z} = [\mathbf{z}(1), \ldots, \mathbf{z}(T)] \in \mathbb{C}^{N \times T}$ a standard (circular) Gaussian random matrix.

3 ESPRIT method for DoA estimation

In this paper, we would like to evaluate the performance of the estimation of Direction of Arrival (DoA) method ESPRIT [**?**], the idea of which is described as follows.

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[1] The normalization by $\sqrt{N}$ here is for notational convenience so that $\mathbf{a}(\theta_\ell)$ is of unit norm, note that this is equivalent to a *rescaling* of the source signal s_ℓ.

ESPRIT: intuition. Note that the *population* covariance of received signal

$$\begin{aligned}\mathbf{C} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}^{\mathsf{H}}(t)] &= \mathbb{E}[(\mathbf{A}\mathbf{s}(t) + \mathbf{z}(t))(\mathbf{A}\mathbf{s}(t) + \mathbf{z}(t))^{\mathsf{H}}] \\ &= \mathbf{A}\mathbb{E}[\mathbf{s}(t)\mathbf{s}^{\mathsf{H}}(t)]\mathbf{A}^{\mathsf{H}} + \mathbb{E}[\mathbf{z}(t)\mathbf{z}^{\mathsf{H}}(z)] \\ &= \mathbf{A}\mathbf{P}(t)\mathbf{A}^{\mathsf{H}} + \sigma^2\mathbf{I}_N\end{aligned}$$

where we used the fact that $\mathbf{z}(t)$ is independent of the signal $\mathbf{s}(t)$ and denote the signal power $\mathbf{P}(t) \equiv \mathbb{E}[\mathbf{s}(t)\mathbf{s}^{\mathsf{H}}(t)]$. Then, for diagonal $\mathbf{P}(t) = \operatorname{diag}\{p_\ell(t)\}_{\ell=1}^k$ (which implies uncorrelated signal in the Gaussian case), one has

$$\mathbf{C} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}^{\mathsf{H}}(t)] = \sum_{k=1}^{K} p_k(t)\mathbf{a}(\theta_k)\mathbf{a}^{\mathsf{H}}(\theta_k) + \sigma^2\mathbf{I}_N = \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}} + \sigma^2\mathbf{I}_N, \tag{4}$$

so that the top subspace of *population* covariance is expected to obtain structure information about the subspace spanned by the steering vectors $\mathbf{a}(\theta_k)$. *If the sample covariance* $\hat{\mathbf{C}}$ *is a good "proxy" of the population* $\mathbf{C}$ in the sense that, e.g.,

$$\|\hat{\mathbf{C}} - \mathbf{C}\| \to 0 \tag{5}$$

in spectral norm, then, one has, by Davis–Kahan theorem that

$$\|\hat{\mathbf{U}}_S - \mathbf{U}_S\|_F \to 0, \tag{6}$$

(in fact, this holds for each individual eigenvector).

On the other hand, using the rotational invariance of the matrix $\mathbf{A}$, we have, for two selection matrices $\mathbf{J}_1, \mathbf{J}_2 \in \mathbb{R}^{n\times N}$ that selection n among the in total N rows of $\mathbf{X}$, with "distance " Δ, that

$$\mathbf{J}_1\mathbf{A}\operatorname{diag}\{e^{\imath\omega\Delta\cdot\sin(\theta_\ell)}\}_{k=1}^K = \mathbf{J}_2\mathbf{A} \tag{7}$$

with

$$\mathbf{J}_1 = \begin{bmatrix} \mathbf{e}_k^{\mathsf{T}} \\ \cdots \\ \mathbf{e}_{n+k-1}^{\mathsf{T}} \end{bmatrix} \in \mathbb{R}^{n\times N}, \quad \mathbf{J}_2 = \begin{bmatrix} \mathbf{e}_{k+\Delta}^{\mathsf{T}} \\ \cdots \\ \mathbf{e}_{n+k+\Delta-1}^{\mathsf{T}} \end{bmatrix} \in \mathbb{R}^{n\times N} \tag{8}$$

for $\mathbf{e}_k$ the canonical vector of $\mathbb{R}^N$ with $[\mathbf{e}_k]_i = \delta_{ij}$. We take, without loss of generality, $k = 1$ here so that

$$\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1 = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{N-n} \\ \mathbf{0}_{N-n} & \mathbf{0}_{N-n} \end{bmatrix}, \quad \mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2 = \begin{bmatrix} \mathbf{0}_{n\times\Delta} & \mathbf{I}_n & \mathbf{0}_{n\times(N-n-\Delta)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{9}$$

so that with $\mathbf{U}_S = \mathbf{A}\mathbf{Q}^{-1}$ for some invertible $\mathbf{Q} \in \mathbb{C}^{K\times K}$ and $\hat{\mathbf{U}}_S \simeq \mathbf{U}_S$, we obtain

$$\mathbf{Q}\operatorname{diag}\{e^{\imath\omega\Delta\cdot\sin(\theta_\ell)}\}_{\ell=1}^k\mathbf{Q}^{-1} = (\mathbf{J}_1\mathbf{U}_S)^\dagger\mathbf{J}_2\mathbf{U}_S \simeq (\mathbf{J}_1\hat{\mathbf{U}}_S)^\dagger\mathbf{J}_2\hat{\mathbf{U}}_S. \tag{10}$$

ESPRIT: algorithm.

1. define two selection matrices $\mathbf{J}_1, \mathbf{J}_2 \in \mathbb{R}^{n\times N}$ that selection n among the in total N rows of $\mathbf{X}$, with "distance " Δ, for instance, $\mathbf{J}_1$ from row i to $i+n$, and $\mathbf{J}_2$ from $i+\Delta$ to $i+\Delta+n$;

2. compute the sample covariance matrix $\hat{\mathbf{C}} = \frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} = \frac{1}{T}\sum_{t=1}^T \mathbf{x}(t)\mathbf{x}^{\mathsf{H}}(t)$, and denote $\hat{\mathbf{U}}_S$ the top subspace composed of the eigenvectors associated to the largest-k eigenvalues, or the so-called *signal subspace*;

3. compute $\mathbf{\Phi} = (\mathbf{J}_1 \hat{\mathbf{U}}_S)^\dagger \mathbf{J}_2 \hat{\mathbf{U}}_S \in \mathbb{C}^{K \times K}$, where we denote $\mathbf{A}^\dagger$ the Moore–Penrose pseudoinverse of $\mathbf{A}$, note that the resulting matrix $\mathbf{\Phi}$ is, in general, non-Hermitian;

4. the estimate of the angles of $\hat{\theta}_k$ are given by

$$\hat{\theta}_k = \arcsin(\arg(\lambda_k(\mathbf{\Phi}))/\omega/\Delta), \tag{11}$$

with λ_k the kth (complex) eigenvalues of $\mathbf{\Phi}$.

4 Characterization of ESPRIT method for large linear array

It follows from (**??**) that, for selection matrix $\mathbf{J}_1$ such that $\mathbf{J}_1 \mathbf{U}_S$ has linearly independent columns so that the inverse $(\mathbf{U}_S^H \mathbf{J}_1^H \mathbf{J}_1 \mathbf{U}_S)^{-1}$ is well defined, we have

$$\operatorname{diag}\{e^{\imath \omega \Delta \cdot \sin(\theta_\ell)}\}_{k=1}^K = (\mathbf{A}^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_1 \mathbf{A})^{-1} \mathbf{A}^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_2 \mathbf{A}$$

where we assume that the selection matrix It thus suffices to evaluate the two terms $\mathbf{A}^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_1 \mathbf{A}$ and $\mathbf{A}^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_2 \mathbf{A}$ so as to retrieve the DoA θ_k as desired.

We positive ourselves under the following large array scenario.

Assumption 1 (Large array). *As $T \to \infty$, we have that*

$$0 < \liminf_T N/T < \limsup_T N/T < \infty, \quad 0 < \liminf_T n/N < \limsup_T n/N < 1. \tag{12}$$

Assumption 2 (Widely spaced DoA). *All DoA angles $\theta_1, \ldots, \theta_K$ are* fixed *as $N, T \to \infty$.*

The *widely spaced* DoA scenario as **??** practically arises, e.g., when the DoA have an angular separation much larger than a beamwidth [**?**], by considering the case of all DoAs $\theta_1, \ldots, \theta_K$ are *fixed* with respect to N large. In this case, we have in particular that

$$\begin{aligned}
[\mathbf{A}^\mathsf{H} \mathbf{A}]_{ij} = \mathbf{a}(\theta_i)^\mathsf{H} \mathbf{a}(\theta_j) &= \frac{1}{N} \sum_{\ell=1}^N e^{-\imath \omega (\ell-1)(\sin(\theta_j) - \sin(\theta_i))} \\
&= \begin{cases} 1 & \text{for } i = j \\ \frac{1}{N} \frac{1 - e^{-\imath\omega(\ell-1)N(\sin(\theta_j)-\sin(\theta_i))}}{1 - e^{-\imath\omega(\ell-1)(\sin(\theta_j)-\sin(\theta_i))}} = O(N^{-1}) & \text{for } i \neq j. \end{cases}
\end{aligned}$$

$$[\mathbf{A}^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_1 \mathbf{A}]_{ij} = \mathbf{a}(\theta_i)^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_1 \mathbf{a}(\theta_j) = \begin{cases} \frac{n}{N}, & \text{for } i = j; \\ O(N^{-1}), & \text{for } i \neq j \end{cases} \tag{13}$$

as well as

$$[\mathbf{A}^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_2 \mathbf{A}]_{ij} = \mathbf{a}(\theta_i)^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_1 \mathbf{a}(\theta_j) = \begin{cases} \frac{n}{N} e^{-\imath\omega\Delta \sin(\theta_i)}, & \text{for } i = j; \\ O(N^{-1}), & \text{for } i \neq j. \end{cases} \tag{14}$$

As such, under **??** and **??**, we have, in matrix form that

$$\mathbf{A}^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_1 \mathbf{A} = \begin{bmatrix} \mathbf{a}(\theta_1)^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_1 \mathbf{a}(\theta_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \mathbf{a}(\theta_K)^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_1 \mathbf{a}(\theta_K) \end{bmatrix} + O_{\|\cdot\|}(N^{-1}), \tag{15}$$

and similarly

$$\mathbf{A}^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_2 \mathbf{A} = \begin{bmatrix} \mathbf{a}(\theta_1)^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_2 \mathbf{a}(\theta_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \mathbf{a}(\theta_K)^\mathsf{H} \mathbf{J}_1^\mathsf{H} \mathbf{J}_2 \mathbf{a}(\theta_K) \end{bmatrix} + O_{\|\cdot\|}(N^{-1}). \tag{16}$$

As a consequence, we get

$$\mathbf{Q}^{-1}\,\mathrm{diag}\{e^{\imath\omega\Delta\cdot\sin(\theta_k)}\}_{k=1}^{K}\mathbf{Q} = (\mathbf{Q}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_1\mathbf{A}\mathbf{Q})^{-1}\mathbf{Q}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_2\mathbf{A}\mathbf{Q} = (\mathbf{U}_S^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_1\mathbf{U}_S)^{-1}\mathbf{U}_S^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_2\mathbf{U}_S, \tag{17}$$

holds for some invertible matrix $\mathbf{Q} \in \mathbb{C}^{K\times K}$ such that

$$\mathbf{U}_S = \mathbf{A}\mathbf{Q}. \tag{18}$$

[Zhenyu: The above claim to clarify!]

The above result illustrates two cases, one is when N tends to infinity, the elements on the off-diagonal are 0 and the division of the elements on the diagonal is the angle we want

$$\begin{aligned}\mathbf{Q}\,\mathrm{diag}\{e^{\imath\omega\Delta\cdot\sin(\theta_\ell)}\}_{\ell=1}^{k}\mathbf{Q}^{-1} &= (\mathbf{J}_1\mathbf{U}_S)^{\dagger}\mathbf{J}_2\mathbf{U}_S.\\ &= (diag(\mathbf{U}_S^{H}\mathbf{J}_1^{H}\mathbf{J}_1\mathbf{U}_S))\backslash diag(\mathbf{U}_S^{H}\mathbf{J}_1^{H}\mathbf{J}_2\mathbf{U}_S)\end{aligned}$$

In practice we cannot correctly estimate the u_i,and we tend to approximate u_i by $\hat{u}_i$,which leads to an error.

We need to find two functions $u_i^H\mathbf{J}_1^H\mathbf{J}_1u_i = f_1(\hat{u}_i^H\mathbf{J}_1\mathbf{J}_1^H\hat{u}_i), u_i^H\mathbf{J}_1^H\mathbf{J}_2u_i = f_2(\hat{u}_i^H\mathbf{J}_1^H\mathbf{J}_2\hat{u}_i)$, We need to approximate the real eigenvector by these two functions

If we can perfectly estimate the true eigenvector,

$$(diag(\mathbf{A}^H\mathbf{J}_1^H\mathbf{J}_1\mathbf{A}))\backslash diag(\mathbf{A}^H\mathbf{J}_1^H\mathbf{J}_2\mathbf{A}) = \mathrm{diag}\{e^{\imath\omega\Delta\cdot\sin(\theta_\ell)}\}_{\ell=1}^{k}$$

5 Simplified model with single DoA

In the case of $k = 1$ with angle θ, we have the following simplified model

$$\mathbf{X} = \mathbf{a}(\theta)[s(1),\ldots,s(T)] + \mathbf{Z} \equiv \mathbf{a}(\theta)\mathbf{s}^{\mathsf{H}} + \mathbf{Z} \tag{19}$$

so that

$$\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} = \frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} + \begin{bmatrix}\mathbf{a} & \frac{\mathbf{Z}\mathbf{s}}{T}\end{bmatrix}\begin{bmatrix}\rho & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\mathbf{a}^{\mathsf{H}}\\ \frac{(\mathbf{Z}\mathbf{s})^{\mathsf{H}}}{T}\end{bmatrix} \equiv \frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} + \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{\mathsf{H}} \tag{20}$$

where we denote $\rho = \frac{1}{T}\|\mathbf{s}\|^2$ the (limit of the normalized) signal strength and $\mathbf{V} = \begin{bmatrix}\mathbf{a} & \frac{\mathbf{Z}\mathbf{s}}{T}\end{bmatrix} \in \mathbb{C}^{N\times 2}$.

As a consequence,

$$\begin{aligned}\left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} &= \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} + \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1}\\ &= \mathbf{Q} - \mathbf{Q}\mathbf{V}\boldsymbol{\Lambda}(\mathbf{I}_2 + \mathbf{V}^{\mathsf{H}}\mathbf{Q}\mathbf{V}\boldsymbol{\Lambda})^{-1}\mathbf{V}^{\mathsf{H}}\mathbf{Q}\end{aligned}$$

where we denoted $\mathbf{Q}(z) = \mathbf{Q} = (\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}$ and used Woodbury matrix identity. Now, since

$$\mathbf{V}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{V} = \begin{bmatrix}\mathbf{a}^{\mathsf{H}}\\ \frac{(\mathbf{Z}\mathbf{s})^{\mathsf{H}}}{T}\end{bmatrix}\mathbf{Q}(z)\begin{bmatrix}\mathbf{a} & \frac{\mathbf{Z}\mathbf{s}}{T}\end{bmatrix} = \begin{bmatrix}\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{a} & 0\\ 0 & \frac{1}{T}\mathbf{s}^{\mathsf{H}}\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{Z}\mathbf{s}\end{bmatrix} + o(1) \tag{21}$$

with

$$\frac{1}{T}\mathbf{s}^{\mathsf{H}}\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{Z}\mathbf{s} = \frac{1}{T}\mathbf{s}^{\mathsf{H}}\tilde{\mathbf{Q}}(z)\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Z}\mathbf{s} = \frac{1}{T}\mathbf{s}^{\mathsf{H}}(\mathbf{I}_T + z\tilde{\mathbf{Q}}(z))\mathbf{s} = \rho + \frac{z}{T}\mathbf{s}^{\mathsf{H}}\tilde{\mathbf{Q}}(z)\mathbf{s} \tag{22}$$

for co-resolvent $\tilde{\mathbf{Q}}(z) = \left(\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Z} - z\mathbf{I}_T\right)^{-1}$.

Since

$$\mathbf{Q}(z) \leftrightarrow \bar{\mathbf{Q}}(z) = m(z)\mathbf{I}_N = \left(\frac{1}{1+cm(z)} - z\right)^{-1}\mathbf{I}_N, \quad \tilde{\mathbf{Q}}(z) = -\left(\frac{1}{zm(z)} + 1\right)\mathbf{I}_T \tag{23}$$

for $c = \lim N/T$ and $m(z)$ the unique solution of the Marčenko-Pastur equation

$$zcm^2(z) - (1-c-z)m(z) + 1 = 0, \tag{24}$$

Therefore

$$(\mathbf{I}_2 + \mathbf{V}^{\mathsf{H}}\mathbf{Q}\mathbf{V}\boldsymbol{\Lambda})^{-1} = \begin{bmatrix} 1+\rho m(z) & m(z) \\ \rho\left(1 - z - \frac{1}{m(z)}\right) & 1 \end{bmatrix}^{-1} + o(1) \tag{25}$$

and

$$\mathbf{V}\boldsymbol{\Lambda}(\mathbf{I}_2 + \mathbf{V}^{\mathsf{H}}\mathbf{Q}\mathbf{V}\boldsymbol{\Lambda})^{-1}\mathbf{V}^{\mathsf{H}} = \begin{bmatrix}\mathbf{a} & \frac{\mathbf{Zs}}{T}\end{bmatrix} \frac{1}{1+\rho+\rho z m(z)} \begin{bmatrix} \rho z + \frac{\rho}{m(z)} & \mathsf{H} \\ \mathsf{H} & \mathsf{H} \end{bmatrix} \begin{bmatrix} \mathbf{a}^{\mathsf{H}} \\ \frac{(\mathbf{Zs})^{\mathsf{H}}}{T} \end{bmatrix} + o_{\|\cdot\|}(1) \tag{26}$$

so that it suffices to evaluate the following expectations:

1. $\mathbb{E}[\mathbf{Q}(z)\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)] = m^2(z)\mathbf{a}\mathbf{a}^{\mathsf{H}} + o_{\|\cdot\|}(1)$;
2. $\frac{1}{T}\mathbb{E}[\mathbf{Q}(z)\mathbf{a}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{T}}\mathbf{Q}(z)] = o_{\|\cdot\|}(1)$ and it Hermitian transpose.

This thus allows to conclude that

$$\left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} \leftrightarrow m(z)\mathbf{I}_N - \frac{\rho m(z)(zm(z)+1)}{1+\rho(zm(z)+1)}\mathbf{a}\mathbf{a}^{\mathsf{H}}. \tag{27}$$

5.1 Random matrix analysis

In the case of single DoA, our object of interest is the following *complex* random variable

$$Z = \frac{\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}}{\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}}, \tag{28}$$

with $\hat{\mathbf{u}} \in \mathbb{C}^N$ the dominant eigenvector of the sample covariance matrix $\hat{\mathbf{C}} = \frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}}$. Note that $\|\mathbf{J}_i^{\mathsf{T}}\mathbf{J}_j\| = O(1)$ for $i, j \in \{1, 2\}$, we have

According to (**??**), we need to evaluate the random variable of the form $\mathbf{y}_1^{\mathsf{H}}\mathbf{Q}(z)\mathbf{y}_2$ for $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{C}^N$ of bounded Euclidean norm. To this end, we introduce the following deterministic equivalent result.

Theorem 1 (Deterministic Equivalent)**.**

$$\left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} \leftrightarrow m(z)\mathbf{I}_N - \frac{\rho m(z)(zm(z)+1)}{1+\rho(zm(z)+1)}\mathbf{a}\mathbf{a}^{\mathsf{H}}. \tag{29}$$

As a consequence, we have, in the notation of (**??**),

$$\mathbf{e}_j^{\mathsf{H}}(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}\mathbf{e}_j \leftrightarrow m(z) - \frac{\rho m(z)(zm(z)+1)}{1+\rho(zm(z)+1)}(\mathbf{e}_j^{\mathsf{H}}\mathbf{a})^2 \tag{30}$$

and

$$\mathbf{e}_{j+\Delta}^{\mathsf{H}}(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}\mathbf{e}_j \leftrightarrow m(z)\delta_{\Delta=0} - \frac{\rho m(z)(zm(z)+1)}{1+\rho(zm(z)+1)}(\mathbf{e}_{j+\Delta}^{\mathsf{H}}\mathbf{a})(\mathbf{a}^{\mathsf{H}}\mathbf{e}_j) \tag{31}$$

It follows from Cauchy's integral formula and residue theorem that, for Γ_S circling around the isolated eigenvalue, we have

$$\begin{aligned} &-\frac{1}{2\pi\imath}\oint_{\Gamma_S}\mathbf{e}_j^{\mathsf{H}}(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}\mathbf{e}_j\,dz \simeq \frac{1}{2\pi\imath}\oint_{\Gamma_S}\frac{m(z)}{1+\rho(zm(z)+1)}dz\cdot(\mathbf{e}_j^{\mathsf{H}}\mathbf{a})^2 \\ &= -\lim_{z\to\lambda_S}(z-\lambda_S)\frac{m(z)}{1+\rho(zm(z)+1)}\cdot(\mathbf{e}_j^{\mathsf{H}}\mathbf{a})^2 = -\frac{m(\lambda_S)}{\rho(m(\lambda_S)+\lambda_S m'(\lambda_S))}\cdot(\mathbf{e}_j^{\mathsf{H}}\mathbf{a})^2 \end{aligned}$$

with $\lambda_S \equiv 1+\rho+c\frac{1+\rho}{\rho}$ the (asymptotic) position of the isolated eigenvalue and $m'(z) = \frac{m^2(z)}{1-\frac{cm^2(z)}{(1+cm(z))^2}}$ obtained by differentiating the Marčenko-Pastur equation, where we used the fact that Γ_S does not contain any pole of $m(z)$.

And similarly that

$$-\frac{1}{2\pi\imath}\oint_{\Gamma_S}\mathbf{e}_{j+\Delta}^{\mathsf{H}}(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}\mathbf{e}_j\,dz \simeq -\frac{m(\lambda_S)}{\rho(m(\lambda_S)+\lambda_S m'(\lambda_S))}\cdot\mathbf{e}_{j+\Delta}^{\mathsf{H}}\mathbf{a}\cdot\mathbf{a}^{\mathsf{H}}\mathbf{e}_j \tag{32}$$

To-do list:

1. simulate the different between $\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{M}\hat{\mathbf{u}}$ and its deterministic equivalent, for spiked covariance model and big matrix $\mathbf{M}$ with large rank, Note that two things happen at the same time: i) $\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{e}\mathbf{e}^{\mathsf{H}}\hat{\mathbf{u}} = O(1)$ and ii) $\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{I}_N\hat{\mathbf{u}} = 1 = O(1)$.

6 Proof of ESPIRIT in the case of single DoA

To investigate the performance of ESPIRIT method, one needs to evaluate the statistic of the dominant eigenvector $\hat{\mathbf{u}}_S$ of the spiked random matrix model $\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}}$, for $\mathbf{X} = \mathbf{a}\mathbf{s}^{\mathsf{H}} + \mathbf{Z}$ with $\mathbf{a} \in \mathbb{C}^N$ such that $\|\mathbf{a}\| = 1$, $\mathbf{s} \in \mathbb{C}^T$ a standard (circular) Gaussian random vector, and $\mathbf{Z} \in \mathbb{C}^{N\times T}$ a standard (circular) Gaussian random matrix.

Denote $(\hat{\lambda}, \hat{\mathbf{u}})$ the pair of largest eigenvalue-eigenvector pair of $\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}}$, and thus satisfies

$$\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{T}}\hat{\mathbf{u}} = \hat{\lambda}\hat{\mathbf{u}} = \frac{\|\mathbf{s}\|^2}{T}\mathbf{a}\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}} + \frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}}\hat{\mathbf{u}} + \frac{1}{T}\left(\mathbf{a}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}} + \mathbf{Z}\mathbf{s}\mathbf{a}^{\mathsf{H}}\right)\hat{\mathbf{u}}. \tag{33}$$

Denote $\mathbf{Q}(z) = (\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N)$, for $z \in \mathbb{C}$ not an eigenvalue of $\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{T}}$ (which is known to have eigenvalues lying within the MP support as $N, T \to \infty$), we obtain

$$\begin{aligned} &\mathbf{0} = \frac{\|\mathbf{s}\|^2}{T}\mathbf{a}\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}} + \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - \hat{\lambda}\mathbf{I}_N\right)\hat{\mathbf{u}} + \frac{1}{T}\left(\mathbf{a}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}} + \mathbf{Z}\mathbf{s}\mathbf{a}^{\mathsf{H}}\right)\hat{\mathbf{u}} \\ &\Leftrightarrow -\hat{\mathbf{u}} = \frac{\|\mathbf{s}\|^2}{T}\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}}\cdot\mathbf{Q}(\hat{\lambda})\mathbf{a} + \frac{1}{T}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\hat{\mathbf{u}}\cdot\mathbf{Q}(\hat{\lambda})\mathbf{a} + \mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}}\cdot\frac{1}{T}\mathbf{Q}(\hat{\lambda})\mathbf{Z}\mathbf{s} \\ &\Rightarrow \sqrt{N}[\hat{\mathbf{u}}]_i = \sqrt{N}\mathbf{e}_i^{\mathsf{T}}\hat{\mathbf{u}} = -\frac{\|\mathbf{s}\|^2}{T}\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}}\cdot\sqrt{N}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{a} - \frac{1}{T}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\hat{\mathbf{u}}\cdot\sqrt{N}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{a} - \mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}}\cdot\frac{\sqrt{N}}{T}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{Z}\mathbf{s} \end{aligned}$$

Note that till now no asymptotic approximation has been performed, we have only used linear algebraic results.

Proof to-do list:

(i) establish the asymptotic *complex* limit of $\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}} =? + o(1)$; and

(ii) establish the asymptotic *complex* limit of $\sqrt{N}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{a} =? + o(1)$; and

(iii) show that $\frac{1}{T}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\hat{\mathbf{u}} \to 0$ almost surely (this, together with item (ii), allows us to asymptotic discard the term $\frac{1}{T}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\hat{\mathbf{u}} \cdot \sqrt{N}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{a}$);

this allows us to conclude that the i-th entry of $\hat{\mathbf{u}}$ satisfies

$$\sqrt{N}[\hat{\mathbf{u}}]_i = -\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}}\left(\underbrace{\frac{\|\mathbf{s}\|^2}{T}\sqrt{N}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{a}}_{\text{deterministic } O(1)+o(1)} + \underbrace{\frac{\sqrt{N}}{T}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{Z}\mathbf{s}}_{\text{Gaussian fluctuation } O(1)}\right) + o(1). \tag{34}$$

This further leads to

$$N[\hat{\mathbf{u}}]_i^2 = +o(1),$$
$$N\overline{[\hat{\mathbf{u}}]_i}[\hat{\mathbf{u}}]_j = +o(1).$$

We thus obtain

$$(\mathbf{J}_1\hat{\mathbf{u}})^{\dagger}\mathbf{J}_2\hat{\mathbf{u}} = \frac{\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_2\hat{\mathbf{u}}}{\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_1\hat{\mathbf{u}}} = \frac{\sum_{j=i}^{i+n}\overline{[\hat{\mathbf{u}}]_j}[\hat{\mathbf{u}}]_{j+\Delta}}{\sum_{j=i}^{i+n}[\hat{\mathbf{u}}]_j^2} \tag{35}$$

7 Alternative proof of ESPIRIT in the case of single DoA: random signal case

In the case of (proper) complex Gaussian signal $\mathbf{s} \sim \mathcal{CN}(\mathbf{0}, \rho^2\mathbf{I}_T)$ with signal strength ρ^2, we have that the observation matrix $\mathbf{X} \in \mathbb{C}^{N\times}$ is equivalently given by

$$\mathbf{X} = \left(\mathbf{I}_N + \rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{\frac{1}{2}}\mathbf{Z}, \tag{36}$$

for standard complex Gaussian $\mathbf{Z} \in \mathbb{C}^{N\times T}$.

Let us first consider the form $\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}$, with $\hat{\mathbf{u}}$ the dominant eigenvector of the SCM $\hat{\mathbf{C}} = \frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}}$. We have, for Γ circling around the isolated eigenvalue of $\hat{\mathbf{C}}$, that

$$\begin{aligned}
\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}} &= \sum_{i=1}^{n}\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{e}_i\mathbf{e}_{i+\Delta}^{\mathsf{T}}\hat{\mathbf{u}} \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_{\Gamma}\mathbf{e}_{i+\Delta}^{\mathsf{T}}(\hat{\mathbf{C}} - z\mathbf{I}_N)^{-1}\mathbf{e}_i\,dz \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_{\Gamma}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N + \rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\left(\mathbf{I}_N + \rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-1}\right)^{-1}\left(\mathbf{I}_N + \rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz, \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_{\Gamma}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N + \rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N + z\frac{\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}}{1+\rho^2}\right)^{-1}\left(\mathbf{I}_N + \rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz, \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_{\Gamma}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N + \rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\left(\mathbf{Q}(z) - \frac{\rho^2}{1+\rho^2}\frac{z\mathbf{Q}(z)\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)}{1 + \frac{\rho^2}{1+\rho^2}\cdot z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{a}}\right)\left(\mathbf{I}_N + \rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz, \\
&= \frac{1}{2\pi\imath}\frac{\rho^2}{1+\rho^2}\sum_{i=1}^{n}\oint_{\Gamma}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N + \rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\frac{z\mathbf{Q}(z)\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)}{1 + \frac{\rho^2}{1+\rho^2}\cdot z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{a}}\left(\mathbf{I}_N + \rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz,
\end{aligned}$$

with the resolvent

$$\mathbf{Q}(z) \equiv \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1}. \tag{37}$$

This leads to

$$\begin{aligned}
\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}} &= \frac{1}{2\pi\imath}\frac{\rho^2}{1+\rho^2}\sum_{i=1}^{n}\oint_{\Gamma}\frac{z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{a}}{1+\frac{\rho^2}{1+\rho^2}\cdot z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{a}}\,dz,\\
&= \frac{1}{2\pi\imath}\frac{\rho^2}{1+\rho^2}\oint_{\Gamma}\frac{z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{a}}{1+\frac{\rho^2}{1+\rho^2}\cdot z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{a}}\,dz,
\end{aligned}$$

Note that the only pole in this case is $\lambda \in \mathbb{R}$ such that

$$z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(\lambda)\mathbf{a} = -\frac{1+\rho^2}{\rho^2}. \tag{38}$$

Since $\mathbf{Q}(z) \leftrightarrow \bar{\mathbf{Q}}(z) = m(z)\mathbf{I}_N$, we have

$$h(\lambda) \equiv \lambda m(\lambda) = -1-\rho^{-2}, \tag{39}$$

with

$$\lambda = 1+\rho^2+c\frac{1+\rho^2}{\rho^2}, \tag{40}$$

and therefore the following first-order result

$$\begin{aligned}
&\frac{1}{2\pi\imath}\frac{\rho^2}{1+\rho^2}\oint_{\Gamma}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\frac{z\mathbf{Q}(z)\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)}{1+\frac{\rho^2}{1+\rho^2}\cdot z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{a}}\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz\\
&\simeq \frac{1}{2\pi\imath}\frac{\rho^2}{1+\rho^2}\oint_{\Gamma}\frac{zm^2(z)\,dz}{1+\rho^2+\rho^2zm(z)}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{e}_i + O(N^{-1/2})\\
&= -\operatorname{Res}\left(\frac{zm^2(z)\,dz}{1+\rho^2+\rho^2zm(z)}\right)\frac{\rho^2}{1+\rho^2}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{e}_i + O(N^{-1/2})\\
&= \frac{m(\lambda)(1+h(\lambda))}{h'(\lambda)}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{e}_i + O(N^{-1/2}).
\end{aligned}$$

Since $m(z)$ is the solution to

$$zcm^2(z) - (1-c-z)m(z) + 1 = 0, \tag{41}$$

we have

$$m'(z) = \frac{m^2(z)}{1-\frac{cm^2(z)}{(1+cm(z))^2}}, \tag{42}$$

so that

$$h'(z) = m(z) + zm'(z). \tag{43}$$

Further note that

$$\sum_{i=1}^{n}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{e}_i = \mathbf{a}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{a}. \tag{44}$$

8 Proof of ESPIRIT in the case of multiple DoAs: random signal case

In this case, we have

$$\mathbf{X} = (\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}})^{\frac{1}{2}}\mathbf{Z}, \quad \mathbf{APA}^{\mathsf{H}} = \mathbf{U\Lambda U}^{\mathsf{H}}, \quad \mathbf{U} \in \mathbb{C}^{N\times K}, \quad \mathbf{\Lambda} = \operatorname{diag}\{\rho_i\}_{i=1}^{K}, \tag{45}$$

for standard complex Gaussian $\mathbf{Z} \in \mathbb{C}^{N\times T}$.

Let us first consider the diagonal entries of the form $\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_k$, with $\hat{\mathbf{u}}_k$ the kth dominant eigenvector of the SCM $\hat{\mathbf{C}} = \frac{1}{T}\mathbf{XX}^{\mathsf{H}}$. We have, for Γ circling around the isolated eigenvalue of $\hat{\mathbf{C}}$, that

$$\begin{aligned}
&\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_k = \sum_{i=1}^{n}\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{e}_i\mathbf{e}_{i+\Delta}^{\mathsf{T}}\hat{\mathbf{u}}_k \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}(\hat{\mathbf{C}} - z\mathbf{I}_N)^{-1}\mathbf{e}_i\,dz \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\left(\frac{1}{T}\mathbf{ZZ}^{\mathsf{H}} - z\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-1}\right)^{-1}\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz, \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\left(\frac{1}{T}\mathbf{ZZ}^{\mathsf{H}} - z\mathbf{I}_N + z\mathbf{A}(\mathbf{P}^{-1} + \mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{H}}\right)^{-1}\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz, \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\left(\mathbf{Q}(z) - z\mathbf{Q}(z)\mathbf{A}\left(\mathbf{P}^{-1} + \mathbf{A}^{\mathsf{H}}\mathbf{A} + z\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\right)\left(\mathbf{I}_N + \mathbf{AP}\right. \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\cdot z\mathbf{Q}(z)\mathbf{A}\left(\mathbf{P}^{-1} + \mathbf{A}^{\mathsf{H}}\mathbf{A} + z\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,d \\
&= -\frac{1}{2\pi\imath}\oint_\Gamma z\operatorname{tr}\left(\left(\mathbf{P}^{-1} + \mathbf{A}^{\mathsf{H}}\mathbf{A} + z\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{A}\right)\,d
\end{aligned}$$

with the resolvent

$$\mathbf{Q}(z) \equiv \left(\frac{1}{T}\mathbf{ZZ}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1}. \tag{46}$$

We have the following first and second deterministic equivalent results.

$$\mathbf{Q}(z) \leftrightarrow \bar{\mathbf{Q}}(z) = m(z)\mathbf{I}_N, \quad \mathbf{Q}(z)\mathbf{BQ}(z) \leftrightarrow \bar{\mathbf{Q}}(z)\mathbf{B}\bar{\mathbf{Q}}(z) + \frac{1}{T}\operatorname{tr}(\mathbf{B})\frac{m'(z)m^2(z)}{(1+cm(z))^2}\mathbf{I}_N, \tag{47}$$

for any deterministic matrix $\mathbf{B} \in \mathbb{C}^{N\times N}$ of bounded operator norm.

As such,

$$z\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{A} = zm(z)\mathbf{A}^{\mathsf{H}}\mathbf{A} + o_{\|\cdot\|}(1), \tag{48}$$

and

$$\begin{aligned}
&\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{A} \\
&= m^2(z)\mathbf{A}^{\mathsf{H}}\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{A} \\
&+ \frac{1}{T}\operatorname{tr}\left(\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\left(\mathbf{I}_N + \mathbf{APA}^{\mathsf{H}}\right)^{-1}\right)\frac{m'(z)m^2(z)}{(1+cm(z))^2}\mathbf{A}^{\mathsf{H}}\mathbf{A} + o_{\|\cdot\|}(1),
\end{aligned}$$

[Zhenyu: Note that we have in general $\frac{1}{T}\operatorname{tr}\left(\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-1}\right)=o(1)$ and $\mathbf{A}^{\mathsf{H}}\mathbf{A}=\mathbf{I}_K+o_{\|\cdot\|}(1)$. This leads to]

For the non-diagonal entries with $k \neq \ell$, we have instead

$$
\begin{aligned}
&|\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_\ell|^2 = \hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_\ell\hat{\mathbf{u}}_\ell^{\mathsf{H}}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}_k\\
&= -\frac{1}{2\pi\imath}\oint_{\Gamma_1}\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2(\hat{\mathbf{C}}-z_1\mathbf{I}_N)^{-1}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}_k\,dz_1\\
&= -\frac{1}{2\pi\imath}\oint_{\Gamma_1}\sum_{i=1}^n\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{e}_i\mathbf{e}_{i+\Delta}^{\mathsf{T}}(\hat{\mathbf{C}}-z_1\mathbf{I}_N)^{-1}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}_k\,dz_1\\
&= -\frac{1}{2\pi\imath}\oint_{\Gamma_1}\sum_{i=1}^n\mathbf{e}_{i+\Delta}^{\mathsf{T}}(\hat{\mathbf{C}}-z_1\mathbf{I}_N)^{-1}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}_k\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{e}_i\,dz_1\\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\sum_{i=1}^n\mathbf{e}_{i+\Delta}^{\mathsf{T}}(\hat{\mathbf{C}}-z_1\mathbf{I}_N)^{-1}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1(\hat{\mathbf{C}}-z_2\mathbf{I}_N)^{-1}\mathbf{e}_i\,dz_1\,dz_2\\
&= -\frac{1}{4\pi^2}\sum_{i=1}^n\oint_{\Gamma_1}\oint_{\Gamma_2}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\left(\frac{1}{T}\mathbf{ZZ}^{\mathsf{H}}-z_1\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-1}\right)^{-1}\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\\
&\times\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\left(\frac{1}{T}\mathbf{ZZ}^{\mathsf{H}}-z_2\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-1}\right)^{-1}\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz_1\,dz_2\\
&= -\frac{1}{4\pi^2}\sum_{i=1}^n\oint_{\Gamma_1}\oint_{\Gamma_2}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}z_1\mathbf{Q}(z_1)\mathbf{A}\left(\mathbf{P}^{-1}+\mathbf{A}^{\mathsf{H}}\mathbf{A}+z_1\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_1)\Big(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\\
&\times\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}z_2\mathbf{Q}(z_2)\mathbf{A}\left(\mathbf{P}^{-1}+\mathbf{A}^{\mathsf{H}}\mathbf{A}+z_2\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_2)\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz_1\,dz_2\\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}z_1z_2\operatorname{tr}\Big(\left(\mathbf{P}^{-1}+\mathbf{A}^{\mathsf{H}}\mathbf{A}+z_1\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{A}\right)^{-1}\times\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_1)\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\\
&\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{Q}(z_2)\mathbf{A}\times\left(\mathbf{P}^{-1}+\mathbf{A}^{\mathsf{H}}\mathbf{A}+z_2\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{A}\right)^{-1}\\
&\times\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_2)\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\left(\mathbf{I}_N+\mathbf{APA}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{Q}(z_1)\mathbf{A}\Big)\,dz_1\,dz_2
\end{aligned}
$$

with

9 Proof of ESPIRIT in the case of multiple DoAs: deterministic signal case

In this case, we have

$$
\mathbf{X}=\mathbf{AS}+\mathbf{Z}, \tag{49}
$$

with $\mathbf{X}=[\mathbf{x}(1),\ldots,\mathbf{x}(T)]\in\mathbb{C}^{N\times T}$, $\mathbf{A}=[\mathbf{a}(\theta_1),\ldots,\mathbf{a}(\theta_k)]\in\mathbb{C}^{N\times K}$, $\mathbf{S}$=$[\mathbf{s}(1),\ldots,\mathbf{s}(T)]\in\mathbb{C}^{K\times T}$, column vector $\mathbf{s}(t)=[s_1(t),\ldots,s_k(t)]^{\mathsf{T}}\in\mathbb{C}^K$ and $\mathbf{Z}=[\mathbf{z}(1),\ldots,\mathbf{z}(T)]\in\mathbb{C}^{N\times T}$ a standard (circular) Gaussian random matrix.

Let us first consider the form $\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_k$, with $\hat{\mathbf{u}}$ the kth dominant eigenvector of the

SCM $\hat{\mathbf{C}} = \frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}}$. We have, for Γ circling around the kth isolated eigenvalue of $\hat{\mathbf{C}}$, that

$$\begin{aligned}
\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_k &= \sum_{i=1}^{n} \hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{e}_i\mathbf{e}_{i+\Delta}^{\mathsf{T}}\hat{\mathbf{u}}_k \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}}(\hat{\mathbf{C}} - z\mathbf{I}_N)^{-1}\mathbf{e}_i \, dz \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N + \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathsf{H}}\right)^{-1}\mathbf{e}_i \, dz,
\end{aligned}$$

with $\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{Z}$ and that

$$\mathbf{U} = \begin{bmatrix}\mathbf{A} & \frac{1}{T}\mathbf{Z}\mathbf{S}^{\mathsf{H}}\end{bmatrix} \in \mathbb{C}^{N\times 2K}, \quad \mathbf{\Lambda} = \begin{bmatrix}\mathbf{P} & \mathbf{I}_K \\ \mathbf{I}_K & \mathbf{0}_K\end{bmatrix} \in \mathbb{R}^{2K\times 2K}, \mathbf{P} = \lim \frac{1}{T}\mathbf{S}\mathbf{S}^{\mathsf{H}}. \tag{50}$$

As such, we get, by Woodbury identity that

$$\begin{aligned}
\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_k &= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{Q}(z) - \mathbf{Q}(z)\mathbf{U}\left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\right)\mathbf{e}_i \, dz, \\
&= \frac{1}{2\pi\imath}\sum_{i=1}^{n}\operatorname{tr}\left(\oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{Q}(z)\mathbf{U}\left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{e}_i \, dz\right), \\
&= \frac{1}{2\pi\imath}\operatorname{tr}\left(\oint_{\Gamma}\left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z)\mathbf{U} \, dz\right),
\end{aligned}$$

with

$$\left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1} = \begin{bmatrix} (z + \frac{1}{m(z)})(\mathbf{P}^{-1} + (zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A}))^{-1} & (\mathbf{I}_K + (zm(z)+1)\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A}) \\ (\mathbf{I}_K + (zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A}\mathbf{P}) & -m(z)((\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1} + (zm(z)+1)\mathbf{P})^{-1}\end{bmatrix} \tag{51}$$

$$\mathbf{Q}(z) \equiv \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1}, \tag{52}$$

the resolvent.

As such, we have

$$\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z)\mathbf{U} = \begin{bmatrix}\mathbf{A}^{\mathsf{H}} \\ \frac{1}{T}\mathbf{S}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\end{bmatrix}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z)\begin{bmatrix}\mathbf{A} & \frac{1}{T}\mathbf{Z}\mathbf{S}\end{bmatrix} \simeq \left[\begin{matrix}\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z)\mathbf{A} & \mathbf{0}_K \\ \mathbf{0}_K & \frac{1}{T}\mathbf{S}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z)\frac{1}{T}\end{matrix}\right. \tag{53}$$

with

$$\frac{1}{T}\mathbf{S}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z)\frac{1}{T}\mathbf{Z}\mathbf{S} = [\text{Zhenyu} : \frac{1}{\mathsf{T}}\mathbf{S}^{\mathsf{H}}\tilde{\mathbf{Q}}(\mathsf{z})\frac{1}{\mathsf{T}}\mathbf{Z}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Z}\tilde{\mathbf{Q}}(\mathsf{z})\mathbf{S}] \tag{54}$$

for the co-resolvent

$$\tilde{\mathbf{Q}}(z) \equiv \left(\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Z} - z\mathbf{I}_T\right)^{-1}. \tag{55}$$

For the non-diagonal entries with $k \neq \ell$, we have instead

$$
\begin{aligned}
&|\hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \hat{\mathbf{u}}_\ell|^2 = \hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \hat{\mathbf{u}}_\ell \hat{\mathbf{u}}_\ell^{\mathsf{H}} \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \hat{\mathbf{u}}_k \\
&= -\frac{1}{2\pi\imath} \oint_{\Gamma_1} \hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 (\hat{\mathbf{C}} - z_1 \mathbf{I}_N)^{-1} \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \hat{\mathbf{u}}_k \, dz_1 \\
&= -\frac{1}{2\pi\imath} \oint_{\Gamma_1} \sum_{i=1}^{n} \hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{e}_i \mathbf{e}_{i+\Delta}^{\mathsf{T}} (\hat{\mathbf{C}} - z_1 \mathbf{I}_N)^{-1} \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \hat{\mathbf{u}}_k \, dz_1 \\
&= -\frac{1}{2\pi\imath} \oint_{\Gamma_1} \sum_{i=1}^{n} \mathbf{e}_{i+\Delta}^{\mathsf{T}} (\hat{\mathbf{C}} - z_1 \mathbf{I}_N)^{-1} \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \hat{\mathbf{u}}_k \hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{e}_i \, dz_1 \\
&= -\frac{1}{4\pi^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \sum_{i=1}^{n} \mathbf{e}_{i+\Delta}^{\mathsf{T}} (\hat{\mathbf{C}} - z_1 \mathbf{I}_N)^{-1} \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 (\hat{\mathbf{C}} - z_2 \mathbf{I}_N)^{-1} \mathbf{e}_i \, dz_1 \, dz_2 \\
&= -\frac{1}{4\pi^2} \sum_{i=1}^{n} \oint_{\Gamma_1} \oint_{\Gamma_2} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \left(\mathbf{Q}(z_1) - \mathbf{Q}(z_1) \mathbf{U} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z_1) \mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z_1) \right) \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \left(\mathbf{Q}(z_2) - \mathbf{Q}(z_2) \mathbf{U} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}} \right.\right. \\
&= -\frac{1}{4\pi^2} \sum_{i=1}^{n} \oint_{\Gamma_1} \oint_{\Gamma_2} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \mathbf{Q}(z_1) \mathbf{U} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z_1) \mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z_1) \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \mathbf{Q}(z_2) \mathbf{U} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z_2) \mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z_2) \mathbf{e}_i \\
&= -\frac{1}{4\pi^2} \operatorname{tr} \left(\oint_{\Gamma_1} \oint_{\Gamma_2} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z_1) \mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z_1) \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \mathbf{Q}(z_2) \mathbf{U} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z_2) \mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z_2) \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \mathbf{Q}(z_1) \mathbf{U} \, dz \right.
\end{aligned}
$$

with

——oooooooooooooooooooooooooooooooooooooooooo——————-

$$
\begin{aligned}
\hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \hat{\mathbf{u}}_k &= -\frac{1}{2\pi\imath} \sum_{i=1}^{n} \oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \left(\mathbf{Q}(z) - \mathbf{Q}(z) \mathbf{U} \left(\mathbf{\Lambda}^{-1} + \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z) \mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z) \right) \mathbf{e}_i \, dz, \\
&= \frac{1}{2\pi\imath} \sum_{i=1}^{n} \operatorname{tr} \left(\oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \mathbf{Q}(z) \mathbf{U} \left(\mathbf{\Lambda}^{-1} + \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z) \mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z) \mathbf{e}_i \, dz \right), \\
&= \frac{1}{2\pi\imath} \operatorname{tr} \left(\oint_{\Gamma} \left(\mathbf{\Lambda}^{-1} + \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z) \mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z) \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \mathbf{Q}(z) \mathbf{U} \, dz \right),
\end{aligned}
$$

$$
\left(\mathbf{\Lambda}^{-1} + \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z) \mathbf{U} \right)^{-1} = \begin{bmatrix} (z + \frac{1}{m(z)})(\mathbf{P}^{-1} + (zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1} & (\mathbf{I}_K + (zm(z)+1)\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1} \\ (\mathbf{I}_K + (zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A}\mathbf{P})^{-1} & \frac{-m(z)\mathbf{P}^{-1}}{zm(z)+1}(\mathbf{I}_K - (\mathbf{I}_K + (zm(z)+1)\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1} \end{bmatrix} \tag{56}
$$

set $\mathsf{H}_1 = (z + \frac{1}{m(z)})(\mathbf{P}^{-1} + (zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}$, $\mathsf{H}_2 = \frac{-m(z)\mathbf{P}^{-1}}{zm(z)+1}(\mathbf{I}_K - (\mathbf{I}_K + (zm(z)+1)\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1})$,
then

$$
\left(\mathbf{\Lambda}^{-1} + \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z) \mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}} \mathbf{Q}(z) \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \mathbf{Q}(z) \mathbf{U} = \begin{bmatrix} \mathsf{H}_1 \mathbf{A}^{\mathsf{H}} \mathbf{Q} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \mathbf{Q} \mathbf{A} & \mathbf{0}_K \\ \mathbf{0}_K & \mathsf{H}_2 \frac{1}{T} \mathbf{S}^{\mathsf{H}} \mathbf{Z}^{\mathsf{H}} \mathbf{Q} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \mathbf{Q} \frac{1}{T} \mathbf{Z} \mathbf{S} \end{bmatrix}
$$

then

$$
\hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \hat{\mathbf{u}}_k = \frac{1}{2\pi\imath} \oint_{\Gamma} \operatorname{tr} \left(\mathsf{H}_1 \mathbf{A}^{\mathsf{H}} \mathbf{Q} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \mathbf{Q} \mathbf{A} \right) + \operatorname{tr} \left(\mathsf{H}_2 \frac{1}{T} \mathbf{S}^{\mathsf{H}} \mathbf{Z}^{\mathsf{H}} \mathbf{Q} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \mathbf{Q} \frac{1}{T} \mathbf{Z} \mathbf{S} \right) dz.
$$

where

$$
\begin{aligned}
&\frac{1}{2\pi\imath}\oint_\Gamma \operatorname{tr}\left(\mathsf{H}_1\mathbf{A}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}\mathbf{A}\right)dz = \frac{1}{2\pi\imath}\oint_\Gamma \operatorname{tr}\left((z+\frac{1}{m(z)})(\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}\mathbf{A}\right)dz \\
&= \frac{1}{2\pi\imath}\oint_\Gamma (zm^2(z)+m(z))\operatorname{tr}\left((\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{A}\right)dz \\
&= \frac{1}{2\pi\imath}\sum_{i=l}^{l+n-1}\oint_\Gamma (zm^2(z)+m(z))\mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{A}(\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\mathbf{A}^H\mathbf{e}_i\,dz \\
&= \frac{1}{2\pi\imath}\sum_{i=l}^{l+n-1}\sum_{j=1}^{K}\oint_\Gamma \frac{zm^2(z)+m(z)}{zm(z)+1+l_j^{-1}}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{A}\mathbf{u}_j\mathbf{u}_j^H\mathbf{A}^{\mathsf{H}}\mathbf{e}_i\,dz \\
&= \lim_{z\to\bar\lambda_k}\frac{(z-\bar\lambda_k)(zm^2(z)+m(z))}{zm(z)+1+l_k^{-1}}\operatorname{tr}\left(\mathbf{A}\mathbf{u}_k\mathbf{u}_k^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_2\right) \\
&= \frac{1-cl_k^{-2}}{1+cl_k^{-1}}\mathbf{u}_k^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_2\mathbf{A}\mathbf{u}_k
\end{aligned}
$$

where we consider the eigendecomposition of $\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A} = \mathbf{U}\mathbf{V}\mathbf{U}^H$.

$$
\frac{1}{2\pi\imath}\oint_\Gamma \operatorname{tr}\left(\mathsf{H}_2\frac{1}{T}\mathbf{S}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}\frac{1}{T}\mathbf{Z}\mathbf{S}\right)dz = 0?
$$

so we have $\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_k = -\frac{1-cl_k^{-2}}{1+cl_k^{-1}}\mathbf{u}_k^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_2\mathbf{A}\mathbf{u}_k + o(1)$.

For the non-diagonal entries with $k\neq\ell$, we have instead

$$
\begin{aligned}
&|\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_\ell|^2 = -\frac{1}{4\pi^2}\operatorname{tr}\left(\oint_{\Gamma_1}\oint_{\Gamma_2}\left(\mathbf{\Lambda}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}(z_2)\mathbf{U}\left(\mathbf{\Lambda}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{J}_1^{\mathsf{T}}\right. \\
&= -\frac{1}{4\pi^2}\operatorname{tr}\left(\oint_{\Gamma_1}\oint_{\Gamma_2}\mathsf{H}_1(z_1)\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}(z_2)\mathbf{A}\mathsf{H}_1(z_2)\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z_1)\mathbf{A}\,dz_1\,dz_2\right) \\
&= -\frac{1}{4\pi^2}\sum_{i=l}^{l+n-1}\oint_{\Gamma_1}\oint_{\Gamma_2}\sum_{n=1}^{K}\sum_{m=1}^{K}\frac{zm^2(z_1)+m(z_1)}{zm(z_1)+1+l_m^{-1}}\cdot\frac{zm^2(z_2)+m(z_2)}{zm(z_2)+1+l_n^{-1}}\mathbf{e}_i^T\mathbf{A}\mathbf{u}_n\mathbf{u}_n^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_2\mathbf{A}\mathbf{u}_m\mathbf{u}_m^H\mathbf{A}^{\mathsf{H}}\mathbf{e}_{i+\Delta} \\
&= \frac{1-cl_k^{-2}}{1+cl_k^{-1}}\cdot\frac{1-cl_l^{-2}}{1+cl_l^{-1}}\mathbf{u}_l^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_2\mathbf{A}\mathbf{u}_k\mathbf{u}_k^H\mathbf{A}^H\mathbf{J}_2^T\mathbf{J}_1\mathbf{A}\mathbf{u}_l.
\end{aligned}
$$

The conclusion about the diagonal elements and the nondiagonal elements are the same as $\mathbf{\Phi}_2$ in the case of random signals.

$$
\begin{aligned}
\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}_k &= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{Q}(z)-\mathbf{Q}(z)\mathbf{U}\left(\mathbf{\Lambda}^{-1}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\right)\mathbf{e}_i\,dz, \\
&= \frac{1}{2\pi\imath}\sum_{i=1}^{n}\operatorname{tr}\left(\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{Q}(z)\mathbf{U}\left(\mathbf{\Lambda}^{-1}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{e}_i\,dz\right), \\
&= \frac{1}{2\pi\imath}\operatorname{tr}\left(\oint_\Gamma \left(\mathbf{\Lambda}^{-1}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}(z)\mathbf{U}\,dz\right),
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{1}{2\pi\imath}\oint_{\Gamma}\operatorname{tr}\left(\mathsf{H}_1\mathbf{A}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}\mathbf{A}\right)dz=\frac{1}{2\pi\imath}\oint_{\Gamma}\operatorname{tr}\left((z+\frac{1}{m(z)})(\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}\mathbf{A}\right)dz\\
&=\frac{1}{2\pi\imath}\oint_{\Gamma}\operatorname{tr}\left((z+\frac{1}{m(z)})(\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}[m^2(z)\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{A}+\frac{n}{T}\cdot\frac{m'(z)m^2(z)}{(1+cm(z))^2}\mathbf{I}_K]\right)dz\\
&=\frac{1-cl_k^{-2}}{1+cl_k^{-1}}\mathbf{u}_k^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_k+\frac{1}{2\pi\imath}\oint_{\Gamma}\operatorname{tr}\left(\frac{n}{T}\frac{(zm(z)+1)m'(z)m(z)}{(1+cm(z))^2}(\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\right)dz\\
&=\frac{1-cl_k^{-2}}{1+cl_k^{-1}}\mathbf{u}_k^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_k+\frac{1}{2\pi\imath}\oint_{\Gamma}\sum_{i=1}^{K}\frac{n}{T}\frac{(zm(z)+1)m'(z)m(z)}{(1+cm(z))^2}\cdot\frac{\mathbf{u}_i^H\mathbf{u}_i}{zm(z)+1+l_i^{-1}}dz\\
&=\frac{1-cl_k^{-2}}{1+cl_k^{-1}}\mathbf{u}_k^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_k+\lim_{z\to\bar{\lambda}_k}\frac{n}{T}\frac{(z-\bar{\lambda}_k)(zm(z)+1)m'(z)m(z)}{(1+cm(z))^2(zm(z)+1+l_k^{-1})}\\
&=\frac{1-cl_k^{-2}}{1+cl_k^{-1}}\mathbf{u}_k^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_k+\frac{n}{T}\frac{1}{l_k^2-cl_k}+o(1)(????)
\end{aligned}
$$

Similarly for off-diagonal entries:

$$
\begin{aligned}
&|\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}_\ell|^2=-\frac{1}{4\pi^2}\operatorname{tr}\left(\oint_{\Gamma_1}\oint_{\Gamma_2}\mathsf{H}_1(z_1)\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}(z_2)\mathbf{A}\mathsf{H}_1(z_2)\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}(z_1)\mathbf{A}\,dz_1\,dz_2\right)\\
&=-\frac{1}{4\pi^2}\operatorname{tr}\left(\oint_{\Gamma_1}\oint_{\Gamma_2}\mathsf{H}_1(z_1)(m(z_1)m(z_2)\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}+\eta(z_1,z_2)\frac{n}{T}\mathbf{I}_K)\mathsf{H}_1(z_2)(m(z_1)m(z_2)\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}+\eta(z_1,z_2)\frac{n}{T}\mathbf{I}\right.\\
&=o(1).
\end{aligned}
$$

10 Proof of ESPIRIT in the case of multiple DoAs: uncorrelated signal case

$$
\mathbf{T}_1=\mathbf{P}^{-1}+(1+zm(z))\mathbf{I}_K+o_{\|\cdot\|}(1),\quad \mathbf{T}_2=m^2(z)\mathbf{A}^{\mathsf{H}}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{A}+o_{\|\cdot\|}(1). \tag{57}
$$

Considering the spectral decomposition $\mathbf{P}=\mathbf{U}L\mathbf{U}^H$ with $L=\operatorname{diag}\{l_i,\cdots,l_k\}$ and $\mathbf{U}=[\mathbf{u}_1,\cdots,\mathbf{u}_k]$. Then we have

$$
\begin{aligned}
(\mathbf{T}_1)^{-1}&=\frac{1}{1+zm(z)}\mathbf{I}_K-\frac{1}{(1+zm(z))^2}\mathbf{U}(\mathbf{I}_K+\frac{1}{1+zm(z)}Ł^{-1})^{-1}\mathbf{U}^H\\
&=\frac{1}{1+zm(z)}\mathbf{I}_K-\sum_{i=1}^{K}\frac{1}{(1+zm(z))^2}\mathbf{u}_i\frac{(1+zm(z))l_i^{-1}}{1+zm(z)+l_i^{-1}}\mathbf{u}_i^H
\end{aligned}
$$

Then

$$
\begin{aligned}
[\mathbf{\Phi}_2]_{11} &= -\frac{1}{2\pi\imath}\oint_{\Gamma_1} z\,\mathrm{tr}(\mathbf{T}_1^{-1}\mathbf{T}_2)\,dz \\
&= \frac{1}{2\pi\imath}\oint_{\Gamma_1}\sum_{i=1}^{K}\frac{zm^2(z)}{1+zm(z)}\frac{l_i^{-1}}{1+zm(z)l_i^{-1}}\,\mathrm{tr}(\mathbf{u}_i\mathbf{A}^{\mathsf{H}}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{A}\mathbf{u}_i^H)dz \\
&= \lim_{z\to\bar{\lambda}_1}\frac{(z-\bar{\lambda}_1)zm^2(z)}{1+zm(z)}\frac{l_1^{-1}}{1+zm(z)+l_1^{-1}}\mathbf{u}_1^H\mathbf{A}^{\mathsf{H}}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{A}\mathbf{u}_1 \\
&= \lim_{z\to\bar{\lambda}_1}\frac{(z-\bar{\lambda}_1)zm^2(z)}{1+zm(z)+l_1^{-1}}\mathbf{u}_1^H\mathbf{A}^{\mathsf{H}}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{A}\mathbf{u}_1+o(1) \\
&= \frac{1-cl_1^{-2}}{1+cl_1^{-1}}\cdot\mathbf{u}_1^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{A}\mathbf{u}_1+o(1)
\end{aligned}
$$

The conclusion is similar to the previous formula, except that $\mathbf{a}(\theta_1)$ is replaced by eigenvector $\mathbf{u}_i^H\mathbf{a}(\theta_1)^H$.

$$
\mathbf{T}_2(z_1,z_2) = m(z_1)m(z_2)\mathbf{A}^{\mathsf{H}}\mathbf{B}\mathbf{A}+o_{\|\cdot\|}(1)
$$

where $\mathbf{B}=\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\in\mathbb{C}^{N\times N}$.

Similarly,

$$
\begin{aligned}
|[\mathbf{\Phi}_2]_{12}|^2 &= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2} z_1z_2\,\mathrm{tr}(\mathbf{T}_2^{\mathsf{H}}(z_1,z_2)\mathbf{T}_1^{-1}(z_1)\mathbf{T}_2(z_1,z_2)\mathbf{T}_1^{-1}(z_2))\,dz_1\,dz_2, \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\sum_{i=1}^{K}\sum_{j=1}^{K}\frac{z_1m^2(z_1)l_i^{-1}z_2m^2(z_2)l_j^{-1}\,\mathrm{tr}(\mathbf{u}_j\mathbf{u}_j^H\mathbf{A}^H\mathbf{B}^H\mathbf{A}\mathbf{u}_i\mathbf{u}_i^H\mathbf{A}^H\mathbf{B}\mathbf{A})\quad dz_1dz_2}{(1+z_1m(z_1))(1+z_1m(z_1)+l_i^{-1})(1+z_2m(z_2))(1+z_2m(z_2)+l_j^{-1})} \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\frac{z_1m^2(z_1)l_1^{-1}z_2m^2(z_2)l_2^{-1}\mathbf{u}_2^H\mathbf{A}^H\mathbf{B}^H\mathbf{A}\mathbf{u}_1\mathbf{u}_1^H\mathbf{A}^H\mathbf{B}\mathbf{A}\mathbf{u}_2\quad dz_1dz_2}{(1+z_1m(z_1))(1+z_1m(z_1)+l_1^{-1})(1+z_2m(z_2))(1+z_2m(z_2)+l_2^{-1})} \\
&= \frac{1-cl_1^{-2}}{1+cl_1^{-1}}\cdot\frac{1-cl_2^{-2}}{1+cl_2^{-1}}\mathbf{u}_2^H\mathbf{A}^H\mathbf{J}_2^T\mathbf{J}_1\mathbf{A}\mathbf{u}_1\mathbf{u}_1^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_2\mathbf{A}\mathbf{u}_2.
\end{aligned}
$$

where we use the fact that $\mathbf{u}_2^H\mathbf{A}^H\mathbf{B}^H\mathbf{A}\mathbf{u}_1\mathbf{u}_1^H\mathbf{A}^H\mathbf{A}\mathbf{u}_2=\bar{\lambda}_1\bar{\lambda}_2\mathbf{u}_2^H\mathbf{A}^H\mathbf{J}_2^T\mathbf{J}_1\mathbf{A}\mathbf{u}_1\mathbf{u}_1^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_2\mathbf{A}\mathbf{u}_2$ in the last line.

$$
\mathbf{T}_3 = m^2(z)\mathbf{A}^{\mathsf{H}}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{A}+\frac{n}{T}\frac{m^4(z)\mathbf{I}_K}{(1+cm(z))^2-cm^2(z)}+o_{\|\cdot\|}(1)
$$

As such, we have

$$
\begin{aligned}
[\mathbf{\Phi}_1]_{11} &= -\frac{1}{2\pi\imath}\oint_{\Gamma_1} z\,\mathrm{tr}(\mathbf{T}_1^{-1}\mathbf{T}_3)\,dz \\
&= \frac{1}{2\pi\imath}\oint_{\Gamma_1}\sum_{i=1}^{K}\frac{zm^2(z)}{1+zm(z)}\frac{l_i^{-1}}{1+zm(z)l_i^{-1}}\,\mathrm{tr}(\mathbf{u}_i\mathbf{u}_i^H\mathbf{A}^\mathsf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^\mathsf{T}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{A}) \\
&+ z\,\mathrm{tr}(\sum_{i=1}^{K}\frac{zm^2(z)}{1+zm(z)}\frac{l_i^{-1}}{1+zm(z)l_i^{-1}}\frac{n}{T}\frac{m^4(z)}{(1+cm(z))^2-cm^2(z)}\cdot\mathbf{u}_i\mathbf{u}_i^H)dz \\
&= \frac{1-cl_1^{-2}}{1+cl_1^{-1}}\cdot\mathbf{u}_1^H\mathbf{A}^\mathsf{H}\mathbf{J}_1^\mathsf{T}\mathbf{J}_1\mathbf{A}\mathbf{u}_1+\frac{n}{T}\frac{1+\rho_1^{-1}}{c+\rho_1}+o(1) \\
&= \frac{1-cl_1^{-2}}{1+cl_1^{-1}}\cdot\frac{n}{N}+\frac{n}{T}\frac{1+\rho_1^{-1}}{c+\rho_1}+o(1).
\end{aligned}
$$

Similarly for off-diagonal entries of $\mathbf{\Phi}_1$ as

$$
\begin{aligned}
&|[\mathbf{\Phi}_1]_{12}|^2 = -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2} z_1z_2\,\mathrm{tr}(\mathbf{T}_4(z_2,z_1)\mathbf{T}_1^{-1}(z_1)\mathbf{T}_4(z_1,z_2)\mathbf{T}_1^{-1}(z_2))\,dz_1\,dz_2, \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\mathrm{tr}(\sum_{i=1}^{K}\sum_{j=1}^{K}\frac{z_1m^2(z_1)l_i^{-1}z_2m^2(z_2)l_j^{-1}\,\mathrm{tr}(\mathbf{T}_4(z_2,z_1)u_ju_j^H\mathbf{T}_4(z_1,z_2)u_iu_i^H)\quad dz_1dz_2}{(1+z_1m(z_1))(1+z_1m(z_1)+l_i^{-1})(1+z_2m(z_2))(1+z_2m(z_2)+l_j^{-1})} \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\frac{z_1m^2(z_1)l_1^{-1}z_2m^2(z_2)l_2^{-1}\mathbf{u}_2^H\mathbf{A}^H\mathbf{H}^H\mathbf{A}\mathbf{u}_1\mathbf{u}_1^H\mathbf{A}^H\mathbf{H}\mathbf{A}\mathbf{u}_2\quad dz_1dz_2}{(1+z_1m(z_1))(1+z_1m(z_1)+l_1^{-1})(1+z_2m(z_2))(1+z_2m(z_2)+l_2^{-1})} \\
&= \frac{1-cl_1^{-2}}{1+cl_1^{-1}}\cdot\frac{1-cl_2^{-2}}{1+cl_2^{-1}}\mathbf{u}_2^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_1\mathbf{u}_1^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_2+o(1) \\
&= o(1)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathbf{T}_4(z_1,z_2) &= \mathbf{A}^\mathsf{H}\mathbf{Q}(z_1)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^\mathsf{T}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z_2)\mathbf{A} \\
&= m(z_1)m(z_2)\mathbf{A}^\mathsf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^\mathsf{T}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{A}+\frac{n}{T}\eta(z_1,z_2)\mathbf{I}_K+o_{\|\cdot\|}(1)=\mathbf{T}_4(z_2,z_1),
\end{aligned}
$$

and $\mathbf{H}=\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^\mathsf{T}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\in\mathbb{C}^{N\times N}$. The last line is because $\mathbf{u}_2^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_1=\mathbf{u}_2^H\frac{n}{N}\mathbf{I}_K\mathbf{u}_1=0$. So we can draw the conclusion that:

$$
\bar{\mathbf{\Phi}}_1 = \mathrm{diag}\left\{\frac{n}{N}\frac{1-cl_k^{-2}}{1+cl_k^{-1}}+\frac{n}{N}\frac{1+l_k^{-1}}{1+l_k/c}\right\}_{k=1}^{K}, \tag{58}
$$

$$
[\bar{\mathbf{\Phi}}_2]_{kk} = \frac{1-cl_1^{-2}}{1+cl_1^{-1}}\cdot\mathbf{u}_k^H\mathbf{A}^\mathsf{H}\mathbf{J}_1^\mathsf{T}\mathbf{J}_2\mathbf{A}\mathbf{u}_k \tag{59}
$$

However, we can only get the modulus of the off-diagonal elements of $\bar{\mathbf{\Phi}}_2$.

$$
|[\mathbf{\Phi}_2]_{mn}|^2 = \frac{1-cl_m^{-2}}{1+cl_m^{-1}}\cdot\frac{1-cl_n^{-2}}{1+cl_n^{-1}}\mathbf{u}_m^H\mathbf{A}^H\mathbf{J}_2^T\mathbf{J}_1\mathbf{A}\mathbf{u}_n\mathbf{u}_n^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_2\mathbf{A}\mathbf{u}_m.
$$

11 Closely DoA

Settings: $\mathbf{a}(\theta)=\frac{1}{\sqrt{N}}[1,e^{\imath\theta},\dots,e^{\imath(N-1)\theta}]^\mathsf{T}$.

We set that $\theta_2 = \theta_1 + \frac{\alpha}{T}$, where T is the number of time slots, then:

$$\begin{aligned}
\mathbf{a}^{\mathsf{H}}(\theta_1)\mathbf{a}(\theta_2) &= \frac{1}{N}\sum_{n=0}^{N-1} e^{\imath n\alpha/T} = \frac{1}{N}\frac{1-e^{\imath N\alpha/T}}{1-e^{\imath\alpha/T}} = \lim_{N,T\to\infty}\frac{\frac{1}{T}(1-e^{\imath N\alpha/T})}{\frac{N}{T}(1-e^{\imath\alpha/T})} \\
&= \lim_{N,T\to\infty}\frac{-\frac{1}{T^2}(1-e^{\imath\alpha c})}{\frac{1}{T^2}\imath\alpha c e^{\imath\frac{\alpha}{T}}} = \imath\frac{1-e^{\imath c\alpha}}{c\alpha}
\end{aligned}$$

where the j-th entry of $\mathbf{a}(\theta)$ is given by

$$[\mathbf{a}(\theta_\ell)]_j = \frac{1}{\sqrt{N}}e^{\imath\frac{2\pi d}{\lambda_0}(j-1)\theta_\ell} \equiv \frac{1}{\sqrt{N}}e^{\imath\omega(j-1)\theta_\ell}, \quad \omega \equiv \frac{2\pi d}{\lambda_0} \tag{60}$$

So we have:

$$\mathbf{A}^{\mathsf{H}}\mathbf{A} \to \begin{bmatrix} 1 & e^{\imath\alpha c/2}sinc(\alpha c/2) \\ e^{-\imath\alpha c/2}sinc(\alpha c/2) & 1 \end{bmatrix}$$

with its eigenvalue decomposition $\mathbf{V}L\mathbf{V}^H$ with $\Sigma = \operatorname{diag}\{1+\operatorname{sinc}\frac{\alpha c}{2}, 1-\operatorname{sinc}\frac{\alpha c}{2}\}$ and $\mathbf{U} = [\mathbf{v}_1, \mathbf{v}_2]$, where $\mathbf{v}_1 = \frac{1}{\sqrt{2}}[e^{\frac{\imath\alpha c}{2}}, 1]^T$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}}[1, -e^{-\frac{\imath\alpha c}{2}}]^T$.

In this part, we have:

$$\mathbf{T}_1 = \mathbf{P}^{-1} + \mathbf{A}^{\mathsf{H}}\mathbf{A} + z\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{A} = \mathbf{P}^{-1} + (1_z m(z))\mathbf{A}^{\mathsf{H}}\mathbf{A}, \tag{61}$$

and

$$\mathbf{T}_1^{-1} = \mathbf{P} - (1+zm(z))\mathbf{P}\mathbf{A}^{\mathsf{H}}(\mathbf{I}_N + (1+zm(z))\mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}})^{-1}\mathbf{A}\mathbf{P}.$$

Then, we have

$$\begin{aligned}
[\mathbf{\Phi}_2]_{kk} &= \frac{1}{2\pi\imath}\oint_{\Gamma_k} z\operatorname{tr}(\mathbf{T}_1^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{C}^{\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{\frac{1}{2}}\mathbf{Q}(z)\mathbf{A})\,dz \\
&= -\frac{1}{2\pi\imath}\oint_{\Gamma_k} z\operatorname{tr}((1+zm(z))\mathbf{P}\mathbf{A}^{\mathsf{H}}(\mathbf{I}_N + (1+zm(z))\mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}})^{-1}\mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{A})\,dz \\
&= -\frac{1}{2\pi\imath}\oint_{\Gamma_k} z\operatorname{tr}(\mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}(\mathbf{I}_N - (\mathbf{I}_N + (1+zm(z))\mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}})^{-1})\mathbf{Q}(z)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z))\,dz \\
&= \frac{1}{2\pi\imath}\oint_{\Gamma_k} z\operatorname{tr}(\mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}(\mathbf{I}_N + (1+zm(z))\mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}})^{-1}\mathbf{Q}(z)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z))\,dz
\end{aligned}$$

we define the spectral decomposition $\mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}} = \mathbf{U}L\mathbf{U}^H$ with $L = \operatorname{diag}\{\rho_i, \cdots, \rho_N\}$ and $\mathbf{U} = [\mathbf{u}_1, \cdots, \mathbf{u}_N]$. Then:

$$\begin{aligned}
[\mathbf{\Phi}_2]_{kk} &= \frac{1}{2\pi\imath}\oint_{\Gamma_k}\sum_{i=1}^{N}\frac{z}{1+(1+zm(z)\rho_i)}\operatorname{tr}(\mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{u}_i\mathbf{u}_i^H\mathbf{Q}(z)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z))\,dz \\
&= \frac{1}{2\pi\imath}\oint_{\Gamma_k}\sum_{i=1}^{N}\frac{z\rho_k}{1+(1+zm(z)\rho_i)}\mathbf{u}_k^H\mathbf{Q}(z)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{u}_k\,dz \\
&= \frac{1}{2\pi\imath}\oint_{\Gamma_k}\frac{zm^2(z)}{\rho_i^{-1}+1+zm(z)}\mathbf{u}_i^H\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{u}_i\,dz \\
&= \frac{1-c\rho_k^{-2}}{1+c\rho_k^{-1}}\cdot\mathbf{u}_K^H\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{u}_k + o(1)
\end{aligned}$$

the conclusion is the same as $[\mathbf{\Phi}_2]_{kk}$ in the case of widely DoA.

Similarly for $[\mathbf{\Phi}_1]_{kk}$ as:

$$
\begin{aligned}
[\mathbf{\Phi}_1]_{kk} &= \frac{1}{2\pi\imath}\oint_{\Gamma_k} \frac{z}{\rho_k^{-1}+1+zm(z)} \mathbf{u}_k^H \mathbf{Q}(z)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{u}_k \, dz \\
&= \frac{1-c\rho_k^{-2}}{1+c\rho_k^{-1}} \cdot \mathbf{u}_K^H\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{u}_k + \frac{1}{2\pi\imath}\oint_{\Gamma_k} \frac{z}{\rho_k^{-1}+1+zm(z)} \frac{n}{T}\frac{m'(z)m^2(z)}{(1+cm(z))^2}\, dz \\
&= \frac{1-c\rho_k^{-2}}{1+c\rho_k^{-1}} \cdot \mathbf{u}_K^H\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{u}_k + \frac{n}{T}\frac{1+\rho_k^{-1}}{c+\rho_k} + o(1).
\end{aligned}
$$

where $\mathbf{u}_k$ and ρ_k represent the k-th eigenvetor and eigenvalue of $\mathbf{APA}^{\mathsf{H}}$, respectively. The extra bias term is the same as the previous conclusions.

For the off-diagonal term:

$$
\begin{aligned}
|[\mathbf{\Phi}_2]_{12}|^2 &= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2} z_1 z_2 \,\mathrm{tr}(\mathbf{T}_2^{\mathsf{H}}(z_1,z_2)\mathbf{T}_1^{-1}(z_1)\mathbf{T}_2(z_1,z_2)\mathbf{T}_1^{-1}(z_2))\, dz_1\, dz_2, \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\sum_{i=1}^{K}\sum_{j=1}^{K} \frac{z_1 m^2(z_1) z_2 m^2(z_2)\mathbf{u}_j^H\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{u}_j\mathbf{u}_i^H\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{u}_i \quad dz_1 dz_2}{(1+z_1 m(z_1)+\rho_i^{-1})(1+z_2 m(z_2)+\rho_j^{-1})} \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2} \frac{z_1 m^2(z_1) l_1^{-1} z_2 m^2(z_2) l_2^{-1}\mathbf{u}_1^H\mathbf{J}_2^T\mathbf{J}_1\mathbf{u}_2\mathbf{u}_2^H\mathbf{J}_1^T\mathbf{J}_2\mathbf{u}_1 \quad dz_1 dz_2}{(1+z_1 m(z_1)+\rho_1^{-1})(1+z_2 m(z_2)+\rho_2^{-1})} \\
&= \frac{1-c\rho_1^{-2}}{1+c\rho_1^{-1}}\cdot\frac{1-c\rho_2^{-2}}{1+c\rho_2^{-1}}\mathbf{u}_1^H\mathbf{J}_2^T\mathbf{J}_1\mathbf{u}_2\mathbf{u}_2^H\mathbf{J}_1^T\mathbf{J}_2\mathbf{u}_1.
\end{aligned}
$$

Similarly for $[\mathbf{\Phi}_1]_{12}$ as:

$$
\begin{aligned}
|[\mathbf{\Phi}_1]_{12}|^2 &= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2} z_1 z_2 \,\mathrm{tr}(\mathbf{T}_3^{\mathsf{H}}(z_1,z_2)\mathbf{T}_1^{-1}(z_1)\mathbf{T}_3(z_1,z_2)\mathbf{T}_1^{-1}(z_2))\, dz_1\, dz_2, \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\sum_{i=1}^{K}\sum_{j=1}^{K} \frac{z_1 m^2(z_1) z_2 m^2(z_2)\mathbf{u}_j^H\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{u}_j\mathbf{u}_i^H\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{u}_i \quad dz_1 dz_2}{(1+z_1 m(z_1)+\rho_i^{-1})(1+z_2 m(z_2)+\rho_j^{-1})} \\
&= \frac{1-c\rho_1^{-2}}{1+c\rho_1^{-1}}\cdot\frac{1-c\rho_2^{-2}}{1+c\rho_2^{-1}}\mathbf{u}_1^H\mathbf{J}_1^T\mathbf{J}_1\mathbf{u}_2\mathbf{u}_2^H\mathbf{J}_1^T\mathbf{J}_1\mathbf{u}_1.
\end{aligned}
$$

where $\mathbf{T}_3^{\mathsf{H}}(z_1,z_2) = m(z_1)m(z_2)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}} + \frac{n}{T}\eta(z_1,z_2)\mathbf{I}_N$

For the special case $\mathbf{P} = \mathbf{I}_K$, setting that $\mathbf{x}$ is the eigenvector of $\mathbf{A}^{\mathsf{H}}\mathbf{AP}$, then $(\mathbf{AP})\mathbf{x}$ is the eigenvector of $\mathbf{APA}^{\mathsf{H}}$. So that $\mathbf{u}_1$ and $\mathbf{u}_2$ have the following form:

$$
\begin{aligned}
\mathbf{u}_1 = \mathbf{A}\mathbf{x}_1 &= \frac{1}{\sqrt{2}}[e^{\frac{\imath\alpha c}{2}}\mathbf{a}(\theta_1) + \mathbf{a}(\theta_2)]^{\mathsf{T}} \\
\mathbf{u}_2 = \mathbf{A}\mathbf{x}_2 &= \frac{1}{\sqrt{2}}[\mathbf{a}(\theta_1) - e^{-\frac{\imath\alpha c}{2}}\mathbf{a}(\theta_2)]^{\mathsf{T}}
\end{aligned}
$$

Then $\mathbf{u}_1\mathbf{J}_1^T\mathbf{J}_2\mathbf{u}_1$ has the form as:

$$
\mathbf{u}_1\mathbf{J}_1^T\mathbf{J}_2\mathbf{u}_1 = \frac{1}{2}[\frac{n}{N}e^{\imath\Delta\theta_1} + \frac{n}{N}e^{\imath\Delta\theta_2} + \frac{1-e^{\imath\alpha c n/N}}{-\alpha c}e^{\frac{\imath\alpha c}{2}}\frac{n}{N}e^{-\imath\Delta\theta_2} + \frac{1-e^{\imath\alpha c n/N}}{-\imath\alpha c}e^{-\frac{\imath\alpha c}{2}}\frac{n}{N}e^{\imath\Delta\theta_2}]
$$

12 Widely spaced DoA(P is not diagonal)

Signal Mode123l :

$$\mathbf{X} = (\mathbf{APA}^H + \mathbf{I}_k)^{\frac{1}{2}}\mathbf{Z}$$

$$\mathbf{C} = (\mathbf{APA}^H + \mathbf{I}_k)$$

$$P = BLB^H, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, B^H B = BB^H = I, \mathbf{U}_s = \mathbf{AB} \tag{62}$$

Q is

We aim to get the eigenvalue of this $\Phi_1^{-1}\Phi_2$,(denoted as $\lambda_1 ... \lambda_k$)

$$\overline{\Phi}_1 = \frac{n}{N} I_k, \Phi_2 = \begin{pmatrix} \hat{u}_1^H J_1^H J_2 \hat{u}_1 & \hat{u}_1^H J_1^H J_2 \hat{u}_2 \\ \hat{u}_2^H J_1^H J_2 \hat{u}_1 & \hat{u}_2^H J_1^H J_2 \hat{u}_2 \end{pmatrix} \tag{63}$$

We do not have direct access to the deterministic equivalent for Φ_2 referring to $\overline{\Phi}_2$,but an asymptotic eigenvalue equivalence(denoted as $\overline{\lambda}_1 ... \overline{\lambda}_k$) can be obtained.

$$Eig(\Phi_1^{-1}\Phi_2) = Eig(\overline{\Phi}_1^{-1}\Phi_2) \tag{64}$$

$$= \frac{n}{N} Eig(\Phi_2) \tag{65}$$

The above equation is the solution of the following equation

$$\begin{aligned}
\det(\Phi_2 - \lambda I_2) &= (\hat{u}_1^H J_1^H J_2 \hat{u}_1 - \lambda)(\hat{u}_2^H J_1^H J_2 \hat{u}_2 - \lambda) - \hat{u}_1^H J_1^H J_2 \hat{u}_2 \hat{u}_2^H J_1^H J_2 \hat{u}_1 \\
&= \lambda^2 - \lambda(\hat{u}_1^H J_1^H J_2 \hat{u}_1 + \hat{u}_2^H J_1^H J_2 \hat{u}_2) + \hat{u}_1^H J_1^H J_2 \hat{u}_1 \hat{u}_2^H J_1^H J_2 \hat{u}_2 - \hat{u}_1^H J_1^H J_2 \hat{u}_2 \hat{u}_2^H J_1^H J_2 \hat{u}_1 \\
&= \overline{\lambda}^2 - \overline{\lambda}(g_1 u_1^H J_1^H J_2 u_1 + g_2 u_2^H J_1^H J_2 u_2) + g_1 g_2 u_1^H J_1^H J_2 u_1 u_2^H J_1^H J_2 u_2 - g_1 g_2 u_1^H J_1^H J_2 u_2 u_2^H J_1^H J_2 u_1 \\
&= \overline{\lambda}^2 - \overline{\lambda}(g_1 u_1^H J_1^H J_2 u_1 + g_2 u_2^H J_1^H J_2 u_2) + g_1 g_2 Det(\mathbf{U}_s^H \mathbf{J}_1^H \mathbf{J}_2 \mathbf{U}_s) \\
&= \overline{\lambda}^2 - \overline{\lambda}(g_1(b_{11}^2 \frac{n}{N} e^{i\theta_1} + b_{12}^2 \frac{n}{N} e^{i\theta_1}) + g_2(b_{21}^2 \frac{n}{N} e^{i\theta_1} + b_{22}^2 \frac{n}{N} e^{i\theta_2})) + g_1 g_2 Det(\mathbf{A}^H \mathbf{J}_1^H \mathbf{J}_2 \mathbf{A}) \\
&= \overline{\lambda}^2 - \overline{\lambda}((g_1 b_{11}^2 + g_2 b_{21}^2) \frac{n}{N} e^{i\theta_1} + (g_1 b_{12}^2 + g_2 b_{22}^2) \frac{n}{N} e^{i\theta_2})) + g_1 g_2 (\frac{n}{N})^2 e^{i\theta_1} e^{i\theta_2} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\alpha_1 &= (g_1 b_{11}^2 + g_2 b_{21}^2) \frac{n}{N} e^{i\theta_1} + (g_1 b_{12}^2 + g_2 b_{22}^2) \frac{n}{N} e^{i\theta_2}) \quad g_i = (1 - c l_i^{-2})/(1 + c l_i^{-1}) \\
\alpha_2 &= g_1 g_2 (\frac{n}{N})^2 e^{i\theta_1} e^{i\theta_2} \\
\overline{\lambda}_1 &= \frac{\alpha_1 + \sqrt{\Delta}}{2} \\
\overline{\lambda}_2 &= \frac{\alpha_1 - \sqrt{\Delta}}{2} \\
\Delta &= \alpha_1^2 - 4\alpha_2 \\
\overline{\lambda}_1 &= \frac{(g_1 b_{11}^2 + g_2 b_{21}^2) \frac{n}{N} e^{i\theta_1} + (g_1 b_{12}^2 + g_2 b_{22}^2) \frac{n}{N} e^{i\theta_2}) + \sqrt{((g_1 b_{11}^2 + g_2 b_{21}^2) \frac{n}{N} e^{i\theta_1} + (g_1 b_{12}^2 + g_2 b_{22}^2) \frac{n}{N} e^{i\theta_2}))^2 - 4 *}}{2}
\end{aligned}$$

I guess the two roots of the equation are $(g_1 b_{11}^2 + g_2 b_{21}^2) \frac{n}{N} e^{i\theta_1}$, $(g_1 b_{21}^2 + g_2 b_{22}^2) \frac{n}{N} e^{i\theta_2}$

Specially,P is diagonal,$B = \mathbf{I}_k$,$b_{11} = b_{22} = 1, b_{12} = b_{21} = 0$,$\overline{\lambda}_1 = g_1 \frac{n}{N} e^{i\theta_1}$, $\overline{\lambda}_2 = g_2 \frac{n}{N} e^{i\theta_2}$(consistent with the results of our paper)

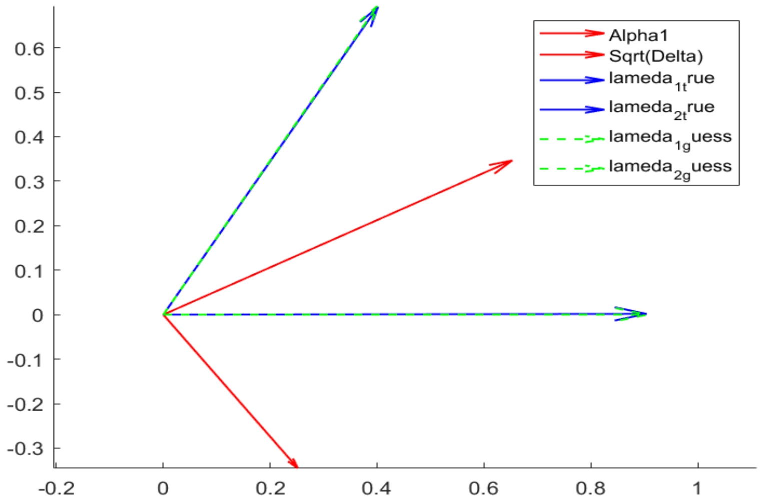

Figure 1: The caption

$$g_1 g_2 = (g_1 b_{11}^2 + g_2 b_{21}^2)(g_1 b_{21}^2 + g_2 b_{22}^2) - b_{11}^2 b_{12}^2 (g_1 - g_2)^2 \quad (67)$$

If we can prove that the second item ($b_{11}^2 b_{12}^2 (g_1 - g_2)^2$) asymptotically($N, T \to \infty$) converges to 0(**Note** : the item is equal to 0 absolutely when P is diagonal, due to $b_{12} = 0$).

$$\begin{aligned} \overline{\lambda}_1 &= (g_1 b_{11}^2 + g_2 b_{21}^2) \frac{n}{N} e^{i\theta_1} \\ \overline{\lambda}_2 &= (g_1 b_{21}^2 + g_2 b_{22}^2) \frac{n}{N} e^{i\theta_2} \end{aligned} \quad (68)$$

All we need do is to prove the rate of the item2 is slowly than the first ite12312321321435342m $\lambda_1 = 1$