Notes on ESPRIT me21321ods for large array processing

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1 Introduction

In this pa123123per, we would like to evaluate the performance of the estimation of Direction of Arrival (DoA) method ESPRIT [1], the idea of which is described in the following section.

2 System Model

For a Unitary Linear Array (ULA) of size N, we consider the following model for the signal received at time $t = 1, \ldots, T$

$$\mathbf{x}(t) = \sum_{\ell=1}^{K} \mathbf{a}(\theta_\ell) s_\ell(t) + \mathbf{z}(t) \in \mathbb{C}^N \tag{1}$$

with $\mathbf{a}(\theta_\ell) \in \mathbb{C}^N$ the steering vector of source s_ℓ at angle of arrival θ_ℓ, its j-th entry given by[1]

$$[\mathbf{a}(\theta_\ell)]_j = \frac{1}{\sqrt{N}} e^{\imath \frac{2\pi d}{\lambda_0}(j-1)\sin(\theta_\ell)} \equiv \frac{1}{\sqrt{N}} e^{\imath \omega (j-1)\sin(\theta_\ell)}, \quad \omega \equiv \frac{2\pi d}{\lambda_0} \tag{2}$$

where there is in total k signal sources $\{s_\ell\}_{\ell=1}^K$, at angle $\{\theta_\ell\}_{\ell=1}^K$ for some $K \ll \min(N, T)$, as well as some independent Gaussian noise $\mathbf{z}(t) \overset{i.i.d.}{\sim} \mathcal{CN}(\mathbf{0}, \mathbf{I}_N)$ for all t.

The above signal model can be rewritten in matrix model by cascading the total T observations as

$$\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{Z} \tag{3}$$

with $\mathbf{X} = [\mathbf{x}(1), \ldots, \mathbf{x}(T)] \in \mathbb{C}^{N\times T}$, $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_k)] \in \mathbb{C}^{N\times K}$, $\mathbf{S} = [\mathbf{s}(1), \ldots, \mathbf{s}(T)] \in \mathbb{C}^{K\times T}$, column vector $\mathbf{s}(t) = [s_1(t), \ldots, s_k(t)]^\mathsf{T} \in \mathbb{C}^K$ and $\mathbf{Z} = [\mathbf{z}(1), \ldots, \mathbf{z}(T)] \in \mathbb{C}^{N\times T}$ a standard (circular) Gaussian random matrix.

3 ESPRIT method for DoA estimation

In this paper, we would like to evaluate the performance of the estimation of Direction of Arrival (DoA) method ESPRIT [1], the idea of which is described as follows.

[1] The normalization by $\sqrt{N}$ here is for notational convenience so that $\mathbf{a}(\theta_\ell)$ is of unit norm, note that this is equivalent to a *rescaling* of the source signal s_ℓ.

ESPRIT: intuition. Note that the *population* covariance of received signal

$$\begin{aligned}\mathbf{C} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}^{\mathsf{H}}(t)] &= \mathbb{E}[(\mathbf{A}\mathbf{s}(t)+\mathbf{z}(t))(\mathbf{A}\mathbf{s}(t)+\mathbf{z}(t))^{\mathsf{H}}] \\ &= \mathbf{A}\mathbb{E}[\mathbf{s}(t)\mathbf{s}^{\mathsf{H}}(t)]\mathbf{A}^{\mathsf{H}} + \mathbb{E}[\mathbf{z}(t)\mathbf{z}^{\mathsf{H}}(z)] \\ &= \mathbf{A}\mathbf{P}(t)\mathbf{A}^{\mathsf{H}} + \sigma^2 \mathbf{I}_N\end{aligned}$$

where we used the fact that $\mathbf{z}(t)$ is independent of the signal $\mathbf{s}(t)$ and denote the signal power $\mathbf{P}(t) \equiv \mathbb{E}[\mathbf{s}(t)\mathbf{s}^{\mathsf{H}}(t)]$. Then, for diagonal $\mathbf{P}(t) = \operatorname{diag}\{p_\ell(t)\}_{\ell=1}^{k}$ (which implies uncorrelated signal in the Gaussian case), one has

$$\mathbf{C} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}^{\mathsf{H}}(t)] = \sum_{k=1}^{K} p_k(t)\mathbf{a}(\theta_k)\mathbf{a}^{\mathsf{H}}(\theta_k) + \sigma^2\mathbf{I}_N = \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}} + \sigma^2\mathbf{I}_N, \tag{4}$$

so that the top subspace of *population* covariance is expected to obtain structure information about the subspace spanned by the steering vectors $\mathbf{a}(\theta_k)$. *If the sample covariance* $\hat{\mathbf{C}}$ *is a good "proxy" of the population* $\mathbf{C}$ in the sense that, e.g.,

$$\|\hat{\mathbf{C}} - \mathbf{C}\| \to 0 \tag{5}$$

in spectral norm, then, one has, by Davis–Kahan theorem that

$$\|\hat{\mathbf{U}}_S - \mathbf{U}_S\|_F \to 0, \tag{6}$$

(in fact, this holds for each individual eigenvector).

On the other hand, using the rotational invariance of the matrix $\mathbf{A}$, we have, for two selection matrices $\mathbf{J}_1, \mathbf{J}_2 \in \mathbb{R}^{n\times N}$ that selection n among the in total N rows of $\mathbf{X}$, with "distance " Δ, that

$$\mathbf{J}_1\mathbf{A}\operatorname{diag}\{e^{\imath\omega\Delta\cdot\sin(\theta_\ell)}\}_{k=1}^{K} = \mathbf{J}_2\mathbf{A} \tag{7}$$

with

$$\mathbf{J}_1 = \begin{bmatrix} \mathbf{e}_k^{\mathsf{T}} \\ \cdots \\ \mathbf{e}_{n+k-1}^{\mathsf{T}} \end{bmatrix} \in \mathbb{R}^{n\times N}, \quad \mathbf{J}_2 = \begin{bmatrix} \mathbf{e}_{k+\Delta}^{\mathsf{T}} \\ \cdots \\ \mathbf{e}_{n+k+\Delta-1}^{\mathsf{T}} \end{bmatrix} \in \mathbb{R}^{n\times N} \tag{8}$$

for $\mathbf{e}_k$ the canonical vector of $\mathbb{R}^N$ with $[\mathbf{e}_k]_i = \delta_{ij}$. We take, without loss of generality, $k = 1$ here so that

$$\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1 = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{N-n} \\ \mathbf{0}_{N-n} & \mathbf{0}_{N-n} \end{bmatrix}, \quad \mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2 = \begin{bmatrix} \mathbf{0}_{n\times\Delta} & \mathbf{I}_n & \mathbf{0}_{n\times(N-n-\Delta)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{9}$$

so that with $\mathbf{U}_S = \mathbf{A}\mathbf{Q}^{-1}$ for some invertible $\mathbf{Q} \in \mathbb{C}^{K\times K}$ and $\hat{\mathbf{U}}_S \simeq \mathbf{U}_S$, we obtain

$$\mathbf{Q}\operatorname{diag}\{e^{\imath\omega\Delta\cdot\sin(\theta_\ell)}\}_{\ell=1}^{k}\mathbf{Q}^{-1} = (\mathbf{J}_1\mathbf{U}_S)^{\dagger}\mathbf{J}_2\mathbf{U}_S \simeq (\mathbf{J}_1\hat{\mathbf{U}}_S)^{\dagger}\mathbf{J}_2\hat{\mathbf{U}}_S. \tag{10}$$

ESPRIT: algorithm.

1. define two selection matrices $\mathbf{J}_1, \mathbf{J}_2 \in \mathbb{R}^{n\times N}$ that selection n among the in total N rows of $\mathbf{X}$, with "distance " Δ, for instance, $\mathbf{J}_1$ from row i to $i+n$, and $\mathbf{J}_2$ from $i+\Delta$ to $i+\Delta+n$;

2. compute the sample covariance matrix $\hat{\mathbf{C}} = \frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}(t)\mathbf{x}^{\mathsf{H}}(t)$, and denote $\hat{\mathbf{U}}_S$ the top subspace composed of the eigenvectors associated to the largest-k eigenvalues, or the so-called *signal subspace*;

3. compute $\mathbf{\Phi} = (\mathbf{J}_1 \hat{\mathbf{U}}_S)^\dagger \mathbf{J}_2 \hat{\mathbf{U}}_S \in \mathbb{C}^{K \times K}$, where we denote $\mathbf{A}^\dagger$ the Moore–Penrose pseudoinverse of $\mathbf{A}$, note that the resulting matrix $\mathbf{\Phi}$ is, in general, non-Hermitian;

4. the estimate of the angles of $\hat{\theta}_k$ are given by

$$\hat{\theta}_k = \arcsin(\arg(\lambda_k(\mathbf{\Phi}))/\omega/\Delta), \tag{11}$$

with λ_k the kth (complex) eigenvalues of $\mathbf{\Phi}$.

4 Characterization of ESPRIT method for large linear array

It follows from (7) that, for selection matrix $\mathbf{J}_1$ such that $\mathbf{J}_1 \mathbf{U}_S$ has linearly independent columns so that the inverse $(\mathbf{U}_S^H \mathbf{J}_1^H \mathbf{J}_1 \mathbf{U}_S)^{-1}$ is well defined, we have

$$\operatorname{diag}\{e^{\imath\omega\Delta\cdot\sin(\theta_\ell)}\}_{k=1}^K = (\mathbf{A}^\mathsf{H}\mathbf{J}_1^\mathsf{H}\mathbf{J}_1\mathbf{A})^{-1}\mathbf{A}^\mathsf{H}\mathbf{J}_1^\mathsf{H}\mathbf{J}_2\mathbf{A}$$

where we assume that the selection matrix It thus suffices to evaluate the two terms $\mathbf{A}^\mathsf{H}\mathbf{J}_1^\mathsf{H}\mathbf{J}_1\mathbf{A}$ and $\mathbf{A}^\mathsf{H}\mathbf{J}_1^\mathsf{H}\mathbf{J}_2\mathbf{A}$ so as to retrieve the DoA θ_k as desired.

We positive ourselves under the following large array scenario.

Assumption 1 (Large array). *As $T \to \infty$, we have that*

$$0 < \liminf_T N/T < \limsup_T N/T < \infty, \quad 0 < \liminf_T n/N < \limsup_T n/N < 1. \tag{12}$$

Assumption 2 (Widely spaced DoA). *All DoA angles $\theta_1, \ldots, \theta_K$ are* fixed *as $N, T \to \infty$.*

The *widely spaced* DoA scenario as Assumption 2 practically arises, e.g., when the DoA have an angular separation much larger than a beamwidth [2], by considering the case of all DoAs $\theta_1, \ldots, \theta_K$ are *fixed* with respect to N large. In this case, we have in particular that

$$\begin{aligned}
[\mathbf{A}^\mathsf{H}\mathbf{A}]_{ij} = \mathbf{a}(\theta_i)^\mathsf{H}\mathbf{a}(\theta_j) &= \frac{1}{N}\sum_{\ell=1}^N e^{-\imath\omega(\ell-1)(\sin(\theta_j)-\sin(\theta_i))} \\
&= \begin{cases} 1 & \text{for } i = j \\ \frac{1}{N}\frac{1-e^{-\imath\omega(\ell-1)N(\sin(\theta_j)-\sin(\theta_i))}}{1-e^{-\imath\omega(\ell-1)(\sin(\theta_j)-\sin(\theta_i))}} = O(N^{-1}) & \text{for } i \neq j. \end{cases}
\end{aligned}$$

$$[\mathbf{A}^\mathsf{H}\mathbf{J}_1^\mathsf{H}\mathbf{J}_1\mathbf{A}]_{ij} = \mathbf{a}(\theta_i)^\mathsf{H}\mathbf{J}_1^\mathsf{H}\mathbf{J}_1\mathbf{a}(\theta_j) = \begin{cases} \frac{n}{N}, & \text{for } i = j; \\ O(N^{-1}), & \text{for } i \neq j \end{cases} \tag{13}$$

as well as

$$[\mathbf{A}^\mathsf{H}\mathbf{J}_1^\mathsf{H}\mathbf{J}_2\mathbf{A}]_{ij} = \mathbf{a}(\theta_i)^\mathsf{H}\mathbf{J}_1^\mathsf{H}\mathbf{J}_1\mathbf{a}(\theta_j) = \begin{cases} \frac{n}{N}e^{-\imath\omega\Delta\sin(\theta_i)}, & \text{for } i = j; \\ O(N^{-1}), & \text{for } i \neq j. \end{cases} \tag{14}$$

As such, under Assumption 1 and 2, we have, in matrix form that

$$\mathbf{A}^\mathsf{H}\mathbf{J}_1^\mathsf{H}\mathbf{J}_1\mathbf{A} = \begin{bmatrix} \mathbf{a}(\theta_1)^\mathsf{H}\mathbf{J}_1^\mathsf{H}\mathbf{J}_1\mathbf{a}(\theta_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \mathbf{a}(\theta_K)^\mathsf{H}\mathbf{J}_1^\mathsf{H}\mathbf{J}_1\mathbf{a}(\theta_K) \end{bmatrix} + O_{\|\cdot\|}(N^{-1}), \tag{15}$$

and similarly

$$\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_2\mathbf{A} = \begin{bmatrix} \mathbf{a}(\theta_1)^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_2\mathbf{a}(\theta_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \mathbf{a}(\theta_K)^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_2\mathbf{a}(\theta_K) \end{bmatrix} + O_{\|\cdot\|}(N^{-1}). \tag{16}$$

As a consequence, we get

$$\mathbf{Q}^{-1}\operatorname{diag}\{e^{\imath\omega\Delta\cdot\sin(\theta_k)}\}_{k=1}^{K}\mathbf{Q} = (\mathbf{Q}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_1\mathbf{A}\mathbf{Q})^{-1}\mathbf{Q}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_2\mathbf{A}\mathbf{Q} = (\mathbf{U}_S^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_1\mathbf{U}_S)^{-1}\mathbf{U}_S^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_2\mathbf{U}_S, \tag{17}$$

holds for some invertible matrix $\mathbf{Q} \in \mathbb{C}^{K\times K}$ such that

$$\mathbf{U}_S = \mathbf{A}\mathbf{Q}. \tag{18}$$

[Zhenyu: The above claim to clarify!]

The above result illustrates two cases, one is when N tends to infinity, the elements on the off-diagonal are 0 and the division of the elements on the diagonal is the angle we want

$$\begin{aligned}\mathbf{Q}\operatorname{diag}\{e^{\imath\omega\Delta\cdot\sin(\theta_\ell)}\}_{\ell=1}^{k}\mathbf{Q}^{-1} &= (\mathbf{J}_1\mathbf{U}_S)^{\dagger}\mathbf{J}_2\mathbf{U}_S. \\ &= (diag(\mathbf{U}_S^{H}\mathbf{J}_1^{H}\mathbf{J}_1\mathbf{U}_S))\backslash diag(\mathbf{U}_S^{H}\mathbf{J}_1^{H}\mathbf{J}_2\mathbf{U}_S)\end{aligned}$$

In practice we cannot correctly estimate the u_i,and we tend to approximate u_i by $\hat{u}_i$,which leads to an error.

We need to find two functions $u_i^H\mathbf{J}_1^H\mathbf{J}_1u_i = f_1(\hat{u}_i^H\mathbf{J}_1\mathbf{J}_1^H\hat{u}_i), u_i^H\mathbf{J}_1^H\mathbf{J}_2u_i = f_2(\hat{u}_i^H\mathbf{J}_1^H\mathbf{J}_2\hat{u}_i)$, We need to approximate the real eigenvector by these two functions

If we can perfectly estimate the true eigenvector,

$$(diag(\mathbf{A}^H\mathbf{J}_1^H\mathbf{J}_1\mathbf{A}))\backslash diag(\mathbf{A}^H\mathbf{J}_1^H\mathbf{J}_2\mathbf{A}) = \operatorname{diag}\{e^{\imath\omega\Delta\cdot\sin(\theta_\ell)}\}_{\ell=1}^{k}$$

5 Simplified model with single DoA

In the case of $k = 1$ with angle θ, we have the following simplified model

$$\mathbf{X} = \mathbf{a}(\theta)[s(1), \ldots, s(T)] + \mathbf{Z} \equiv \mathbf{a}(\theta)\mathbf{s}^{\mathsf{H}} + \mathbf{Z} \tag{19}$$

so that

$$\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} = \frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} + \begin{bmatrix}\mathbf{a} & \frac{\mathbf{Z}\mathbf{s}}{T}\end{bmatrix}\begin{bmatrix}\rho & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}\mathbf{a}^{\mathsf{H}} \\ \frac{(\mathbf{Z}\mathbf{s})^{\mathsf{H}}}{T}\end{bmatrix} \equiv \frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{\mathsf{H}} \tag{20}$$

where we denote $\rho = \frac{1}{T}\|\mathbf{s}\|^2$ the (limit of the normalized) signal strength and $\mathbf{V} = \begin{bmatrix}\mathbf{a} & \frac{\mathbf{Z}\mathbf{s}}{T}\end{bmatrix} \in \mathbb{C}^{N\times 2}$.

As a consequence,

$$\begin{aligned}\left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} &= \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} \\ &= \mathbf{Q} - \mathbf{Q}\mathbf{V}\mathbf{\Lambda}(\mathbf{I}_2 + \mathbf{V}^{\mathsf{H}}\mathbf{Q}\mathbf{V}\mathbf{\Lambda})^{-1}\mathbf{V}^{\mathsf{H}}\mathbf{Q}\end{aligned}$$

where we denoted $\mathbf{Q}(z) = \mathbf{Q} = (\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}$ and used Woodbury matrix identity. Now, since

$$\mathbf{V}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{V} = \begin{bmatrix} \mathbf{a}^{\mathsf{H}} \\ \frac{(\mathbf{Z}\mathbf{s})^{\mathsf{H}}}{T} \end{bmatrix} \mathbf{Q}(z) \begin{bmatrix} \mathbf{a} & \frac{\mathbf{Z}\mathbf{s}}{T} \end{bmatrix} = \begin{bmatrix} \mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{a} & 0 \\ 0 & \frac{1}{T}\mathbf{s}^{\mathsf{H}}\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{Z}\mathbf{s} \end{bmatrix} + o(1) \tag{21}$$

with

$$\frac{1}{T}\mathbf{s}^{\mathsf{H}}\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{Z}\mathbf{s} = \frac{1}{T}\mathbf{s}^{\mathsf{H}}\tilde{\mathbf{Q}}(z)\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Z}\mathbf{s} = \frac{1}{T}\mathbf{s}^{\mathsf{H}}(\mathbf{I}_T + z\tilde{\mathbf{Q}}(z))\mathbf{s} = \rho + \frac{z}{T}\mathbf{s}^{\mathsf{H}}\tilde{\mathbf{Q}}(z)\mathbf{s} \tag{22}$$

for co-resolvent $\tilde{\mathbf{Q}}(z) = \left(\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Z} - z\mathbf{I}_T\right)^{-1}$.

Since

$$\mathbf{Q}(z) \leftrightarrow \bar{\mathbf{Q}}(z) = m(z)\mathbf{I}_N = \left(\frac{1}{1+cm(z)} - z\right)^{-1}\mathbf{I}_N, \quad \tilde{\mathbf{Q}}(z) = -\left(\frac{1}{zm(z)} + 1\right)\mathbf{I}_T \tag{23}$$

for $c = \lim N/T$ and $m(z)$ the unique solution of the Marčenko-Pastur equation

$$zcm^2(z) - (1-c-z)m(z) + 1 = 0, \tag{24}$$

Therefore

$$(\mathbf{I}_2 + \mathbf{V}^{\mathsf{H}}\mathbf{Q}\mathbf{V}\mathbf{\Lambda})^{-1} = \begin{bmatrix} 1+\rho m(z) & m(z) \\ \rho\left(1 - z - \frac{1}{m(z)}\right) & 1 \end{bmatrix}^{-1} + o(1) \tag{25}$$

and

$$\mathbf{V}\mathbf{\Lambda}(\mathbf{I}_2 + \mathbf{V}^{\mathsf{H}}\mathbf{Q}\mathbf{V}\mathbf{\Lambda})^{-1}\mathbf{V}^{\mathsf{H}} = \begin{bmatrix} \mathbf{a} & \frac{\mathbf{Z}\mathbf{s}}{T} \end{bmatrix} \frac{1}{1+\rho+\rho z m(z)} \begin{bmatrix} \rho z + \frac{\rho}{m(z)} & \mathsf{H} \\ \mathsf{H} & \mathsf{H} \end{bmatrix} \begin{bmatrix} \mathbf{a}^{\mathsf{H}} \\ \frac{(\mathbf{Z}\mathbf{s})^{\mathsf{H}}}{T} \end{bmatrix} + o_{\|\cdot\|}(1) \tag{26}$$

so that it suffices to evaluate the following expectations:

1. $\mathbb{E}[\mathbf{Q}(z)\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)] = m^2(z)\mathbf{a}\mathbf{a}^{\mathsf{H}} + o_{\|\cdot\|}(1)$;
2. $\frac{1}{T}\mathbb{E}[\mathbf{Q}(z)\mathbf{a}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{T}}\mathbf{Q}(z)] = o_{\|\cdot\|}(1)$ and it Hermitian transpose.

This thus allows to conclude that

$$\left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} \leftrightarrow m(z)\mathbf{I}_N - \frac{\rho m(z)(zm(z)+1)}{1+\rho(zm(z)+1)}\mathbf{a}\mathbf{a}^{\mathsf{H}}. \tag{27}$$

5.1 Random matrix analysis

In the case of single DoA, our object of interest is the following *complex* random variable

$$Z = \frac{\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}}{\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}}, \tag{28}$$

with $\hat{\mathbf{u}} \in \mathbb{C}^N$ the dominant eigenvector of the sample covariance matrix $\hat{\mathbf{C}} = \frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}}$. Note that $\|\mathbf{J}_i^{\mathsf{T}}\mathbf{J}_j\| = O(1)$ for $i,j \in \{1,2\}$, we have

According to (**??**), we need to evaluate the random variable of the form $\mathbf{y}_1^{\mathsf{H}}\mathbf{Q}(z)\mathbf{y}_2$ for $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{C}^N$ of bounded Euclidean norm. To this end, we introduce the following deterministic equivalent result.

Theorem 1 (Deterministic Equivalent)**.**

$$\left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} \leftrightarrow m(z)\mathbf{I}_N - \frac{\rho m(z)(zm(z)+1)}{1+\rho(zm(z)+1)}\mathbf{a}\mathbf{a}^{\mathsf{H}}. \tag{29}$$

As a consequence, we have, in the notation of (**??**),

$$\mathbf{e}_j^{\mathsf{H}}(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}\mathbf{e}_j \leftrightarrow m(z) - \frac{\rho m(z)(zm(z)+1)}{1+\rho(zm(z)+1)}(\mathbf{e}_j^{\mathsf{H}}\mathbf{a})^2 \tag{30}$$

and

$$\mathbf{e}_{j+\Delta}^{\mathsf{H}}(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}\mathbf{e}_j \leftrightarrow m(z)\delta_{\Delta=0} - \frac{\rho m(z)(zm(z)+1)}{1+\rho(zm(z)+1)}(\mathbf{e}_{j+\Delta}^{\mathsf{H}}\mathbf{a})(\mathbf{a}^{\mathsf{H}}\mathbf{e}_j) \tag{31}$$

It follows from Cauchy's integral formula and residue theorem that, for Γ_S circling around the isolated eigenvalue, we have

$$\begin{aligned}
&-\frac{1}{2\pi\imath}\oint_{\Gamma_S}\mathbf{e}_j^{\mathsf{H}}(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}\mathbf{e}_j\,dz \simeq \frac{1}{2\pi\imath}\oint_{\Gamma_S}\frac{m(z)}{1+\rho(zm(z)+1)}dz\cdot(\mathbf{e}_j^{\mathsf{H}}\mathbf{a})^2\\
&= -\lim_{z\to\lambda_S}(z-\lambda_S)\frac{m(z)}{1+\rho(zm(z)+1)}\cdot(\mathbf{e}_j^{\mathsf{H}}\mathbf{a})^2 = -\frac{m(\lambda_S)}{\rho(m(\lambda_S)+\lambda_S m'(\lambda_S))}\cdot(\mathbf{e}_j^{\mathsf{H}}\mathbf{a})^2
\end{aligned}$$

with $\lambda_S \equiv 1+\rho+c\frac{1+\rho}{\rho}$ the (asymptotic) position of the isolated eigenvalue and $m'(z) = \frac{m^2(z)}{1-\frac{cm^2(z)}{(1+cm(z))^2}}$ obtained by differentiating the Marčenko-Pastur equation, where we used the fact that Γ_S does not contain any pole of $m(z)$.

And similarly that

$$-\frac{1}{2\pi\imath}\oint_{\Gamma_S}\mathbf{e}_{j+\Delta}^{\mathsf{H}}(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}\mathbf{e}_j\,dz \simeq -\frac{m(\lambda_S)}{\rho(m(\lambda_S)+\lambda_S m'(\lambda_S))}\cdot\mathbf{e}_{j+\Delta}^{\mathsf{H}}\mathbf{a}\cdot\mathbf{a}^{\mathsf{H}}\mathbf{e}_j \tag{32}$$

To-do list:

1. simulate the different between $\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{M}\hat{\mathbf{u}}$ and its deterministic equivalent, for spiked covariance model and big matrix $\mathbf{M}$ with large rank, Note that two things happen at the same time: i) $\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{e}\mathbf{e}^{\mathsf{H}}\hat{\mathbf{u}} = O(1)$ and ii) $\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{I}_N\hat{\mathbf{u}} = 1 = O(1)$.

6 Proof of ESPIRIT in the case of single DoA

To investigate the performance of ESPIRIT method, one needs to evaluate the statistic of the dominant eigenvector $\hat{\mathbf{u}}_S$ of the spiked random matrix model $\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}}$, for $\mathbf{X} = \mathbf{a}\mathbf{s}^{\mathsf{H}} + \mathbf{Z}$ with $\mathbf{a} \in \mathbb{C}^N$ such that $\|\mathbf{a}\| = 1$, $\mathbf{s} \in \mathbb{C}^T$ a standard (circular) Gaussian random vector, and $\mathbf{Z} \in \mathbb{C}^{N\times T}$ a standard (circular) Gaussian random matrix.

Denote $(\hat{\lambda}, \hat{\mathbf{u}})$ the pair of largest eigenvalue-eigenvector pair of $\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}}$, and thus satisfies

$$\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{T}}\hat{\mathbf{u}} = \hat{\lambda}\hat{\mathbf{u}} = \frac{\|\mathbf{s}\|^2}{T}\mathbf{a}\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}} + \frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}}\hat{\mathbf{u}} + \frac{1}{T}\left(\mathbf{a}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}} + \mathbf{Z}\mathbf{s}\mathbf{a}^{\mathsf{H}}\right)\hat{\mathbf{u}}. \tag{33}$$

Denote $\mathbf{Q}(z) = (\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N)$, for $z \in \mathbb{C}$ not an eigenvalue of $\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{T}}$ (which is known to have eigenvalues lying within the MP support as $N, T \to \infty$), we obtain

$$\begin{aligned}
&\mathbf{0} = \frac{\|\mathbf{s}\|^2}{T}\mathbf{a}\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}} + \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - \hat{\lambda}\mathbf{I}_N\right)\hat{\mathbf{u}} + \frac{1}{T}\left(\mathbf{a}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}} + \mathbf{Z}\mathbf{s}\mathbf{a}^{\mathsf{H}}\right)\hat{\mathbf{u}} \\
&\Leftrightarrow -\hat{\mathbf{u}} = \frac{\|\mathbf{s}\|^2}{T}\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}} \cdot \mathbf{Q}(\hat{\lambda})\mathbf{a} + \frac{1}{T}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\hat{\mathbf{u}} \cdot \mathbf{Q}(\hat{\lambda})\mathbf{a} + \mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}} \cdot \frac{1}{T}\mathbf{Q}(\hat{\lambda})\mathbf{Z}\mathbf{s} \\
&\Rightarrow \sqrt{N}[\hat{\mathbf{u}}]_i = \sqrt{N}\mathbf{e}_i^{\mathsf{T}}\hat{\mathbf{u}} = -\frac{\|\mathbf{s}\|^2}{T}\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}} \cdot \sqrt{N}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{a} - \frac{1}{T}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\hat{\mathbf{u}} \cdot \sqrt{N}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{a} - \mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}} \cdot \frac{\sqrt{N}}{T}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{Z}\mathbf{s}
\end{aligned}$$

Note that till now no asymptotic approximation has been performed, we have only used linear algebraic results.

Proof to-do list:

(i) establish the asymptotic *complex* limit of $\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}} = ? + o(1)$; and

(ii) establish the asymptotic *complex* limit of $\sqrt{N}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{a} = ? + o(1)$; and

(iii) show that $\frac{1}{T}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\hat{\mathbf{u}} \to 0$ almost surely (this, together with item (ii), allows us to asymptotic discard the term $\frac{1}{T}\mathbf{s}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\hat{\mathbf{u}} \cdot \sqrt{N}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{a}$);

this allows us to conclude that the i-th entry of $\hat{\mathbf{u}}$ satisfies

$$\sqrt{N}[\hat{\mathbf{u}}]_i = -\mathbf{a}^{\mathsf{H}}\hat{\mathbf{u}}\left(\underbrace{\frac{\|\mathbf{s}\|^2}{T}\sqrt{N}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{a}}_{\text{deterministic } O(1)+o(1)} + \underbrace{\frac{\sqrt{N}}{T}\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}(\hat{\lambda})\mathbf{Z}\mathbf{s}}_{\text{Gaussian fluctuation } O(1)}\right) + o(1). \tag{34}$$

This further leads to

$$\begin{aligned}
N[\hat{\mathbf{u}}]_i^2 &= +o(1), \\
N\overline{[\hat{\mathbf{u}}]_i}[\hat{\mathbf{u}}]_j &= +o(1).
\end{aligned}$$

We thus obtain

$$(\mathbf{J}_1\hat{\mathbf{u}})^{\dagger}\mathbf{J}_2\hat{\mathbf{u}} = \frac{\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_2\hat{\mathbf{u}}}{\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{H}}\mathbf{J}_1\hat{\mathbf{u}}} = \frac{\sum_{j=i}^{i+n}\overline{[\hat{\mathbf{u}}]_j}[\hat{\mathbf{u}}]_{j+\Delta}}{\sum_{j=i}^{i+n}[\hat{\mathbf{u}}]_j^2} \tag{35}$$

7 Alternative proof of ESPIRIT in the case of single DoA: random signal case

In the case of (proper) complex Gaussian signal $\mathbf{s} \sim \mathcal{CN}(\mathbf{0}, \rho^2\mathbf{I}_T)$ with signal strength ρ^2, we have that the observation matrix $\mathbf{X} \in \mathbb{C}^{N\times}$ is equivalently given by

$$\mathbf{X} = \left(\mathbf{I}_N + \rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{\frac{1}{2}}\mathbf{Z}, \tag{36}$$

for standard complex Gaussian $\mathbf{Z} \in \mathbb{C}^{N\times T}$.

Let us first consider the form $\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}$, with $\hat{\mathbf{u}}$ the dominant eigenvector of the SCM $\hat{\mathbf{C}} = \frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}}$. We have, for Γ circling around the isolated eigenvalue of $\hat{\mathbf{C}}$, that

$$
\begin{aligned}
\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}} &= \sum_{i=1}^{n} \hat{\mathbf{u}}^{\mathsf{H}}\mathbf{e}_i\mathbf{e}_{i+\Delta}^{\mathsf{T}}\hat{\mathbf{u}} \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}(\hat{\mathbf{C}} - z\mathbf{I}_N)^{-1}\mathbf{e}_i\,dz \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-1}\right)^{-1}\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz, \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N + z\frac{\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}}{1+\rho^2}\right)^{-1}\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz, \\
&= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\left(\mathbf{Q}(z) - \frac{\rho^2}{1+\rho^2}\frac{z\mathbf{Q}(z)\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)}{1+\frac{\rho^2}{1+\rho^2}\cdot z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{a}}\right)\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz, \\
&= \frac{1}{2\pi\imath}\frac{\rho^2}{1+\rho^2}\sum_{i=1}^{n}\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\frac{z\mathbf{Q}(z)\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)}{1+\frac{\rho^2}{1+\rho^2}\cdot z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{a}}\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz,
\end{aligned}
$$

with the resolvent

$$
\mathbf{Q}(z) \equiv \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1}. \tag{37}
$$

This leads to

$$
\begin{aligned}
\hat{\mathbf{u}}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}} &= \frac{1}{2\pi\imath}\frac{\rho^2}{1+\rho^2}\sum_{i=1}^{n}\oint_\Gamma \frac{z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{a}}{1+\frac{\rho^2}{1+\rho^2}\cdot z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{a}}\,dz, \\
&= \frac{1}{2\pi\imath}\frac{\rho^2}{1+\rho^2}\oint_\Gamma \frac{z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{a}}{1+\frac{\rho^2}{1+\rho^2}\cdot z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{a}}\,dz,
\end{aligned}
$$

Note that the only pole in this case is $\lambda \in \mathbb{R}$ such that

$$
z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(\lambda)\mathbf{a} = -\frac{1+\rho^2}{\rho^2}. \tag{38}
$$

Since $\mathbf{Q}(z) \leftrightarrow \bar{\mathbf{Q}}(z) = m(z)\mathbf{I}_N$, we have

$$
h(\lambda) \equiv \lambda m(\lambda) = -1 - \rho^{-2}, \tag{39}
$$

with

$$
\lambda = 1 + \rho^2 + c\frac{1+\rho^2}{\rho^2}, \tag{40}
$$

and therefore the following first-order result

$$
\begin{aligned}
&\frac{1}{2\pi\imath}\frac{\rho^2}{1+\rho^2}\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\frac{z\mathbf{Q}(z)\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)}{1+\frac{\rho^2}{1+\rho^2}\cdot z\mathbf{a}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{a}}\left(\mathbf{I}_N+\rho^2\mathbf{a}\mathbf{a}^{\mathsf{H}}\right)^{-\frac{1}{2}}\mathbf{e}_i\,dz\\
&\simeq \frac{1}{2\pi\imath}\frac{\rho^2}{1+\rho^2}\oint_\Gamma \frac{zm^2(z)\,dz}{1+\rho^2+\rho^2 zm(z)}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{e}_i+O(N^{-1/2})\\
&= -\mathrm{Res}\left(\frac{zm^2(z)\,dz}{1+\rho^2+\rho^2 zm(z)}\right)\frac{\rho^2}{1+\rho^2}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{e}_i+O(N^{-1/2})\\
&= \frac{m(\lambda)(1+h(\lambda))}{h'(\lambda)}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{e}_i+O(N^{-1/2}).
\end{aligned}
$$

Since $m(z)$ is the solution to

$$
zcm^2(z)-(1-c-z)m(z)+1=0, \tag{41}
$$

we have

$$
m'(z)=\frac{m^2(z)}{1-\frac{cm^2(z)}{(1+cm(z))^2}}, \tag{42}
$$

so that

$$
h'(z)=m(z)+zm'(z). \tag{43}
$$

Further note that

$$
\sum_{i=1}^{n}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{a}\mathbf{a}^{\mathsf{H}}\mathbf{e}_i=\mathbf{a}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{a}. \tag{44}
$$

8 Proof of ESPIRIT in the case of multiple DoAs: random signal case

In this case, we have

$$
\mathbf{X}=(\mathbf{I}_N+\mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}})^{\frac{1}{2}}\mathbf{Z},\quad \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}=\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathsf{H}},\quad \mathbf{U}\in\mathbb{C}^{N\times K},\quad \boldsymbol{\Lambda}=\mathrm{diag}\{\rho_i\}_{i=1}^{K}, \tag{45}
$$

for standard complex Gaussian $\mathbf{Z}\in\mathbb{C}^{N\times T}$.

Let us first consider the diagonal entries of the form $\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_k$, with $\hat{\mathbf{u}}_k$ the kth dominant eigenvector of the SCM $\hat{\mathbf{C}}=\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}}$. We have, for Γ circling around the isolated eigenvalue of $\hat{\mathbf{C}}$, that

$$
\begin{aligned}
\hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \hat{\mathbf{u}}_k &= \sum_{i=1}^{n} \hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{e}_i \mathbf{e}_{i+\Delta}^{\mathsf{T}} \hat{\mathbf{u}}_k \\
&= -\frac{1}{2\pi\imath} \sum_{i=1}^{n} \oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}} (\hat{\mathbf{C}} - z\mathbf{I}_N)^{-1} \mathbf{e}_i \, dz \\
&= -\frac{1}{2\pi\imath} \sum_{i=1}^{n} \oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-1} \right)^{-1} \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{e}_i \, dz, \\
&= -\frac{1}{2\pi\imath} \sum_{i=1}^{n} \oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N + z\mathbf{A}(\mathbf{P}^{-1} + \mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{H}} \right)^{-1} \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{e}_i \, dz, \\
&= -\frac{1}{2\pi\imath} \sum_{i=1}^{n} \oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \left(\mathbf{Q}(z) - z\mathbf{Q}(z)\mathbf{A}\left(\mathbf{P}^{-1} + \mathbf{A}^{\mathsf{H}}\mathbf{A} + z\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{A}\right)^{-1} \mathbf{A}^{\mathsf{H}}\mathbf{Q}(z) \right) \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\right. \\
&= -\frac{1}{2\pi\imath} \sum_{i=1}^{n} \oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \cdot z\mathbf{Q}(z)\mathbf{A}\left(\mathbf{P}^{-1} + \mathbf{A}^{\mathsf{H}}\mathbf{A} + z\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{A}\right)^{-1} \mathbf{A}^{\mathsf{H}}\mathbf{Q}(z) \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{e}_i \, d \\
&= -\frac{1}{2\pi\imath} \oint_{\Gamma} z \operatorname{tr} \left(\left(\mathbf{P}^{-1} + \mathbf{A}^{\mathsf{H}}\mathbf{A} + z\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{A}\right)^{-1} \mathbf{A}^{\mathsf{H}}\mathbf{Q}(z) \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2 \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{Q}(z)\mathbf{A} \right) d
\end{aligned}
$$

with the resolvent

$$
\mathbf{Q}(z) \equiv \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N \right)^{-1}. \tag{46}
$$

We have the following first and second deterministic equivalent results.

$$
\mathbf{Q}(z) \leftrightarrow \bar{\mathbf{Q}}(z) = m(z)\mathbf{I}_N, \quad \mathbf{Q}(z)\mathbf{B}\mathbf{Q}(z) \leftrightarrow \bar{\mathbf{Q}}(z)\mathbf{B}\bar{\mathbf{Q}}(z) + \frac{1}{T}\operatorname{tr}(\mathbf{B})\frac{m'(z)m^2(z)}{(1+cm(z))^2}\mathbf{I}_N, \tag{47}
$$

for any deterministic matrix $\mathbf{B} \in \mathbb{C}^{N \times N}$ of bounded operator norm.

As such,

$$
z\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{A} = zm(z)\mathbf{A}^{\mathsf{H}}\mathbf{A} + o_{\|\cdot\|}(1), \tag{48}
$$

and

$$
\begin{aligned}
&\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z) \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2 \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{Q}(z)\mathbf{A} \\
&= m^2(z)\mathbf{A}^{\mathsf{H}} \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2 \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{A} \\
&+ \frac{1}{T}\operatorname{tr}\left(\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2 \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-1} \right) \frac{m'(z)m^2(z)}{(1+cm(z))^2}\mathbf{A}^{\mathsf{H}}\mathbf{A} + o_{\|\cdot\|}(1),
\end{aligned}
$$

[Zhenyu: Note that we have in general $\frac{1}{T}\operatorname{tr}\left(\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2 \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-1} \right) = o(1)$ and $\mathbf{A}^{\mathsf{H}}\mathbf{A} = \mathbf{I}_K + o_{\|\cdot\|}(1)$. This leads to]

For the non-diagonal entries with $k \neq \ell$, we have instead

$$
\begin{aligned}
&|\hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \hat{\mathbf{u}}_\ell|^2 = \hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \hat{\mathbf{u}}_\ell \hat{\mathbf{u}}_\ell^{\mathsf{H}} \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \hat{\mathbf{u}}_k \\
&= -\frac{1}{2\pi\imath} \oint_{\Gamma_1} \hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 (\hat{\mathbf{C}} - z_1 \mathbf{I}_N)^{-1} \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \hat{\mathbf{u}}_k \, dz_1 \\
&= -\frac{1}{2\pi\imath} \oint_{\Gamma_1} \sum_{i=1}^{n} \hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{e}_i \mathbf{e}_{i+\Delta}^{\mathsf{T}} (\hat{\mathbf{C}} - z_1 \mathbf{I}_N)^{-1} \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \hat{\mathbf{u}}_k \, dz_1 \\
&= -\frac{1}{2\pi\imath} \oint_{\Gamma_1} \sum_{i=1}^{n} \mathbf{e}_{i+\Delta}^{\mathsf{T}} (\hat{\mathbf{C}} - z_1 \mathbf{I}_N)^{-1} \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \hat{\mathbf{u}}_k \hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{e}_i \, dz_1 \\
&= -\frac{1}{4\pi^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \sum_{i=1}^{n} \mathbf{e}_{i+\Delta}^{\mathsf{T}} (\hat{\mathbf{C}} - z_1 \mathbf{I}_N)^{-1} \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 (\hat{\mathbf{C}} - z_2 \mathbf{I}_N)^{-1} \mathbf{e}_i \, dz_1 \, dz_2 \\
&= -\frac{1}{4\pi^2} \sum_{i=1}^{n} \oint_{\Gamma_1} \oint_{\Gamma_2} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z_1 \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-1}\right)^{-1} \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \\
&\times \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z_2 \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-1}\right)^{-1} \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{e}_i \, dz_1 \, dz_2 \\
&= -\frac{1}{4\pi^2} \sum_{i=1}^{n} \oint_{\Gamma_1} \oint_{\Gamma_2} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} z_1 \mathbf{Q}(z_1) \mathbf{A} \left(\mathbf{P}^{-1} + \mathbf{A}^{\mathsf{H}}\mathbf{A} + z_1 \mathbf{A}^{\mathsf{H}} \mathbf{Q}(z_1) \mathbf{A}\right)^{-1} \mathbf{A}^{\mathsf{H}} \mathbf{Q}(z_1) \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right. \\
&\times \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} z_2 \mathbf{Q}(z_2) \mathbf{A} \left(\mathbf{P}^{-1} + \mathbf{A}^{\mathsf{H}}\mathbf{A} + z_2 \mathbf{A}^{\mathsf{H}} \mathbf{Q}(z_2) \mathbf{A}\right)^{-1} \mathbf{A}^{\mathsf{H}} \mathbf{Q}(z_2) \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{e}_i \, dz_1 \, dz_2 \\
&= -\frac{1}{4\pi^2} \oint_{\Gamma_1} \oint_{\Gamma_2} z_1 z_2 \operatorname{tr} \left(\left(\mathbf{P}^{-1} + \mathbf{A}^{\mathsf{H}}\mathbf{A} + z_1 \mathbf{A}^{\mathsf{H}} \mathbf{Q}(z_1) \mathbf{A}\right)^{-1} \times \mathbf{A}^{\mathsf{H}} \mathbf{Q}(z_1) \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{J}_2^{\mathsf{T}} \mathbf{J}_1 \right. \\
&\left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{Q}(z_2) \mathbf{A} \times \left(\mathbf{P}^{-1} + \mathbf{A}^{\mathsf{H}}\mathbf{A} + z_2 \mathbf{A}^{\mathsf{H}} \mathbf{Q}(z_2) \mathbf{A}\right)^{-1} \\
&\left. \times \mathbf{A}^{\mathsf{H}} \mathbf{Q}(z_2) \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \left(\mathbf{I}_N + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{H}}\right)^{-\frac{1}{2}} \mathbf{Q}(z_1) \mathbf{A} \right) dz_1 \, dz_2
\end{aligned}
$$

with

9 Proof of ESPIRIT in the case of multiple DoAs: deterministic signal case

In this case, we have

$$
\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{Z}, \tag{49}
$$

with $\mathbf{X} = [\mathbf{x}(1), \ldots, \mathbf{x}(T)] \in \mathbb{C}^{N \times T}$, $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_k)] \in \mathbb{C}^{N \times K}$, $\mathbf{S}$=$[\mathbf{s}(1), \ldots, \mathbf{s}(T)] \in \mathbb{C}^{K \times T}$, column vector $\mathbf{s}(t) = [s_1(t), \ldots, s_k(t)]^{\mathsf{T}} \in \mathbb{C}^K$ and $\mathbf{Z} = [\mathbf{z}(1), \ldots, \mathbf{z}(T)] \in \mathbb{C}^{N \times T}$ a standard (circular) Gaussian random matrix.

Let us first consider the form $\hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \hat{\mathbf{u}}_k$, with $\hat{\mathbf{u}}$ the kth dominant eigenvector of the SCM $\hat{\mathbf{C}} = \frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}}$. We have, for Γ circling around the kth isolated eigenvalue of $\hat{\mathbf{C}}$, that

$$
\begin{aligned}
\hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \hat{\mathbf{u}}_k &= \sum_{i=1}^{n} \hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{e}_i \mathbf{e}_{i+\Delta}^{\mathsf{T}} \hat{\mathbf{u}}_k \\
&= -\frac{1}{2\pi\imath} \sum_{i=1}^{n} \oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}} (\hat{\mathbf{C}} - z \mathbf{I}_N)^{-1} \mathbf{e}_i \, dz \\
&= -\frac{1}{2\pi\imath} \sum_{i=1}^{n} \oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N + \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathsf{H}}\right)^{-1} \mathbf{e}_i \, dz,
\end{aligned}
$$

with $\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{Z}$ and that

$$\mathbf{U} = \begin{bmatrix} \mathbf{A} & \frac{1}{T}\mathbf{Z}\mathbf{S}^{\mathsf{H}} \end{bmatrix} \in \mathbb{C}^{N \times 2K}, \quad \mathbf{\Lambda} = \begin{bmatrix} \mathbf{P} & \mathbf{I}_K \\ \mathbf{I}_K & \mathbf{0}_K \end{bmatrix} \in \mathbb{R}^{2K \times 2K}, \mathbf{P} = \lim \frac{1}{T}\mathbf{S}\mathbf{S}^{\mathsf{H}}. \tag{50}$$

As such, we get, by Woodbury identity that

$$\begin{aligned} \hat{\mathbf{u}}_k^{\mathsf{H}} \mathbf{J}_1^{\mathsf{T}} \mathbf{J}_2 \hat{\mathbf{u}}_k &= -\frac{1}{2\pi\imath} \sum_{i=1}^{n} \oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \left(\mathbf{Q}(z) - \mathbf{Q}(z)\mathbf{U} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z) \right) \mathbf{e}_i \, dz, \\ &= \frac{1}{2\pi\imath} \sum_{i=1}^{n} \operatorname{tr} \left(\oint_{\Gamma} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \mathbf{Q}(z)\mathbf{U} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{e}_i \, dz \right), \\ &= \frac{1}{2\pi\imath} \operatorname{tr} \left(\oint_{\Gamma} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z)\mathbf{U} \, dz \right), \end{aligned}$$

with

$$\left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U} \right)^{-1} = \begin{bmatrix} (z + \frac{1}{m(z)})(\mathbf{P}^{-1} + (zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A}))^{-1} & (\mathbf{I}_K + (zm(z)+1)\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A}) \\ (\mathbf{I}_K + (zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A}\mathbf{P}) & -m(z)((\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1} + (zm(z)+1)\mathbf{P})^{-1} \end{bmatrix} \tag{51}$$

$$\mathbf{Q}(z) \equiv \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N \right)^{-1}, \tag{52}$$

the resolvent.

As such, we have

$$\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z)\mathbf{U} = \begin{bmatrix} \mathbf{A}^{\mathsf{H}} \\ \frac{1}{T}\mathbf{S}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}} \end{bmatrix} \mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z) \begin{bmatrix} \mathbf{A} & \frac{1}{T}\mathbf{Z}\mathbf{S} \end{bmatrix} \simeq \begin{bmatrix} \mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z)\mathbf{A} & \mathbf{0}_K \\ \mathbf{0}_K & \frac{1}{T}\mathbf{S}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z)\frac{1}{T} \end{bmatrix} \tag{53}$$

with

$$\frac{1}{T}\mathbf{S}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z)\frac{1}{T}\mathbf{Z}\mathbf{S} = [\text{Zenyu}: \frac{1}{T}\mathbf{S}^{\mathsf{H}}\tilde{\mathbf{Q}}(z)\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Z}\tilde{\mathbf{Q}}(z)\mathbf{S}] \tag{54}$$

for the co-resolvent

$$\tilde{\mathbf{Q}}(z) \equiv \left(\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Z} - z\mathbf{I}_T \right)^{-1}. \tag{55}$$

For the non-diagonal entries with $k \neq \ell$, we have instead

$$\begin{aligned} &|\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_\ell|^2 = \hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_\ell\hat{\mathbf{u}}_\ell^{\mathsf{H}}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}_k \\ &= -\frac{1}{2\pi\imath}\oint_{\Gamma_1} \hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2(\hat{\mathbf{C}} - z_1\mathbf{I}_N)^{-1}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}_k \, dz_1 \\ &= -\frac{1}{2\pi\imath}\oint_{\Gamma_1} \sum_{i=1}^{n} \hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{e}_i\mathbf{e}_{i+\Delta}^{\mathsf{T}}(\hat{\mathbf{C}} - z_1\mathbf{I}_N)^{-1}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}_k \, dz_1 \\ &= -\frac{1}{2\pi\imath}\oint_{\Gamma_1} \sum_{i=1}^{n} \mathbf{e}_{i+\Delta}^{\mathsf{T}}(\hat{\mathbf{C}} - z_1\mathbf{I}_N)^{-1}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}_k\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{e}_i \, dz_1 \\ &= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2} \sum_{i=1}^{n} \mathbf{e}_{i+\Delta}^{\mathsf{T}}(\hat{\mathbf{C}} - z_1\mathbf{I}_N)^{-1}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1(\hat{\mathbf{C}} - z_2\mathbf{I}_N)^{-1}\mathbf{e}_i \, dz_1 \, dz_2 \\ &= -\frac{1}{4\pi^2}\sum_{i=1}^{n}\oint_{\Gamma_1}\oint_{\Gamma_2} \mathbf{e}_{i+\Delta}^{\mathsf{T}} \left(\mathbf{Q}(z_1) - \mathbf{Q}(z_1)\mathbf{U} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_1) \right) \mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1 \left(\mathbf{Q}(z_2) - \mathbf{Q}(z_2)\mathbf{U} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}} \right.\right. \\ &= -\frac{1}{4\pi^2}\sum_{i=1}^{n}\oint_{\Gamma_1}\oint_{\Gamma_2} \mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{Q}(z_1)\mathbf{U} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}(z_2)\mathbf{U} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{e}_i \\ &= -\frac{1}{4\pi^2} \operatorname{tr} \left(\oint_{\Gamma_1}\oint_{\Gamma_2} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}(z_2)\mathbf{U} \left(\mathbf{\Lambda} + \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{U} \right)^{-1} \mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z_1)\mathbf{U} \, d \right. \end{aligned}$$

with

——ooooooooooooooooooooooooooooooooooooooooooooo——————-

$$
\begin{aligned}
\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_k &= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_{\Gamma}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{Q}(z)-\mathbf{Q}(z)\mathbf{U}\left(\mathbf{\Lambda}^{-1}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\right)\mathbf{e}_i\,dz,\\
&= \frac{1}{2\pi\imath}\sum_{i=1}^{n}\operatorname{tr}\left(\oint_{\Gamma}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{Q}(z)\mathbf{U}\left(\mathbf{\Lambda}^{-1}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{e}_i\,dz\right),\\
&= \frac{1}{2\pi\imath}\operatorname{tr}\left(\oint_{\Gamma}\left(\mathbf{\Lambda}^{-1}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z)\mathbf{U}\,dz\right),
\end{aligned}
$$

$$
\left(\mathbf{\Lambda}^{-1}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1} = \begin{bmatrix}(z+\frac{1}{m(z)})(\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1} & (\mathbf{I}_K+(zm(z)+1)\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\\ (\mathbf{I}_K+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A}\mathbf{P})^{-1} & \frac{-m(z)\mathbf{P}^{-1}}{zm(z)+1}(\mathbf{I}_K-(\mathbf{I}_K+(zm(z)+1)\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\end{bmatrix} \tag{56}
$$

set $\mathsf{H}_1=(z+\frac{1}{m(z)})(\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}$, $\mathsf{H}_2=\frac{-m(z)\mathbf{P}^{-1}}{zm(z)+1}(\mathbf{I}_K-(\mathbf{I}_K+(zm(z)+1)\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1})$, then

$$
\left(\mathbf{\Lambda}^{-1}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z)\mathbf{U} = \begin{bmatrix}\mathsf{H}_1\mathbf{A}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}\mathbf{A} & \mathbf{0}_K\\ \mathbf{0}_K & \mathsf{H}_2\frac{1}{T}\mathbf{S}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}\frac{1}{T}\mathbf{Z}\mathbf{S}\end{bmatrix}
$$

then

$$
\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_k = \frac{1}{2\pi\imath}\oint_{\Gamma}\operatorname{tr}\left(\mathsf{H}_1\mathbf{A}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}\mathbf{A}\right)+\operatorname{tr}\left(\mathsf{H}_2\frac{1}{T}\mathbf{S}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}\frac{1}{T}\mathbf{Z}\mathbf{S}\right)\,dz.
$$

where

$$
\begin{aligned}
&\frac{1}{2\pi\imath}\oint_{\Gamma}\operatorname{tr}\left(\mathsf{H}_1\mathbf{A}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}\mathbf{A}\right)dz = \frac{1}{2\pi\imath}\oint_{\Gamma}\operatorname{tr}\left((z+\frac{1}{m(z)})(\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}\mathbf{A}\right)dz\\
&= \frac{1}{2\pi\imath}\oint_{\Gamma}(zm^2(z)+m(z))\operatorname{tr}\left((\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{A}\right)dz\\
&= \frac{1}{2\pi\imath}\sum_{i=l}^{l+n-1}\oint_{\Gamma}(zm^2(z)+m(z))\mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{A}(\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\mathbf{A}^{H}\mathbf{e}_i\,dz\\
&= \frac{1}{2\pi\imath}\sum_{i=l}^{l+n-1}\sum_{j=1}^{K}\oint_{\Gamma}\frac{zm^2(z)+m(z)}{zm(z)+1+l_j^{-1}}\mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{A}\mathbf{u}_j\mathbf{u}_j^{H}\mathbf{A}^{\mathsf{H}}\mathbf{e}_i\,dz\\
&= \lim_{z\to\bar{\lambda}_k}\frac{(z-\bar{\lambda}_k)(zm^2(z)+m(z))}{zm(z)+1+l_k^{-1}}\operatorname{tr}\left(\mathbf{A}\mathbf{u}_k\mathbf{u}_k^{H}\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{T}\mathbf{J}_2\right)\\
&= \frac{1-cl_k^{-2}}{1+cl_k^{-1}}\mathbf{u}_k^{H}\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{T}\mathbf{J}_2\mathbf{A}\mathbf{u}_k
\end{aligned}
$$

where we consider the eigendecomposition of $\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A}=\mathbf{U}\mathbf{V}\mathbf{U}^{H}$.

$$
\frac{1}{2\pi\imath}\oint_{\Gamma}\operatorname{tr}\left(\mathsf{H}_2\frac{1}{T}\mathbf{S}^{\mathsf{H}}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}\frac{1}{T}\mathbf{Z}\mathbf{S}\right)dz = 0?
$$

so we have $\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_k = -\frac{1-cl_k^{-2}}{1+cl_k^{-1}}\mathbf{u}_k^{H}\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{T}\mathbf{J}_2\mathbf{A}\mathbf{u}_k+o(1)$.

For the non-diagonal entries with $k \neq \ell$, we have instead

$$|\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\hat{\mathbf{u}}_\ell|^2 = -\frac{1}{4\pi^2}\operatorname{tr}\left(\oint_{\Gamma_1}\oint_{\Gamma_2}\left(\boldsymbol{\Lambda}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}(z_2)\mathbf{U}\left(\boldsymbol{\Lambda}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{J}_1^{\mathsf{T}}$$

$$= -\frac{1}{4\pi^2}\operatorname{tr}\left(\oint_{\Gamma_1}\oint_{\Gamma_2}\mathsf{H}_1(z_1)\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}(z_2)\mathbf{A}\mathsf{H}_1(z_2)\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{Q}(z_1)\mathbf{A}\,dz_1\,dz_2\right)$$

$$= -\frac{1}{4\pi^2}\sum_{i=l}^{l+n-1}\oint_{\Gamma_1}\oint_{\Gamma_2}\sum_{n=1}^{K}\sum_{m=1}^{K}\frac{zm^2(z_1)+m(z_1)}{zm(z_1)+1+l_m^{-1}}\cdot\frac{zm^2(z_2)+m(z_2)}{zm(z_2)+1+l_n^{-1}}\mathbf{e}_i^T\mathbf{A}\mathbf{u}_n\mathbf{u}_n^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_2\mathbf{A}\mathbf{u}_m\mathbf{u}_m^H\mathbf{A}^{\mathsf{H}}\mathbf{e}_{i+\Delta}$$

$$= \frac{1-cl_k^{-2}}{1+cl_k^{-1}}\cdot\frac{1-cl_l^{-2}}{1+cl_l^{-1}}\mathbf{u}_l^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_2\mathbf{A}\mathbf{u}_k\mathbf{u}_k^H\mathbf{A}^H\mathbf{J}_2^T\mathbf{J}_1\mathbf{A}\mathbf{u}_l.$$

The conclusion about the diagonal elements and the nondiagonal elements are the same as $\boldsymbol{\Phi}_2$ in the case of random signals.

$$\begin{aligned}
\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}_k &= -\frac{1}{2\pi\imath}\sum_{i=1}^{n}\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}\left(\mathbf{Q}(z)-\mathbf{Q}(z)\mathbf{U}\left(\boldsymbol{\Lambda}^{-1}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\right)\mathbf{e}_i\,dz,\\
&= \frac{1}{2\pi\imath}\sum_{i=1}^{n}\operatorname{tr}\left(\oint_\Gamma \mathbf{e}_{i+\Delta}^{\mathsf{T}}\mathbf{Q}(z)\mathbf{U}\left(\boldsymbol{\Lambda}^{-1}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{e}_i\,dz\right),\\
&= \frac{1}{2\pi\imath}\operatorname{tr}\left(\oint_\Gamma\left(\boldsymbol{\Lambda}^{-1}+\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}(z)\mathbf{U}\,dz\right),
\end{aligned}$$

Similarly,

$$\frac{1}{2\pi\imath}\oint_\Gamma\operatorname{tr}\left(\mathsf{H}_1\mathbf{A}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}\mathbf{A}\right)dz = \frac{1}{2\pi\imath}\oint_\Gamma\operatorname{tr}\left((z+\frac{1}{m(z)})(\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{Q}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}\mathbf{A}\right)dz$$

$$= \frac{1}{2\pi\imath}\oint_\Gamma\operatorname{tr}\left((z+\frac{1}{m(z)})(\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}[m^2(z)\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{A}+\frac{n}{T}\cdot\frac{m'(z)m^2(z)}{(1+cm(z))^2}\mathbf{I}_K]\right)dz$$

$$= \frac{1-cl_k^{-2}}{1+cl_k^{-1}}\mathbf{u}_k^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_k + \frac{1}{2\pi\imath}\oint_\Gamma\operatorname{tr}\left(\frac{n}{T}\frac{(zm(z)+1)m'(z)m(z)}{(1+cm(z))^2}(\mathbf{P}^{-1}+(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\right)dz$$

$$= \frac{1-cl_k^{-2}}{1+cl_k^{-1}}\mathbf{u}_k^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_k + \frac{1}{2\pi\imath}\oint_\Gamma\sum_{i=1}^{K}\frac{n}{T}\frac{(zm(z)+1)m'(z)m(z)}{(1+cm(z))^2}\cdot\frac{\mathbf{u}_i^H\mathbf{u}_i}{zm(z)+1+l_i^{-1}}dz$$

$$= \frac{1-cl_k^{-2}}{1+cl_k^{-1}}\mathbf{u}_k^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_k + \lim_{z\to\bar{\lambda}_k}\frac{n}{T}\frac{(z-\bar{\lambda}_k)(zm(z)+1)m'(z)m(z)}{(1+cm(z))^2(zm(z)+1+l_k^{-1})}$$

$$= \frac{1-cl_k^{-2}}{1+cl_k^{-1}}\mathbf{u}_k^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_k + \frac{n}{T}\frac{1}{l_k^2-cl_k}+o(1)(????)$$

Similarly for off-diagonal entries:

$$|\hat{\mathbf{u}}_k^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\hat{\mathbf{u}}_\ell|^2 = -\frac{1}{4\pi^2}\operatorname{tr}\left(\oint_{\Gamma_1}\oint_{\Gamma_2}\mathsf{H}_1(z_1)\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}(z_2)\mathbf{A}\mathsf{H}_1(z_2)\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_2)\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{Q}(z_1)\mathbf{A}\,dz_1\,dz_2\right)$$

$$= -\frac{1}{4\pi^2}\operatorname{tr}\left(\oint_{\Gamma_1}\oint_{\Gamma_2}\mathsf{H}_1(z_1)(m(z_1)m(z_2)\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}+\eta(z_1,z_2)\frac{n}{T}\mathbf{I}_K)\mathsf{H}_1(z_2)(m(z_1)m(z_2)\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}+\eta(z_1,z_2)\frac{n}{T}\mathbf{I}$$

$$= o(1).$$

10 Proof of ESPIRIT in the case of multiple DoAs: uncorrelated signal case

$$\mathbf{T}_1 = \mathbf{P}^{-1} + (1 + zm(z))\mathbf{I}_K + o_{\|\cdot\|}(1), \quad \mathbf{T}_2 = m^2(z)\mathbf{A}^{\mathsf{H}}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{A} + o_{\|\cdot\|}(1). \tag{57}$$

Considering the spectral decomposition $\mathbf{P} = \mathbf{U}L\mathbf{U}^H$ with $L = \text{diag}\{l_i, \cdots, l_k\}$ and $\mathbf{U} = [\mathbf{u}_1, \cdots, \mathbf{u}_k]$. Then we have

$$\begin{aligned}
(\mathbf{T}_1)^{-1} &= \frac{1}{1+zm(z)}\mathbf{I}_K - \frac{1}{(1+zm(z))^2}\mathbf{U}(\mathbf{I}_K + \frac{1}{1+zm(z)}Ł^{-1})^{-1}\mathbf{U}^H \\
&= \frac{1}{1+zm(z)}\mathbf{I}_K - \sum_{i=1}^{K}\frac{1}{(1+zm(z))^2}\mathbf{u}_i\frac{(1+zm(z))l_i^{-1}}{1+zm(z)+l_i^{-1}}\mathbf{u}_i^H
\end{aligned}$$

Then

$$\begin{aligned}
[\mathbf{\Phi}_2]_{11} &= -\frac{1}{2\pi\imath}\oint_{\Gamma_1} z\,\text{tr}(\mathbf{T}_1^{-1}\mathbf{T}_2)\,dz \\
&= \frac{1}{2\pi\imath}\oint_{\Gamma_1}\sum_{i=1}^{K}\frac{zm^2(z)}{1+zm(z)}\frac{l_i^{-1}}{1+zm(z)l_i^{-1}}\,\text{tr}(\mathbf{u}_i\mathbf{A}^{\mathsf{H}}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{A}\mathbf{u}_i^H)dz \\
&= \lim_{z\to\bar{\lambda}_1}\frac{(z-\bar{\lambda}_1)zm^2(z)}{1+zm(z)}\frac{l_1^{-1}}{1+zm(z)+l_1^{-1}}\mathbf{u}_1^H\mathbf{A}^{\mathsf{H}}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{A}\mathbf{u}_1 \\
&= \lim_{z\to\bar{\lambda}_1}\frac{(z-\bar{\lambda}_1)zm^2(z)}{1+zm(z)+l_1^{-1}}\mathbf{u}_1^H\mathbf{A}^{\mathsf{H}}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{A}\mathbf{u}_1 + o(1) \\
&= \frac{1-cl_1^{-2}}{1+cl_1^{-1}}\cdot\mathbf{u}_1^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{A}\mathbf{u}_1 + o(1)
\end{aligned}$$

The conclusion is similar to the previous formula, except that $\mathbf{a}(\theta_1)$ is replaced by eigenvector $\mathbf{u}_i^H\mathbf{a}(\theta_1)^H$.

$$\mathbf{T}_2(z_1, z_2) = m(z_1)m(z_2)\mathbf{A}^{\mathsf{H}}\mathbf{B}\mathbf{A} + o_{\|\cdot\|}(1)$$

where $\mathbf{B} = \mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}} \in \mathbb{C}^{N\times N}$.

Similarly,

$$\begin{aligned}
|[\mathbf{\Phi}_2]_{12}|^2 &= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2} z_1z_2\,\text{tr}(\mathbf{T}_2^{\mathsf{H}}(z_1,z_2)\mathbf{T}_1^{-1}(z_1)\mathbf{T}_2(z_1,z_2)\mathbf{T}_1^{-1}(z_2))\,dz_1\,dz_2, \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\sum_{i=1}^{K}\sum_{j=1}^{K}\frac{z_1m^2(z_1)l_i^{-1}z_2m^2(z_2)l_j^{-1}\,\text{tr}(\mathbf{u}_j\mathbf{u}_j^H\mathbf{A}^H\mathbf{B}^H\mathbf{A}\mathbf{u}_i\mathbf{u}_i^H\mathbf{A}^H\mathbf{B}\mathbf{A})\quad dz_1dz_2}{(1+z_1m(z_1))(1+z_1m(z_1)+l_i^{-1})(1+z_2m(z_2))(1+z_2m(z_2)+l_j^{-1})} \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\frac{z_1m^2(z_1)l_1^{-1}z_2m^2(z_2)l_2^{-1}\mathbf{u}_2^H\mathbf{A}^H\mathbf{B}^H\mathbf{A}\mathbf{u}_1\mathbf{u}_1^H\mathbf{A}^H\mathbf{B}\mathbf{A}\mathbf{u}_2\quad dz_1dz_2}{(1+z_1m(z_1))(1+z_1m(z_1)+l_1^{-1})(1+z_2m(z_2))(1+z_2m(z_2)+l_2^{-1})} \\
&= \frac{1-cl_1^{-2}}{1+cl_1^{-1}}\cdot\frac{1-cl_2^{-2}}{1+cl_2^{-1}}\mathbf{u}_2^H\mathbf{A}^H\mathbf{J}_2^T\mathbf{J}_1\mathbf{A}\mathbf{u}_1\mathbf{u}_1^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_2\mathbf{A}\mathbf{u}_2.
\end{aligned}$$

where we use the fact that $\mathbf{u}_2^H\mathbf{A}^H\mathbf{B}^H\mathbf{A}\mathbf{u}_1\mathbf{u}_1^H\mathbf{A}^H\mathbf{A}\mathbf{u}_2 = \bar{\lambda}_1\bar{\lambda}_2\mathbf{u}_2^H\mathbf{A}^H\mathbf{J}_2^T\mathbf{J}_1\mathbf{A}\mathbf{u}_1\mathbf{u}_1^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_2\mathbf{A}\mathbf{u}_2$ in the last line.

$$\mathbf{T}_3 = m^2(z)\mathbf{A}^{\mathsf{H}}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{A} + \frac{n}{T}\frac{m^4(z)\mathbf{I}_K}{(1+cm(z))^2 - cm^2(z)} + o_{\|\cdot\|}(1)$$

As such, we have

$$\begin{aligned}
[\mathbf{\Phi}_1]_{11} &= -\frac{1}{2\pi\imath}\oint_{\Gamma_1} z\,\mathrm{tr}(\mathbf{T}_1^{-1}\mathbf{T}_3)\,dz \\
&= \frac{1}{2\pi\imath}\oint_{\Gamma_1}\sum_{i=1}^{K}\frac{zm^2(z)}{1+zm(z)}\frac{l_i^{-1}}{1+zm(z)l_i^{-1}}\,\mathrm{tr}(\mathbf{u}_i\mathbf{u}_i^H\mathbf{A}^{\mathsf{H}}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{A}) \\
&+ z\,\mathrm{tr}(\sum_{i=1}^{K}\frac{zm^2(z)}{1+zm(z)}\frac{l_i^{-1}}{1+zm(z)l_i^{-1}}\frac{n}{T}\frac{m^4(z)}{(1+cm(z))^2-cm^2(z)}\cdot\mathbf{u}_i\mathbf{u}_i^H)dz \\
&= \frac{1-cl_1^{-2}}{1+cl_1^{-1}}\cdot\mathbf{u}_1^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{A}\mathbf{u}_1 + \frac{n}{T}\frac{1+\rho_1^{-1}}{c+\rho_1} + o(1) \\
&= \frac{1-cl_1^{-2}}{1+cl_1^{-1}}\cdot\frac{n}{N} + \frac{n}{T}\frac{1+\rho_1^{-1}}{c+\rho_1} + o(1).
\end{aligned}$$

Similarly for off-diagonal entries of $\mathbf{\Phi}_1$ as

$$\begin{aligned}
|[\mathbf{\Phi}_1]_{12}|^2 &= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2} z_1z_2\,\mathrm{tr}(\mathbf{T}_4(z_2,z_1)\mathbf{T}_1^{-1}(z_1)\mathbf{T}_4(z_1,z_2)\mathbf{T}_1^{-1}(z_2))\,dz_1\,dz_2, \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\mathrm{tr}(\sum_{i=1}^{K}\sum_{j=1}^{K}\frac{z_1m^2(z_1)l_i^{-1}z_2m^2(z_2)l_j^{-1}\,\mathrm{tr}(\mathbf{T}_4(z_2,z_1)u_ju_j^H\mathbf{T}_4(z_1,z_2)u_iu_i^H)\quad dz_1dz_2}{(1+z_1m(z_1))(1+z_1m(z_1)+l_i^{-1})(1+z_2m(z_2))(1+z_2m(z_2)+l_j^{-1})} \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\frac{z_1m^2(z_1)l_1^{-1}z_2m^2(z_2)l_2^{-1}\mathbf{u}_2^H\mathbf{A}^H\mathbf{H}^H\mathbf{A}\mathbf{u}_1\mathbf{u}_1^H\mathbf{A}^H\mathbf{H}\mathbf{A}\mathbf{u}_2\quad dz_1dz_2}{(1+z_1m(z_1))(1+z_1m(z_1)+l_1^{-1})(1+z_2m(z_2))(1+z_2m(z_2)+l_2^{-1})} \\
&= \frac{1-cl_1^{-2}}{1+cl_1^{-1}}\cdot\frac{1-cl_2^{-2}}{1+cl_2^{-1}}\mathbf{u}_2^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_1\mathbf{u}_1^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_2 + o(1) \\
&= o(1)
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{T}_4(z_1,z_2) &= \mathbf{A}^{\mathsf{H}}\mathbf{Q}(z_1)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z_2)\mathbf{A} \\
&= m(z_1)m(z_2)\mathbf{A}^{\mathsf{H}}\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{A} + \frac{n}{T}\eta(z_1,z_2)\mathbf{I}_K + o_{\|\cdot\|}(1) = \mathbf{T}_4(z_2,z_1),
\end{aligned}$$

and $\mathbf{H} = \mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}} \in \mathbb{C}^{N\times N}$. The last line is because $\mathbf{u}_2^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_1\mathbf{A}\mathbf{u}_1 = \mathbf{u}_2^H\frac{n}{N}\mathbf{I}_K\mathbf{u}_1 = 0$. So we can draw the conclusion that:

$$\bar{\mathbf{\Phi}}_1 = \mathrm{diag}\left\{\frac{n}{N}\frac{1-cl_k^{-2}}{1+cl_k^{-1}} + \frac{n}{N}\frac{1+l_k^{-1}}{1+l_k/c}\right\}_{k=1}^{K}, \tag{58}$$

$$[\bar{\mathbf{\Phi}}_2]_{kk} = \frac{1-cl_1^{-2}}{1+cl_1^{-1}}\cdot\mathbf{u}_k^H\mathbf{A}^{\mathsf{H}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{A}\mathbf{u}_k \tag{59}$$

However, we can only get the modulus of the off-diagonal elements of $\bar{\mathbf{\Phi}}_2$.

$$|[\mathbf{\Phi}_2]_{mn}|^2 = \frac{1-cl_m^{-2}}{1+cl_m^{-1}}\cdot\frac{1-cl_n^{-2}}{1+cl_n^{-1}}\mathbf{u}_m^H\mathbf{A}^H\mathbf{J}_2^T\mathbf{J}_1\mathbf{A}\mathbf{u}_n\mathbf{u}_n^H\mathbf{A}^H\mathbf{J}_1^T\mathbf{J}_2\mathbf{A}\mathbf{u}_m.$$

11 Closely DoA

Settings: $\mathbf{a}(\theta) = \frac{1}{\sqrt{N}}[1, e^{\imath\theta}, \dots, e^{\imath(N-1)\theta}]^\mathsf{T}$.

We set that $\theta_2 = \theta_1 + \frac{\alpha}{T}$, where T is the number of time slots, then:

$$\begin{aligned}\mathbf{a}^\mathsf{H}(\theta_1)\mathbf{a}(\theta_2) &= \frac{1}{N}\sum_{n=0}^{N-1} e^{\imath n\alpha/T} = \frac{1}{N}\frac{1-e^{\imath N\alpha/T}}{1-e^{\imath\alpha/T}} = \lim_{N,T\to\infty}\frac{\frac{1}{T}(1-e^{\imath N\alpha/T})}{\frac{N}{T}(1-e^{\imath\alpha/T})}\\ &= \lim_{N,T\to\infty}\frac{-\frac{1}{T^2}(1-e^{\imath\alpha c})}{\frac{1}{T^2}\imath\alpha c e^{\imath\frac{\alpha}{T}}} = \imath\frac{1-e^{\imath c\alpha}}{c\alpha}\end{aligned}$$

where the j-th entry of $\mathbf{a}(\theta)$ is given by

$$[\mathbf{a}(\theta_\ell)]_j = \frac{1}{\sqrt{N}}e^{\imath\frac{2\pi d}{\lambda_0}(j-1)\theta_\ell} \equiv \frac{1}{\sqrt{N}}e^{\imath\omega(j-1)\theta_\ell}, \quad \omega \equiv \frac{2\pi d}{\lambda_0} \tag{60}$$

So we have:

$$\mathbf{A}^\mathsf{H}\mathbf{A} \to \begin{bmatrix} 1 & e^{\imath\alpha c/2}sinc(\alpha c/2)\\ e^{-\imath\alpha c/2}sinc(\alpha c/2) & 1\end{bmatrix}$$

with its eigenvalue decomposition $\mathbf{V}L\mathbf{V}^H$ with $\Sigma = \operatorname{diag}\{1+\operatorname{sinc}\frac{\alpha c}{2}, 1-\operatorname{sinc}\frac{\alpha c}{2}\}$ and $\mathbf{U} = [\mathbf{v}_1, \mathbf{v}_2]$, where $\mathbf{v}_1 = \frac{1}{\sqrt{2}}[e^{\frac{\imath\alpha c}{2}}, 1]^T$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}}[1, -e^{-\frac{\imath\alpha c}{2}}]^T$.

In this part, we have:

$$\mathbf{T}_1 = \mathbf{P}^{-1} + \mathbf{A}^\mathsf{H}\mathbf{A} + z\mathbf{A}^\mathsf{H}\mathbf{Q}(z)\mathbf{A} = \mathbf{P}^{-1} + (1_z m(z))\mathbf{A}^\mathsf{H}\mathbf{A}, \tag{61}$$

and

$$\mathbf{T}_1^{-1} = \mathbf{P} - (1+zm(z))\mathbf{P}\mathbf{A}^\mathsf{H}(\mathbf{I}_N + (1+zm(z))\mathbf{A}\mathbf{P}\mathbf{A}^\mathsf{H})^{-1}\mathbf{A}\mathbf{P}.$$

Then, we have

$$\begin{aligned}[\mathbf{\Phi}_2]_{kk} &= \frac{1}{2\pi\imath}\oint_{\Gamma_k} z\,\mathrm{tr}(\mathbf{T}_1^{-1}\mathbf{A}^\mathsf{H}\mathbf{Q}(z)\mathbf{C}^{\frac{1}{2}}\mathbf{J}_1^\mathsf{T}\mathbf{J}_2\mathbf{C}^{\frac{1}{2}}\mathbf{Q}(z)\mathbf{A})\,dz\\ &= -\frac{1}{2\pi\imath}\oint_{\Gamma_k} z\,\mathrm{tr}((1+zm(z))\mathbf{P}\mathbf{A}^\mathsf{H}(\mathbf{I}_N+(1+zm(z))\mathbf{A}\mathbf{P}\mathbf{A}^\mathsf{H})^{-1}\mathbf{A}\mathbf{P}\mathbf{A}^\mathsf{H}\mathbf{Q}(z)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^\mathsf{T}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{A})\,dz\\ &= -\frac{1}{2\pi\imath}\oint_{\Gamma_k} z\,\mathrm{tr}(\mathbf{A}\mathbf{P}\mathbf{A}^\mathsf{H}(\mathbf{I}_N - (\mathbf{I}_N+(1+zm(z))\mathbf{A}\mathbf{P}\mathbf{A}^\mathsf{H})^{-1})\mathbf{Q}(z)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^\mathsf{T}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z))\,dz\\ &= \frac{1}{2\pi\imath}\oint_{\Gamma_k} z\,\mathrm{tr}(\mathbf{A}\mathbf{P}\mathbf{A}^\mathsf{H}(\mathbf{I}_N+(1+zm(z))\mathbf{A}\mathbf{P}\mathbf{A}^\mathsf{H})^{-1}\mathbf{Q}(z)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^\mathsf{T}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z))\,dz\end{aligned}$$

we define the spectral decomposition $\mathbf{A}\mathbf{P}\mathbf{A}^\mathsf{H} = \mathbf{U}L\mathbf{U}^H$ with $L = \operatorname{diag}\{\rho_i, \cdots, \rho_N\}$ and $\mathbf{U} = [\mathbf{u}_1, \cdots, \mathbf{u}_N]$. Then:

$$\begin{aligned}[\mathbf{\Phi}_2]_{kk} &= \frac{1}{2\pi\imath}\oint_{\Gamma_k}\sum_{i=1}^N \frac{z}{1+(1+zm(z)\rho_i)}\,\mathrm{tr}(\mathbf{A}\mathbf{P}\mathbf{A}^\mathsf{H}\mathbf{u}_i\mathbf{u}_i^H\mathbf{Q}(z)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^\mathsf{T}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z))\,dz\\ &= \frac{1}{2\pi\imath}\oint_{\Gamma_k}\sum_{i=1}^N \frac{z\rho_k}{1+(1+zm(z)\rho_i)}\mathbf{u}_k^H\mathbf{Q}(z)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^\mathsf{T}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{u}_k\,dz\\ &= \frac{1}{2\pi\imath}\oint_{\Gamma_k}\frac{zm^2(z)}{\rho_i^{-1}+1+zm(z)}\mathbf{u}_i^H\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^\mathsf{T}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{u}_i\,dz\\ &= \frac{1-c\rho_k^{-2}}{1+c\rho_k^{-1}}\cdot\mathbf{u}_K^H\mathbf{J}_1^\mathsf{T}\mathbf{J}_2\mathbf{u}_k + o(1)\end{aligned}$$

the conclusion is the same as $[\mathbf{\Phi}_2]_{kk}$ in the case of widely DoA.

Similarly for $[\mathbf{\Phi}_1]_{kk}$ as:

$$
\begin{aligned}
[\mathbf{\Phi}_1]_{kk} &= \frac{1}{2\pi\imath}\oint_{\Gamma_k}\frac{z}{\rho_k^{-1}+1+zm(z)}\mathbf{u}_k^H\mathbf{Q}(z)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{u}_k\,dz \\
&= \frac{1-c\rho_k^{-2}}{1+c\rho_k^{-1}}\cdot\mathbf{u}_K^H\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{u}_k+\frac{1}{2\pi\imath}\oint_{\Gamma_k}\frac{z}{\rho_k^{-1}+1+zm(z)}\frac{n}{T}\frac{m'(z)m^2(z)}{(1+cm(z))^2}dz \\
&= \frac{1-c\rho_k^{-2}}{1+c\rho_k^{-1}}\cdot\mathbf{u}_K^H\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{u}_k+\frac{n}{T}\frac{1+\rho_k^{-1}}{c+\rho_k}+o(1).
\end{aligned}
$$

where $\mathbf{u}_k$ and ρ_k represent the k-th eigenvetor and eigenvalue of $\mathbf{APA}^{\mathsf{H}}$, respectively. The extra bias term is the same as the previous conclusions.

For the off-diagonal term:

$$
\begin{aligned}
|[\mathbf{\Phi}_2]_{12}|^2 &= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2} z_1z_2\,\mathrm{tr}(\mathbf{T}_2^{\mathsf{H}}(z_1,z_2)\mathbf{T}_1^{-1}(z_1)\mathbf{T}_2(z_1,z_2)\mathbf{T}_1^{-1}(z_2))\,dz_1\,dz_2, \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\sum_{i=1}^{K}\sum_{j=1}^{K}\frac{z_1m^2(z_1)z_2m^2(z_2)\mathbf{u}_j^H\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{u}_j\mathbf{u}_i^H\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{u}_i\quad dz_1dz_2}{(1+z_1m(z_1)+\rho_i^{-1})(1+z_2m(z_2)+\rho_j^{-1})} \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\frac{z_1m^2(z_1)l_1^{-1}z_2m^2(z_2)l_2^{-1}\mathbf{u}_1^H\mathbf{J}_2^T\mathbf{J}_1\mathbf{u}_2\mathbf{u}_2^H\mathbf{J}_1^T\mathbf{J}_2\mathbf{u}_1\quad dz_1dz_2}{(1+z_1m(z_1)+\rho_1^{-1})(1+z_2m(z_2)+\rho_2^{-1})} \\
&= \frac{1-c\rho_1^{-2}}{1+c\rho_1^{-1}}\cdot\frac{1-c\rho_2^{-2}}{1+c\rho_2^{-1}}\mathbf{u}_1^H\mathbf{J}_2^T\mathbf{J}_1\mathbf{u}_2\mathbf{u}_2^H\mathbf{J}_1^T\mathbf{J}_2\mathbf{u}_1.
\end{aligned}
$$

Similarly for $[\mathbf{\Phi}_1]_{12}$ as:

$$
\begin{aligned}
|[\mathbf{\Phi}_1]_{12}|^2 &= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2} z_1z_2\,\mathrm{tr}(\mathbf{T}_3^{\mathsf{H}}(z_1,z_2)\mathbf{T}_1^{-1}(z_1)\mathbf{T}_3(z_1,z_2)\mathbf{T}_1^{-1}(z_2))\,dz_1\,dz_2, \\
&= -\frac{1}{4\pi^2}\oint_{\Gamma_1}\oint_{\Gamma_2}\sum_{i=1}^{K}\sum_{j=1}^{K}\frac{z_1m^2(z_1)z_2m^2(z_2)\mathbf{u}_j^H\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_2^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}\mathbf{u}_j\mathbf{u}_i^H\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_2\mathbf{C}^{-\frac{1}{2}}\mathbf{u}_i\quad dz_1dz_2}{(1+z_1m(z_1)+\rho_i^{-1})(1+z_2m(z_2)+\rho_j^{-1})} \\
&= \frac{1-c\rho_1^{-2}}{1+c\rho_1^{-1}}\cdot\frac{1-c\rho_2^{-2}}{1+c\rho_2^{-1}}\mathbf{u}_1^H\mathbf{J}_1^T\mathbf{J}_1\mathbf{u}_2\mathbf{u}_2^H\mathbf{J}_1^T\mathbf{J}_1\mathbf{u}_1.
\end{aligned}
$$

where $\mathbf{T}_3^{\mathsf{H}}(z_1,z_2)=m(z_1)m(z_2)\mathbf{C}^{-\frac{1}{2}}\mathbf{J}_1^{\mathsf{T}}\mathbf{J}_1\mathbf{C}^{-\frac{1}{2}}+\frac{n}{T}\eta(z_1,z_2)\mathbf{I}_N$

For the special case $\mathbf{P}=\mathbf{I}_K$, setting that $\mathbf{x}$ is the eigenvector of $\mathbf{A}^{\mathsf{H}}\mathbf{AP}$, then $(\mathbf{AP})\mathbf{x}$ is the eigenvector of $\mathbf{APA}^{\mathsf{H}}$. So that $\mathbf{u}_1$ and $\mathbf{u}_2$ have the following form:

$$
\begin{aligned}
\mathbf{u}_1 = \mathbf{Ax}_1 &= \frac{1}{\sqrt{2}}[e^{\frac{\imath\alpha c}{2}}\mathbf{a}(\theta_1)+\mathbf{a}(\theta_2)]^{\mathsf{T}} \\
\mathbf{u}_2 = \mathbf{Ax}_2 &= \frac{1}{\sqrt{2}}[\mathbf{a}(\theta_1)-e^{-\frac{\imath\alpha c}{2}}\mathbf{a}(\theta_2)]^{\mathsf{T}}
\end{aligned}
$$

Then $\mathbf{u}_1\mathbf{J}_1^T\mathbf{J}_2\mathbf{u}_1$ has the form as:

$$
\mathbf{u}_1\mathbf{J}_1^T\mathbf{J}_2\mathbf{u}_1=\frac{1}{2}[\frac{n}{N}e^{\imath\Delta\theta_1}+\frac{n}{N}e^{\imath\Delta\theta_2}+\frac{1-e^{\imath\alpha cn/N}}{-\alpha c}e^{\frac{\imath\alpha c}{2}}\frac{n}{N}e^{-\imath\Delta\theta_2}+\frac{1-e^{\imath\alpha cn/N}}{-\imath\alpha c}e^{-\frac{\imath\alpha c}{2}}\frac{n}{N}e^{\imath\Delta\theta_2}]
$$

12 Widely spaced DoA(P is not diagonal)

Signal Mode123l :

$$\mathbf{X} = (\mathbf{APA}^H + \mathbf{I}_k)^{\frac{1}{2}}\mathbf{Z}$$
$$\mathbf{C} = (\mathbf{APA}^H + \mathbf{I}_k)$$
$$P = BLB^H, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, B^H B = BB^H = I, \mathbf{U}_s = \mathbf{AB} \tag{62}$$

Q is

We aim to get the eigenvalue of this $\Phi_1^{-1}\Phi_2$,(denoted as $\lambda_1 ... \lambda_k$)

$$\overline{\Phi}_1 = \frac{n}{N} I_k, \Phi_2 = \begin{pmatrix} \hat{u}_1^H J_1^H J_2 \hat{u}_1 & \hat{u}_1^H J_1^H J_2 \hat{u}_2 \\ \hat{u}_2^H J_1^H J_2 \hat{u}_1 & \hat{u}_2^H J_1^H J_2 \hat{u}_2 \end{pmatrix} \tag{63}$$

We do not have dir21312312321312321321321ect access to the deterministic equivalent for Φ_2 referring to $\overline{\Phi}_2$,but an asymptotic eigenvalue equivalence(denoted as $\overline{\lambda}_1 ... \overline{\lambda}_k$) can be obtained.

$$Eig(\Phi_1^{-1}\Phi_2) = Eig(\overline{\Phi}_1^{-1}\Phi_2) \tag{64}$$
$$= \frac{n}{N} Eig(\Phi_2) \tag{65}$$

The above equation is the solution of the following equation

$$\begin{aligned}
\det(\Phi_2 - \lambda I_2) &= (\hat{u}_1^H J_1^H J_2 \hat{u}_1 - \lambda)(\hat{u}_2^H J_1^H J_2 \hat{u}_2 - \lambda) - \hat{u}_1^H J_1^H J_2 \hat{u}_2 \hat{u}_2^H J_1^H J_2 \hat{u}_1 \\
&= \lambda^2 - \lambda(\hat{u}_1^H J_1^H J_2 \hat{u}_1 + \hat{u}_2^H J_1^H J_2 \hat{u}_2) + \hat{u}_1^H J_1^H J_2 \hat{u}_1 \hat{u}_2^H J_1^H J_2 \hat{u}_2 - \hat{u}_1^H J_1^H J_2 \hat{u}_2 \hat{u}_2^H J_1^H J_2 \hat{u}_1 \\
&= \overline{\lambda}^2 - \overline{\lambda}(g_1 u_1^H J_1^H J_2 u_1 + g_2 u_2^H J_1^H J_2 u_2) + g_1 g_2 u_1^H J_1^H J_2 u_1 u_2^H J_1^H J_2 u_2 - g_1 g_2 u_1^H J_1^H J_2 u_2 u_2^H J_1^H J_2 u_1 \\
&= \overline{\lambda}^2 - \overline{\lambda}(g_1 u_1^H J_1^H J_2 u_1 + g_2 u_2^H J_1^H J_2 u_2) + g_1 g_2 Det(\mathbf{U}_s^H \mathbf{J}_1^H \mathbf{J}_2 \mathbf{U}_s) \\
&= \overline{\lambda}^2 - \overline{\lambda}(g_1(b_{11}^2 \frac{n}{N} e^{i\theta_1} + b_{12}^2 \frac{n}{N} e^{i\theta_1}) + g_2(b_{21}^2 \frac{n}{N} e^{i\theta_1} + b_{22}^2 \frac{n}{N} e^{i\theta_2})) + g_1 g_2 Det(\mathbf{A}^H \mathbf{J}_1^H \mathbf{J}_2 \mathbf{A}) \\
&= \overline{\lambda}^2 - \overline{\lambda}((g_1 b_{11}^2 + g_2 b_{21}^2)\frac{n}{N} e^{i\theta_1} + (g_1 b_{12}^2 + g_2 b_{22}^2)\frac{n}{N} e^{i\theta_2})) + g_1 g_2 (\frac{n}{N})^2 e^{i\theta_1} e^{i\theta_2} \\
&= 0
\end{aligned}$$

$$\alpha_1 = (g_1 b_{11}^2 + g_2 b_{21}^2)\frac{n}{N} e^{i\theta_1} + (g_1 b_{12}^2 + g_2 b_{22}^2)\frac{n}{N} e^{i\theta_2}) \quad g_i = (1 - c l_i^{-2})/(1 + c l_i^{-1})$$
$$\alpha_2 = g_1 g_2 (\frac{n}{N})^2 e^{i\theta_1} e^{i\theta_2}$$
$$\overline{\lambda}_1 = \frac{\alpha_1 + \sqrt{\Delta}}{2}$$
$$\overline{\lambda}_2 = \frac{\alpha_1 - \sqrt{\Delta}}{2}$$
$$\Delta = \alpha_1^2 - 4\alpha_2$$
$$\overline{\lambda}_1 = \frac{(g_1 b_{11}^2 + g_2 b_{21}^2)\frac{n}{N} e^{i\theta_1} + (g_1 b_{12}^2 + g_2 b_{22}^2)\frac{n}{N} e^{i\theta_2}) + \sqrt{((g_1 b_{11}^2 + g_2 b_{21}^2)\frac{n}{N} e^{i\theta_1} + (g_1 b_{12}^2 + g_2 b_{22}^2)\frac{n}{N} e^{i\theta_2}))^2 - 4 *}}{2}$$

I guess the two roots of the equation are $(g_1 b_{11}^2 + g_2 b_{21}^2)\frac{n}{N} e^{i\theta_1}$, $(g_1 b_{21}^2 + g_2 b_{22}^2)\frac{n}{N} e^{i\theta_2}$

Specially,P is diagonal,$B = \mathbf{I}_k$,$b_{11} = b_{22} = 1, b_{12} = b_{21} = 0$,$\overline{\lambda}_1 = g_1 \frac{n}{N} e^{i\theta_1}, \overline{\lambda}_2 = g_2 \frac{n}{N} e^{i\theta_2}$(consistent with the results of our paper)

If we can prove that the second item $(b_{11}^2 b_{12}^2 (g_1 - g_2)^2)$ asymptotically$(N, T \to \infty)$ converges to 0(**Note** : the item is equal to 0 absolutely when P is diagonal, due to $b_{12} = 0$).

$$\begin{aligned} \overline{\lambda}_1 &= (g_1 b_{11}^2 + g_2 b_{21}^2) \frac{n}{N} e^{i\theta_1} \\ \overline{\lambda}_2 &= (g_1 b_{21}^2 + g_2 b_{22}^2) \frac{n}{N} e^{i\theta_2} \end{aligned} \tag{66}$$

All we need do is to prove the rate of the item2 is slowly than the first ite12312321321435342m $\lambda_1 = 1$

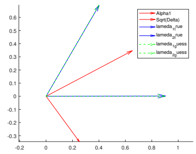

Figure 1: figure title

13 Multiple angle detection

Let us get back to the general case and consider $(\hat{\mathbf{U}}_S^{\mathsf{H}} \mathbf{J}_1^{\mathsf{H}} \mathbf{J}_1 \hat{\mathbf{U}}_S)^{-1} \cdot \hat{\mathbf{U}}_S^{\mathsf{H}} \mathbf{J}_1^{\mathsf{H}} \mathbf{J}_2 \hat{\mathbf{U}}_S$. Note that the two matrices of interest are of small dimensions $(k \times k)$ and can be shown to also establish asymptotically deterministic behavior.

In the case of a fixed number k of signal sources with angle $\theta_1, \ldots, \theta_k$, we have the following generic model

$$\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{Z} \tag{67}$$

with $\mathbf{X} = [\mathbf{x}(1), \ldots, \mathbf{x}(T)] \in \mathbb{C}^{N \times T}$, $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_k)] \in \mathbb{C}^{N \times k}$, $\mathbf{S} = [\mathbf{s}(1), \ldots, \mathbf{s}(T)] \in \mathbb{C}^{k \times T}$, column vector $\mathbf{s}(t) = [s_1(t), \ldots, s_k(t)]^{\mathsf{T}} \in \mathbb{C}^k$ and $\mathbf{Z} = [\mathbf{z}(1), \ldots, \mathbf{z}(T)] \in \mathbb{C}^{N \times T}$ a standard (circular) Gaussian random matrix. As a consequence,

$$\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} = \frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} + \begin{bmatrix} \mathbf{A} & \frac{\mathbf{Z}\mathbf{S}^{\mathsf{H}}}{T} \end{bmatrix} \begin{bmatrix} \frac{1}{T}\mathbf{S}\mathbf{S}^{\mathsf{H}} & \mathbf{I}_k \\ \mathbf{I}_k & \mathbf{0}_k \end{bmatrix} \begin{bmatrix} \mathbf{A}^{\mathsf{H}} \\ \frac{(\mathbf{Z}\mathbf{S}^{\mathsf{H}})^{\mathsf{H}}}{T} \end{bmatrix} \equiv \frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{\mathsf{H}} \tag{68}$$

where we assume the the signals are uncorrelated and denote the diagonal $\mathbf{P} = \lim_{T\to\infty} \frac{1}{T}\mathbf{S}\mathbf{S}^{\mathsf{H}}$ the (limit of the normalized) signal strength and $\mathbf{V} = \begin{bmatrix} \mathbf{A} & \frac{\mathbf{Z}\mathbf{S}^{\mathsf{H}}}{T} \end{bmatrix} \in \mathbb{C}^{N \times 2k}$.

As a consequence,

$$\begin{aligned} \left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} &= \left(\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} \\ &= \mathbf{Q} - \mathbf{Q}\mathbf{V}\mathbf{\Lambda}(\mathbf{I}_{2k} + \mathbf{V}^{\mathsf{H}}\mathbf{Q}\mathbf{V}\mathbf{\Lambda})^{-1}\mathbf{V}^{\mathsf{H}}\mathbf{Q} \end{aligned}$$

where we denoted $\mathbf{Q}(z) = \mathbf{Q} = (\frac{1}{T}\mathbf{Z}\mathbf{Z}^{\mathsf{H}} - z\mathbf{I}_N)^{-1}$ and used Woodbury matrix identity. Now, since

$$\mathbf{V}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{V} = \begin{bmatrix} \mathbf{A}^{\mathsf{H}} \\ \frac{(\mathbf{Z}\mathbf{S}^{\mathsf{H}})^{\mathsf{H}}}{T} \end{bmatrix} \mathbf{Q}(z) \begin{bmatrix} \mathbf{A} & \frac{\mathbf{Z}\mathbf{S}^{\mathsf{H}}}{T} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \frac{1}{T}\mathbf{S}\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{Z}\mathbf{S}^{\mathsf{H}} \end{bmatrix} + o_{\|\cdot\|}(1) \tag{69}$$

with

$$\frac{1}{T}\mathbf{S}\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Q}(z)\mathbf{Z}\mathbf{S}^{\mathsf{H}} = \frac{1}{T}\mathbf{S}\tilde{\mathbf{Q}}(z)\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Z}\mathbf{S}^{\mathsf{H}} = \frac{1}{T}\mathbf{S}(\mathbf{I}_T + z\tilde{\mathbf{Q}}(z))\mathbf{S}^{\mathsf{H}} = \mathbf{P} + \frac{z}{T}\mathbf{S}\tilde{\mathbf{Q}}(z)\mathbf{S}^{\mathsf{H}} \tag{70}$$

for co-resolvent $\tilde{\mathbf{Q}}(z) = \left(\frac{1}{T}\mathbf{Z}^{\mathsf{H}}\mathbf{Z} - z\mathbf{I}_T\right)^{-1}$.

Since

$$\mathbf{Q}(z) \leftrightarrow \bar{\mathbf{Q}}(z) = m(z)\mathbf{I}_N = \left(\frac{1}{1+cm(z)} - z\right)^{-1} \mathbf{I}_N, \quad \tilde{\mathbf{Q}}(z) = -\left(\frac{1}{zm(z)} + 1\right)\mathbf{I}_T \tag{71}$$

for $c = \lim N/T$ and $m(z)$ the unique solution of the Marčenko-Pastur equation

$$zcm^2(z) - (1-c-z)m(z) + 1 = 0, \tag{72}$$

Therefore

$$(\mathbf{I}_{2k} + \mathbf{V}^{\mathsf{H}}\mathbf{Q}\mathbf{V}\boldsymbol{\Lambda})^{-1} = \begin{bmatrix} \mathbf{I}_k + m(z)\mathbf{A}^{\mathsf{H}}\mathbf{A}\mathbf{P} & m(z)\mathbf{A}^{\mathsf{H}}\mathbf{A} \\ \left(1 - z - \frac{1}{m(z)}\right)\mathbf{P} & \mathbf{I}_k \end{bmatrix}^{-1} + o_{\|\cdot\|}(1) \tag{73}$$

and

$$\mathbf{V}\boldsymbol{\Lambda}(\mathbf{I}_2 + \mathbf{V}^{\mathsf{H}}\mathbf{Q}\mathbf{V}\boldsymbol{\Lambda})^{-1}\mathbf{V}^{\mathsf{H}} = \begin{bmatrix}\mathbf{A} & \frac{\mathbf{Z}\mathbf{S}^{\mathsf{H}}}{T}\end{bmatrix} \begin{bmatrix} (z + \frac{1}{m(z)})(\mathbf{P}^{-1} + (zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1} & \mathsf{H} \\ \mathsf{H} & \mathsf{H} \end{bmatrix} \begin{bmatrix} \mathbf{A}^{\mathsf{H}} \\ \frac{(\mathbf{Z}\mathbf{S}^{\mathsf{H}})^{\mathsf{H}}}{T} \end{bmatrix} + o_{\|\cdot\|}(1) \tag{74}$$

so that it suffices to evaluate the following expectations:

1. $\mathbb{E}[\mathbf{Q}(z)\mathbf{A}(\cdot)\mathbf{A}^{\mathsf{H}}\mathbf{Q}(z)] = (zm^2(z) + m(z))\mathbf{A}(\mathbf{P}^{-1} + (zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{H}} + o_{\|\cdot\|}(1)$;
2. $\frac{1}{T}\mathbb{E}[\mathbf{Q}(z)\mathbf{a}\mathbf{S}^{\mathsf{H}}\mathbf{Z}^{\mathsf{T}}\mathbf{Q}(z)] = o_{\|\cdot\|}(1)$ and it Hermitian transpose.

This thus allows to conclude that

$$\left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} \leftrightarrow m(z)\mathbf{I}_N - (zm^2(z) + m(z))\mathbf{A}(\mathbf{P}^{-1} + (zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{H}}. \tag{75}$$

With the simplification of $\mathbf{A}^{\mathsf{H}}\mathbf{A} = \mathbf{I}_k$ as $N \to \infty$ in the LUA setting, we have

$$\left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} \leftrightarrow m(z)\mathbf{I}_N - (zm^2(z) + m(z))\mathbf{A}(\mathbf{P}^{-1} + (zm(z)+1)\mathbf{I}_k)^{-1}\mathbf{A}^{\mathsf{H}}. \tag{76}$$

In particular, for $k = 1$, the above relation becomes

$$\left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1} \leftrightarrow m(z)\mathbf{I}_N - \frac{\rho m(z)(zm(z)+1)}{1+\rho(zm(z)+1)}\mathbf{a}\mathbf{a}^{\mathsf{H}}. \tag{77}$$

for $\mathbf{P} = \rho$ and where we recall that $\mathbf{a}^{\mathsf{H}}\mathbf{a} = \|\mathbf{a}\|^2 = 1$.

13.1 Asymptotic location of isolated eigenvalues

The asymptotic location of (the possible) isolated eigenvalues correspond to $z \in \mathbb{R}$ such that

$$\det(\mathbf{P}^{-1} + (zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A}) = 0 \Leftrightarrow \det((zm(z)+1)^{-1}\mathbf{I}_k + \mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A}) = 0 \tag{78}$$

where we used the fact that $\det(\mathbf{P}^{-1}) \neq 0$ and $zm(z) + 1 \neq 0$, that is,

$$\frac{1}{zm(z)+1} = -\lambda_i(\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A}) \Leftrightarrow zm(z) = -1 - \frac{1}{\lambda_i(\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A})} \tag{79}$$

where $\lambda_i(\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A}) \geq 0$ denotes the eigenvalues of $\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A}$.

First note that the function $x \mapsto xm(x) = \int \frac{x}{t-x}\mu(dt)$ is increasing and that $xm(x) \to -1$ as $x \to \infty$. Using the Marčenko-Pastur equation

$$zcm^2(z) - (1-c-z)m(z) + 1 = 0 \Leftrightarrow zm(z) = -1 + \frac{1}{1-z-czm(z)} \tag{80}$$

we particularly obtain $\lim_{x\downarrow(1+\sqrt{c})^2} xm(x) = -1 - \frac{1}{\sqrt{c}}$. So one must have $\ell_i = \lambda_i(\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A}) \geq \sqrt{c}$. **This is the so-called separation condition**. Plugging $xm(z) = -1 - \frac{1}{\ell_i}$ back into (80), one deduces

$$\hat{\lambda}_i \to \bar{\lambda}_i = 1 + \ell_i + c\frac{1+\ell_i}{\ell_i}, \quad \ell_i \equiv \lambda_i(\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A}) \geq \sqrt{c}. \tag{81}$$

This can be (equivalently) derived by considering, for $z \in \mathbb{R}$,

$$0 = \det\left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N\right) = \ldots$$

13.2 ESPRIT performance analysis

To analyze the performance of the ESPIRIT method, one needs to evaluate the following quantities

$$T_1 = \mathbf{A}^{\mathsf{H}}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i^{\mathsf{H}}\mathbf{A} = (\mathbf{A}^{\mathsf{H}}\hat{\mathbf{u}}_i)(\mathbf{A}^{\mathsf{H}}\hat{\mathbf{u}}_i)^{\mathsf{H}} \in \mathbb{C}^{k\times k}, \quad T_2 = \mathbf{A}^{\mathsf{H}}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i^{\mathsf{H}}\mathbf{M}\hat{\mathbf{u}}_j\hat{\mathbf{u}}_j^{\mathsf{H}}\mathbf{A} \in \mathbb{C}^{k\times k}. \tag{82}$$

Let us first consider the term T_1. For $(\hat{\lambda}_i, \hat{\mathbf{u}}_i)$ an isolated eigenvalue-eigenvector pair of $\frac{1}{n}\mathbf{X}\mathbf{X}^{\mathsf{H}}$ (so in particular, one must have $\ell_i = \lambda_i(\mathbf{P}\mathbf{A}^{\mathsf{H}}\mathbf{A}) \geq \sqrt{c}$). Then, by Cauchy's integral formula

$$T_1 = -\frac{1}{2\pi\imath}\oint_{\Gamma_{\bar{\lambda}_i}} \mathbf{A}^{\mathsf{H}}\left(\frac{1}{T}\mathbf{X}\mathbf{X}^{\mathsf{H}} - z\mathbf{I}_N\right)^{-1}\mathbf{A}\,dz = -\frac{1}{2\pi\imath}\oint_{\Gamma_{\bar{\lambda}_i}} m(z)\mathbf{A}^{\mathsf{H}}\mathbf{A}\,dz \tag{83}$$

$$+ \frac{1}{2\pi\imath}\oint_{\Gamma_{\bar{\lambda}_i}} m(z)(zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A}(\mathbf{P}^{-1} + (zm(z)+1)\mathbf{A}^{\mathsf{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{A}\,dz + o(1) \tag{84}$$

$$= -\frac{1}{2\pi\imath}\oint_{\Gamma_{\bar{\lambda}_i}} m(z)\left(\mathbf{I}_k + (zm(z)+1)\mathbf{P}^{\frac{1}{2}}\mathbf{A}^{\mathsf{H}}\mathbf{A}\mathbf{P}^{\frac{1}{2}}\right)^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{A}\,dz + o(1) \tag{85}$$

$$= \mathrm{Res}_{\bar{\lambda}_i} - m(z)\left(\mathbf{I}_k + (zm(z)+1)\mathbf{P}^{\frac{1}{2}}\mathbf{A}^{\mathsf{H}}\mathbf{A}\mathbf{P}^{\frac{1}{2}}\right)^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{A} + o(1) \tag{86}$$

for $\Gamma_{\bar{\lambda}_i}$ a positively oriented contour circling around the asymptotic location of the i-th eigenvalue $\hat{\lambda}_i$. It thus suffices to consider the eigendecomposition of $\mathbf{P}^{\frac{1}{2}}\mathbf{A}^{\mathsf{H}}\mathbf{A}\mathbf{P}^{\frac{1}{2}} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathsf{H}}$ and

$$\begin{aligned} T_1 &= \mathrm{Res}_{\bar{\lambda}_i}\mathbf{U}\,\mathrm{diag}\left\{\frac{-m(z)}{1+(zm(z)+1)\ell_j}\right\}_{j=1}^{k}\mathbf{U}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}\mathbf{A} + o_{\|\cdot\|}(1) \\ &= \frac{-m(\bar{\lambda}_i)}{\ell_i(m(\bar{\lambda}_i) + \bar{\lambda}_i m'(\bar{\lambda}_i))}\mathbf{u}_i\mathbf{u}_i^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}\mathbf{A} + o_{\|\cdot\|}(1). \end{aligned}$$

We in fact only care about the diagonal entries of T_1.

In the case of diagonal $\mathbf{A}^{\mathsf{H}}\mathbf{A} = \mathbf{I}_k$ (in the case of LUA), one obtains in particular,

$$T_1 = -\frac{m(\bar{\lambda}_i)}{\ell_i(m(\bar{\lambda}_i) + \bar{\lambda}_i m'(\bar{\lambda}_i))}\mathbf{u}_i\mathbf{u}_i^{\mathsf{H}} + o_{\|\cdot\|}(1). \tag{87}$$

In the special case uncorrelated sources with diagonal $\mathbf{P} = \operatorname{diag}\{P_{\ell\ell}\}_{\ell=1}^k$, the $\mathbf{u}_i$ are canonical vectors $\mathbf{e}_i$ and

$$[T_1]_{\ell\ell} = -\frac{1}{P_{\ell\ell}} \frac{m(\bar{\lambda}_i)}{m(\bar{\lambda}_i) + \bar{\lambda}_i m'(\bar{\lambda}_i)}. \tag{88}$$

Let us move on and consider the (slightly) more involved term T_2. First note that, from the derivation of T_1, we have, for $\mathbf{a}, \mathbf{b}$ *any* (i'-th and j'-th) column of $\mathbf{A}$ that

$$\mathbf{a}^{\mathsf{H}} \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^{\mathsf{H}} \mathbf{b} = [T_1]_{i'j'} = -\frac{m(\bar{\lambda}_i)}{\ell_i(m(\bar{\lambda}_i) + \bar{\lambda}_i m'(\bar{\lambda}_i))} [\mathbf{u}_i \mathbf{u}_i^{\mathsf{H}}]_{i'j'} + o(1). \tag{89}$$

Our objective is to, for any $\mathbf{a}, \mathbf{b}$, estimate the *complex* numbers $\mathbf{a}^{\mathsf{H}} \hat{\mathbf{u}}_i$ and $\mathbf{b}^{\mathsf{H}} \hat{\mathbf{u}}_i$.

For the convenience of further use, we can similarly derive

$$\mathbf{A}^{\mathsf{T}} \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^{\mathsf{H}} \bar{\mathbf{A}} = \mathbf{A}^{\mathsf{T}} \bar{\hat{\mathbf{u}}}_i \bar{\hat{\mathbf{u}}}_i^{\mathsf{H}} \bar{\mathbf{A}} \tag{90}$$

14 MUSIC and its extension in random linear array setting

In this section, we consider the more realistic Random Linear Array (RLA) setting of same N, which differs from the ULA setting (60) in the fact that the steering vector $\mathbf{a}(\theta_\ell)$ of source ℓ and of arriving angle θ_ℓ is now given by[2]

$$[\mathbf{a}_\epsilon(\theta_\ell)]_j = \frac{1}{\sqrt{N}} e^{\imath \frac{2\pi d_j}{\lambda_0} \sin(\theta_\ell)} = \frac{1}{\sqrt{N}} e^{\imath \frac{2\pi d (j-1)}{\lambda_0} \sin(\theta_\ell)} \cdot e^{\imath \frac{2\pi d \epsilon_j}{\lambda_0} \sin(\theta_\ell)} = \frac{1}{\sqrt{N}} e^{\imath (j-1)\omega \sin(\theta_\ell)} \cdot e^{\imath \omega \epsilon_j \sin(\theta_\ell)} \tag{91}$$

for $\omega \equiv \frac{2\pi d}{\lambda_0}$. More precisely, the "spacing" d_j between different receivers are no longer assumes to be strictly uniform, but only "close to" uniform such that

$$d_j = (j-1)d + d\epsilon_j, \quad \epsilon_j \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) \tag{92}$$

with some small perturbation ϵ_j of zero mean and variance σ^2. As a consequence, we have

$$\mathbf{a}_\epsilon(\theta_\ell) = \mathbf{a}(\theta_\ell) \odot e^{\imath \omega \sin(\theta_\ell) \epsilon} \tag{93}$$

for the (standard) ULA steering vector $\mathbf{a}(\theta_\ell)$ defined in (60). Let us consider the following Taylor expansion of the nonlinear complex function $\exp(\imath \omega \sin \theta_\ell \cdot \epsilon)$ for $\epsilon \sim \mathcal{N}(0, \sigma^2) \ll 1$ (so in essence the assumption should be $\sigma^2 = o(1)$):

$$\begin{aligned} \exp(\imath \omega \sin \theta_\ell \cdot \epsilon) &= 1 + \imath \omega \sin \theta_\ell \epsilon - \frac{1}{2} \omega^2 \sin^2 \theta_\ell \epsilon^2 - \frac{\imath}{6} \omega^3 \sin^3 \theta_\ell \epsilon^3 + o(\epsilon^3) & (94) \\ &= 1 - \frac{1}{2} \omega^2 \sin^2 \theta_\ell \epsilon^2 + \imath \cdot \omega \sin \theta_\ell \epsilon \cdot \left(1 - \frac{1}{6} \omega^2 \sin^2 \theta_\ell \epsilon^2 \right) + o(\epsilon^3) & (95) \end{aligned}$$

(Note interestingly that the approximation resulting from Taylor expansion entails that the expression is, in its first order, real.)

We consider again the following model for the signal received at time $t = 1, \ldots, T$

$$\mathbf{x}(t) = \sum_{\ell=1}^k \mathbf{a}_\epsilon(\theta_\ell) s_\ell(t) + \mathbf{z}(t) = \sum_{\ell=1}^k \mathbf{a}(\theta_\ell) \odot e^{\imath \omega \sin(\theta_\ell) \epsilon} \cdot s_\ell(t) + \mathbf{z}(t) \in \mathbb{C}^N \tag{96}$$

[2] The normalization by $\sqrt{N}$ here is for notational convenience so that $\mathbf{a}(\theta_\ell)$ is of unit norm, note that this is equivalent to a *rescaling* of the source signal s_ℓ.

which can be rewritten in matrix model by cascading the total T observations as

$$\mathbf{X} = \mathbf{A}_\epsilon \mathbf{S} + \mathbf{Z} = \left(\mathbf{A} \odot [e^{\imath\omega \sin(\theta_1)\epsilon}, \ldots, e^{\imath\omega \sin(\theta_k)\epsilon}] \right) \mathbf{S} + \mathbf{Z} \tag{97}$$

with $\mathbf{X} = [\mathbf{x}(1), \ldots, \mathbf{x}(T)] \in \mathbb{C}^{N\times T}$, $\mathbf{A}_\epsilon = [\mathbf{a}_\epsilon(\theta_1), \ldots, \mathbf{a}_\epsilon(\theta_k)] \in \mathbb{C}^{N\times k}$, $\mathbf{S} = [\mathbf{s}(1), \ldots, \mathbf{s}(T)] \in \mathbb{C}^{k\times T}$, column vector $\mathbf{s}(t) = [s_1(t), \ldots, s_k(t)]^\mathsf{T} \in \mathbb{C}^k$ and $\mathbf{Z} = [\mathbf{z}(1), \ldots, \mathbf{z}(T)] \in \mathbb{C}^{N\times T}$ a standard (circular) Gaussian random matrix.

14.1 Simplified model with single DoA

In the case of $k = 1$ with angle θ, we have the following simplified model

$$\mathbf{X} = \mathbf{a}_\epsilon(\theta)[s(1), \ldots, s(T)] + \mathbf{Z} \equiv \mathbf{a}_\epsilon(\theta)\mathbf{s}^\mathsf{H} + \mathbf{Z} \tag{98}$$

so that

$$\frac{1}{T}\mathbf{X}\mathbf{X}^\mathsf{H} = \frac{1}{T}\mathbf{Z}\mathbf{Z}^\mathsf{H} + \begin{bmatrix} \mathbf{a}_\epsilon & \frac{\mathbf{Z}\mathbf{s}}{T} \end{bmatrix} \begin{bmatrix} \rho & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_\epsilon^\mathsf{H} \\ \frac{(\mathbf{Z}\mathbf{s})^\mathsf{H}}{T} \end{bmatrix} \equiv \frac{1}{T}\mathbf{Z}\mathbf{Z}^\mathsf{H} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\mathsf{H} \tag{99}$$

where we denote $\rho = \frac{1}{T}\|\mathbf{s}\|^2$ the (limit of the normalized) signal strength and $\mathbf{V} = \begin{bmatrix} \mathbf{a}_\epsilon & \frac{\mathbf{Z}\mathbf{s}}{T} \end{bmatrix} \in \mathbb{C}^{N\times 2}$. Taking the expectation with respect to $\mathbf{Z}$, one obtains

$$\frac{1}{T}\mathbb{E}_\mathbf{X}[\mathbf{X}\mathbf{X}^\mathsf{H}] = \mathbf{I}_N + \rho \cdot \mathbf{a}_\epsilon \mathbf{a}_\epsilon^\mathsf{H} \tag{100}$$

further taking the expectation with respect to ϵ, the (i,j) entry of which is given by

$$\frac{1}{T}\mathbb{E}[\mathbf{X}\mathbf{X}^\mathsf{H}]_{ij} = \delta_{ij} + \frac{\rho}{N} e^{\imath(i-1)\omega\sin(\theta)} \cdot e^{-\imath(j-1)\omega\sin(\theta)} \cdot \mathbb{E}[e^{\imath\omega\epsilon_i \sin(\theta)} \cdot e^{-\imath\omega\epsilon_j \sin(\theta)}] \tag{101}$$

$$= \delta_{ij} + \frac{\rho}{N} e^{\imath(i-1)\omega\sin(\theta)} \cdot e^{-\imath(j-1)\omega\sin(\theta)} \cdot e^{-\frac{1}{2}\omega^2\sigma^2\sin^2\theta} \cdot e^{-\frac{1}{2}\omega^2\sigma^2\sin^2\theta} \tag{102}$$

$$= \delta_{ij} + \frac{\rho}{N} e^{\imath(i-1)\omega\sin(\theta)} \cdot e^{-\imath(j-1)\omega\sin(\theta)} \cdot e^{-\omega^2\sigma^2\sin^2\theta} \tag{103}$$

for $i \neq j$, so that

$$\frac{1}{T}\mathbb{E}[\mathbf{X}\mathbf{X}^\mathsf{H}] = (1 + \frac{\rho}{N} - \frac{\rho}{N} e^{-\omega^2\sigma^2\sin^2\theta})\mathbf{I}_N + \frac{\rho}{N}\mathbf{a}\mathbf{a}^\mathsf{H} \cdot e^{-\omega^2\sigma^2\sin^2\theta} \tag{104}$$

simulation seems OK.

It can be further checked that

$$\lambda_1(\mathbb{E}[\mathbf{X}\mathbf{X}^\mathsf{H}]/T) = 1 + \frac{\rho}{N} \tag{105}$$

that is, the MUSIC method is *robust* in the random linear array setting, in the sense that its performance is independent of the noise level. The same conclusion can be reached by a RMT analysis, from which the MUSIC performance can be shown to only depend on the eigenvalue of $\mathbf{A}\mathbf{P}\mathbf{A}^\mathsf{H}$, and not the specific form of the steering vector $\mathbf{a}$.

15 ESPIRIT approach in random linear array setting

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