# A Proximal Linearization-based Decentralized Method for Nonconvex Problems with Nonlinear Constraints

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Abstract—Decentralized optimization for non-convex problems are now demanding by many emerging applications (e.g., smart grids, smart building, etc.). Though dramatic progress has been achieved in convex problems, the results for non-convex cases, especially with non-linear constraints, are still largely unexplored. This is mainly due to the challenges imposed by the non-linearity and non-convexity, which makes establishing the convergence conditions bewildered. This paper investigates decentralized optimization for a class of structured non-convex problems characterized by: (i) nonconvex global objective function (possibly nonsmooth) and (ii) coupled nonlinear constraints and local bounded convex constraints w.r.t. the agents. For such problems, a decentralized approach called Proximal Linearizationbased Decentralized Method (PLDM) is proposed. Different from the traditional (augmented) Lagrangian-based methods which usually require the exact (local) optima at each iteration, the proposed method leverages a proximal linearization-based technique to update the decision variables iteratively, which makes it computationally efficient and viable for the non-linear cases. Under some standard conditions, the PLDM's global convergence and local convergence rate to the  $\epsilon$ -critical points are studied based on the Kurdyka-ojasiewicz property which holds for most analytical functions. Finally, the performance and efficacy of the method are illustrated through a numerical example and an application to multi-zone heating, ventilation and air-conditioning (HVAC) control.

Index Terms—decentralized optimization, nonconvex problems, coupled nonlinear constraints, proximal linearization, augmented Lagrangian-based method.

### I. INTRODUCTION

The Internet of Things (IoT) and smart sensors have promoted the emergence of large-scale networked systems, which are eager for scalable and efficient computation. Consequently, the past decades have witnessed the revived interests and dramatic progress in decentralized optimization, especially for convex problems (see for examples, [1–3]). Nevertheless, the presence of complex dynamic systems and big data (see [5, 6]

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and references therein) nowadays requests for decentralized approaches that work for non-convex and non-linear context, which still remains an open question and has not been well established. This is mainly due to the intrinsic challenges imposed by the non-linearity and non-convexity, which lead to the NP-hard complexity as the existence of multiple optima, the lack of structure properties to guarantee the presence of optima (e.g., convexity and strong convexity), and the deficit of conditions to investigate convergence (e.g., sufficient global optimality conditions).

#### A. Related Works

In general, the available results on (decentralized) nonconvex optimization can be categorized based on the problem structures: (i) unconstrained non-convex problems, (ii) linearly constrained non-convex problem, and (iii) non-linearly constrained non-convex problems. Unless specified, we refer the constraints above to the coupled constraints among different agents. For unconstrained problems, the most straightforward approaches are (sub-)gradient methods, whose convergence has been early established for convex problems (e.g., see [8]). Recently, their extension [9] and variations, such as Frank-Wolf algorithm [10], proximal gradient method [11], have been discussed for non-convex cases. The broad results are that some local optima (i.e., critical points) can be approached with diminishing step-sizes. Generally, the above works are mainly focused on investigating the convergence of such methods for non-convex situations rather than achieving decentralized computation. On that basis, some decentralized paradigms by combining (proximal) gradient-based methods with alternating minimization technique (i.e, Gauss-Seidel update) have been proposed, which include proximal alternating linearized minimization (PALM) [12, 13], inexact proximal gradient methods (IPG) [14] and some variations [15, 17].

As the noteworthy performance of the methods of multipliers (MMs) or augmented Lagrangian-based methods, such as alternating direction multiplier method (ADMM) [3], has been thoroughly observed and understood in tackling coupled (linear) constraints in convex context, their extension to linearly coupled non-convex problems seems natural. The recent years have witnessed the widespread discussions on decentralized optimization for non-convex problems subject to coupled linear constraints under the Lagrangian-based framework, especially ADMM and its variations (see [18–20] and the references therein). In principle, ADMM can leverage the

fast convergence feature of augmented Lagrangian methods and the separable structure of dual decomposition based on the alternating minimization technique (i.e, Gauss-Seidel update). Nevertheless, in contrast to convex cases (see [18–20]), it's not straightforward or trivial to achieve their performance guarantee in non-convex context. Though favorable performance of ADMM and its variations does have been observed and reported in various applications involving non-convexity (see for examples, [21, 22]), the theoretical understandings are still fairly limited and deficient except for [23–26], where some results have been established for special structured problems.

With the prevalence of complex multi-agent dynamic systems, growing demand has been raised for decentralized nonconvex optimization in the presence of general coupled nonlinear constraints. Such instances are ubiquitous nowadays, including the optimal power flow (OPF) problem in smart grids [27] and the multi-zone heating, ventilation and airconditioning (HVAC) control in buildings [28], etc. The nonlinear constraints generally arise from the coupled complex system dynamics, which are pivotal while designing decentralized controllers. However, such non-linear couplings further compound the difficulties and challenges to develop decentralized computing paradigms. The complexities mainly stem from i) the lack of standard framework to deal with such general non-linear constraints; ii) the difficulties to ensure feasibility of the coupled nonlinear constraints while performing decentralized computing; iii) the challenges to investigate the convergence of a specific decentralized algorithm without any structure properties (e.g., convexity, strong convexity, etc). Though the augmented Lagrangian methods are appealing in deal with coupled constraints, their extension to general non-linear constraints are not straightforward or well-founded. Moreover, the existing standard augmented Lagrangian-based framework generally require the exact optimization (e.g., local optima) at each iteration, which are not viable for nonlinear and nonconvex cases practically. Surprisingly, though lack of theoretical foundations, the favorable performance of some Lagrangian-based methods also has been observed in some applications (e.g. matrix completion and factorization [22], optimal power flow [21, 29]). This has inspired some recent exploratory studies on their theoretical understandings [31-33]. For example, [31] studied the direct extension of ADMM to two-block convex problems with coupled non-linear but separable constraints. [34] proposed a tailored penalty dual decomposition (PPD) method by combining penalty method and augmented Lagrangian method to tackle non-convex problems with non-linear constraints. Except those, there exist another two excellent recent works that have shed some light on such situations. One is [33] which i) thoroughly and systematically investigated the intrinsic challenges to establish convergence guarantee for ADMM in such situations; and ii) resorted to a two-level nested framework as a remedy. However, [33] generally requires the joint optimization of multi-block nonlinear problem at each iteration in the inner loop, which are not viable or attainable in practice. Another noteworthy work is [16] which investigated the general conditions for Lagrangian-based framework to achieve global convergence guarantee in non-convex context. Rather than proposing a

specific algorithm, [16] seems more focused on establish a general framework and leaves the algorithm design open.

Overall, two key points from the status quo that may necessitate our attention. First, the above recent progress on nonconvex optimization are mainly attributed to the establishment of Kurdyka-ojasiewicz properties hold by many analytical functions, which was first proposed in [35, 36] and later extended in [37, 38]. The KL properties are powerful as they make it possible to characterize sequence around critical points without convexity. Second, the key ideas to investigate the methods' convergence are mainly twofolds: i) studying the convergence of primal and dual sequences by inspecting a tailored Lyapunov function; ii) investigating the local convergence of the algorithm based on the KL properties. However, a typical and difficult problem is that the augmented Lagrangian function generally oscillates in the non-convex context, which makes the use and design of Lyapunov functions particularly difficult [16].

#### B. Our Contributions

Motivated by the recent progress on non-convex optimization, this paper seeks to investigate decentralized optimization for a class of structured problems with (i) nonconvex global objective function (possibly nonsmooth) and (ii) local bounded convex constraints and coupled nonlinear constraints w.r.t. the agents.

In general, to develop a viable decentralized method, we need to overcome two main challenges. First, we need to realize the difficulties of calculating local optima of nonlinear problems at each iteration as required by most existing decentralized paradigms [34]. Second, we need to figure out the convergence conditions to achieve performance guarantee. To address such issues, this paper proposes a *Proximal* Linearization-based Decentralized Method (PLDM). The main ideas are twofolds. First, considering the difficulties to ensure the feasibility of the non-linear coupled constraints, we first introduce some consensus variables to eliminate the nonlinear couplings. Second, to overcome the intrinsic challenges to guarantee convergence as explained in [33], we solve a relaxed non-convex problems in a decentralized manner by combing the augmented Lagrangian-based framework, the alternating minimization (i.e., i.e, Gauss-Seidel), and proximal linearization [12, 39] to approximate the solutions of the original problem. In particular, different from the traditional MMs and augmented Lagrangian-based methods where the exact local optima of the problems are required at each iteration, the proposed method leverages proximal linearization-based technique to update decision variables at each iteration, which makes it computationally efficient and viable for the nonconvex and nonlinear cases. The main contributions of this paper are outlined, i.e.,

- We propose a PLDM for a class of structured nonconvex problems subject to coupled nonlinear constraints and local bounded convex constraints.
- The global convergence and local convergence rate of the method to the ε-critical points are studied by inspecting a tailored Lyapunov function and the ojasiewicz property of the AL function.

The performance of the decentralized method is illustrated by presenting a numeric example and an application to multi-zone heating, ventilation and airconditioning (HVAC) control.

The remainder of this paper is structured. In Section II, the main notations and the problem are presented. In Section III, the PLDM is introduced. In Section IV, the global convergence and local convergence rate of the method are investigated. In Section IV, the performance of the method is evaluated through a numeric example and an application. In Section V, we briefly conclude and discuss this paper.

#### II. THE PROBLEM AND MAIN NOTATIONS

#### A. Notations

Throughout this paper, we use  $\mathbb{N}$ ,  $\mathbb{R}^n$ ,  $\mathbb{R}^n$ , and  $\mathbb{R}^{m \times n}$  to denote the spaces of integers, reals, n-dimensional (positive) real vectors, and  $m \times n$ -dimensional real matrices, respectively. The superscript T denotes the transpose operator.  $I_n$  and  $O_n$  denote the n-dimensional identify and zero matrices. Without specification,  $\|\cdot\|$  denotes  $\ell_2$  norm.  $P_{\mathcal{X}}[\cdot]$  represents the projection operation on the set  $\mathcal{X}$ . We use  $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right)^T$  to denote the gradients of  $f: \mathbb{R}^n \to \mathbb{R}$  w.r.t its entities. If  $\mathbf{h}: \mathbb{R}^n \to \mathbb{R}^m$ , i.e.,  $\mathbf{h}(\mathbf{x}) = (h_1(\mathbf{x}), h_2(\mathbf{x}), \cdots, h_m(\mathbf{x}))^T$  with  $h_i: \mathbb{R}^n \to \mathbb{R}$ , we have  $\nabla \mathbf{h} = \left(\nabla h_1, \nabla h_2, \cdots, \nabla h_m\right)^T$ .  $\prod_{i=1}^N \mathcal{X}_i$  denotes the Cartesian product of the sets  $\mathcal{X}_i$   $(i=1,2,\cdots,N)$ . The other notations are standard and follow the literature [11, 34].

#### B. The Problem

This paper focuses on a class of problems given by

$$\min_{\boldsymbol{x}_{i}, i=1, 2, \dots, N} \sum_{i=1}^{N} f_{i}(\boldsymbol{x}_{i}) + \sum_{i=1}^{N} \phi_{i}(\{x_{j}\}_{j \in \mathcal{N}_{i}})$$

$$s.t. \quad h_{i}(\{x_{j}\}_{j \in \mathcal{N}_{i}}) = 0, \quad \forall i \in \mathcal{N}.$$

$$\boldsymbol{x}_{i} \in \boldsymbol{\mathcal{X}}_{i}, \quad \forall i \in \mathcal{N}.$$
(1)

where i denotes the index of the agents from the set  $\mathcal{N} = \{1, 2, \dots, N\}$ . Here  $\mathbf{x}_i \in \mathbb{R}^{n_i}$  denotes the local decision component of Agent i.  $\mathcal{N}_i$  is alluded to the collection of agent i and its neighbors.

Note that problem (1) has global objective function which is composed by the separable parts  $f_i:\mathbb{R}^{n_i}\to\mathbb{R}$  and the composite parts  $\phi_i:\mathbb{R}^{\bar{n}_i}\to\mathbb{R}$   $(\bar{n}_i=\sum_{i\in\mathcal{N}_i}n_i)$  w.r.t. agents. Wherein the objective function  $f_i$  and  $\phi_i$  may be nonconvex (possibly nonsmooth).  $h_i:\mathbb{R}^{\bar{n}_i}\to\mathbb{R}^{m_i}$  denotes the coupled nonlinear constraints pertaining to Agent i, which are smooth and differentiable. As inequality constraints could be transformed to equality constraints by introducing slack variables, this paper only investigate equality constraints. Besides, there exist local bounded convex constraints represented by  $\mathcal{X}_i$   $(i\in\mathcal{N})$  for the agents. In addition, we make the following assumptions for problem (1) in our analysis, i.e.,

(A1) The equality constraints  $h_i$  are continuously differentiable over  $\mathcal{X}_i$  ( $h_i$  and  $\nabla h_i$  are Lipschitz continuous with constants  $L_{h_i}$  and  $M_{h_i}$ ).

- (A2) The separable parts  $f_i$  and  $\nabla f_i$  are Lipschitz continuous with constants  $L_{f_i}$  and  $M_{f_i}$  over  $\mathcal{X}_i$ , respectively.
- (A3) The coupled parts  $\phi_i$  and  $\nabla \phi_i$  are Lipschitz continuous with constants  $L_{\phi_i}$  and  $M_{\phi_i}$  over  $\prod_{j \in \mathcal{N}_i} \mathcal{X}_i$ , respectively.

#### C. Problem Reformulation

As aforementioned, it's generally challenging to tackle problem (1) with non-linearity and non-convexity both in the objective and the constraints with performance guarantee. Therefore, this part presents some reformulations of the problem as a necessary preparation for the following study. First, to handle the coupled nonlinear couplings, we introduce a block of consensus variables  $\mathbf{Z} = ((z_1)^T, (z_2)^T, \cdots, (z_N)^T)^T \in$  $\mathbb{R}^{\sum_{i\in\mathcal{N}}n_i}$ , which represents the hypothetical copy of the collected decision components for all the agents. As displayed in Fig. 1, each Agent i will hold a local copy of the augmented decision variables denoted by  $X_i = (x_i, \{x_i^j\}_{j \in \mathcal{N}_i \setminus \{i\}})^T \in \mathbb{R}^{\bar{n}_i}$  $(\bar{n}_i = \sum_{j \in \mathcal{N}_i} n_j)$ , in which  $x_i^j$  denotes the local copy of the decision component for its neighboring Agent j. Meanwhile, we assume there is a virtual Agent 0, who is obligated to manage the block of consensus decision variables Z. Intuitively, for an algorithm to converge, we require

$$x_i = z_i, \quad x_i^j = z_j. \tag{2a}$$

$$x_j^i = z_i, \quad x_j = z_j. \tag{2b}$$

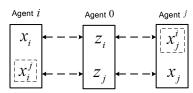


Fig. 1. The local and global (consensus) decision variables.

In this case, problem (1) can be restated as

$$\min_{\boldsymbol{X}_{i},i=1,2,\cdots,N.\boldsymbol{Z}} \quad \sum_{i=1}^{N} \tilde{f}_{i}(\boldsymbol{X}_{i}) + \sum_{i=1}^{N} \phi_{i}(\boldsymbol{X}_{i})$$
(3)

s.t. 
$$h_i(\mathbf{X}_i) = \mathbf{0}, \ \forall i \in \mathcal{N}.$$
 (3a)

$$X_i = E_i Z, \quad \forall i \in \mathcal{N}.$$
 (3b)

$$Z \in \mathcal{X}$$
. (3c)

where we have  $\mathcal{X} = \prod_{i \in \mathcal{N}} \mathcal{X}_i$ .  $\tilde{f}_i : \mathbb{R}^{\bar{n}_i} \to \mathbb{R}$  denotes the extended function of  $f_i$ .  $E_i = diag\{I_{n_j}\}_{j \in \mathcal{N}_i} \in \mathbb{R}^{\bar{n}_i \times \sum\limits_{j \in \mathcal{N}} n_j}$  are constant matrices. (3b) denotes the compact form of the equality constraints (2a) and (2b).

Observe (3), we note that the objective function is now decomposable w.r.t. the agents. However, there appear three types of constraints: (i) local nonlinear and nonconvex constraints (3a), (ii) coupled linear constraints (3b), and (iii) local bounded convex constraints (3c). There exist two blocks of decision variables, i.e., the primal and consensus decision variables  $X = [X_i^T]_{i \in \mathcal{N}}^T$  and Z. Nevertheless, we note that the local constraints are now not regular as  $[(\nabla_{X_i} h_i(X_i))^T, I_{\bar{n}_i}]^T$  is not full row rank. In fact, this presents one of the intrinsic challenges to investigate the convergence of a general decentralized

methods for such problems (this may be comprehended in the rigorous analysis later). Here this may be understood that even a particular (decentralized) algorithm is able to force Z to some (local) optima  $Z^*$ , no  $X^*$  that satisfies the constraints (3a) and (3c) simultaneously can be found. To address such a challenging issue, we introduce some slack variables  $Y_i \in \mathbb{R}^{\bar{n}_i}$   $(i \in \mathcal{N})$  and restate problem (3) as

$$\min_{\bar{\boldsymbol{X}}_{i}, i \in \mathcal{N}.\boldsymbol{Z}} \quad \sum_{i=1}^{N} \tilde{f}_{i}(\boldsymbol{X}_{i}) + \sum_{i=1}^{N} \phi_{i}(\boldsymbol{X}_{i})$$
(4)

s.t. 
$$h_i(\mathbf{X}_i) = \mathbf{0}, \quad \forall i \in \mathcal{N}.$$
 (4a)

$$A_i \bar{X}_i = E_i Z, \quad \forall i \in \mathcal{N}.$$
 (4b)

$$Y_i = 0, \quad \forall i \in \mathcal{N}.$$
 (4c)

$$Z \in \mathcal{X},$$
 (4d)

where  $A_i = [I_{\bar{n}_i}, I_{\bar{n}_i}] \in \mathbb{R}^{\bar{n}_i \times 2\bar{n}_i}$ .  $\bar{X}_i = ((X_i)^T, (Y_i)^T)^T$   $(i \in \mathcal{N})$  denotes the extended local decision variable hold by Agent i.

Note that problem (3) is equivalent to (4) as the slack variables  $Y_i$  are forced to be *zero* in the constraints. However, the local constraints w.r.t the extended local decision variables  $\bar{X}_i$  ( $i \in \mathcal{N}$ ) are still not regular, therefore we have to resort to the following relaxed problem:

$$\min_{\bar{X}_{i}, i \in \mathcal{N}.Z} \quad \sum_{i=1}^{N} \tilde{f}_{i}(X_{i}) + \sum_{i=1}^{N} \phi_{i}(X_{i}) + \sum_{i=1}^{N} M_{i} \|Y_{i}\|^{2} \quad (5)$$

$$s.t.$$
  $h_i(\mathbf{X}_i) = \mathbf{0}, \quad \forall i \in \mathcal{N}.$  (5a)

$$A_i \bar{X}_i = E_i Z, \quad \forall i \in \mathcal{N}.$$
 (5b)

$$Z \in \mathcal{X},$$
 (5c)

where  $M_i > 0$   $(i \in \mathcal{N})$  denotes some positive penalty parameter. In contrast of (5) with (4), one may note that the constraints  $\mathbf{Y}_i = \mathbf{0}$  for the slack variables have been softened by appending some penalty terms  $M_i \| \mathbf{Y}_i \|^2$  in the global objective function. As i) problem (5) contains all the feasible solutions of (4); ii)  $P^{(5),*} \leq P^{(4),*}$   $(P^{(\cdot),*}$  denotes the optima of the problems), problem (5) can be regarded as an relaxation of problem (4). Moreover, we note that with  $M_i \to 0$   $(i \in \mathcal{N})$ , the well-posedness optimal solutions of (5) (with bounded objective value) are exactly those for the original problem (4) as we will have  $\mathbf{Y}_i^* \to \mathbf{0}$   $(\forall i \in \mathcal{N})$  otherwise  $P^{(5),*} \to \infty$ .

The main ideas of this paper are twofolds. *First*, we investigate decentralized method for problem (5) with performance guarantee. *Second*, we prove that the attained solutions are the  $\epsilon$ -critical points of (4) under some mild conditions. Before that, we first make the following extra assumptions.

(A4)  $M_i > 0$   $(i \in \mathcal{N})$  are sufficiently large.

**Remark 1.** Generally, rigorous analysis requires  $M_i \to +\infty$  to guarantee the equivalence of problem (4) and (5). However, a sufficiently large positive value is enough to guarantee the sub-optimality in practice.

(A5) 
$$F_i(\bar{X}_i) = \begin{pmatrix} \nabla_{X_i} h_i(X_i) & O_{n_i} \\ A_i \end{pmatrix}$$
 is uniformly regular with constant  $\theta$  over the bounded set  $\bar{\mathcal{X}}_i^{\eta}$  (see **Definition** 3), where  $\bar{\mathcal{X}}_i^{\eta} = P_{\mathbb{R}^{n_i}}[\mathcal{X}^{\eta}]$ , with  $\mathcal{X}^{\eta} = P_{\mathbb{R}^{n_i}}[\mathcal{X}^{\eta}]$ 

 $\{\bar{X} \in \mathbb{R}^{\sum_{i \in \mathcal{N}} 2\overline{n}_i} \mid \sum_{i=1}^N ||h_i(X_i)|| \le \eta\}$  ( $\eta$  is a positive threshold).

**Remark 2.** (A5) is standard and can generally be satisfied through regulating the dimension of the slack variables.

## III. PROXIMAL LINEARIZATION-BASED DECENTRALIZED METHOD

This section first presents a decentralized method called PLDM for problem (5) and gives the main results on its convergence. After that we prove that the attained solutions are the  $\epsilon$ -critical points of the original problem (4). In particular, the proposed method is established under the general augmented Lagrangian-based framework for nonconvex context (see [16, 40] for examples ) and contains the three standard steps: (i) primal update based on proximal linearization, (ii) dual update and (iii) adaptive step for penalty factors.

To handle the local nonlinear constraints (4a) and the coupled linear constraints (4b), we define the Augmented Lagrangian (AL) function as

$$\mathbb{L}_{\rho}(\bar{\boldsymbol{X}}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{i=1}^{N} \tilde{f}_{i}(\boldsymbol{X}_{i}) + \sum_{i=1}^{N} \phi_{i}(\boldsymbol{X}_{i}) + \sum_{i=1}^{N} M_{i} \|\boldsymbol{Y}_{i}\|^{2}$$

$$+ \sum_{i=1}^{N} (\boldsymbol{\lambda}_{i})^{T} h_{i}(\boldsymbol{X}_{i}) + \sum_{i=1}^{N} \frac{\rho}{2} \|h_{i}(\boldsymbol{X}_{i})\|^{2}$$

$$+ \sum_{i=1}^{N} (\boldsymbol{\mu}_{i})^{T} (\boldsymbol{A}_{i} \bar{\boldsymbol{X}}_{i} - \boldsymbol{E}_{i} \boldsymbol{Z}) + \sum_{i=1}^{N} \frac{\rho}{2} \|\boldsymbol{A}_{i} \bar{\boldsymbol{X}}_{i} - \boldsymbol{E}_{i} \boldsymbol{Z}\|^{2}$$

$$(6)$$

where  $\bar{X}=[\bar{X}_i]_{i\in\mathcal{N}}\in\mathbb{R}^{\sum_{i\in\mathcal{N}}2\bar{n}_i}$  denotes the augmented decision variable for the problem.  $\pmb{\lambda}=[\pmb{\lambda}_i]_{i\in\mathcal{N}}\in\mathbb{R}^{\sum_{i\in\mathcal{N}}m_i}$  and  $\pmb{\mu}=[\pmb{\mu}_i]_{i\in\mathcal{N}}\in\mathbb{R}^{\sum_{i\in\mathcal{N}}2\bar{n}_i}$  are Lagrangian multipliers.  $\rho>0$  denotes the penalty factor.

For given Lagrangian multipliers  $\lambda$  and  $\mu$ , the primal problem needs to be solved is given by

$$\min_{\bar{\boldsymbol{X}}, \boldsymbol{Z}} \quad \mathbb{L}_{\rho}(\bar{\boldsymbol{X}}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \\
s.t. \quad \boldsymbol{Z} \in \boldsymbol{\mathcal{X}}.$$
(7)

Primal Update: observe that (7) has two blocks of decision variables (i.e.,  $\bar{X}$  and Z). For general MMs, a joint optimization of the two decision variable blocks is usually required [41]. However, the primal problem (7) is nonlinear and nonconvex, even obtaining a local optima is difficult with existing approaches. To handle this, we achieve primal update at each iteration in a decentralized manner by performing two steps: (i) proximal linearization of the nonlinear subproblems and (ii) Gauss-seidal update w.r.t. the decision variable blocks (i.e.,  $\bar{X}$ , Z). Observing that AL function (6) is decomposable w.r.t. the agents with the extended local decision variable  $\bar{X}_i$  ( $\forall i \in \mathcal{N}$ ), we define the subproblems for each Agent i as

$$\min_{\bar{\boldsymbol{X}}_{i}} \quad \mathbb{L}_{\rho}^{i}(\bar{\boldsymbol{X}}_{i}, \boldsymbol{Z}, \boldsymbol{\lambda}_{i}, \boldsymbol{\mu}_{i}) = \tilde{f}_{i}(\boldsymbol{X}_{i}) + \phi_{i}(\boldsymbol{X}_{i}) + M_{i} \|\boldsymbol{Y}_{i}\|^{2} 
+ (\boldsymbol{\lambda}_{i})^{T} h_{i}(\boldsymbol{X}_{i}) + \frac{\rho}{2} \|h_{i}(\boldsymbol{X}_{i})\|^{2} + (\boldsymbol{\mu}_{i})^{T} (\boldsymbol{A}_{i} \bar{\boldsymbol{X}}_{i} - \boldsymbol{E}_{i} \boldsymbol{Z})$$
(8)  

$$+ \frac{\rho}{2} \|\boldsymbol{A}_{i} \bar{\boldsymbol{X}}_{i} - \boldsymbol{E}_{i} \boldsymbol{Z}\|^{2}, \quad \forall i \in \mathcal{N}.$$

By performing proximal linearization on the nonlinear and nonconvex subproblems (8) at each iteration k, we have

$$\begin{split} & \min_{\bar{\boldsymbol{X}}_i} \tilde{\mathbb{L}}_{\rho_k}^i(\bar{\boldsymbol{X}}_i, \bar{\boldsymbol{X}}_i^k, \boldsymbol{Z}^k, \boldsymbol{\lambda}_i^k, \boldsymbol{\mu}_i^k) = (\boldsymbol{\mu}_i^k)^T (\boldsymbol{A}_i \bar{\boldsymbol{X}}_i - \boldsymbol{E}_i \boldsymbol{Z}^{k+1}) \\ & + \frac{\rho_k}{2} \|\boldsymbol{A}_i \bar{\boldsymbol{X}}_i - \boldsymbol{E}_i \boldsymbol{Z}^{k+1}\|^2 + \langle \nabla_{\boldsymbol{X}_i} g_i(\bar{\boldsymbol{X}}_i^k, \boldsymbol{\lambda}_i^k, \rho_k), \bar{\boldsymbol{X}}_i - \bar{\boldsymbol{X}}_i^k \rangle \\ & + \frac{c_i^k}{2} \|\bar{\boldsymbol{X}}_i - \bar{\boldsymbol{X}}_i^k\|^2, \ \ \, \forall i \in \mathcal{N}. \end{split}$$

where  $c_i^k$  denotes the step-size for subproblem i at iteration k. Besides, we define  $g_i(\bar{X}_i, \lambda_i, \rho) = \tilde{f}_i(X_i) + \phi_i(X_i) + (\lambda_i)^T h_i(X_i) + \frac{\rho}{2} \|h_i(X_i)\|^2 + M_i \|Y_i\|^2$ .

**Remark 3.** If the objective function  $\tilde{f}_i$  or  $\phi_i$  is non-smooth w.r.t the local decision variable  $\bar{X}_i$ , we can remove it from  $g_i(\bar{X}_i^k, \lambda_i^k, \rho_k)$  and keep them in  $\tilde{\mathbb{L}}_{\rho_k}^i(\bar{X}_i, \bar{X}_i^k, Z^k, \lambda_i^k, \mu_i^k)$ .

For the consensus variable Z, the subproblem is given by

$$\min_{\boldsymbol{Z}} \quad \mathbb{L}_{\rho}(\boldsymbol{Z}, \bar{\boldsymbol{X}}, \boldsymbol{\mu}) = \sum_{i=1}^{N} (\boldsymbol{\mu}_{i})^{T} (\boldsymbol{A}_{i} \bar{\boldsymbol{X}}_{i} - \boldsymbol{E}_{i} \boldsymbol{Z}) 
+ \sum_{i=1}^{N} \frac{\rho}{2} ||\boldsymbol{A}_{i} \bar{\boldsymbol{X}}_{i} - \boldsymbol{E}_{i} \boldsymbol{Z}||^{2}, \quad s.t. \quad \boldsymbol{Z} \in \boldsymbol{\mathcal{X}}.$$
(9)

We note that subproblem (9) is a quadratic programming (QP), which can be solved efficiently.

Dual update & Adaptive step: the dual variables are updated following the standard augmented Lagrangian methods (see [42] for example). However, we introduce an adaptive step to dynamically update the penalty factor  $\rho$ . In particular, we pre-define a sub-feasible region regarding the non-linear constraints of problem (5), i.e.,

$$\bar{\mathcal{X}}^{\eta} \triangleq \left\{ \bar{\mathbf{X}} \in \mathbb{R}^{\sum_{i \in \mathcal{N}} 2\overline{n}_i} \middle| \sum_{i=1}^N \|h_i(\mathbf{X}_i)\| \leq \eta \right\}.$$

We iteratively increase  $\rho_k$  with an increment  $\delta$  until  $\bar{X}^k$  is forced into the pre-defined sub-feasible region  $\bar{\mathcal{X}}^{\eta}$ .

The main steps of the proposed PLDM method for solving problem (5) are summarized in **Algorithm** 1. The algorithm starts by initializing the Lagrangian multipliers, penalty factor, and decision variables. Afterwards, the main steps include the alternative update of the two decision blocks  $(\bar{X}, Z)$  (Step 3-5), the Lagragnain multipliers  $(\lambda, \mu)$  (Step 6), and the penalty factor  $(\rho)$  (Step 7). We note that as the primal problem (7) is decomposable w.r.t the agents, the update of the primal decision variable block  $(\bar{X})$  can be performed in parallel by the agents. In **Algorithm** 1, the stopping criterion is defined as the residual error bound of the constraints, i.e.,

$$R(k) = \sum_{i=1}^{N} \left\{ \|h_i(\boldsymbol{X}_i^{k+1})\| + \|\boldsymbol{A}_i \bar{\boldsymbol{X}}_i^{k+1} - \boldsymbol{E}_i \boldsymbol{Z}^{k+1}\| \right\} \le \epsilon$$

where  $\epsilon$  is a constant threshold.

The algorithm iterates until the stopping criterion is reached. Still, this does not mean the convergence of the algorithm. This needs to be studied in greater detail later.

Algorithm 1 Proximal Linearization-based Decentralized Method (PLDM) for Nonconvex and Nonlinear Problems

- 1: **Initialization:**  $\lambda^0$ ,  $\mu^0$ ,  $\bar{X}^0$ ,  $Z^0$  and  $\rho_0$ , and set  $k \to 0$ .
- 2: Repeat:
- 3: Primal Update:
- 4: Update the consensus variables Z, i.e.,

$$\boldsymbol{Z}^{k+1} = \arg\min_{\boldsymbol{Z} \in \mathcal{X}} \mathbb{L}_{\rho_k}(\boldsymbol{Z}, \bar{\boldsymbol{X}}^k, \boldsymbol{\mu}^k). \tag{10}$$

5: Update the primal decision variables  $\bar{X}$  in parallel, i.e.,

$$\bar{\boldsymbol{X}}_{i}^{k+1} = \arg\min_{\bar{\boldsymbol{X}}_{i}} \tilde{\mathbb{L}}_{\rho_{k}}^{i} (\bar{\boldsymbol{X}}_{i}, \bar{\boldsymbol{X}}_{i}^{k}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}_{i}^{k}, \boldsymbol{\mu}_{i}^{k}),$$

$$\forall i \in \mathcal{N}.$$
(11)

6: **Dual update:** Update Lagrangian multipliers  $\lambda$ ,  $\mu$ , i.e.,

$$\lambda_i^{k+1} = \lambda_i^k + \rho_k h_i(\boldsymbol{X}_i^{k+1}), 
\boldsymbol{\mu}_i^{k+1} = \boldsymbol{\mu}_i^k + \rho_k (\boldsymbol{A}_i \bar{\boldsymbol{X}}_i^{k+1} - \boldsymbol{E}_i \boldsymbol{Z}^{k+1}), 
\forall i \in \mathcal{N}.$$
(12)

- 7: **Adaptive step:** Update the penalty factor  $\rho$ , i.e., if  $\bar{X}^{k+1} \in \mathcal{X}^{\eta}$ , set  $\rho_{k+1} = \rho_k + \delta$ , otherwise  $\rho_{k+1} = \rho_k$ .
- 8: If  $R(k) \le \epsilon$  stop, otherwise set k = k+1 and go to **Step** 2.

## IV. PERFORMANCE AND CONVERGENCE ANALYSIS OF PLDM

This section discusses the performance and convergence of PLDM. First, we present the additional assumptions, basic definitions and propositions required. Then we illustrate the main results in **Theorem** 1 and **Theorem** 2.

#### A. Definitions and Lemmas

**Lemma 1.** (Descent lemma) (see [8], Proposition A.24) Let  $h : \mathbb{R}^d \to \mathbb{R}$  be  $M_h$ -Lipschitz gradient continuous, we have

$$h(u) \le h(v) + \langle u - v, \nabla h(u) \rangle + \frac{M_h}{2} ||u - v||^2, \ \forall u, v \in \mathbb{R}^d.$$

**Lemma 2.** (Sufficient decrease property) (see [12], Lemma 2) Let  $h: \mathbb{R}^d \to \mathbb{R}$  be  $M_h$ -Lipschitz gradient continuous and  $\sigma: \mathbb{R}^d \to \mathbb{R}$  be a proper and lower semicontinuous function with  $\inf_{\mathbb{R}^d} \sigma > -\infty$ . If  $\operatorname{prox}_t^{\sigma} = \arg\min\{\sigma(u) + \frac{t}{2}||u - x||^2\}$  denotes the proximal map associated with  $\sigma$ , then for any fixed  $t > M_h$ ,  $u \in \operatorname{dom} \sigma$ , and  $u^+$  defined by

$$u^{+} \in prox_{t}^{\sigma} \left( u - \frac{1}{t} \nabla h(u) \right) \tag{13}$$

we have  $h(u^+) + \sigma(u^+) \le h(u) + \sigma(u) - \frac{1}{2}(t - M_h) \|u^+ - u\|^2$ .

**Definition 1.** (Normal cone) (see [40]) Let  $\mathcal{X} \subseteq \mathbb{R}^d$  be a convex set, the normal cone of  $\mathcal{X}$  is the set-valued mapping

$$\mathcal{N}_{\mathcal{X}}(\bar{x}) = \left\{ \begin{array}{ll} \left\{ g \in \mathbb{R}^d | \forall x \in \mathcal{X}, g^T(x - \bar{x}) \leq 0 \right\}, & \text{if } \bar{x} \in \mathcal{X} \\ \varnothing, & \text{if } \bar{x} \notin \mathcal{X}. \end{array} \right.$$

**Definition 2.** (*Critical point*) (see [43–45]) Considering the following problem (**P**):

$$(\mathbf{P}) \left\{ \min_{x \in \mathbb{R}^d} f(x) \mid \mathbf{g}(x) \le \mathbf{0}, \mathbf{h}(x) = \mathbf{0}. \right\}$$

where the objective  $f: \mathbb{R}^d \to \mathbb{R}$ , and the constraints  $\mathbf{g} = (g_1, g_2, \cdots, g_m)$  with  $g_i: \mathbb{R}^d \to \mathbb{R}$ ,  $\mathbf{h} = (h_1, h_2, \cdots, h_\ell)$  with  $h_i: \mathbb{R}^d \to \mathbb{R}$  are continuously differentiable. The critical points (i.e., KKT points) of problem (P) denote its feasible points satisfying first-order optimality condition described by

$$\operatorname{crit}(\boldsymbol{P}) = \left\{ \begin{array}{c|c} x \in \mathbb{R}^d \\ \boldsymbol{\lambda} \in \mathbb{R}^m_+ \\ \boldsymbol{\mu} \in \mathbb{R}^\ell \end{array} \middle| \begin{array}{c} \nabla f(x) + (\nabla \boldsymbol{g}(x))^T \boldsymbol{\lambda} + (\nabla h(x))^T \boldsymbol{\mu} = \boldsymbol{0}. \\ \boldsymbol{g}(x) \geq 0, \ \boldsymbol{h}(x) = 0. \\ \boldsymbol{\lambda}_i g_i(x) = 0, \ i = 1, 2, \cdots, m. \end{array} \right\}$$

As a direct extension, we can define the collection of  $\epsilon$ -critical points  $crit_{\epsilon}(P)$  by replacing 0 on the right-hand side of the equalities (inequalities) with  $\epsilon$ .

Remark 4. For convex problems, the critical points (KKT points) are exactly the global optima. However, without convexity, a critical point can be a global optima, a local optima, or a "saddle point". For problem (5) discussed in this paper, the critical points can be described by

$$crit (problem (5)) =$$

$$\left\{\begin{array}{ll} \bar{\boldsymbol{X}} \in \mathbb{R}^{\sum_{i \in \mathcal{N}} \bar{n}_i} \\ \boldsymbol{\lambda} \in \mathbb{R}^{\sum_{i \in \mathcal{N}} m_i} \\ \boldsymbol{\mu} \in \mathbb{R}^{\sum_{i \in \mathcal{N}} 2\bar{n}_i} \end{array} \middle| \begin{array}{l} \nabla_{\bar{\boldsymbol{X}}} \mathbb{L}_{\rho}(\bar{\boldsymbol{X}}, \boldsymbol{Z}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \boldsymbol{0}. \\ \nabla_{\bar{\boldsymbol{X}}} \mathbb{L}_{\rho}(\bar{\boldsymbol{X}}, \boldsymbol{Z}, \boldsymbol{\lambda}, \boldsymbol{\mu}) + \mathcal{N}_{\mathcal{X}}(\boldsymbol{Z}) = \boldsymbol{0}. \\ \boldsymbol{h}_i(\bar{\boldsymbol{X}}_i) = \boldsymbol{0}, \ \forall i \in \mathcal{N}. \end{array} \right\}$$

**Definition 3.** (Uniform Regularity) (see [16]) Let  $\mathcal{X} \subseteq \mathbb{R}^m$ and  $h: \mathbb{R}^m \to \mathbb{R}^n$  be continuously differentiable, we claim h as uniformly regular over X with a positive constant  $\theta$  if

$$\|(\nabla h(x))^T v\| \ge \theta \|v\|, \forall x \in \mathcal{X}, v \in \mathbb{R}^n$$
 (14)

**Remark 5.** For a given  $x \in \mathcal{X}$ , asserting that  $\nabla h(x)$  is uniformly regular with  $\theta$  ( $\theta > 0$ ) is equivalent to

$$\gamma(F,x) = \min_{\|\boldsymbol{v}\|=1} \left\{ \|(\nabla h(x))^T \boldsymbol{v}\| \right\} > 0.$$

Equivalently, the mapping  $(\nabla h(x))^T$  is supposed to be surjective or  $\nabla h(x)(\nabla h(x))^T$  is positive definite. one may note that  $\nabla h(x)(\nabla h(x))^T$  is always positive semidefinite. Therefore, the uniform regularity of  $\nabla h(x)$  requires that the minimum eigenvalue of  $\nabla h(x)(\nabla h(x))^T$  is positive. Besides, we note that if  $\nabla h(x)$  has full row rank,  $\nabla h(x)$  will be uniformly regular with an existing positive constant.

The next definition is a property that many analytical functions hold and plays a central role in the convergence analysis of nonconvex optimization (see [12, 46]).

**Definition 4.** (Kurdyka-ojasiewicz (K) property) (see [12]) We say function  $h: \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$  has the K property at  $x^* \in dom \ \partial h$ , if there exist  $\eta \in (0, +\infty)$ , a neighborhood U of  $x^*$ , and a continuous concave function  $\varphi:[0,\eta)\to\mathbb{R}_+$ 

- (i)  $\varphi(0) = 0$  and  $\varphi$  is differentible on  $(0, \varphi)$ ;
- (ii)  $\forall s \in (0, \eta), \ \varphi'(s) > 0;$
- (iii)  $\forall s \in U \cap \{x : h(x^*) < h(x) < h(x^*) + \eta\}$ . Then the following K inequality holds:

$$\varphi^{'}\big(h(x)-h(x^*)\big)\cdot \mathit{dist}\big(0,\partial h(x)\big)\geq 1,$$

and function  $\varphi$  is called a desingularizing function of h at  $x^*$ .

For Algorithm 1, we need two additional assumptions to guarantee convergence.

- (A6) The Lagrangian multipliers sequences  $\{\lambda_i^k\}_{k\in\mathbb{N}}$   $(i\in\mathcal{N})$ and  $\{\mu_i^k\}_{k\in\mathbb{N}}$   $(i\in\mathcal{N})$  generated by Algorithm 1 are bounded by  $M_{\lambda_i}$  and  $M_{\mu_i}$ , respectively.
- (A7) The original problem (1) is well-defined. Thus for a proper penalty factor  $\rho < +\infty$ , the primal problem in (7) is lower bounded as

$$\inf_{\bar{X}, Z \in \mathcal{X}} \mathbb{L}_{\rho}(\bar{X}, Z, \lambda, \mu) > -\infty.$$
 (15)

#### B. PLDM Convergence

In this subsection, we first illustrate **Propositions** 1-9 that required to prove the convergence of the PLDM. After that we present the main results in **Theorem** 1.

**Proposition 1.** Let  $\bar{\mathcal{X}}_i^{\eta} = P_{\mathbb{R}^{\bar{n}_i}}[\bar{\mathcal{X}}^{\eta}]$   $(i \in \mathcal{N})$ , there exists a finite iteration k such that

$$\begin{array}{ll} \hbox{\it (a)} & \{\bar{\boldsymbol{X}}_i^k\}_{k\in\mathbb{N}, k\geq\underline{k}}\in\bar{\boldsymbol{\mathcal{X}}}_i^{\eta} \ (\forall i\in\mathcal{N}).\\ \hbox{\it (b)} & \rho_k=\rho_{\underline{k}}, \ \forall k\geq\underline{k}. \end{array}$$

(b) 
$$\rho_k = \rho_k, \quad \forall k \geq \underline{k}$$

Proof: Refer to Appendix A.

Remark 6. Proposition 1(a) illustrates that for any predefined sub-feasible region for the non-linear constraints (5a), the local decision variables  $(\bar{X}_i^k)$  for the agents will be forced into the sub-feasible region  $(\bar{X}_i^\eta)$  within finite iterations  $(\underline{k})$ . Meanwhile, the penalty factor  $(\rho)$  will stop increasing as described in **Proposition** 1(b).

**Proposition 2.**  $g_i(\bar{X}_i, \lambda_i^k, \rho_k)$   $(\forall k \in \mathbb{N})$  is Lipschitz continuous w.r.t  $\bar{X}_i$  over  $\bar{X}_i^{\eta}$  with constant  $L_{g_i} = \max\{L_{f_i} + 1\}$  $L_{\phi_i} + M_{\lambda_i} L_{h_i} + \rho_k C_{h_i}, 2M_i$  ( $C_{h_i}$  denotes the upper bound of  $\|(\nabla_{\mathbf{X}_i} h_i(\mathbf{X}_i))^T \overline{h}_i(\mathbf{X}_i)\|$  over  $\mathbf{\bar{X}}_i^{\eta}$ ).

Proof: Refer to Appendix B.

Remark 7. Proposition 2 presents the smoothness of the function  $g_i(\bar{X}_i, \lambda_i^k, \rho_k)$  ( $\forall k \in \mathbb{N}$ ) over the bounded subfeasible region  $\bar{\mathcal{X}}_{i}^{\eta}$ .

**Proposition 3.** Let  $\{\bar{X}^k\}_{k\in\mathbb{N},k\geq\underline{k}}$  and  $\{Z^k\}_{k\in\mathbb{N},k\geq\underline{k}}$  be the sequences generated by **Algorithm** 1, then we have

$$\mathbb{L}_{\rho_{\underline{k}}}(\bar{X}^{k+1}, Z^{k+1}, \lambda^{k}, \mu^{k}) \leq \mathbb{L}_{\rho_{\underline{k}}}(\bar{X}^{k}, Z^{k}, \lambda^{k}, \mu^{k})$$
$$-\sum_{i=1}^{N} \frac{1}{2} (c_{i}^{k} - L_{g_{i}}) \|\bar{X}_{i}^{k+1} - \bar{X}_{i}^{k}\|^{2}$$
(16)

Proof: Refer to Appendix C.

Remark 8. Proposition 3 provides the lower bound for the "decrease" of the AL function w.r.t. the primal updates. One may note that the AL function is non-increasing w.r.t the primal update at each iteration with a step-size  $c_i^k \geq L_{q_i}$ .

**Proposition 4.** Let  $\{\bar{X}^k\}_{k\in\mathbb{N},k\geq\underline{k}}$  and  $\{Z^k\}_{k\in\mathbb{N},k\geq\underline{k}}$  be the sequances generated by Algorithm 1, we have

$$\|\nabla_{\bar{\boldsymbol{X}}_{i}}\mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}^{k}, \boldsymbol{\mu}^{k})\|$$

$$\leq (L_{q_{i}} + c_{i}^{k})\|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\|, \ \forall i \in \mathcal{N}.$$
(17)

Proof: Refer to Appendix D.

**Remark 9.** Proposition 4 provides the upper bound for the subgradients of the AL functions after primal update at each iteration.

**Proposition 5.** Let  $\{\bar{X}^k\}_{k\in\mathbb{N},k\geq\underline{k}}$  be the sequence generated by **Algorithm** 1, we have

$$\|\boldsymbol{\gamma}_{i}^{k+1} - \boldsymbol{\gamma}_{i}^{k}\| \leq \Omega_{1}^{i} \|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\| + \Omega_{2}^{i} \|\bar{\boldsymbol{X}}_{i}^{k} - \bar{\boldsymbol{X}}_{i}^{k-1}\|,$$

$$\forall i \in \mathcal{N}.$$
(18)

where  $\gamma_i = \left( (\lambda_i)^T, (\mu_i)^T \right)^T$  denoting the augmented Lagrangian multipliers for Agent i. We have  $\Omega_1^i = \left( L_{g_i} + c_i^k + L_{f_i} + L_{\phi_i} + M_{h_i} M_{\gamma_i} \right) / \theta$  and  $\Omega_2^i = \left( L_{g_i} + c_i^{k-1} \right) / \theta$  ( $i \in \mathcal{N}$ ).

Proof: Refer to Appendix E.

**Remark 10.** Proposition 5 provides the upper bound for the difference of the Lagrangian multipliers over two successive iterations.

To illustrate the convergence of the proposed PLMD, we need to resort to a Lyapunov function defined as

$$\Phi_{\beta}(\bar{X}, Z, \lambda, \mu, U) = \mathbb{L}_{\rho}(\bar{X}, Z, \lambda, \mu) + \beta \|\bar{X} - U\|^{2},$$

where  $\beta > 0$  and  $U \in \mathbb{R}^{\sum_{i \in \mathcal{N}} \bar{n}_i}$ . One may note that the Lyapunov function closely relates to the AL function except for the extra term  $\beta \|\bar{X} - U\|^2$ . For the Lyapunov function under the sequences generated by **Algorithm** 1, we have the following proposition.

**Proposition 6.** Let  $\left\{ W^k = \left( \bar{X}^k, Z^k, \lambda^k, \mu^k, \bar{X}^{k-1} \right) \right\}_{k \in \mathbb{N}, k \geq \underline{k}}$  be the sequence generated by **Algorithm** 1, we have

$$\Phi_{\beta_k}(\boldsymbol{W}^k) - \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}) 
\geq b_1^k \|\bar{\boldsymbol{X}}^{k+1} - \bar{\boldsymbol{X}}^k\|^2 + b_2^k \|\bar{\boldsymbol{X}}^k - \bar{\boldsymbol{X}}^{k-1}\|^2,$$
(19)

where  $\Phi_{\beta_k}(\boldsymbol{W}^k) = \mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^k, \boldsymbol{Z}^k, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) + \beta_k \|\bar{\boldsymbol{X}}^k - \bar{\boldsymbol{X}}^{k-1}\|^2$ , with  $\beta_k$  a positive parameter. Besides, we have

$$\begin{split} b_1^k &= \min_i \Big\{ \frac{1}{2} (c_i^k - L_{g_i}) - \frac{2(\Omega_1^i)^2}{\rho_{\underline{k}}} - \beta_{k+1} \Big\}, \\ b_2^k &= \min_i \Big\{ \beta_k - \frac{2(\Omega_2^i)^2}{\rho_{\underline{k}}} \Big\}. \end{split}$$

Proof: Refer to Appendix F.

**Remark 11.** From **Proposition** 1, we have  $\{\bar{X}_k\}_{k\in\mathbb{N},k\geq\underline{k}}\in \bar{\mathcal{X}}_i^{\eta}$ . To illustrate the convergence of PLDM over  $k\geq\underline{k}$ , we resort to the Lyaponov function  $\Phi_{\beta_k}(\mathbf{W}^k)$ . **Proposition** 6 provides the lower bound for the decrease of the Lyapunov function over successive iterations after iteration  $\underline{k}$ .

**Proposition 7.** Let  $\{\lambda^k\}_{k\in\mathbb{N}}$  and  $\{\bar{X}^k\}_{k\in\mathbb{N}}$  be the sequence generated by **Algorithm** 1, then we have

$$\lim_{k \to +\infty} \left\| \bar{\boldsymbol{X}}^{k+1} - \bar{\boldsymbol{X}}^{k} \right\| \to 0.$$

$$\lim_{k \to +\infty} \left\| \bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k} \right\| \to 0,$$

$$\lim_{k \to +\infty} \left\| \boldsymbol{\lambda}_{i}^{k+1} - \boldsymbol{\lambda}_{i}^{k} \right\| \to 0,$$

$$\lim_{k \to +\infty} \left\| \boldsymbol{\mu}_{i}^{k+1} - \boldsymbol{\mu}_{i}^{k} \right\| \to 0, \quad \forall i \in \mathcal{N},$$
(20)

provided with **Condition** (a):

$$b_1^k = \min_i \left\{ \frac{1}{2} (c_i^k - L_{g_i}) - \frac{2(\Omega_1^i)^2}{\rho_{\underline{k}}} - \beta_{k+1} \right\} > 0,$$

$$b_2^k = \min_i \left\{ \beta_k - \frac{2(\Omega_2^i)^2}{\rho_k} \right\} > 0.$$

is satisfied.

Proof: Refer to Appendix G.

**Remark 12.** Proposition 7 illustrates the boundedness of primal and dual sequences generated by Algorithm 1 under Condition (a), which can be satisfied by selecting  $\beta_k$  that  $\max_i \left\{ \frac{2(\Omega_2^i)^2}{\rho_k} \right\} < \beta_k < \min_i \left\{ \frac{1}{2} (c_i^{k-1} - L_{g_i}) - \frac{2(\Omega_1^i)^2}{\rho_k} \right\}.$ 

**Proposition 8.** Let  $\{W^k\}_{k\in\mathbb{N}, k\geq\underline{k}}$  be the sequence generated by Algorithm 1, we have

$$\|\nabla \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1})\| \le b_3^k \sum_{i=1}^N \|\bar{\boldsymbol{X}}_i^{k+1} - \bar{\boldsymbol{X}}_i^k\| + b_4^k \sum_{i=1}^N \|\bar{\boldsymbol{X}}_i^k - \bar{\boldsymbol{X}}_i^{k-1}\|$$
(21)

where we have  $B = \sup_{k \geq \underline{k}} \| \mathbf{F}_i(\bar{\mathbf{X}}_i^{k+1}) \|$ .  $b_3^k = \max_i \{ L_{g_i} + c_i^k + \Omega_1^i B + 4\beta_{k+1} + \rho_{\underline{k}} + \frac{\Omega_1^i}{\rho_k} \}$  and  $b_4^k = \max_i \{ \Omega_2^i B + \frac{\Omega_2^i}{\rho_k} \}$ .

Proof: Refer to Appendix H.

**Remark 13.** Proposition 8 provides the upper bound for the subgradient of the Lyapunov function  $\Phi_{\beta_k}$ .

Based on **Proposition**1-9, we have the following results for the convergence of the PLDM.

**Theorem 1.** (Convergence) The PLDM described in Algorithm 1 converges to the critical points of problem (5) provided with a step-size  $c_k^i$  satisfying Condition (a) in Proposition 7.

*Proof.* Based on *Proposition* 7 and *Proposition* 8, we have

$$\lim_{k \to +\infty} \left\| \nabla \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}) \right\| \to 0. \tag{22}$$

Specifically, we have

$$\nabla_{\bar{\boldsymbol{X}}} \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}) = \nabla_{\bar{\boldsymbol{X}}} \mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}^{k+1}, \boldsymbol{\mu}^{k+1}) + 2\beta_{k+1}(\bar{\boldsymbol{X}}^{k+1} - \bar{\boldsymbol{X}}^{k})$$

$$\nabla_{\boldsymbol{Z}} \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}) = \nabla_{\boldsymbol{Z}} \mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}^{k+1}, \boldsymbol{\mu}^{k+1}) + \mathcal{N}_{\mathcal{X}}(\boldsymbol{Z}^{k+1})$$

$$\nabla_{\boldsymbol{\gamma}} \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}) = \nabla_{\boldsymbol{\gamma}} \mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}^{k+1}, \boldsymbol{\mu}^{k+1})$$

$$\nabla_{\boldsymbol{U}} \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}) = 2\beta_{k+1}(\bar{\boldsymbol{X}}^{k+1} - \bar{\boldsymbol{X}}^{k})$$

$$(23)$$

By combining (22) with (23), we have

$$\lim_{k \to +\infty} \nabla_{\bar{\boldsymbol{X}}} \mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}^{k+1}, \boldsymbol{\mu}^{k+1}) = 0.$$

$$\lim_{k \to +\infty} \nabla_{\boldsymbol{Z}} \mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}^{k+1}, \boldsymbol{\mu}^{k+1}) + \mathcal{N}_{\mathcal{X}}(\boldsymbol{Z}^{k+1}) = 0.$$

$$\lim_{k \to +\infty} h_{i}(\bar{\boldsymbol{X}}_{i}^{k+1}) = 0, \ \forall i \in \mathcal{N}.$$

$$\lim_{k \to +\infty} \left(\boldsymbol{A}_{i}\bar{\boldsymbol{X}}_{i}^{k+1} - \boldsymbol{E}_{i}\boldsymbol{Z}^{k+1}\right) = 0, \ \forall i \in \mathcal{N}.$$

According to **Definition** 2, the above implies that PLDM will converge to the critical points of problem (4).

**Theorem 2.** The critical points of problem (5) obtained from Algorithm 1 are the  $\epsilon$ -critical points of problem (4).

*Proof.* We first denote the critical points of problem (5) as  $W^* = (\bar{X}^*, Z^*, \lambda^*, \mu^*, \bar{X}^*)$ . Based on **Theorem** 1, we have  $W^k \to W^*$  with  $k \to +\infty$ . Based on the definition of  $\epsilon$ -critical points (see **Definition** 2), we only need to prove  $\|Y_i^*\| \le \epsilon$  to illustrate the results.

This can be illustrated in twofolds.

- i) according to **proposition** 6, we have  $\Phi_{\beta_k}(\boldsymbol{W}^k)$  is non-increasing w.r.t the iteration k. Therefore, we have  $\Phi_{\bar{\beta}}(\boldsymbol{W}^*) \leq \Phi_{\beta_0}(\boldsymbol{W}^0)$  with  $k \in \mathbb{N}$  (bounded), where we denote  $\beta_k \to \bar{\beta}$  with  $k \to +\infty$ .
- $\begin{array}{ll} \emph{ii)} \ \ \text{Based} \ \ \text{on} \ \ \mathbf{Theorem} \quad 1, \ \ \text{we have} \ \ \Phi_{\bar{\beta}}(\boldsymbol{W}^*) = \\ \sum_{i=1}^N \tilde{f}_i(\boldsymbol{X}_i^*) + \sum_{i=1}^N \phi_i(\boldsymbol{X}_i^*) + \sum_{i=1}^N M_i \|\boldsymbol{Y}_i^*\|^2 \ \text{as} \ h_i(\bar{\boldsymbol{X}}_i^*) = \\ 0 \ \ \text{and} \ \ \left(\boldsymbol{A}_i\bar{\boldsymbol{X}}_i^* \boldsymbol{E}_i\boldsymbol{Z}^*\right) = \boldsymbol{0} \ (\forall i \in \mathcal{N}). \end{array}$

Thus by combining i) and ii), we imply

$$\|Y_i^*\| \le \sqrt{\frac{\Phi_{\beta_0}(W^0) - \sum_{i=1}^N \tilde{f}(X_i^*) - \sum_{i=1}^N \phi_i(X_i^*)}{\min_i M_i \cdot N}} \le \epsilon,$$

 $\forall i \in \mathcal{N}$  with  $M_i$  sufficiently large.

#### C. Convergence Rate

With the global convergence of the PLDM studied, this section discusses the local convergence rate of the method. Before we present the main results, we illustrate the following propositions that referred to.

**Proposition 9.** Let  $\{W^k\}_{k \in \mathbb{N}, k \geq \underline{k}}$  be the sequance generated by **Algorithm** 1, then we have

$$\left[ dist \left( \nabla \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}), 0 \right) \right]^{2} \\
\leq 2N\nu_{k} \left( \Phi_{\beta_{k}}(\boldsymbol{W}^{k}) - \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}) \right) \tag{24}$$

provided with the step-size  $c_i^k$  and the parameter  $\beta_k$  satisfying **Condition** (b):

$$\nu_k = \frac{(b_4^k)^2}{b_2^k} \le \frac{(b_3^k)^2}{b_1^k}$$

where  $\nu_k$  is the positive parameter that need to be decided for **Algorithm** 1.

Remark 14. Proposition 9 gaps the subgradients of the Lyapunov function  $\Phi_{\beta_{k+1}}(\mathbf{W}^{k+1})$  in terms of its value decrease over successive iterations. To guarantee Condition (b), the step-size  $c_i^k$  and the parameter  $\beta_k$  can be selected following the procedures below:

Based on Proposition 6 and Proposition 8, we have

$$\begin{split} b_1^k &= \min_i \left\{ \frac{1}{2} (c_i^k - L_{g_i}) - \frac{2(\Omega_1^i)^2}{\rho_{\underline{k}}} - \beta_{k+1} \right\} \\ &= \frac{1}{2} c_i^k - \max_i \left\{ \frac{1}{2} L_{g_i} \right) + \frac{2(\Omega_1^i)^2}{\rho_{\underline{k}}} + \beta_{k+1} \right\} \\ &= \frac{1}{2} c_i^k - M \\ b_2^k &= \min_i \left\{ \beta_k - \frac{2(\Omega_2^i)^2}{\rho_{\underline{k}}} \right\} \\ b_3^k &= \max_i \left\{ L_{g_i} + c_i^k + \Omega_1^i B + 4\beta_{k+1} + \rho_{\underline{k}} + \frac{\Omega_1^i}{\rho_{\underline{k}}} \right\} \\ &= c_i^k + \max_i \left\{ L_{g_i} + \Omega_1^i B + 4\beta_{k+1} + \rho_{\underline{k}} + \frac{\Omega_1^i}{\rho_{\underline{k}}} \right\} \\ &= c_i^k + L \\ b_4^k &= \max_i \left\{ (B + \frac{1}{\rho_{\underline{k}}}) \Omega_2^i \right\} \end{split}$$

where we have  $M=\max_i \left\{\frac{1}{2}L_{g_i}+\frac{2(\Omega_1^i)^2}{\rho_{\underline{k}}}+\beta_{k+1}\right\}$  and  $L=\max_i \left\{L_{g_i}+\Omega_1^iB+4\beta_{k+1}+\rho_{\underline{k}}+\frac{\Omega_1^i}{\rho_{\underline{k}}}\right\}$ .

We first select a large enough positive parameter  $\nu_k > 0$ . By letting  $\nu_k = \frac{(b_4^k)^2}{b_2^k} = \frac{\max\limits_i \left\{ \left(B + \frac{1}{\rho_{\underline{k}}}\right)^2 (\Omega_2^i)^2 \right\}}{\min\limits_i \left\{ \beta_k - \frac{2(\Omega_2^i)^2}{\rho_{\underline{k}}} \right\}}$ , we can set  $\beta_k = 2 \left( \frac{(B + \frac{1}{\rho_k})^2}{\nu_k} + \frac{1}{\rho_{\underline{k}}} \right) \max\limits_i \left\{ (\Omega_2^i)^2 \right\}$  $= 2 \frac{(1 + B\rho_{\underline{k}})^2 + \nu_k}{\nu_k \rho_k} \max\limits_i \left\{ (\Omega_2^i)^2 \right\}.$ 

Besides, if we select  $\nu_k = \nu_{\underline{k}}$ , we have  $\beta_k = \beta_{\underline{k}} \quad \forall k \geq \underline{k}$ . Afterwards, by letting  $\nu_k \geq (b_3^k)^2/b_1^k$ , i.e.,

$$\nu_k \ge \frac{(c_i^k + L)^2}{\frac{1}{2}c_i^k - M},\tag{25}$$

we can select the step-size  $c_i^k$  at each iteration k as  $c_i^k \in \left[\frac{(\nu_k-4M)-\sqrt{(\nu_k)^2-(8M+16L)\nu_k}}{4}, \frac{(\nu_k-4M)+\sqrt{(\nu_k)^2-(8M+16L)\nu_k}}{4}\right] \cap (0,+\infty)$ . Besides, to ensure that at least one positive step-size exists, the parameter  $\nu_k$  should be selected as  $\nu_k \geq (8M+16L) \ (\forall k \geq \underline{k})$ .

**Theorem 3.** (Convergence rate) Suppose the Lynapunov function  $\Phi_{\beta_k}(\mathbf{W}^k)$  satisfy the KL property with the desin-

<sup>&</sup>lt;sup>1</sup>Note that KL property is general for most analytical functions

gularising function of the form  $\varphi(t) = \frac{C}{\theta}t^{\theta}$  for some C > 0,  $\theta \in (0,1]$  ([46]) and  $c_i^k$ ,  $\beta_k$  are selected satisfying **Condition** (b) in **Proposition** 9, we have:

- (i) If  $\theta = 1$ , Algorithm 1 terminates in finite iterations.
- (ii) If  $\theta \in (0, 1/2)$ ,  $\exists k_0, k_1, k_2 \in \mathbb{R}$ , such that  $\forall \max\{k_0, k_1\} \le k \le k_2$ , we have

$$\Delta^{k} \le \left(\frac{2NC^{2}\nu_{k-1}}{1 + 2NC^{2}\nu_{k-1}}\right)^{\frac{1}{1+2\theta}} \Delta^{k-1}.$$

 $\forall k \geq \max\{k_0, k_1, k_2\}$ , we have

$$\Delta^{k} \le \left(\frac{2NC^{2}\nu_{k-1}}{1 + 2NC^{2}\nu_{k-1}}\right)^{\frac{1}{2-2\theta}} \left(\Delta^{k-1}\right)^{\frac{1}{2-2\theta}}.$$

(iii) If  $\theta \in [1/2, 1)$ ,  $\exists k_3 \in \mathbb{N}$ ,  $\forall k \geq k_3$ , such that

$$\Delta^k \leq \frac{2NC^2\nu_{k-1}}{1+2NC^2\nu_{k-1}} \cdot \Delta^{k-1}$$

where  $\Delta^k = \Phi_{\beta_k}(\mathbf{W}^k) - \Phi^*$  denotes the "suboptimality" (critial points or KKT points) at iteration k, with  $\Phi^*$  as the "optimal" value of the Lynapunov function.

*Proof.* We define the set of critical point for the Lyapunov function  $\Phi_{\beta_k}(\boldsymbol{W}^k)$  as  $\omega(\boldsymbol{W}^0)$  with  $\boldsymbol{W}^0=(\boldsymbol{X}^0,\boldsymbol{X}^0,\boldsymbol{Z}^0,\boldsymbol{\lambda}^0,\boldsymbol{\mu}^0)$  as any given start point.

According to **Theorem** 1, we conclude that the sequence  $\{\boldsymbol{W}^k\}_{k\in\mathbb{N}}$  generated by **Algorithm** 1 from any given initial point  $\boldsymbol{W}^0$  will converge to the set of critical points denoted by  $\omega(\boldsymbol{W}^0) \neq \varnothing$ .

Based on **Proposition** 7,  $\{W^k\}_{k\in\mathbb{N}}$  is bounded (convergent), and for any given initial point  $W^0$ ,  $k\in +\infty$ , there exist a limit point  $\overline{W}=(\bar{X},\bar{Z},\bar{\lambda},\bar{\mu},\bar{X})$ .

Since the Lyapunov function is continuous, we have

$$\lim_{k \to +\infty} \Phi_{\beta_k}(\boldsymbol{W}^k) = \Phi_{\bar{\beta}}(\overline{\boldsymbol{W}})$$
 (26)

As illustrated,  $\Phi_{\beta_k}(\boldsymbol{W}^k)$  is lower bounded, i.e.,  $\lim_{k\to +\infty} \Phi_{\beta_k}(\boldsymbol{W}^k) > -\infty$  and non-increasing w.r.t the iteration k (**Proposition** 6), thus we have

$$\lim_{k \to +\infty} \Phi_{\beta_k}(\boldsymbol{W}^k) = \Phi_{\bar{\beta}}(\bar{\boldsymbol{W}}) = \underline{\Phi}$$
 (27)

where  $\underline{\Phi}$  denotes the limit of  $\Phi_{\beta_k}({m W}^k)$  from the initial point  ${m W}^0$ 

The above implies that  $\Phi_{\beta_k}(\mathbf{W}^k)$  is constant on  $\omega(\mathbf{W}^0)$ . From **Theorem** 1, we have

$$\lim_{k \to +\infty} \operatorname{dist}(\nabla \Phi_{\beta_k}(\boldsymbol{W}^k), 0) = 0$$

That means  $\forall \varepsilon$ ,  $\exists k_0 \in \mathbb{N}$  that  $\forall k \geq k_0$ ,

$$\operatorname{dist}\!\left(\nabla\Phi_{\beta_k}(\boldsymbol{W}^k),0\right)\leq\epsilon$$

For notation, we denote  $\Phi^*$  as the limit of  $\Phi_{\beta_k}(\boldsymbol{W}^k)$  on  $\omega(\boldsymbol{W}^0)$ .  $\Phi_{\beta_k}(\boldsymbol{W}^k)$  is non-increasing w.r.t. the iteration k, thus  $\forall \eta \geq 0$ , there exist  $k_1 \in \mathbb{N}$  such that  $\forall k \geq k_1$ , we have

$$\Phi_{\beta_k}(\boldsymbol{W}^k) \leq \Phi^* + \eta$$

Further,  $\forall k \geq \max\{k_0, k_1\}$ , we have

$$\mathbf{W}^{k} \in \left\{ \mathbf{W} : \operatorname{dist}(\nabla \Phi_{\beta_{k}}(\mathbf{W}^{k}), 0) \leq \epsilon \right\}$$

$$\cap \left\{ \mathbf{W} : \Phi^{*} < \Phi_{\beta_{k}}(\mathbf{W}^{k}) < \Phi^{*} + \eta \right\} = \Omega_{\varepsilon, \eta}$$
(28)

Since the Lyapunov function  $\Phi_{\beta_k}(\boldsymbol{W}^k)$  posess the KL property, based on **Definition** 4, we have

$$1 \le \left[ \varphi' \left( \Phi_{\beta_k}(\mathbf{W}^k) - \Phi^* \right) \right]^2 \left[ \operatorname{dist} \left( 0, \nabla \Phi_{\beta_k}(\mathbf{W}^k) \right) \right]^2 \quad (29)$$

By combining (29) with Proposition 9, we have

$$1 \le \left[\varphi'\left(\Phi_{\beta_k}(\boldsymbol{W}^k) - \Phi^*\right)\right]^2 2N\nu_{k-1} \left[\Phi_{\beta_k}(\boldsymbol{W}^{k-1}) - \Phi_{\beta_{k+1}}(\boldsymbol{W}^k)\right]$$
(30)

Rearrange (30) based on  $\Delta^k = \Phi_{\beta_k}(\boldsymbol{W}^k) - \Phi^*$ , we have

$$1 \le [\varphi'(\Delta^k)]^2 \cdot 2N\nu_{k-1} [\Delta^{k-1} - \Delta^k] \tag{31}$$

Regarding the desingularising function, we have  $\varphi'(t) = Ct^{\theta-1}$ . Therefore, we can derive from (31) that

$$1 \le 2NC^2 \nu_{k-1} (\Delta^k)^{2(\theta-1)} \cdot \left[ \Delta^{k-1} - \Delta^k \right].$$

(i) If  $\theta = 1$ , we have

$$1 \le 2NC^2 \nu_{k-1} \cdot \left[ \Delta^{k-1} - \Delta^k \right] \le 0 \tag{32}$$

This above inequality is a contradiction and implies that the set  $\Omega_{\varepsilon,\eta}=\varnothing$ . In other word,  $\forall k\geq \max\{k_0,k_1\}$ , we have  $\Delta^k=0$ , i.e., **Algorithm** 1 terminates in finite iterations. (ii)  $\theta\in(0,1/2)$ , we have  $1<2-2\theta<2$ .

$$[\Delta^{k}]^{2} \leq 2NC^{2}\nu_{k-1} \cdot [\Delta^{k-1} - \Delta^{k}] \cdot (\Delta^{k})^{2\theta}$$
  
$$\leq 2NC^{2}\nu_{k-1}\Delta^{k-1}(\Delta^{k})^{2\theta} - 2NC^{2}\nu_{k-1}(\Delta^{k})^{1+2\theta}$$
(33)

Thus, we have

$$[\Delta^{k}]^{2} + 2NC^{2}\nu_{k-1}(\Delta^{k})^{1+2\theta}$$

$$\leq 2NC^{2}\nu_{k-1}\Delta^{k-1} \cdot (\Delta^{k})^{2\theta}$$
(34)

Since  $\Delta^k$  is non-increasing (**Proposition** 6), there exist  $k_3 \in \mathbb{R}$  that  $\forall k \leq k_2$ , we have  $\Delta^{k-1} \geq \Delta^k \geq 1$ .

In this case, it's easy to figure out that

$$1 < 1 + 2\theta < 2, \quad 0 < 2\theta < 1$$
  
 $(\Delta^k)^2 \ge (\Delta^k)^{1+2\theta}$   
 $(\Delta^k)^{2\theta} \le (\Delta^{k-1})^{2\theta}$ 

Therefore, we can derive from (34) that

$$(1 + 2NC^{2}\nu_{k-1})(\Delta^{k})^{1+2\theta} \leq [\Delta^{k}]^{2} + 2NC^{2}\nu_{k-1}(\Delta^{k})^{1+2\theta} \leq 2NC^{2}\nu_{k-1}\Delta^{k-1}(\Delta^{k})^{2\theta} \leq 2NC^{2}\nu_{k-1}(\Delta^{k-1})^{1+2\theta}$$

In this case,  $\forall \max\{k_0, k_1\} \le k \le k_2$ , we have

$$\Delta^{k} \le \left(\frac{2NC^{2}\nu_{k-1}}{1 + 2NC^{2}\nu_{k-1}}\right)^{\frac{1}{1+2\theta}} \Delta^{k-1} \tag{35}$$

This implies **Algorithm** 1 shows linear convergence rate.

As mentioned,  $\forall k \geq k_2$ , we have  $0 \leq \Delta_k < 1$ . In this case, it's straight forward that

$$1 < 1 + 2\theta < 2$$
$$(\Delta^k)^2 \le (\Delta^k)^{1+2\theta}$$

Similarly, we can derive from (34) that

$$(1 + 2NC^{2}\nu_{k-1})(\Delta^{k})^{2} \leq [\Delta^{k}]^{2} + 2NC^{2}\nu_{k-1}(\Delta^{k})^{1+2\theta}$$
  
$$\leq 2NC^{2}\nu_{k-1}\Delta^{k-1}(\Delta^{k})^{2\theta}$$
(36)

This implies that

$$(1 + 2NC^2\nu_{k-1})(\Delta^k)^{2-2\theta} \le 2NC^2\nu_{k-1}\Delta^{k-1}$$
 (37)

Equivalently,  $\forall k \geq \max\{k_0, k_1, k_2\}$ , we have

$$\Delta^{k} \le \left(\frac{2NC^{2}\nu_{k-1}}{1 + 2NC^{2}\nu_{k-1}}\right)^{\frac{1}{2-2\theta}} \left(\Delta^{k-1}\right)^{\frac{1}{2-2\theta}} \tag{38}$$

(iii)  $\theta \in [1/2, 1)$ , we have  $0 < 2 - 2\theta \le 1$ . As  $\Delta^k$  is non-increasing, there exist  $k_3 \in \mathbb{R}$  that  $\forall k \ge k_3$ , we have

$$(\Delta^k)^{2(1-\theta)} \ge \Delta^k$$

Therefore,  $\forall k \geq \max\{k_0, k_1, k_3\}$ , we have

$$\Delta^k \le 2NC^2\nu_{k-1} \cdot \left(\Delta^{k-1} - \Delta^k\right)$$

i.e.,

$$\Delta^k \leq \frac{2NC^2\nu_{k-1}}{1+2NC^2\nu_{k-1}} \cdot \Delta^{k-1}$$

In this case, **Algorithm** 1 presents linear convergence rate.

According to **Theorem** 1, the step-size should be selected to guarantee **Condition** (a). Nevertheless this is a challenging task to achieve efficiently in practice. To reduce computation and facilitate implementation, we can select the step-size by the backward linesearch procedures in **Algorithm** 2.  $\alpha$  denotes a small positive value that can be selected adaptively according to the problems.

## Algorithm 2 Backward Linesearch for Stepsize

- 1: Initialization:  $c_i^0$ .
- 2: Repeat:
- 3: If

$$g_{i}(\bar{\boldsymbol{X}}_{i}^{k+1}, \boldsymbol{\lambda}^{k}, \rho_{k}) + \alpha \|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\|^{2} > g_{i}(\bar{\boldsymbol{X}}_{i}^{k}, \boldsymbol{\lambda}^{k}, \rho_{k}) + \langle \nabla_{\bar{\boldsymbol{X}}_{i}} g_{i}(\bar{\boldsymbol{X}}_{i}^{k}, \boldsymbol{\lambda}^{k}, \rho_{k}), \bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k} \rangle + \frac{c_{i}^{k}}{2} \|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\|^{2}$$
(39)

set  $c_i^{k+1} = c_k^i/\delta$  and go to **Step** 2, otherwise stop.

4: Output  $c_i^k$ .

**Corollary 1.** Suppose the step-size  $c_i^k$  at each iteration k is selected according to **Algorithm** 2, the PLDM converges to the critical points of Problem (5).

*Proof.* Refer to **Appendix** C. 
$$\Box$$

#### V. NUMERIC EXPERIMENTS

This section illustrates the PLDM's performance on (i) a simple numerical example with 2 agents and (ii) the application to multi-zone HVAC control. The numerical example helps illustrate the theoretical analysis and the application demonstrates its practical capabilities.

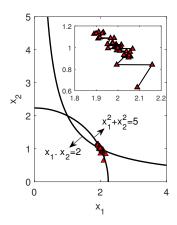


Fig. 2. the trajectory of the sequence  $\{X_1\}_{k\in\mathbb{N}}$  (the feasible points include (1,2) and (2,1) with the (local) optima (2,1)).

#### A. A Numerical Example

We consider a nonlinear and non-convex example with 2 agents given by

$$\min_{x_1, x_2} \quad x_1 + x_2 + \frac{1}{2}x_1 \cdot x_2^2 + \frac{1}{2}x_1 \cdot x_2^2$$

$$s.t. \quad x_1 \cdot x_2 = 2.$$

$$x_1^2 + x_2^2 = 5.$$

$$0 \le x_1 \le 4.$$

$$0 \le x_2 \le 5.$$
(40)

The problem is well-posed with the (local) optima (2,1). The problem is solved by using PLDM in a decentralized manner and our analysis is presented here.

$$\begin{aligned} \boldsymbol{F}_{i}(\bar{\boldsymbol{X}}_{i}) &= \begin{pmatrix} \nabla_{\boldsymbol{X}_{i}} h_{i}(\boldsymbol{X}_{i}) & \boldsymbol{O}_{n_{i}} \\ \boldsymbol{A}_{i} & \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{X}_{i}(2) & \boldsymbol{X}_{i}(1) & 0 & 0 \\ 2\boldsymbol{X}_{i}(1) & 2\boldsymbol{X}_{i}(2) & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \ i = \{1, 2\}. \end{aligned}$$

Note that  $F_i(\bar{X}_i)$  has row full rank and regular with  $X_i \neq 0$ . This implies the satisfaction of assumption (A4), which closely related to the convergence. The trajectory of the local decision variable sequence  $\{X_1\}_{k\in\mathbb{R}}$  is shown in Fig. 2. Fig. 3 shows the subgradients of AL function, i.e.,  $\|\nabla \mathbb{L}_{\rho_k}(\bar{X}^k, Z^k, \lambda^k, \mu^k)\|$  and the Lyapunov function, i.e.,  $\|\nabla \Phi_{\beta_k}(W^k)\|$  converging to zero w.r.t the iteration k. This additionally demonstrates the PLDM's convergence property illustrated in **Theorem** 1.

In addition, we study the PLDM's convergence rate by closely inspecting the Lyapunov function  $\Phi_{\beta_k}(\boldsymbol{W}^k)$  w.r.t the iteration k. As shown in Fig. 4, we see that the Lyapunov function  $\Phi_{\beta_k}(\boldsymbol{W}^k)$  approximately decrease w.r.t the iteration k until the optima  $\Phi^*$  is reached.

#### B. Application: Multi-zone HVAC Control

This section presents the PLDM's application to multizone HVAC control, which has raised extensive discussion among the research communities. The general formulation for designing a model predictive controller (MPC) to minimize

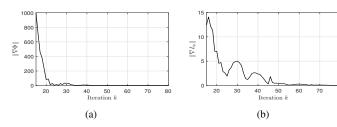


Fig. 3. (a) the subgradients  $\|\nabla \Phi_{\beta_k}(\boldsymbol{W}^k)\|$  w.r.t iteration k. (b) the subgradients  $\|\nabla \mathbb{L}_{\rho_k}(\bar{\boldsymbol{X}}^k, \boldsymbol{Z}^k, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k)\|$  w.r.t iteration k.

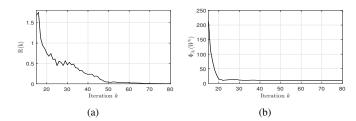


Fig. 4. (a) the residual error R(k) w.r.t iteration k. (b) the Lynapunov function  $\Phi_{\beta_k}({\bf W}^k)$  w.r.t iteration k.

the HVAC's energy cost while guaranteeing thermal comfort is exhibited in (40) with constraints imposed by (i) zone thermal dynamics (coupled, nonlinear and nonvex) (41a), (ii) comfortable temperature ranges (41b), and (iii) operation limits of the local variable air box (41c). Readers are referred to [28, 47, 48] for the detailed problem formulation and notations.

$$\min_{m_t^{zi}, T_t^i} J = \sum_{t=0}^{H-1} c_t \cdot \left\{ c_p (1 - d_r) \sum_{i=1}^{I} m_t^{zi} (T_t^o - T_t^c) + c_p d_r \sum_{i=1}^{I} m_t^{zi} (T_t^i - T_t^c) + \kappa_f \cdot (\sum_{i=1}^{I} m_t^{zi})^3 \right\} \cdot \Delta_t \quad (40)$$

$$s.t. \ T_{t+1}^{i} = A^{ii}T_{t}^{i} + \sum_{j \in \mathcal{N}_{i}} A^{ij}T_{t}^{j} + C^{ii}m_{t}^{zi}(T_{t}^{i} - T_{t}^{c}) + D_{t}^{ii}, \quad (41a)$$

$$\underline{T}^i \le T_t^i \le \overline{T}^i, \tag{41b}$$

$$m^{zi} < m_t^{zi} < \overline{m}^{zi}, \quad \forall i \in \mathcal{I}, \ t = 0, 1, \cdots, H.$$
 (41c)

One can see that the control of the multi-zone HVAC system requires solving a nonlinear and nonconvex optimization problem with coupled nonlinear constraints. Centralized optimization methods are not scalable or computationally viable and therefore decentralized methods become imperative. However, the existing decentralized methods can not be adapted due to the non-linearity and non-convexity both from the objective function and the constraints.

In this part, we resort to the proposed PLDM. We first consider a building with I=10 zones and randomly generate a network to denote the thermal couplings among the different zones. The comfortable zone temperature ranges and the maximum zone mass flow rate are set as  $[24,26]^{o}C$  and  $\bar{m}^{zi}=0.5kg/s$ . The other parameter settings can refer to [47]. When the PLDM method is applied, the zone temperature and zone mass flow rates for two randomly picked zones (Zone 1 and 8) are shown in Fig. 5 (a) and 5 (b). We note that both the zone temperature and zone mass flow rates are maintained within the setting ranges. This implies the feasibility of the

solutions. Besides, by observing the residual error w.r.t the iterations in Fig. 5 (c), we can imply the fast convergence rate of the PLDM. Further, to evaluate the sub-optimality of local optima under the PLDM with some initial start points, we compare the results with a centralized method in case studies with  $I \in \{10, 20\}$  zones. In the centralized method, the optimal solutions can be obtained by solving the non-linear and non-convex problem (40) using Ipopt solver. As shown in Fig. 5 (d), while benchmarking the centralized method (optimal cost is 1), the sub-optimality of the local optima under the PLDM is less than 10%. This implies that the PLDM can approach a satisfactory local optima for non-linear and nonconvex problems for some randomly picked initial points. In principle, the global optima for the non-linear problems can be approached by scattering enough initial points. However, some local optima with favorable performance and feasibility guarantee are generally enough in practice.

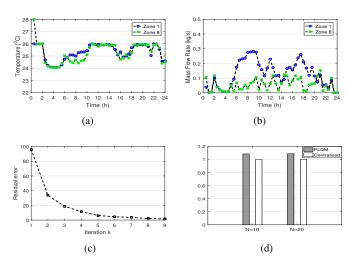


Fig. 5. (a)Zone temperature. (b) Zone mass flow rate. (c) Residual error of the constraints. (d) Comparison of PLDM vs. Centralized Method.

## VI. CONCLUSION

This paper investigated decentralized optimization on a class of non-convex problems structured by: (i) nonconvex global objective function (possibly nonsmooth) and (ii) local bounded convex constraints and coupled nonlinear constraints. Such problems arise from a variety of applications (e.g., smart grid and smart buildings, etc), which demand for efficient and scalable computation. To solve such challenging problems, we proposed a Proximal Linearization-based Decentralized Method (PLDM) under the augmented Lagrangian-based framework. Deviating from conventional augmented Lagrangian-based methods which require exact joint optimization of different decision variable blocks (local optima), the proposed method capitalized on proximal linearization technique to update the decision variables at each iteration, which makes it computationally efficient in the presence of non-linearity. Both the global convergence and the local convergence rate of the PLDM's to the  $\epsilon$ -critical points of the problem were discussed based on the Kurdyka-ojasiewicz property and some standard assumptions. In addition, the PLDM's performance

was illustrated on academic and application examples in which the convergence was observed. Applying PLDM to emerging applications and eliminating consensus variables are the future work to be explored.

## APPENDIX A PROOF OF **PROPOSITION** 1-8

## A. Proof of Proposition 1

*Proof.* We illustrate the proposition by contradiction. We assume  $\forall k \in \mathbb{N}$ , we have

$$\sum_{i=1}^{N} \|h_i(\boldsymbol{X}_i^k)\| > \eta > 0 \tag{42}$$

In this case, we have  $\rho_k$  kept increasing, i.e,  $\rho_k \to +\infty$ . Based on  $\lambda_i^k = \lambda_i^{k-1} + \rho_{k-1} h_i(\boldsymbol{X}_i^k)$ , we have

$$\sum_{i=1}^{N} \|h_i(\boldsymbol{X}_i^k)\| = \frac{1}{\rho_{k-1}} \sum_{i=1}^{N} \|\boldsymbol{\lambda}_i^k - \boldsymbol{\lambda}_i^{k-1}\| \le \frac{\sum_{i=1}^{N} M_{\boldsymbol{\lambda}_i}}{\rho_{k-1}}$$

 $(\{\lambda_i^k\}_{k\in\mathcal{N}} \text{ is assumed bounded, see } (A6))$ 

The above implies that

$$\lim_{k \to +\infty} \sum_{i=1}^{N} \|h_i(\boldsymbol{X}_i^k)\| \to 0 \tag{43}$$

One may note that (42) and (43) contradict with each other. Thus, we conclude that there exist k such that

$$\sum_{i=1}^{N} \|h_i(\boldsymbol{X}_i^k)\| \le \frac{\sum_{i=1}^{N} M_{\boldsymbol{\lambda}_i}}{\rho_{k-1}} \le \eta, \quad \forall k \ge \underline{k}.$$
$$\{\bar{\boldsymbol{X}}_i^k\}_{k \in \mathbb{N}, k \ge \underline{k}} \in \bar{\boldsymbol{\mathcal{X}}}_i^{\eta} \ (i \in \mathcal{N}).$$

Further, based on the adaptive procedure of PLDM, we have

$$\rho_k = \rho_k, \quad \forall k \ge \underline{k}.$$

## B. Proof of Proposition 2

*Proof.* Since  $h_i$  is twice differentiable, we have  $h_i(\boldsymbol{X}_i)$  and  $\nabla_{\boldsymbol{X}_i} h_i(\boldsymbol{X}_i)$  are continuous and bounded over the bounded set  $\bar{\boldsymbol{X}}_i^\eta$ . In addition, we have  $\rho_k \leq \rho_k$  ( $\forall k \in \mathbb{N}$ ), thus

$$\|\rho_k(\nabla_{\mathbf{X}_i}h_i(\mathbf{X}_i))^T h_i(\mathbf{X}_i)\| \le \rho_k C_{h_i} \le +\infty \quad (\forall k \in \mathbb{N})$$

where  $C_{h_i}$  is upper bound of  $\|(\nabla_{\boldsymbol{X}_i}h_i(\boldsymbol{X}_i))^Th_i(\boldsymbol{X}_i)\|$  over  $\bar{\boldsymbol{\mathcal{X}}}_i^{\eta}$ . Besides, we have

$$\nabla_{\boldsymbol{X}_{i}}g_{i}(\bar{\boldsymbol{X}}_{i}, \boldsymbol{\lambda}_{i}^{k}, \rho_{k}) = \nabla_{\boldsymbol{X}_{i}}\tilde{f}_{i}(\boldsymbol{X}_{i}) + \nabla_{\boldsymbol{X}_{i}}\phi_{i}(\boldsymbol{X}_{i}) + \left(\nabla_{\boldsymbol{X}_{i}}h_{i}(\boldsymbol{X}_{i})\right)^{T}\boldsymbol{\lambda}_{i}^{k} + \rho_{k}\left(\nabla_{\boldsymbol{X}_{i}}h_{i}(\boldsymbol{X}_{i})\right)^{T}h_{i}(\boldsymbol{X}_{i}).$$
(44) 
$$\nabla_{\boldsymbol{Y}_{i}}g_{i}(\bar{\boldsymbol{X}}_{i}, \boldsymbol{\lambda}_{i}^{k}, \rho_{k}) = 2M_{i}\boldsymbol{Y}_{i}.$$

As  $\tilde{f}_i$ ,  $\phi_i$  and  $h_i$  are Lipschitz continuous w.r.t  $X_i$  (or  $\bar{X}_i$ ) with constant  $L_{f_i}$ ,  $L_{\phi_i}$  and  $L_{h_i}$ ,  $\{\lambda_i^k\}_{k\in\mathbb{N}}$  upper bounded by  $M_{\lambda_i}$  (see (A6)), we conclude that

$$\|\nabla_{\boldsymbol{X}_i} g_i(\bar{\boldsymbol{X}}_i, \boldsymbol{\lambda}_i^k, \rho^k)\| \le L_{g_i} \text{ over } \bar{\boldsymbol{\mathcal{X}}}_i^{\eta}, \ \forall k \in \mathbb{N}.$$

where we have  $L_{g_i} = \max\{L_{f_i} + L_{\phi_i} + M_{\lambda_i}L_{h_i} + \rho_{\underline{k}}C_{h_i}, 2M_i\}$ . That is,  $g_i(\bar{\boldsymbol{X}}_i, \boldsymbol{\lambda}_i^k, \rho_k)$  ( $\forall k \in \mathbb{N}$ ) is Lipschitz continuous w.r.t  $\bar{\boldsymbol{X}}_i$  over  $\bar{\boldsymbol{\mathcal{X}}}_i^\eta$  with constant  $L_{g_i}$ . C. Proof of Proposition 3

Proof. Based on subproblem (10), it's straightforward that

$$\sum_{i=1}^{N} (\boldsymbol{\mu}_{i}^{k})^{T} (\boldsymbol{A}_{i} \bar{\boldsymbol{X}}_{i}^{k} - \boldsymbol{E}_{i} \boldsymbol{Z}^{k+1}) + \sum_{i=1}^{N} \frac{\rho_{\underline{k}}}{2} \left\| \boldsymbol{A}_{i} \bar{\boldsymbol{X}}_{i}^{k} - \boldsymbol{E}_{i} \boldsymbol{Z}^{k+1} \right\|^{2}$$

$$\leq \sum_{i=1}^{N} (\boldsymbol{\mu}_{i}^{k})^{T} (\boldsymbol{A}_{i} \bar{\boldsymbol{X}}_{i}^{k} - \boldsymbol{E}_{i} \boldsymbol{Z}^{k}) + \sum_{i=1}^{N} \frac{\rho_{\underline{k}}}{2} \left\| \boldsymbol{A}_{i} \bar{\boldsymbol{X}}_{i}^{k} - \boldsymbol{E}_{i} \boldsymbol{Z}^{k} \right\|^{2}$$

$$(45)$$

Subproblem (11) is equivalent to

$$\bar{\boldsymbol{X}}_{i}^{k+1} \in prox_{c_{i}^{k}}^{\sigma_{i}} \Big( \bar{\boldsymbol{X}}_{i}^{k} - \frac{1}{c_{i}^{k}} \nabla_{\bar{\boldsymbol{X}}_{i}} g_{i}(\bar{\boldsymbol{X}}_{i}^{k}, \boldsymbol{\lambda}_{i}^{k}, \rho_{\underline{k}}) \Big)$$

where we have  $\sigma_i(\bar{\boldsymbol{X}}_i) = (\boldsymbol{\mu}_i)^T (\boldsymbol{A}_i \bar{\boldsymbol{X}}_i - \boldsymbol{E}_i \boldsymbol{Z}^{k+1}) + \frac{\rho_k}{2} \|\boldsymbol{A}_i \bar{\boldsymbol{X}}_i - \boldsymbol{E}_i \boldsymbol{Z}^{k+1}\|^2$ .

Thus, by invoking Lemma 2, we have

$$g_{i}(\bar{\boldsymbol{X}}_{i}^{k+1}, \boldsymbol{\lambda}_{i}^{k}, \rho_{\underline{k}}) + (\boldsymbol{\mu}_{i}^{k})^{T}(\boldsymbol{A}_{i}\bar{\boldsymbol{X}}_{i}^{k+1} - \boldsymbol{E}_{i}\boldsymbol{Z}^{k+1})$$

$$+ \frac{\rho_{\underline{k}}}{2} \|\boldsymbol{A}_{i}\bar{\boldsymbol{X}}_{i}^{k+1} - \boldsymbol{E}_{i}\boldsymbol{Z}^{k+1}\|^{2}$$

$$\leq g_{i}(\bar{\boldsymbol{X}}_{i}^{k}, \boldsymbol{\lambda}_{i}^{k}, \rho_{\underline{k}}) + (\boldsymbol{\mu}_{i}^{k})^{T}(\boldsymbol{A}_{i}\boldsymbol{X}_{i}^{k} - \boldsymbol{E}_{i}\boldsymbol{Z}^{k+1})$$

$$+ \frac{\rho_{\underline{k}}}{2} \|\boldsymbol{A}_{i}\bar{\boldsymbol{X}}_{i}^{k} - \boldsymbol{E}_{i}\boldsymbol{Z}^{k+1}\|^{2}$$

$$- \frac{1}{2} (c_{i}^{k} - L_{g_{i}}) \|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\|^{2}, \ \forall i \in \mathcal{N}.$$

$$(46)$$

By summing up (45) and (46) for  $\forall i \in \mathcal{N}$ , we have

$$\sum_{i=1}^{N} g_{i}(\bar{X}_{i}^{k+1}, \lambda_{i}^{k}, \rho_{\underline{k}}) + \sum_{i=1}^{N} (\mu_{i}^{k})^{T} (A_{i}\bar{X}_{i}^{k+1} - E_{i}Z^{k+1}) 
+ \sum_{i=1}^{N} \frac{\rho_{\underline{k}}}{2} ||A_{i}\bar{X}_{i}^{k+1} - E_{i}Z^{k+1}||^{2} + \sum_{i=1}^{N} (\mu_{i}^{k})^{T} (A_{i}\bar{X}_{i}^{k} - E_{i}Z^{k+1}) 
+ \sum_{i=1}^{N} \frac{\rho_{\underline{k}}}{2} ||A_{i}\bar{X}_{i}^{k} - E_{i}Z^{k+1}||^{2} 
\leq \sum_{i=1}^{N} g_{i}(\bar{X}_{i}^{k}, \lambda_{i}^{k}, \rho_{\underline{k}}) + \sum_{i=1}^{N} (\mu_{i}^{k})^{T} (A_{i}\bar{X}_{i}^{k} - E_{i}Z^{k+1}) 
+ \sum_{i=1}^{N} \frac{\rho_{\underline{k}}}{2} ||A_{i}\bar{X}_{i}^{k} - E_{i}Z^{k+1}||^{2} + \sum_{i=1}^{N} (\mu_{i}^{k})^{T} (A_{i}\bar{X}_{i}^{k} - E_{i}Z^{k}) 
+ \sum_{i=1}^{N} \frac{\rho_{\underline{k}}}{2} ||A_{i}\bar{X}_{i}^{k} - E_{i}Z^{k}||^{2} 
+ \sum_{i=1}^{N} \frac{1}{2} (c_{i}^{k} - L_{g_{i}}) ||\bar{X}_{i}^{k+1} - \bar{X}_{i}^{k}||^{2}$$

$$(47)$$

Thus, **Proposition** 3 can be concluded by removing the same terms from both sides of (47).

#### D. Proof of Proposition 4

*Proof.* As subproblem (11) is QP, based on the first-order optimality condition and  $\mu_i^{k+1} = \mu_i^k + \rho_k (A_i \bar{X}_i^{k+1} - E_i Z^{k+1})$ , we have

$$\nabla_{\bar{\boldsymbol{X}}_{i}}g_{i}(\bar{\boldsymbol{X}}_{i}^{k},\boldsymbol{\lambda}_{i}^{k},\rho_{k}) + \boldsymbol{\mu}_{i}^{k+1} + c_{i}^{k}(\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}) = 0 \quad (48)$$

Besides, we have

$$\nabla_{\bar{\boldsymbol{X}}_{i}} \mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}^{k}, \boldsymbol{\mu}^{k})$$

$$= \nabla_{\boldsymbol{X}_{i}} g_{i}(\bar{\boldsymbol{X}}_{i}^{k+1}, \boldsymbol{\lambda}_{i}^{k}, \rho_{k}) + \boldsymbol{\mu}_{i}^{k+1}$$

$$(49)$$

By combining (48) and (49), we have

$$\|\nabla_{\bar{\boldsymbol{X}}_{i}}\mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}^{k}, \boldsymbol{\mu}^{k})\|$$

$$= \|\nabla_{\boldsymbol{X}_{i}}g_{i}(\bar{\boldsymbol{X}}_{i}^{k+1}, \boldsymbol{\lambda}_{i}^{k}, \rho_{\underline{k}}) - \nabla_{\bar{\boldsymbol{X}}_{i}}g_{i}(\bar{\boldsymbol{X}}_{i}^{k}, \boldsymbol{\lambda}_{i}^{k}, \rho_{\underline{k}})$$

$$- c_{i}^{k}(\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k})\|$$

$$\leq \|\nabla_{\bar{\boldsymbol{X}}_{i}}g_{i}(\bar{\boldsymbol{X}}_{i}^{k+1}, \boldsymbol{\lambda}_{i}^{k}, \rho_{\underline{k}}) - \nabla_{\bar{\boldsymbol{X}}_{i}}g_{i}(\bar{\boldsymbol{X}}_{i}^{k}, \boldsymbol{\lambda}_{i}^{k}, \rho_{\underline{k}})\|$$

$$+ c_{i}^{k}\|(\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k})\|$$

$$\leq (L_{g_{i}} + c_{i}^{k})\|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\|$$

$$(50)$$

where the last inequality is concluded from **Proposition** 2.  $\Box$ 

### E. Proof of Proposition 5

Proof. For notation, we define

$$\begin{split} & \Delta_k^{\pmb{\lambda}_i} = \left(\nabla_{\pmb{X}_i} h_i(\pmb{X}_i^{k+1}) \ \pmb{O}_{\bar{n}_i}\right)^T \pmb{\lambda}_i^{k+1} - \left(\nabla_{\pmb{X}_i} h_i(\pmb{X}_i^k) \ \pmb{O}_{\bar{n}_i}\right)^T \pmb{\lambda}_i^k \\ & \Delta_k^{\pmb{\mu}^i} = (\pmb{A}_i)^T \pmb{\mu}_i^{k+1} - (\pmb{A}_i)^T \pmb{\mu}_i^k \end{split}$$

First, we have

$$\|\Delta_{k}^{\boldsymbol{\lambda}_{i}} + \Delta_{k}^{\boldsymbol{\mu}^{i}}\| = \|\left(\boldsymbol{F}_{i}(\bar{\boldsymbol{X}}_{i}^{k+1})\right)^{T} \boldsymbol{\gamma}_{i}^{k+1} - \left(\boldsymbol{F}_{i}(\bar{\boldsymbol{X}}_{i}^{k})\right)^{T} \boldsymbol{\gamma}_{i}^{k}\|$$

$$= \|\left(\boldsymbol{F}_{i}(\bar{\boldsymbol{X}}_{i}^{k+1})\right)^{T} \left(\boldsymbol{\gamma}_{i}^{k+1} - \boldsymbol{\gamma}_{i}^{k}\right)$$

$$+ \left(\boldsymbol{F}_{i}(\bar{\boldsymbol{X}}_{i}^{k+1})\right)^{T} - \left(\boldsymbol{F}_{i}(\bar{\boldsymbol{X}}_{i}^{k})\right)^{T} \boldsymbol{\gamma}_{i}^{k}\|$$

$$\geq \|\left(\boldsymbol{F}_{i}(\bar{\boldsymbol{X}}_{i}^{k+1})\right)^{T} \left(\boldsymbol{\gamma}_{i}^{k+1} - \boldsymbol{\gamma}_{i}^{k}\right)\|$$

$$- \|\left(\boldsymbol{F}_{i}(\bar{\boldsymbol{X}}_{i}^{k+1})\right)^{T} - \left(\boldsymbol{F}_{i}(\bar{\boldsymbol{X}}_{i}^{k})\right)^{T} \boldsymbol{\gamma}_{i}^{k}\|$$

$$(51)$$

where we have  $m{F}_i(m{ar{X}}_i)=\left(egin{array}{cc}
abla_{m{X}_i}h_i(m{X}_i) & m{O}_{n_i} \\
m{A}_i\end{array}
ight)$  and  $\gamma_i = ((\lambda_i)^T (\mu_i)^T)^T$  as defined.

According to (A6), we have  $\|\gamma_i^k\|^2 = \|\lambda_i^k\|^2 + \|\mu_i^k\|^2 \le (M_{\lambda_i})^2 + (M_{\mu_i})^2$  ( $\forall k \in \mathbb{N}$ ). Thus

$$\|\boldsymbol{\gamma}_i^k\| \le M_{\boldsymbol{\gamma}_i} \ \forall k \in \mathbb{N} \tag{52}$$

with  $M_{\gamma_i} = \sqrt{(M_{\lambda_i})^2 + (M_{\mu_i})^2}$ . Besides, according to (A1), we have

$$\|\boldsymbol{F}_{i}(\boldsymbol{X}_{i}^{k+1}) - \boldsymbol{F}_{i}(\boldsymbol{X}_{i}^{k})\| = \|\nabla_{\boldsymbol{X}_{i}} h_{i}(\boldsymbol{X}_{i}^{k+1}) - \nabla_{\boldsymbol{X}_{i}} h_{i}(\boldsymbol{X}_{i}^{k})\|$$

$$\leq M_{h_{i}} \|\boldsymbol{X}_{i}^{k+1} - \boldsymbol{X}_{i}^{k}\| \leq M_{h_{i}} \|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\|, \ \forall k \geq \underline{k}.$$
(53)

Based on (52), (53) and (A5), we can derive from (51) that

$$\|\Delta_{k}^{\lambda_{i}} + \Delta_{k}^{\mu^{i}}\| \ge \theta \|\gamma_{i}^{k+1} - \gamma_{i}^{k}\| - M_{h_{i}} M_{\gamma_{i}} \|\bar{X}_{i}^{k+1} - \bar{X}_{i}^{k}\|$$
 (54)

Meanwhile, we have

$$\begin{split} &\|\Delta_{k}^{\boldsymbol{\lambda}_{i}} + \Delta_{k}^{\boldsymbol{\mu}^{i}}\| = \left\| \left(\boldsymbol{F}_{i}(\bar{\boldsymbol{X}}_{i}^{k+1})\right)^{T}\boldsymbol{\gamma}_{i}^{k+1} - \left(\boldsymbol{F}_{i}(\bar{\boldsymbol{X}}_{i}^{k})\right)^{T}\boldsymbol{\gamma}_{i}^{k} \right\| \\ &= \| \left(\nabla_{\bar{\boldsymbol{X}}_{i}}g_{i}(\bar{\boldsymbol{X}}_{i}^{k+1}, \boldsymbol{\lambda}_{i}^{k}, \rho_{\underline{k}}) + \left(\boldsymbol{A}_{i}\right)^{T}\boldsymbol{\mu}^{i,k+1}\right) \\ &- \left(\nabla_{\boldsymbol{X}_{i}}g_{i}(\bar{\boldsymbol{X}}_{i}^{k}, \boldsymbol{\lambda}_{i}^{k-1}, \rho_{\underline{k}}) + \left(\boldsymbol{A}_{i}\right)^{T}\boldsymbol{\mu}^{i,k}\right) \\ &+ \left(\nabla_{\boldsymbol{X}_{i}}\tilde{f}_{i}(\boldsymbol{X}_{i}^{k}) - \nabla_{\boldsymbol{X}_{i}}\tilde{f}_{i}(\boldsymbol{X}_{i}^{k+1})\right) \\ &+ \left(\nabla_{\boldsymbol{X}_{i}}\phi_{i}(\boldsymbol{X}_{i}^{k}) - \nabla_{\boldsymbol{X}_{i}}\phi_{i}(\boldsymbol{X}_{i}^{k+1})\right) \| \end{split}$$

$$\leq \|\nabla_{\bar{\boldsymbol{X}}_{i}}\mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}^{k}, \boldsymbol{\mu}^{k})\| \\
+ \|\nabla_{\bar{\boldsymbol{X}}_{i}}\mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k}, \boldsymbol{Z}^{k}, \boldsymbol{\lambda}^{k-1}, \boldsymbol{\mu}^{k-1})\| \\
+ \|\nabla_{\boldsymbol{X}_{i}}\tilde{f}_{i}(\boldsymbol{X}_{i}^{k}) - \nabla_{\boldsymbol{X}_{i}}\tilde{f}_{i}(\boldsymbol{X}_{i}^{k+1})\| \\
+ \|\nabla_{\boldsymbol{X}_{i}}\phi_{i}(\boldsymbol{X}_{i}^{k}) - \nabla_{\boldsymbol{X}_{i}}\phi_{i}(\boldsymbol{X}_{i}^{k+1})\| \\
\leq \left(L_{g_{i}} + c_{i}^{k} + L_{f_{i}} + L_{\phi_{i}}\right) \|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\| \\
+ \left(L_{g_{i}} + c_{k-1}^{i}\right) \|\bar{\boldsymbol{X}}_{i}^{k} - \bar{\boldsymbol{X}}_{i}^{k-1}\|$$
(55)

where the last equality is derived from Proposition 4 and (A2)-(A3).

By combining (54) and (55), we have

$$\begin{split} \| \pmb{\gamma}_i^{k+1} - \pmb{\gamma}_i^k \| & \leq \Omega_1^i \| \bar{\pmb{X}}_i^{k+1} - \bar{\pmb{X}}_i^k \| + \Omega_2^i \| \bar{\pmb{X}}_i^k - \bar{\pmb{X}}_i^{k-1} \| \\ \text{where } \Omega_1^i &= \left( L_{g_i} + c_i^k + L_{f_i} + L_{\phi_i} + M_{h_i} M_{\pmb{\gamma}_i} \right) / \theta \text{ and } \Omega_2^i = \\ \left( L_{g_i} + c_i^{k-1} \right) / \theta \ (i \in \mathcal{N}). \end{split}$$

## F. Proof of Proposition 6

*Proof.* First, based on  $h_i(\boldsymbol{X}_i^{k+1}) = \frac{1}{\rho_k}(\boldsymbol{\lambda}_i^{k+1} - \boldsymbol{\lambda}_i^k)$  and  $(\boldsymbol{A}_i\bar{\boldsymbol{X}}_i^{k+1} - \boldsymbol{E}_i\boldsymbol{Z}^{k+1}) = \frac{1}{\rho_k}(\boldsymbol{\mu}_i^{k+1} - \boldsymbol{\mu}_i^k)$ , we have

$$\mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}^{k+1}, \boldsymbol{\mu}^{k+1}) - \mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}^{k}, \boldsymbol{\mu}^{k}) \\
= \frac{1}{\rho_{\underline{k}}} \Big( \sum_{i=1}^{N} \|\boldsymbol{\lambda}_{i}^{k+1} - \boldsymbol{\lambda}_{i}^{k}\|^{2} + \sum_{i=1}^{N} \|\boldsymbol{\mu}_{i}^{k+1} - \boldsymbol{\mu}_{i}^{k}\|^{2} \Big) \\
= \frac{1}{\rho_{\underline{k}}} \Big( \sum_{i=1}^{N} \|\boldsymbol{\gamma}_{i}^{k+1} - \boldsymbol{\gamma}_{i}^{k}\|^{2} \Big) \\
\leq \sum_{i=1}^{N} \Big\{ \frac{2(\Omega_{1}^{i})^{2}}{\rho_{\underline{k}}} \|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\|^{2} + \frac{2(\Omega_{2}^{i})^{2}}{\rho_{\underline{k}}} \|\bar{\boldsymbol{X}}_{i}^{k} - \bar{\boldsymbol{X}}_{i}^{k-1}\|^{2} \Big\}$$
(56)

where the last inequality is derived by invoking **Proposition** 5 and the CauchySchwarz inequality  $(a+b)^2 \le 2(a^2+b^2)$ .

Further, by combining (16) with (56), we have

$$\begin{split} & \mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k}, \boldsymbol{Z}^{k}, \boldsymbol{\lambda}^{k}, \boldsymbol{\mu}^{k}) - \mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}^{k+1}, \boldsymbol{\mu}^{k+1}) \\ & \geq \sum_{i=1}^{N} \Big( \frac{1}{2} (c_{i}^{k} - L_{g_{i}}) - \frac{2(\Omega_{1}^{i})^{2}}{\rho_{\underline{k}}} \Big) \|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\|^{2} \\ & + \sum_{i=1}^{N} (-\frac{2(\Omega_{2}^{i})^{2}}{\rho_{\underline{k}}}) \|\bar{\boldsymbol{X}}_{i}^{k} - \bar{\boldsymbol{X}}_{i}^{k-1}\|^{2} \end{split}$$

$$\Phi_{\beta_{k}}(\boldsymbol{W}^{k}) - \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}) \ge \sum_{i=1}^{N} (\beta_{k} - \frac{2(\Omega_{2}^{i})^{2}}{\rho_{\underline{k}}}) \|\bar{\boldsymbol{X}}_{i}^{k} - \bar{\boldsymbol{X}}_{i}^{k-1}\|^{2} + \sum_{i=1}^{N} \left(\frac{1}{2}(c_{i}^{k} - L_{g_{i}}) - \frac{2(\Omega_{1}^{i})^{2}}{\rho_{k}} - \beta_{k+1}\right) \|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\|^{2}$$

## G. Proof of Proposition 7

**Proposition** 6 is concluded.

*Proof.* According to **Proposition** 6 and by summing up (19) for  $k \in [\underline{k}, \overline{k} - 1]$   $(\overline{k} \in \mathbb{R}, \overline{k} > \underline{k})$ , we have

$$\Phi_{\beta_{\underline{k}}}(\boldsymbol{W}^{\underline{k}}) - \Phi_{\beta_{\bar{k}}}(\boldsymbol{W}^{k}) \\
\geq \sum_{k=\underline{k}}^{\bar{k}-1} b_{1}^{k} \|\bar{\boldsymbol{X}}^{k+1} - \bar{\boldsymbol{X}}^{k}\|^{2} + \sum_{k=\underline{k}}^{K} b_{2}^{k} \|\bar{\boldsymbol{X}}^{k} - \bar{\boldsymbol{X}}^{k-1}\|^{2} \\
\geq b_{1} \sum_{k=\underline{k}}^{\bar{k}-1} \|\bar{\boldsymbol{X}}^{k} - \bar{\boldsymbol{X}}^{k-1}\|^{2} + b_{2} \sum_{k=\underline{k}}^{\bar{k}-1} \|\bar{\boldsymbol{X}}^{k+1} - \bar{\boldsymbol{X}}^{k}\|^{2}$$
(57)

where  $b_1 = \min_{k \in [\underline{k}, \overline{k}-1]} b_1^k$  and  $b_2 = \min_{k \in [\underline{k}, \overline{k}-1]} b_2^k$ . Since  $\Phi_{\beta_k}(\underline{\boldsymbol{\Psi}}^k) \geq \inf_{\bar{\boldsymbol{X}}, \underline{\boldsymbol{Z}}} \mathbb{L}_{\rho_k}(\bar{\boldsymbol{X}}^k, \boldsymbol{Z}^k, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) > -\infty$ (see (A6)), let  $\bar{k} \to +\infty$ , we have

$$b_{1} \sum_{k=\underline{k}}^{+\infty} \|\bar{\boldsymbol{X}}^{k+1} - \bar{\boldsymbol{X}}^{k}\|^{2} + b_{2} \sum_{k=\underline{k}}^{+\infty} \|\bar{\boldsymbol{X}}^{k} - \bar{\boldsymbol{X}}^{k-1}\|^{2}$$

$$\leq \Phi_{\beta_{\underline{k}}}(\boldsymbol{W}^{\underline{k}}) - \lim_{k \in +\infty} \Phi_{\beta_{k}}(\boldsymbol{W}^{k}) < +\infty$$
(58)

If **Condition** (a) is satisfied, we have  $b_1, b_2 > 0$ , thus

$$\sum_{k=k}^{+\infty} \|\bar{X}^{k+1} - \bar{X}^k\|^2 < \infty \tag{59}$$

This above implies

$$\lim_{k \to +\infty} \|\bar{\boldsymbol{X}}^{k+1} - \bar{\boldsymbol{X}}^{k}\| \to 0$$

$$\lim_{k \to +\infty} \|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\| \to 0, \forall i \in \mathcal{N}.$$
(60)

Further, based on **Proposition** 5, we have

$$\lim_{k \to +\infty} \left\| \boldsymbol{\gamma}_{i}^{k+1} - \boldsymbol{\gamma}_{i}^{k} \right\| \to 0,$$

$$\lim_{k \to +\infty} \left\| \boldsymbol{\lambda}_{i}^{k+1} - \boldsymbol{\lambda}_{i}^{k} \right\| \to 0,$$

$$\lim_{k \to +\infty} \left\| \boldsymbol{\mu}_{i}^{k+1} - \boldsymbol{\mu}_{i}^{k} \right\| \to 0, \quad \forall i \in \mathcal{N}.$$
(61)

H. Proof of Proposition 8

*Proof.* (i) the subgradients of  $\Phi_{\beta_{k+1}}(\mathbf{W}^{k+1})$  w.r.t  $\bar{\mathbf{X}}_i$ :

$$\|\nabla_{\bar{\boldsymbol{X}}_{i}}\Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1})\|$$

$$=\|\nabla_{\bar{\boldsymbol{X}}_{i}}\mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1},\boldsymbol{Z}^{k+1},\boldsymbol{\lambda}^{k+1},\boldsymbol{\mu}^{k+1})+2\beta_{k+1}(\bar{\boldsymbol{X}}_{i}^{k+1}-\bar{\boldsymbol{X}}_{i}^{k})\|$$

$$=\|\nabla_{\bar{\boldsymbol{X}}_{i}}\mathbb{L}_{\rho_{\underline{k}}}(\bar{\boldsymbol{X}}^{k+1},\boldsymbol{Z}^{k+1},\boldsymbol{\lambda}^{k},\boldsymbol{\mu}^{k})+(\boldsymbol{F}_{i}(\bar{\boldsymbol{X}}_{i}^{k+1}))^{T}(\boldsymbol{\gamma}_{i}^{k+1}-\boldsymbol{\gamma}_{i}^{k})$$

$$+2\beta_{k+1}(\bar{\boldsymbol{X}}_{i}^{k+1}-\bar{\boldsymbol{X}}_{i}^{k})\|$$

$$\leq (L_{g_{i}}+c_{i}^{i})\|\bar{\boldsymbol{X}}_{i}^{k+1}-\bar{\boldsymbol{X}}_{i}^{k}\|$$

$$+B\|\boldsymbol{\gamma}_{i}^{k+1}-\boldsymbol{\gamma}_{i}^{k}\|+2\beta_{k+1}\|\bar{\boldsymbol{X}}_{i}^{k+1}-\bar{\boldsymbol{X}}_{i}^{k}\|$$

$$\leq (L_{g_{i}}+c_{i}^{i}+\Omega_{1}^{i}B+2\beta_{k+1})\|\boldsymbol{X}_{i}^{k+1}-\boldsymbol{X}_{i}^{k}\|$$

$$+\Omega_{2}^{i}B\|\boldsymbol{X}_{i}^{k}-\boldsymbol{X}_{i}^{k-1}\|$$

$$(62)$$

where the first inequality is derived from Proposition 4 and the definition  $B = \sup_{k \ge k} \| \boldsymbol{F}_i(\bar{\boldsymbol{X}}_i^{k+1}) \|$ . The second inequality is derived by invoking **Proposition** 5.

(ii) the subgradients of  $\Phi_{\beta_{k+1}}(\mathbf{W}^{k+1})$  w.r.t.  $\mathbf{Z}$ :

$$\nabla_{\mathbf{Z}} \Phi_{\beta_{k+1}}(\mathbf{W}^{k+1})$$

$$= \nabla_{\mathbf{Z}} \mathbb{L}_{\rho_{k}}(\bar{\mathbf{X}}^{k+1}, \mathbf{Z}^{k+1}, \boldsymbol{\lambda}^{k+1}, \boldsymbol{\mu}^{k+1}) + \mathcal{N}_{\mathcal{X}}(\mathbf{Z})$$

$$= -\sum_{i=1}^{N} (\mathbf{E}_{i})^{T} \boldsymbol{\mu}_{i}^{k+1} + \mathcal{N}_{\mathcal{X}}(\mathbf{Z}^{k+1})$$
(63)

Based on the first-order optimality condition of subproblem (10), we have

$$-\sum_{i=1}^{N} (\boldsymbol{E}_{i})^{T} \boldsymbol{\mu}_{i}^{k} - \sum_{i=1}^{N} \rho_{k} (\boldsymbol{E}_{i})^{T} (\boldsymbol{A}_{i} \bar{\boldsymbol{X}}_{i}^{k} - \boldsymbol{E}_{i} \boldsymbol{Z}^{k+1})$$

$$+ \mathcal{N}_{\mathcal{X}} (\boldsymbol{Z}^{k+1}) = 0$$

$$(64)$$

Further, based on  $\mu_i^{k+1} = \mu_i^k + \rho_k (A_i X_i^{k+1} - E_i Z^{k+1})$  and by rearranging (64), we have

$$-\sum_{i=1}^{N} (\mathbf{E}_{i})^{T} \boldsymbol{\mu}_{i}^{k+1} + \mathcal{N}_{\mathcal{X}}(\mathbf{Z}^{k+1})$$

$$= \sum_{i=1}^{N} \rho_{\underline{k}} (\mathbf{E}_{i})^{T} (\mathbf{A}_{i} \mathbf{X}_{i}^{k+1} - \mathbf{A}_{i} \mathbf{X}_{i}^{k})$$
(65)

By combining (63) with (65), we have

$$\|\nabla_{\mathbf{Z}}\Phi_{\beta_{k+1}}(\mathbf{W}^{k+1})\| = \|\sum_{i=1}^{N} \rho_{\underline{k}}(\mathbf{E}_{i})^{T} \mathbf{A}_{i} (\mathbf{X}_{i}^{k+1} - \mathbf{X}_{i}^{k})\|$$

$$\leq \rho_{\underline{k}} \sum_{i=1}^{N} \|(\mathbf{E}_{i})^{T} \mathbf{A}_{i} (\mathbf{X}_{i}^{k+1} - \mathbf{X}_{i}^{k})\|$$

$$\leq \rho_{\underline{k}} \lambda_{\max} ((\mathbf{E}_{i})^{T} \mathbf{A}_{i}) \sum_{i=1}^{N} \|\mathbf{X}_{i}^{k+1} - \mathbf{X}_{i}^{k}\|$$

$$(66)$$

where  $\lambda_{\max}((m{E}_i)^Tm{A}_i)$  denotes the maximum eigenvalue of matrix  $(\boldsymbol{E}_i)^T \boldsymbol{A}_i$ .

(iii) the subgradients of  $\Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1})$  w.r.t. the Lagrangian multipliers  $\lambda$  and  $\mu$ :

$$\nabla_{\boldsymbol{\lambda}_{i}} \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}) = \rho_{\underline{k}} h_{i}(\boldsymbol{X}_{i}^{k+1}) = \frac{\boldsymbol{\lambda}_{i}^{k+1} - \boldsymbol{\lambda}_{i}^{k}}{\rho_{\underline{k}}}$$

$$\nabla_{\boldsymbol{\mu}_{i}} \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}) = \boldsymbol{A}_{i} \bar{\boldsymbol{X}}_{i}^{k+1} - \boldsymbol{E}_{i} \boldsymbol{Z}^{k+1} = \frac{\boldsymbol{\mu}_{i}^{k+1} - \boldsymbol{\mu}_{i}^{k}}{\rho_{k}}$$
(67)

Based on **Proposition** 5, we have

$$\|\nabla_{\boldsymbol{\gamma}_{i}}\Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1})\| = \|\frac{\boldsymbol{\gamma}_{i}^{k+1} - \boldsymbol{\gamma}_{i}^{k}}{\rho_{\underline{k}}}\|$$

$$\leq \frac{\Omega_{1}^{i}}{\rho_{k}} \|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\| + \frac{\Omega_{2}^{i}}{\rho_{k}} \|\bar{\boldsymbol{X}}_{i}^{k} - \bar{\boldsymbol{X}}_{i}^{k-1}\|$$
(68)

(iv) the subgradients of  $\Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1})$  w.r.t.  $\boldsymbol{U}$ :

$$\|\nabla_{\boldsymbol{U}}\Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1})\| = 2\beta_{k+1}\|\bar{\boldsymbol{X}}^{k+1} - \bar{\boldsymbol{X}}^{k}\|$$
 (69)

By combining (62), (66), (68) and (69), we have

$$\|\nabla \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1})\| \leq \sum_{i=1}^{N} \left( L_{g_i} + c_i^k + \Omega_1^i B + 4\beta_{k+1} + \rho_{\underline{k}} + \frac{\Omega_1^i}{\rho_{\underline{k}}} \right) \|\bar{\boldsymbol{X}}_i^{k+1} - \bar{\boldsymbol{X}}_i^k\|$$
(70)
$$+ \sum_{i=1}^{N} \left( \Omega_2^i B + \frac{\Omega_2^i}{\rho_{\underline{k}}} \right) \|\bar{\boldsymbol{X}}_i^k - \bar{\boldsymbol{X}}_i^{k-1}\|$$

**Proposition** 8 is concluded.

### APPENDIX B PROOF OF **PROPOSITION** 9

Proof. By invoking Proposition 8 and the CauchySchwarz inequality  $(\frac{\sum_{i=1}^{N} x_i}{N})^2 \leq \frac{1}{N} \sum_{i=1}^{N} (x_i)^2$ , we have

$$\left[\operatorname{dist}\left(\nabla\Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}),0\right)\right]^{2} \leq 2N(b_{3}^{k})^{2} \sum_{i=1}^{N} \|\boldsymbol{X}_{i}^{k+1} - \boldsymbol{X}_{i}^{k}\|^{2} + 2N(b_{4}^{k})^{2} \sum_{i=1}^{N} \|\bar{\boldsymbol{X}}_{i}^{k} - \bar{\boldsymbol{X}}_{i}^{k-1}\|^{2} \tag{71}$$

Besides, according to *Proposition* 6, we have

$$\Phi_{\beta_k}(\boldsymbol{W}^k) - \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}) 
\geq b_1^k \sum_{i=1}^N ||\bar{\boldsymbol{X}}_i^{k+1} - \bar{\boldsymbol{X}}_i^k||^2 + b_2^k \sum_{i=1}^N ||\bar{\boldsymbol{X}}_i^k - \bar{\boldsymbol{X}}_i^{k-1}||^2$$
(72)

Thus, by combining (71), (72) and Condition (b), we have

$$\left[\operatorname{dist}\left(\nabla\Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1}),0\right)\right]^{2} \leq 2N\nu_{k}\left(\Phi_{\beta_{k}}(\boldsymbol{W}^{k}) - \Phi_{\beta_{k+1}}(\boldsymbol{W}^{k+1})\right) \tag{73}$$

## APPENDIX C PROOF OF COROLLARY 1

*Proof.* If the stepsize  $c_i^k$  is selected according to **Algorithm** 1, we have

$$g_{i}(\bar{\boldsymbol{X}}_{i}^{k+1}, \boldsymbol{\lambda}^{k}, \rho_{k}) + \alpha \|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\|^{2}$$

$$\leq g_{i}(\bar{\boldsymbol{X}}_{i}^{k}, \boldsymbol{\lambda}^{k}, \rho_{k}) + \langle \nabla_{\bar{\boldsymbol{X}}_{i}} g_{i}(\bar{\boldsymbol{X}}_{i}^{k}, \boldsymbol{\lambda}^{k}, \rho_{k}), \bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k} \rangle$$

$$+ \frac{c_{i}^{k}}{2} \|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\|^{2}$$

$$(74)$$

Based on subproblem (11), we have

$$(\boldsymbol{\mu}_{i}^{k})^{T}(\boldsymbol{A}_{i}\bar{\boldsymbol{X}}_{i}^{k+1} - \boldsymbol{E}_{i}\boldsymbol{Z}^{k+1}) + \frac{\rho_{k}}{2}\|\boldsymbol{A}_{i}\bar{\boldsymbol{X}}_{i}^{k+1} - \boldsymbol{E}_{i}\boldsymbol{Z}^{k+1}\|^{2}$$

$$g_{i}(\bar{\boldsymbol{X}}_{i}^{k+1}, \boldsymbol{\lambda}^{k}, \rho_{k}) + a\|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\|^{2}$$

$$\leq (\boldsymbol{\mu}_{i}^{k})^{T}(\boldsymbol{A}_{i}\bar{\boldsymbol{X}}_{i}^{k} - \boldsymbol{E}_{i}\boldsymbol{Z}^{k+1}) + \frac{\rho_{k}}{2}\|\boldsymbol{A}_{i}\bar{\boldsymbol{X}}_{i}^{k} - \boldsymbol{E}^{i}\boldsymbol{Z}_{j}^{k+1}\|^{2}$$

$$+ g_{i}(\bar{\boldsymbol{X}}_{i}^{k}, \boldsymbol{\lambda}^{k}, \rho_{k})$$

$$(75)$$

By combining (74) with (75), we have

$$(\boldsymbol{\mu}_{i}^{k})^{T}(\boldsymbol{A}_{i}\bar{\boldsymbol{X}}_{i}^{k+1} - \boldsymbol{E}_{i}\boldsymbol{Z}^{k+1}) + \frac{\rho_{k}}{2}\|\boldsymbol{A}_{i}\bar{\boldsymbol{X}}_{i}^{k+1} - \boldsymbol{E}_{j}^{i}\boldsymbol{Z}_{j}^{k+1}\|^{2}$$

$$+ \langle \nabla_{\bar{\boldsymbol{X}}_{i}}g_{i}(\bar{\boldsymbol{X}}_{i}^{k}, \boldsymbol{\lambda}_{i}^{k}, \rho_{k}), \bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k} \rangle + \frac{c_{i}^{k}}{2}\|\bar{\boldsymbol{X}}_{i}^{k+1} - \bar{\boldsymbol{X}}_{i}^{k}\|^{2}$$

$$\leq (\boldsymbol{\mu}_{i}^{k})^{T}(\boldsymbol{A}_{i}\bar{\boldsymbol{X}}_{i}^{k} - \boldsymbol{E}_{i}\boldsymbol{Z}^{k+1}) + \frac{\rho_{k}}{2}\|\boldsymbol{A}_{i}\bar{\boldsymbol{X}}_{i}^{k} - \boldsymbol{E}_{i}\boldsymbol{Z}^{k+1}\|^{2}$$

$$(76)$$

By summing up (76)  $\forall i \in \mathcal{N}$ , we have

$$\mathbb{L}_{\rho_k}(\bar{\boldsymbol{X}}^{k+1}, \boldsymbol{Z}^{k+1}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) + \alpha \|\bar{\boldsymbol{X}}^{k+1} - \bar{\boldsymbol{X}}^k\|^2$$

$$\leq \mathbb{L}_{\rho_k}(\bar{\boldsymbol{X}}^k, \boldsymbol{Z}^k, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k)$$
(77)

In this case, the convergence of PLDM can be illustrate analogously following **Theorem** 1 by replacing  $\frac{1}{2}(c_i^k-L_{g_i})$  with  $\alpha$ . Also, to guarantee **Condition** (a):  $\alpha$  should be selected and satisfy  $\alpha \geq \frac{4(\Omega_1^i)^2}{\rho_k} + 4\beta_{k+1}$ .

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