TOPOLOGY BASICS

This course is about geometry, a field that studies the concrete drawing of an abstract graph on some space (e.g. $\mathbb{R}^2$ and $\mathbb{R}^3$). Besides the apparent connection to graph theory, our topics fundamentally rely upon the notion of topology. For instance, the concept of a "face" or "boundary" could not be stated rigorously without knowledge of topology, not to mention the proof of Euler's Formula. However, topology is such a deep field that we don't expect deriving all the results from first principle. Instead, we axiomise some obvious statements from topology, and prove a few lemmas based on the axioms. The point here is <u>not</u> setting up the notes in a completely rigourous way, but rather giving the reader a taste of topological ideas and convincing them the possibility of formalisation if they wish to do so.

~~In Topology~~

def. open set.

A set $A \subseteq S$ is open if $\forall a \in A.\ \exists r > 0$ such that the neighbourhood $U(a,r) \subseteq A$.

def. closed set.

A set $A \subseteq S$ is closed if $S \setminus A$ is open. Equivalently, $A \subseteq S$ is closed if the limit of any convergent sequence in A is still in A. (why?)

Remark. In the defitions above we use the notion of "neighbourhood" $U(a,r)$, thus also the notion of distance. For this course, we will always consider $S = \mathbb{R}^n$ ($n=2$ mostly) equipped with Euclidean distance, so these defs fit in our head.

But we [redacted] remind the reader that, in pure topological sense, there are other definitions of open/closed sets which don't require distance at all! Such defs are too general for our purpose.

def. **compact set**: both bounded and closed.

Theorem1. A set $A \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ For any family $\mathcal{F} = \{F_i\}$, $F_i \subseteq \mathbb{R}^n$ open, and $\bigcup_{F \in \mathcal{F}} F \supseteq A$, "cover of A" we could find a finite subfamily $\mathcal{F}' \subseteq \mathcal{F} : \bigcup_{F \in \mathcal{F}'} F \supseteq A$.

Now we are prepared for the more "geometry" side of topology.

def. **arc and Jordan curve.**
Let $f: [0,1] \to \mathbb{R}^n$ be a continuous [redacted] function.

(1) If f is injective, then we call its image an arc. $f(0)$ and $f(1)$ are the endpoints.

(2) If $f(0) = f(1)$ and f is injective on $[0,1)$, then we call its image a Jordan curve.

(We shall denote $\mathring{e} := e \setminus$ endpoints of e for arc e.)

[redacted]

(Just a side note: the definition could be rephrased in concise topological terms "an arc is homeomorphic to $[0,1]$; a Jordan curve is homeomorphic to a unit circle".)

We call $a, a' \in A \subseteq \mathbb{R}^n$ connected (denoted $a \overset{A}{\sim} a'$) if $\exists$ an arc $\subseteq A$ whose endpoints are a and a'. Clearly, $\overset{A}{\sim}$ is an equivalence relation, thus classifying A into one or more equivalence classes. We call each of these classes a region. If A is open, then so are its regions. (why?)

def boundary.

The boundary of region R, denoted ∂R, is all the points $x \in \mathbb{R}^n$ s.t.

$$\forall r>0,\ U(x,r)\cap R \neq \emptyset \ \wedge\ U(x,r)\cap(\mathbb{R}^n\setminus R) \neq \emptyset.$$

i.e. lies exactly upon the "separation" of R and $\mathbb{R}^n\setminus R$.

At this point, we don't have much tools except some basic definitions and a well-known theorem. But these are sufficient for the core definitions of this course: plane and planar graphs.

Exercise. Show that if the endpoints of an arc e lie in different regions, say R_1 and R_2, then $e \cap \partial R_1 \neq \emptyset$, $e \cap \partial R_2 \neq \emptyset$.

PLANARITY

For the moment, we shall make a little twist of the concept of graphs:

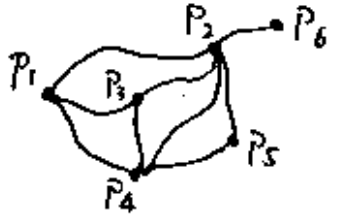
def plane graph.

a plane graph is a pair (V, E) where

(1) $V \subseteq \mathbb{R}^2$ is a finite set of points.

(2) $E \subseteq \mathbb{R}^2$ is a finite set of arcs whose endpoints are always in V. Also, we require that $\forall e, e' \in E$, $\underbrace{\mathring{e}\cap\mathring{e}'=\emptyset}_{\text{no crossing}}$ and $\underbrace{|e\cap e'|\le 1}_{\text{no multi-edge}}$.

So strictly speaking, a plane graph is not a graph, because they are different types of objects. However if we inspect the above definition carefully, we see that there's always an "abstract" graph behind the scenes. Hence we don't actually distinguish a plane graph or its corresponding abstract graph.



a plane graph

$(\{P_1, \cdots, P_6\}, \{P_1P_2, P_1P_3, P_1P_4, P_2P_3, P_2P_4, \cdots\})$

its corresponding abstract graph

Though a plane graph does correspond to an abstract graph, the converse is not necessarily true (and we shall prove that sections later). The reason is that a plane graph imposes the strong restriction that no two edges could cross in the plane, while a general graph doesn't care about ~~that~~ the ~~spe~~ concrete graphical representation.

def. planar graph.

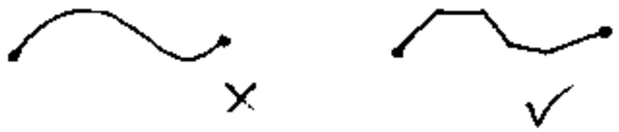
An abstract graph G is planar if we could find a plane graph G' that G corresponds to G'.

If G is planar, then we call the G' above an <u>embedding</u> of G.

not unique.

Proposition 2.

In the definition of plane graph, we could impose an even stronger condition without changing the meaning: "all the arcs are required to be finite consecutive line segments".



proof. The intuitive idea is to cut the arcs (edges) fine enough and straighten piecewise so that no edges would cross. To make the argument rigorous, we appeal to Theorem 1.

For each edge e and $x \in \mathring{e}$, define $d(x)$ to be the minimum distance between x and all other edges. Clearly $d(x) > 0$. Now we draw a square with side length $d(x)$, centered at x. So we collect a family of squares $\mathcal{F}$ after going through all $x \in \mathring{e}$.

Obviously $\mathcal{F}$ covers $\mathring{e}$, So by Theorem 1 we could find a finite subfamily $\mathcal{F}'$ that covers $\mathring{e}$ as well.

Since by our choice of squares, $\left(\bigcup_{F\in\mathcal{F}'} F\right) \cap e' = \emptyset$ for all $e' \neq e$, it's always safe to walk along the sides of these squares without crossing other edges. This gives us the desired "finite linearization" of edge e. ■

def. face.

For a plane graph $G=(V,E)$, the set $\mathbb{R}^2\setminus G$ (which means $\mathbb{R}^2\setminus(V\cup E)$) is open (why?), whose regions are called the faces of G. We use $F(G)$ to denote the set of faces of G.

G

$\mathbb{R}^2\setminus G$

Proposition 3

A plane graph G always has one unbounded face called "outer face". All other faces (if any) are bounded.

proof. Since G is finite, and any arc is bounded (why?), there exists a "bounding box" that contain the entire G. The exterior of the box clearly belongs to the same equivalence class in $\mathbb{R}^2\setminus G$, so $\mathbb{R}^2\setminus G$ has at least one unbounded region, i.e. G has at least one unbounded face. As two faces must be disjoint, the rest of the faces must reside in the bounding box, thus bounded. ■