# DUALITY & LINE ARRANGEMENT

This section studies some naïve-looking problems related to a finite point set $S \subseteq \mathbb{R}^2$.

- Collinearity test: Are there 3 collinear points in $S$?
- Minimum area triangle: From all triangles in $\binom{S}{3}$, find the one with smallest area.

  (Note that the collinearity test reduces to this problem)
- Build rotational system: For every point in $S$, sort the other points in a clockwise order.

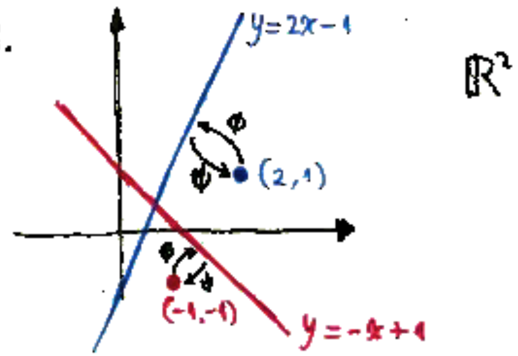
All these problems can be trivially solved in $O(n^3)$ time. But we could do better.

Via a surprisingly simple duality transform to be explained later, the problems translate to their "dual form" concerning with lines (instead of points) in $\mathbb{R}^2$. It turns out that the dual problems are solvable in $O(n^2)$ using the theory of line arrangements.

def duality maps:

$$\begin{array}{c} \text{point} \\ (a,b)\in\mathbb{R}^2 \end{array} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \begin{array}{c} \text{line} \\ \ell: y = ax - b \end{array}$$

e.g.




We remind the readers of a pitfall: A line $\ell$ could be regarded as a monolithic object or as a set of points, So both ~~[illegible]~~ $\psi(\ell)$ and $\phi(\ell)$ are well-defined mathematically. However, keep in mind that $\psi(\ell) \neq \phi(\ell)$!

this gives you a point — this gives you a family of lines

So it is important to remember that, when ~~we~~ people say "the dual of $\ell$" in the literature, they are referring to $\psi(\ell)$, i.e. treating $\ell$ as a monolithic body.

You might be confused why we defined duality maps so arbitrarily — there seems to be little geometric correlation between $p$ and $\phi(p)$. But the lemma below actually extracts some rather handy relations between primal and dual.

**Lemma 31.**

(1) Points $p$ and $q$ have the same $x$-coordinate $\iff$ lines $\phi(p)$ and $\phi(q)$ are parallel.

Moreover, when $p, q$ have the same $x$-coord, $d(p,q) = d(\phi(p), \phi(q))$.

(2) $p$ is on $\ell \iff \psi(\ell)$ is on $\phi(p)$

(3) $p$ is above $\ell \iff \psi(\ell)$ is above $\phi(p)$

(4) The vertical distance ~~between~~ from $p$ ~~and~~ to $\ell$ = the vertical distance from $\psi(\ell)$ to $\phi(p)$.

Proof. (1) Exercise.

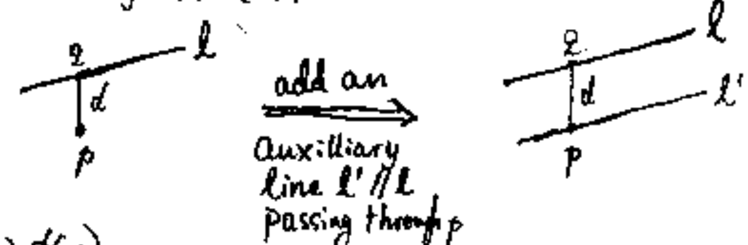
(2) Assume $p = (a, b)$ and $\ell: y = cx - d$.

Then $\phi(p): \; y = ax - b$
$\psi(\ell): \; (c, d)$.

Hence, $p$ is on $\ell \iff b = ac - d$
$\iff d = ac - b$
$\iff \psi(\ell)$ is on $\phi(p)$.

(3) Exercise.

(4) This claim nicely illustrates the use of (2).



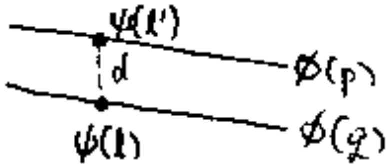
Now look at the dual picture consisting of $\phi(p)$, $\phi(q)$, $\psi(l)$ and $\psi(l')$. From (1) we have

- $\phi(p) \,//\, \phi(q)$.
- $\psi(l)$ and $\psi(l')$ have the same $x$ coordinate. Moreover, their distance is exactly $d$.

From (2) we know

- $\psi(l)$ is on $\phi(q)$.
- $\psi(l')$ is on $\phi(p)$.

So the picture looks like




∎

To test your understanding of duality and the lemma, do the exercise below:

Exercise. Let $I \subseteq \mathbb{R}^2$ be a line segment (regarded as a set of points). Describe the dual $\phi(I)$ in geometry language.

With the guarantee of Lemma 31, it is natural to transform the given point set $S$ to its dual $\phi(S)$, and <u>hope</u> that the geometry properties of interest in $S$ also translate into some corresponding geometry properties in $\phi(S)$. We will illustrate the idea for the problems in our list.

- **Collinearity test.**

There are 3 collinear points in $S$

$\Leftrightarrow$ $\exists$ line $l$ and points $p, q, r \in S$: $p, q, r$ are on line $l$

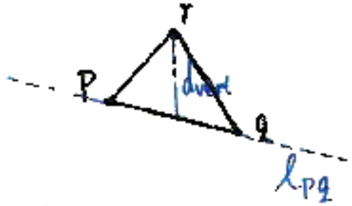
$\overset{\text{Lemma 31(2)}}{\Leftrightarrow}$ $\exists$ point $t$ and lines $l_1, l_2, l_3 \in \phi(S)$: $t$ is on $l_1, l_2, l_3$

$\Leftrightarrow$ There are 3 lines in $\phi(S)$ that intersect at one point

Therefore, the problem transforms into detecting whether there exist 3 lines in $\phi(S)$ that jointly intersect one point.

- **Minimum area triangle**

For any triangle $pqr \in \binom{S}{3}$ we have




$$\operatorname{area}(pqr)$$

$$= \frac{|x_p - x_q|}{2} \cdot d_{vert}(r, l_{pq})$$

$$\overset{\text{Lemma}}{\underset{31(4)}{=}} \frac{|x_p - x_q|}{2} \cdot d_{vert}(\psi(l_{pq}), \phi(r))$$

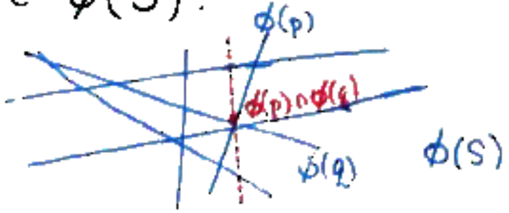
$$= \frac{|x_p - x_q|}{2} \cdot d_{vert}(\phi(p) \cap \phi(q), \phi(r))$$

where $\psi(l_{pq}) = \phi(p) \cap \phi(q)$ because both $p$ and $q$ are on $l_{pq}$ and thus by Lemma 31(2), $\psi(l_{pq})$ must be on both $\phi(p)$ and $\phi(q)$.

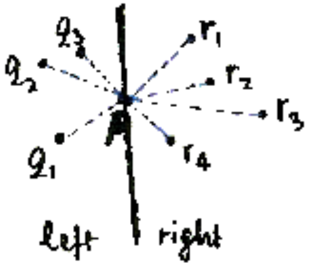
So the problem reduces to computing

$$\min_{r \in S} d_{vert}(\phi(p) \cap \phi(q), \phi(r))$$

for every pair $p, q \in S$. This is a purely geometry problem in the dual picture $\phi(S)$.




- **Build rotational system**



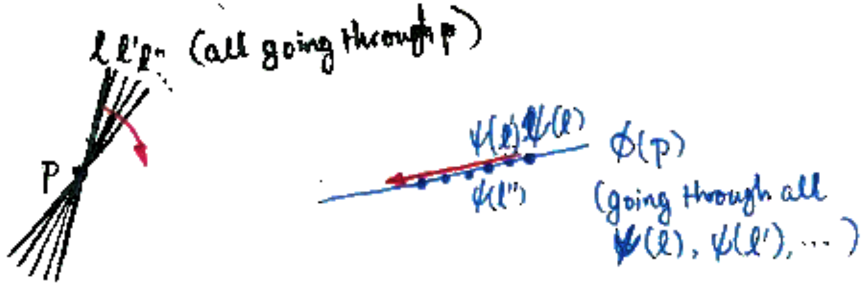

Split $S \setminus \{p\}$ into left part $\{q_1, \cdots, q_k\}$ and right part $\{r_1, \cdots, r_t\}$. Imagine rotating the vertical line clockwise around $p$, until it is vertical again. During the rotation it will sequentially hit $(p_1, p_2, \cdots, p_{n-1})$. Although the $q$'s and $r$'s might be interlacing in the sequence, the relative order inside the $q$'s (resp. the $r$'s) are preserved. e.g.

$$(\underline{r_1}, \underline{q_1}, \underline{r_2}, \underline{r_3}, \underline{q_2}, \underline{r_4}, \underline{q_3})$$

Hence, once we get the sequence $(p_1, \cdots, p_{n-1})$, it's straightforward to recover the rotational system clockwise:

$$(r_1, r_2, \cdots, r_t, q_1, q_2, \cdots, q_k).$$

But the "rotation & hit" procedure has a direct correspondence in the dual picture $\phi(S)$. Rotating a line around $p$ corresponds to moving a point along $\phi(p)$. Since we start with




slope $+\infty$ and ends at slope $-\infty$, the point movement is from the right to the left in the dual.

With this observation, Computing $(p_1, \dots, p_{n-1})$ amounts to reading the intersections of $\phi(p)$ with other dual lines from right to left.

However, the above discussions do not yield directly $O(n^2)$ algorithms for our purposes. To this end, we need to inspect in more detail the structure of the dual picture, or more generally, the arrangement of $n$ lines in $\mathbb{R}^2$.

Given a set $\mathcal{L}$ of $n$ lines in $\mathbb{R}^2$, we may naturally invent the notions of "vertex", "edge" and "face" just as we did for Voronoi diagrams.
The entire incidence structure is called "line arrangement", denoted $A(\mathcal{L})$.


whole structure $= A(\mathcal{L})$

Lemma 32.

Any line arrangement $\mathcal{A}(\mathcal{L})$ contains at most $\binom{n}{2}$ vertices, $n^2$ edges and $\binom{n+1}{2}+1$ faces.

**Proof.** Without loss of generality we assume that no two lines in $\mathcal{L}$ are parallel, and no three ~~points~~ lines in $\mathcal{L}$ intersect at one point. (We could alway perturb the lines a bit to fulfil the requirements and the counts could only increase)

Every two lines intersect exactly once, hence #vertices $=\binom{n}{2}$. Every line intersect $(n-1)$ others, hence is divided into $n$ edges. So #edges $= n \cdot n = n^2$. Finally, we build a planar graph on top of $\mathcal{A}(\mathcal{L})$ by introducing an "infinite vertex" which connects to all infinite edges. By Euler's formula,

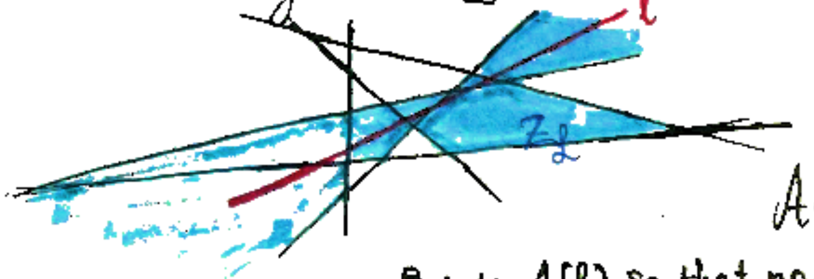
$$(\#\text{vertices}+1) - (\#\text{edges}) + (\#\text{faces}) = 2$$

$$\Rightarrow \#\text{faces} = 2 - \left( \binom{n}{2}+1 \right) + n^2$$
$$= n^2 - \binom{n}{2} + 1$$
$$= \binom{n+1}{2} + 1. \qquad \blacksquare$$

So ~~the~~ line arrangement is not a super simple object to deal with. But still we hope for the best: Can we construct it in (optimal) $O(n^2)$ time? The answer is yes, and the whole magic hides in the ~~fact that lines are straight~~ theorem below. From a high level, lines are straight and they display quite "linear" behaviour when it comes to incidence structure.

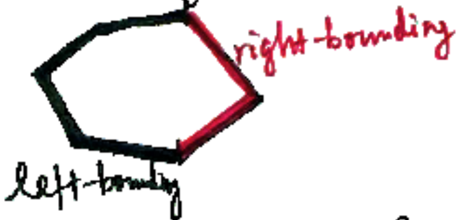
~~Theorem 33 (Zone theorem) For any line arrangement $\mathcal{A}(\mathcal{L})$ and line $\ell$, $\ell$ could be incident to at most~~

**Theorem 33.** (Zone theorem)
Let $A(\mathcal{L})$ be a line arrangement and $\ell$ be a line. Denote by $Z_{\mathcal{L}}$ the collection of faces in $A(\mathcal{L})$ that $\ell$ intersects. ("Z" for "zone") Then the total number of edges in $Z_{\mathcal{L}} \le 10n$.




$A(\mathcal{L})$

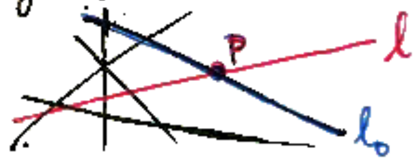
**Proof.** Rotate $A(\mathcal{L})$ so that no line is horizontal. For a face $f$ and an edge $e \in \partial f$, we call $e$ left-bounding (to $f$) if $f$ is completely at its right. Similarly define the notion of right-bounding.




Note that any $e \in \partial f$ is either left- or right-bounding because $f$ is a convex polygon in our context.

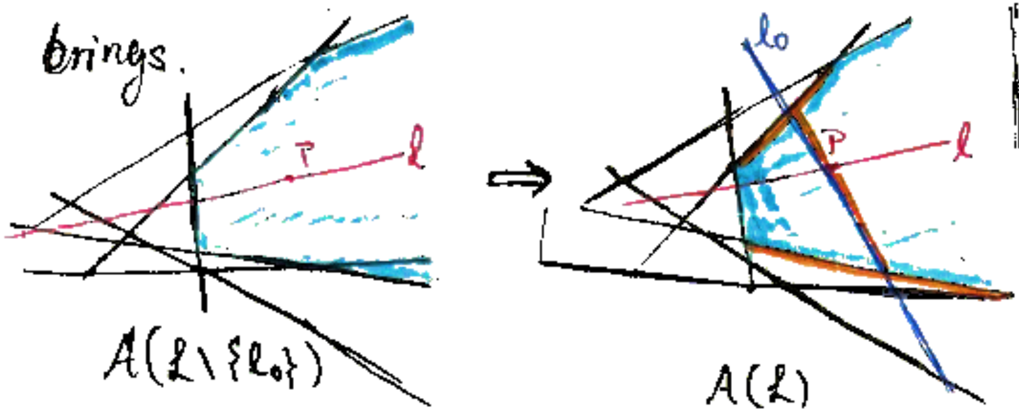
We shall prove by induction that the number of left-bounding edges of faces in $Z_{\mathcal{L}}$ is at most $5n$. A symmetric argument applies to right-bounding edges, so altogether we get the claimed $10n$ upper bound.

When $n=1$ ~~then~~ the claim is trivial. Now we proceed from $n-1$ to $n$. Let $p$ be the rightmost intersection of $\ell$ and $\mathcal{L}$. Assume for the moment that there is only one line, say $\ell_0$, in $\mathcal{L}$ that goes through $p$.



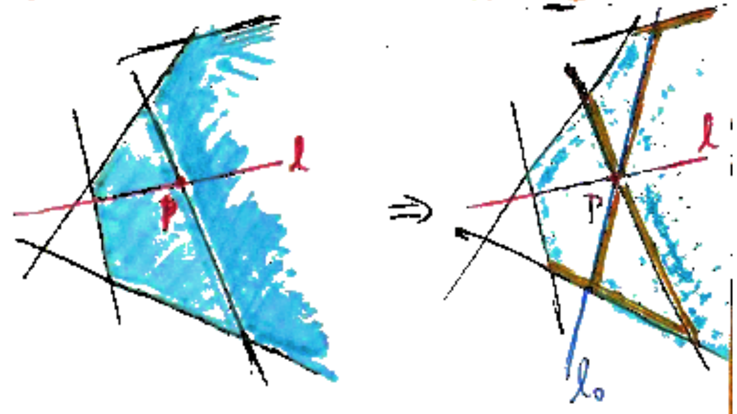

By induction hypothesis, the total number of leftbounding edges of $Z_{\mathcal{L}\setminus\{\ell_0\}}$ is at most $5(n-1)$.

Now we add $\ell_0$ back and discuss how many new left-bounding edges it brings.




Observe that $\ell_0$ could only go through the rightmost face in $\mathcal{Z}_{\mathcal{L}\setminus\{\ell_0\}}$ (shaded blue in the illustration). Since the face is convex, $\ell_0$ intersects it at most twice. Each intersection could split a left-bounding edge into two. Plus that $\ell_0$ itself could be left-bounding, at most 3 new left-bounding edges are added, and then $\# \leq 5(n-1)+3 \leq 5n$.

With this experience, we could now remove our artificial assumption that $\ell_0$ is the only line going through p.




$A(\mathcal{L}\setminus\{\ell_0\})$ $A(\mathcal{L})$

This time, the potential influence spreads over <u>two</u> faces in $\mathcal{Z}$. Still, we have two intersections with one on each side. Plus an additional split at point p and two more edges induced by $\ell_0$ itself. So the total number of addition is 5, then $\# \leq 5(n-1)+5 \leq 5n$. ∎

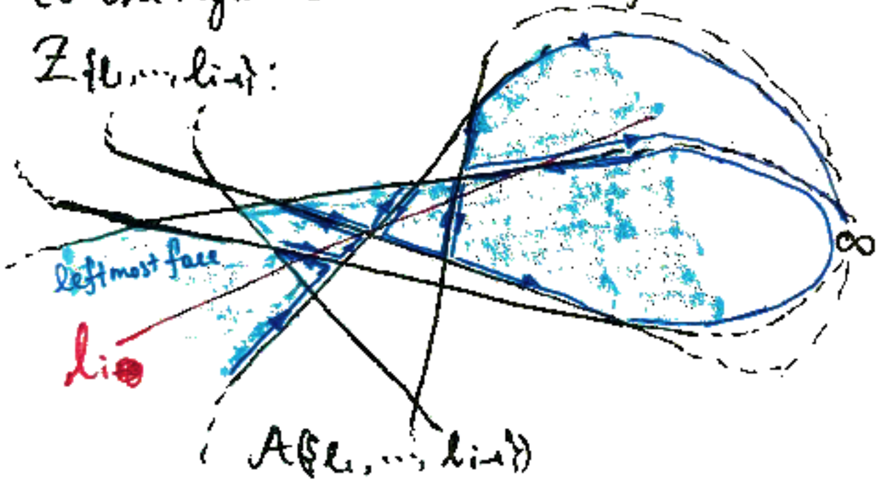
**Corollary 34.**
One may build $A(\mathcal{L})$ in $O(n^2)$ time.

Proof. Assume $\mathcal{L} = \{l_1, \dots, l_n\}$. We incrementally build

$A(\{l_1\}),\ A(\{l_1, l_2\}),\ \dots,\ A(\mathcal{L})$

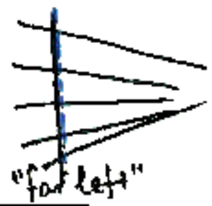
by inserting one line at a time. At step $i$ we insert $l_i$ into $A(\{l_1, \dots, l_{i-1}\})$.

Suppose for now that we could find the leftmost face in $\mathcal{Z}(\{l_1, \dots, l_{i-1}\})$ that $l_i$ intersects. Then we could walk systematically to the right to traverse all faces in $Z\{l_1, \dots, l_{i-1}\}$:




To be specific, we walk along the boundary of the current face in counterclockwise direction, until we detect an intersection with $l_i$. Then we create a new vertex at the intersection and split the edge & face accordingly. Next we "jump" to the other side of the edge to enter a new face. Repeat until no more intersection could be detected.

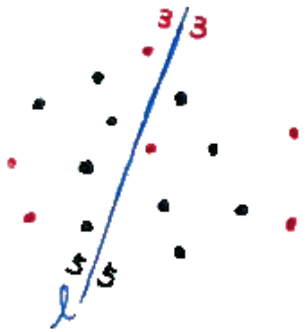
By zone theorem, our traversal takes time $5i = O(n)$. So the only missing piece is to show that we could indeed locate the leftmost face in $Z\{l_1, \dots, l_{i-1}\}$. But this is easy due to the following observation: In the "far left" the lines are sorted by their slope; the ~~higher~~ larger slope it has, the lower the line lives. So the location of $l_i$ could be pinned down in even $O(\log n)$ time. ■



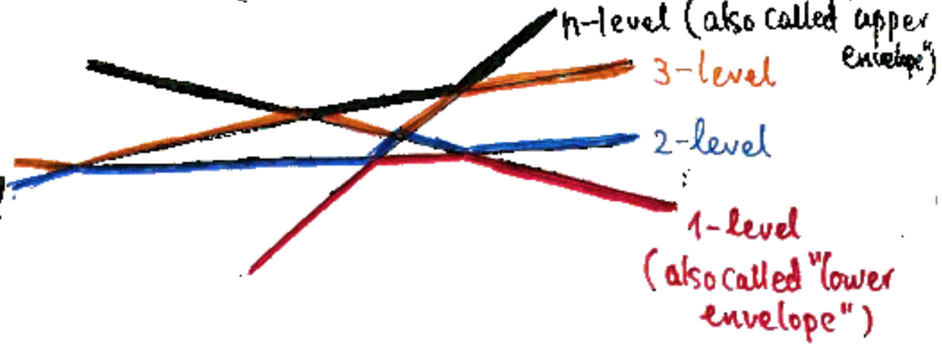
Therefore, all three problems in our list can be effectively solved in $O(n^2)$ time by working in the dual picture!
(Details are left as an exercise.)

We conclude the Section by a beautiful theorem which displays, once again, the power of duality.

**Theorem 35.** (Discrete Ham Sandwich Thm) For any two finite points sets $S, T \subseteq \mathbb{R}^2$, there exists a line $\ell$ that bisects both $S$ and $T$. That is $|S\cap\ell^-|, |S\cap\ell^+| \le |S|/2$ and $|T\cap\ell^-|, |T\cap\ell^+| \le |T|/2$.




def. $k$-level.
The $k$-level of $A(\mathcal{L})$ is the collection of edges that have $(k-1)$ lines below and $(n-k)$ lines above.




Proof. Note that it suffices to prove the theorem for odd $|S|, |T|$ (when they are even we may remove an arbitrary point and apply the result for odd sizes.)

Also, assume without loss of generality that no points in $S\cup T$ have the same x-coordinate. (Otherwise, rotate the system infinitesimally.)

Then we may translate the condition to dual as follows:

$$\exists \ell : \begin{cases} \le \frac{|S|-1}{2} \text{ points in } S \text{ below/above } \ell \\ \le \frac{|T|-1}{2} \text{ points in } T \text{ below/above } \ell \end{cases}$$

Lemma 3(3)

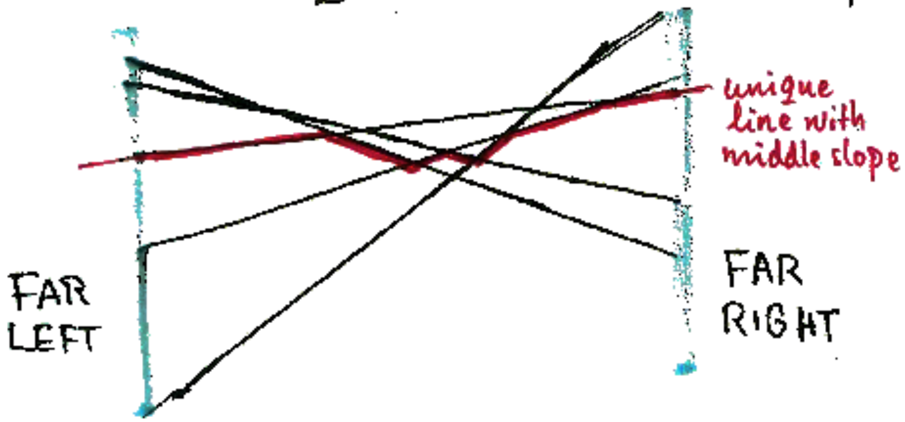
$\iff \exists p: \begin{cases} \le \frac{|S|-1}{2} \text{ lines in } \phi(S) \text{ below/above } p \\ \le \frac{|T|-1}{2} \text{ lines in } \phi(S) \text{ below/above } p \end{cases}$

$\iff \exists p$: $p$ is on the mid-level of $\phi(S)$ and also the mid-level of $\phi(T)$

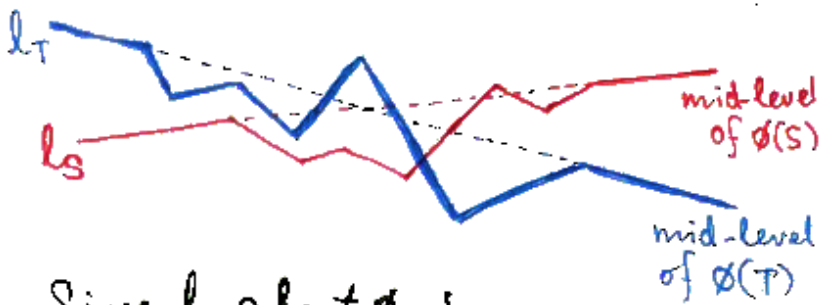
$\iff$ (mid-level of $\phi(S)$) $\cap$ (mid-level of $\phi(T)$) $\neq \emptyset$

Recall the in the "far left" and "far right", the lines are sorted according to slope.

Therefore, the left and right ~~extreme~~ ends of the mid-level of $\phi(S)$ (resp. $\phi(T)$) are both the unique line with middle slope.




So the middle levels of $\phi(S)$ and $\phi(T)$ have the shape




Since $l_T \cap l_S \neq \emptyset$, by continuity we know (mid-level of $\phi(S)$) $\cap$ (mid-level of $\phi(T)$) $\neq \emptyset$, proving the theorem. ∎

Remark. The theorem is a discrete specialisation of a much stronger Ham Sandwich theorem, stating that there exists a hyperplane that bisects both $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^n$. Here "bisection" could be taken with respect to any continuous measure.