Dual Simplex Method

martes, 1 de marzo de 2022 07:43 p. m.

6.4 THE DUAL SIMPLEX METHOD

In this section we describe the *dual simplex* method, which solves the dual problem directly on the (primal) simplex tableau. At each iteration we move from a basic feasible solution of the dual problem to an improved basic feasible solution until optimality of the dual (and also the primal) is reached, or else until we conclude that the dual is unbounded and that the primal is infeasible.

Interpretation of Dual Feasibility on the Primal Simplex Tableau

Consider the following linear programming problem.

$$\begin{aligned} &\text{Minimize } \mathbf{cx} \\ &\text{Subject to } \mathbf{Ax} \geq \mathbf{b} \\ &\qquad\qquad \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Let $\mathbf{B}$ be a basis that is not necessarily feasible and consider the following tableau.

	z	x_1	x_2	$\cdots$	x_n	x_{n+1} (SLACK VARIABLES)	$\cdots$	x_{n+m}	RHS
z	1	$z_1 - c_1$	$z_2 - c_2$	$\cdots$	$z_n - c_n$	$z_{n+1} - c_{n+1}$	$\cdots$	$z_{n+m} - c_{n+m}$	$\mathbf{c}_B\bar{\mathbf{b}}$
x_{B_1}	0	y_{11}	y_{12}	$\cdots$	y_{1n}	$y_{1,n+1}$	$\cdots$	$y_{1,n+m}$	$\bar{b}_1$
x_{B_2}	0	y_{21}	y_{22}	$\cdots$	y_{2n}	$y_{2,n+1}$	$\cdots$	$y_{2,n+m}$	$\bar{b}_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_{B_m}	0	y_{m1}	y_{m2}	$\cdots$	y_{mn}	$y_{m,n+1}$	$\cdots$	$y_{m,n+m}$	$\bar{b}_m$

The tableau presents a primal feasible solution if $\bar{b}_i \geq 0$ for $i = 1, 2, \ldots, m$; that is, if $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$. Furthermore, the tableau is optimal if $z_j - c_j \leq 0$ for $j = 1, 2, \ldots, n + m$. Define $\mathbf{w} = \mathbf{c}_B\mathbf{B}^{-1}$. For $j = 1, 2, \ldots, n$ we have

$$z_j - c_j = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c_j = \mathbf{w}\mathbf{a}_j - c_j$$

Hence $z_j - c_j \leq 0$ for $j = 1, 2, \ldots, n$ implies that $\mathbf{w}\mathbf{a}_j - c_j \leq 0$ for $j = 1, 2, \ldots, n$, which in turn implies that $\mathbf{wA} \leq \mathbf{c}$. Furthermore, note that $\mathbf{a}_{n+i} = -\mathbf{e}_i$ and $c_{n+i} = 0$ for $i = 1, 2, \ldots, m$ and so we have

$$\begin{aligned} z_{n+i} - c_{n+i} &= \mathbf{w}\mathbf{a}_{n+i} - c_{n+i} \\ &= \mathbf{w}(-\mathbf{e}_i) - 0 \\ &= -w_i \qquad i = 1, 2, \ldots, m \end{aligned}$$

In addition, if $z_{n+i} - c_{n+i} \leq 0$ for $i = 1, 2, \ldots, m$, then $w_i \geq 0$ for $i = 1, 2, \ldots, m$ and so $\mathbf{w} \geq \mathbf{0}$. We have just shown that $z_j - c_j \leq 0$ for $j = 1, 2, \ldots, n + m$ implies that $\mathbf{wA} \leq \mathbf{c}$ and $\mathbf{w} \geq \mathbf{0}$, where $\mathbf{w} = \mathbf{c}_B\mathbf{B}^{-1}$. In other words, dual feasibility is precisely the simplex optimality criteria $z_j - c_j \leq 0$ for all j. At optimality $\mathbf{w}^* = \mathbf{c}_B\mathbf{B}^{-1}$ and the dual objective $\mathbf{w}^*\mathbf{b} = (\mathbf{c}_B\mathbf{B}^{-1})\mathbf{b} = \mathbf{c}_B(\mathbf{B}^{-1}\mathbf{b}) = \mathbf{c}_B\bar{\mathbf{b}} = z^*$; that is, the primal and dual objectives are equal. Thus we have the following result.

Lemma 4

At optimality of the primal minimization problem in the canonical form (that is, $z_j - c_j \leq 0$ for all j), $\mathbf{w}^* = \mathbf{c}_B\mathbf{B}^{-1}$ is an optimal solution to the dual problem. Furthermore $w_i^* = -(z_{n+i} - c_{n+i}) = -z_{n+i}$ for $i = 1, 2, \ldots, m$.

The Dual Simplex Method

Consider the following linear programming problem.

$$\begin{aligned} &\text{Minimize } \mathbf{cx} \\ &\text{Subject to } \mathbf{Ax} = \mathbf{b} \\ &\qquad\qquad \mathbf{x} \geq \mathbf{0} \end{aligned}$$

In certain instances it is difficult to find a starting basic solution that is feasible (that is, all $\bar{b}_i \geq 0$) to a linear program without adding artificial variables. In these same instances it is often possible to find a starting basic, but not necessarily feasible, solution that is dual feasible (that is, all $z_j - c_j \leq 0$ for a minimization problem). In such cases it is useful to develop a variant of the simplex method that would produce a series of simplex tableaux that maintain dual feasibility and complementary slackness and strive toward primal feasibility.

	z	x_1	$\cdots$	x_j	$\cdots$	x_k	$\cdots$	x_n	RHS
z	1	$z_1 - c_1$	$\cdots$	$z_j - c_j$	$\cdots$	$z_k - c_k$	$\cdots$	$z_n - c_n$	$\mathbf{c}_B\bar{\mathbf{b}}$
x_{B_1}	0	y_{11}	$\cdots$	y_{1j}	$\cdots$	y_{1k}	$\cdots$	y_{1n}	$\bar{b}_1$
x_{B_2}	0	y_{21}	$\cdots$	y_{2j}	$\cdots$	y_{2k}	$\cdots$	y_{2n}	$\bar{b}_2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_{B_r}	0	y_{r1}	$\cdots$	y_{rj}	$\cdots$	y_{rk}	$\cdots$	y_{rn}	$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_{B_m}	0	y_{m1}	$\cdots$	y_{mj}	$\cdots$	y_{mk}	$\cdots$	y_{mn}	$\bar{b}_m$

Consider the above tableau representing a basic solution at some iteration. Suppose that the tableau is dual feasible (that is, $z_j - c_j \leq 0$ for a minimization problem). Then, if the tableau is also primal feasible (that is, all $\bar{b}_i \geq 0$) then we have the optimal solution. Otherwise, consider some $\bar{b}_r < 0$. By selecting row r as a pivot row and some column k such that $y_{rk} < 0$ as a pivot column we can make the new right-hand side $\bar{b}_r' > 0$. Through a series of such pivots we hope to make all $\bar{b}_i \geq 0$ while maintaining all $z_j - c_j \leq 0$ and thus achieve optimality. The question that remains is how do we select the pivot column so as to maintain dual feasibility after pivoting. The pivot column k is determined by the

Inicialización

Encontrar una base B del Primal tal que

$$z_j - c_j = c_B B^{-1} a_j - c_j \leq 0$$

Pasos

- Si $\bar{b} = B^{-1} b \geq 0$ la solución es optima

  De lo contrario seleccionar "r"

  $$\bar{b}_r < 0 , \quad \bar{b}_r = \min(\bar{b}_i)$$

- Si $y_{rj} \geq 0$ para todas las j el dual es no acotado y el Primal No Factible

  De lo contrario seleccionar k

  $$\frac{z_k - c_k}{y_{rk}} = \min\left(\frac{z_j - c_j}{y_{rj}} : y_{rj} < 0\right)$$

- Pivotear en y_{rk} y regresar al primer paso.

following minimum ratio test.

$$\frac{z_k - c_k}{y_{rk}} = \underset{j}{\text{minimum}} \left\{ \frac{z_j - c_j}{y_{rj}} : y_{rj} < 0 \right\} \tag{6.1}$$

Note that the new entries in row 0 after pivoting are given by:

$$(z_j - c_j)' = (z_j - c_j) - \frac{y_{rj}}{y_{rk}} (z_k - c_k)$$

If $y_{rj} \geq 0$, and since $z_k - c_k \leq 0$ and $y_{rk} < 0$, then $(y_{rj}/y_{rk})(z_k - c_k) \geq 0$ and hence $(z_j - c_j)' \leq z_j - c_j$. Since the previous solution was dual feasible, then $z_j - c_j \leq 0$ and hence $(z_j - c_j)' \leq 0$. Now consider the case where $y_{rj} < 0$. By 6.1 we have:

$$\frac{z_k - c_k}{y_{rk}} \leq \frac{z_j - c_j}{y_{rj}}$$

Multiplying both sides by $y_{rj} < 0$, we get $(z_j - c_j) - (y_{rj}/y_{rk})(z_k - c_k) \leq 0$, that is, $(z_j - c_j)' \leq 0$. To summarize, if the pivot column is chosen according to Equation (6.1), then the new basis obtained by pivoting at y_{rk} is still dual feasible. Moreover, the dual objective after pivoting is given by $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} - (z_k - c_k)\bar{b}_r/y_{rk}$. Since $z_k - c_k \leq 0$, $\bar{b}_r < 0$, and $y_{rk} < 0$, then $-(z_k - c_k)\bar{b}_r/y_{rk} \geq 0$ and the dual objective improves over the current value of $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} = \mathbf{wb}$.

We have just described a procedure that moves from a dual basic feasible solution to an improved (at least not worse) basic dual feasible solution. To complete the analysis we must consider the case when $y_{rj} \geq 0$ for all j and hence no column is eligible to be the pivot column. In this case the ith row reads: $\Sigma_j y_{rj} x_j = \bar{b}_r$. Since $y_{rj} \geq 0$ for all j and x_j is required to be nonnegative, then $\Sigma_j y_{rj} x_j \geq 0$ for any feasible solution. However, $\bar{b}_r < 0$. This contradiction shows that the primal is infeasible and the dual is unbounded (why?). In Exercise 6.31 we ask the reader to show directly that the dual is unbounded by constructing a direction satisfying the unboundedness criterion.