Definición 1. *A **strictly stationary** time series is one for which the probabilistic behavior of every collection of random variables*

$$\{X_{t_1}, X_{t_2}, \dots, X_{t_k}\}$$

is identical to that of the time shifted set

$$\{X_{t_1+h}, X_{t_2+h}, \dots, X_{t_k+h}\}$$

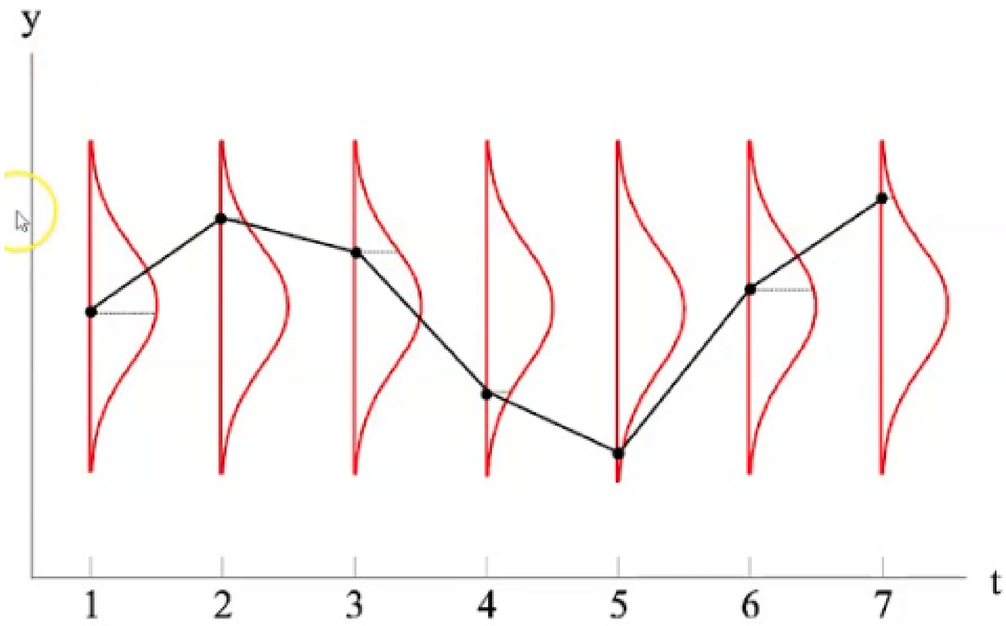
That is,

$$\Pr\{X_{t_1} \leq c_1, X_{t_2} \leq c_2, \dots, X_{t_k} \leq c_k\} = \Pr\{X_{t_1+h}, X_{t_2+h}, \dots, X_{t_k+h}\}$$

for all $k = 1, 2, \dots$, *all time points* $t_1, t_2, \dots, t_k$, *all numbers* $c_1, c_2, \dots, c_k$, *and all time shifts* $h = 0, \pm 1, \pm 2, \dots$

Strict stationarity

$$\Pr\{X_{t_1} \leq c_1, X_{t_2} \leq c_2, \ldots, X_{t_k} \leq c_k\} = \Pr\{X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_k+h}\}$$



Weak stationarity

Definición 2. *A **weakly stationary** time series, X_t, is a finite variance process such that:*

i. the mean value function, μ_t, is constant and does not depend on time t, and

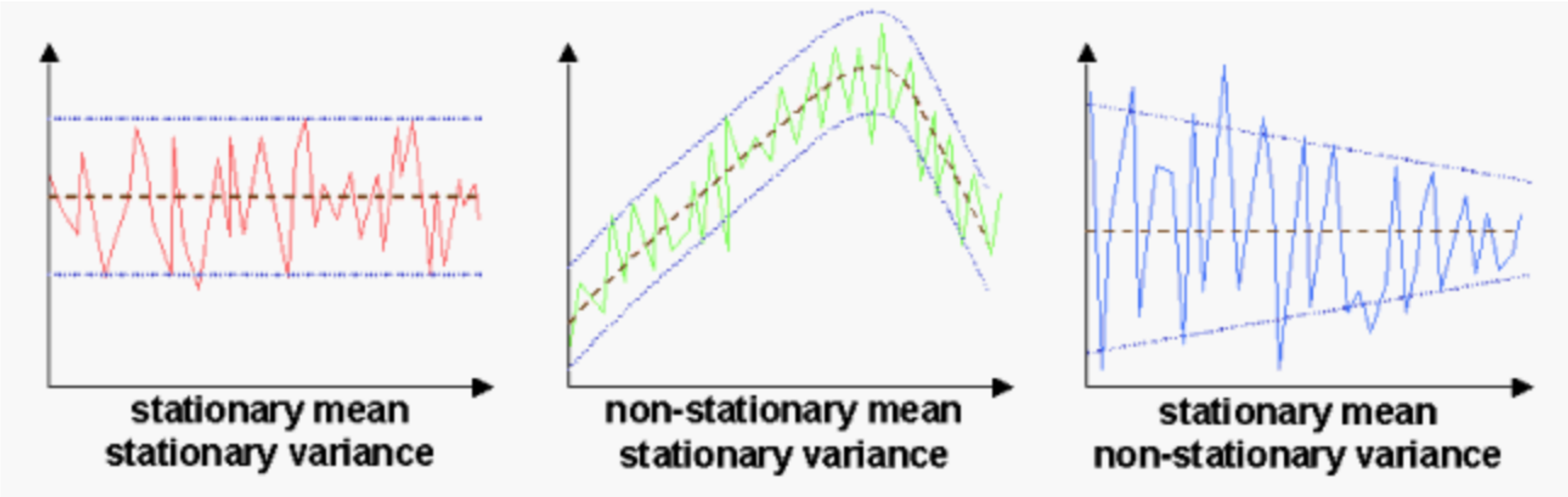
ii. the autocovariance function, $\gamma(s,t)$, depends on s and t only through their difference $|s-t|$.

iii. the variance $\mathbb{E}[(X_t-\mu_X)^2]<\infty$; it means the variance is finite for all t.

*Henceforth, we will use the term **stationary** to mean weakly stationary; if a process is stationary in the strict sense, we will use the term strictly stationary.*

Stationarity requires regularity in the mean and autocorrelation functions so that these quantities (at least) may be estimated by averaging.

Stationarity example



Simplifying the notation

Because the mean function, $\mathbb{E}[X_t] = \mu_t$, of a stationary time series is independent of time t we will write

$$\mu_t = \mu$$

Also, because the autocovariance function, $\gamma(s,t)$, of a stationary time series depends on s and t only through their difference $|s-t|$, we may simplify the notation. Let $s = t+h$, where h represents the time shift or lag. Then

$$\gamma(t+h,t) = \text{cov}(X_{t+h}, X_t) = \text{cov}(X_h, X_0) = \gamma(h,0)$$

because the time difference between times $t+h$ and t is the same as the time difference between times h and 0.

Definición 3. *The **autocovariance function of a stationary time series** will be written as*

$$\gamma(h) = \mathrm{cov}(X_{t+h}, X_t) = \mathbb{E}[(X_{t+h} - \mu)(X_t - \mu)]$$

Definición 4. *The **autocorrelation function (ACF) of a stationary time series** will be written using the definition above as*

$$\rho(h) = \frac{\gamma(t+h,t)}{\sqrt{\gamma(t+h,t+h)\gamma(t,t)}} = \frac{\gamma(h)}{\gamma(0)}$$

The Cauchy-Schwarz inequality shows again that $-1 \leq \rho(h) \leq 1$ for all h.

Stationarity of statistical models

Is the white noise stationary?

i. Is μ_t constant?, and, does it depend on time t?

ii. Does the autocovariance $\gamma(s, t)$ depends on s and t?

Is a random walk with drift stationary?

i. Is μ_t constant?, and, does it depend on time t?

ii. Does the autocovariance $\gamma(s, t)$ depends on s and t?

Trend stationarity

Let $X_t = \alpha + \beta t + Y_t$, where Y_t is stationary.

$$\mathbb{E}[X_t] = \alpha + \beta t + \mu_Y$$

$\mathbb{E}[X_t]$ is not independent of time. Therefore, the process is not stationary. The autocovariance function, however, is independent of time, because

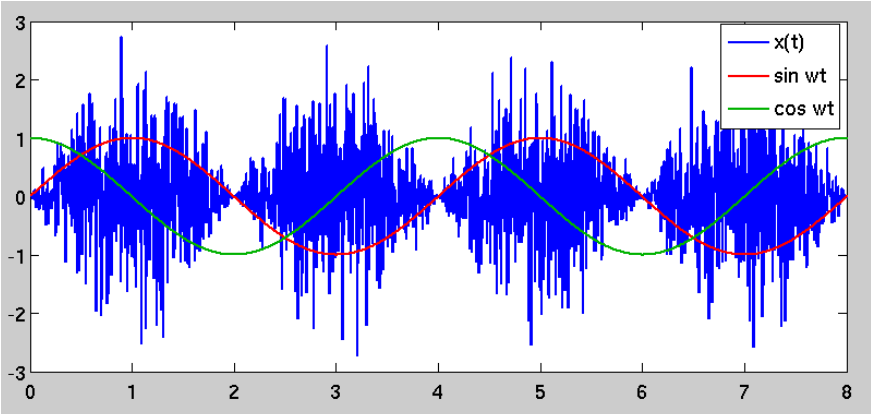
$$\gamma_X(h) = \mathbb{E}[(X_{t+h} - \mu_{X_{t+h}})(X_t - \mu_{X_t})] = \mathbb{E}[(Y_{t+h} - \mu_Y)(Y_t - \mu_Y)] = \gamma_Y(h)$$

Thus, the model may be considered as having stationary behavior around a linear trend; this behavior is sometimes called *trend stationarity*.

Cyclostationarity

A stochastic process is cyclostationary if the joint distribution of any set of samples is invariant over a time shift of $m\,P$, where $m \in \mathbb{Z}$ and $P \in \mathbb{N}$ is the period of the process:

$$F(X_{t_{1+m\,P}}, \ldots, X_{t_{n+\mathrm{mP}}}) = F(X_{t_1}, \ldots, X_{t_n})$$



Autocovariance of a stationary process properties

- $\gamma(h)$ is non-negative definite (positive semi-definite) ensuring that variances of linear combinations of the variates X_t will never be negative.
- The value at $h = 0$, namely, $\gamma(0) = \mathbb{E}[(X_t - \mu)^2]$ is the variance of the time series a the Cauchy-Schwarz inequality implies $|\gamma(h)| \leq \gamma(0)$.
- The autocovariance is symmetric around the origin; that is, $\gamma(h) = \gamma(-h)$.

Time series jointly stationary

Definición 5. *Two time series, say,* X_t *and* Y_t*, are said to be* ***jointly stationary*** *if they are each stationary, and the cross-covariance function*

$$\gamma_{X,Y}(h) = \operatorname{cov}(X_{t+h}, Y_t) = \mathbb{E}[(X_{t+h} - \mu_X)(Y_t - \mu_Y)]$$

is a function only of lag h*.*

Definición 6. *The cross-correlation function (CCF) of jointly stationary time series* X_t *and* Y_t *is defined as*

$$\rho_{X,Y}(h) = \frac{\gamma_{X,Y}(h)}{\sqrt{\gamma_X(0)\gamma_Y(0)}}$$

Importance of stationarity

Although the theoretical autocorrelation and cross-correlation functions are useful for describing the properties of certain hypothesized models, most of the analyses must be performed using sampled data.

From the point of view of classical statistics, this poses a problem because we will typically not have i.i.d. copies of X_t that are available for estimating the covariance and correlation functions.

In the usual situation with only one realization, however, the assumption of stationarity becomes critical.

Estimation of mean

If a time series is stationary, the mean function $\mu_t = \mu$ is constant so that we can estimate it by the *sample mean*:

$$\bar{X} = \frac{1}{n}\sum_{t=1}^{n} X_t$$

In our case, $\mathbb{E}[\bar{X}] = \mu$, and the standard deviation of the estimate is the square root of $\mathrm{var}(\bar{X})$.

Variance of estimation

$$
\begin{aligned}
\mathrm{var}(\bar{X}) &= \mathrm{var}\left(\frac{1}{n}\sum_{t=1}^{n} X_t\right) = \frac{1}{n^2}\mathrm{var}\left(\sum_{t=1}^{n} X_t\right) = \frac{1}{n^2}\mathrm{cov}\left(\sum_{t=1}^{n} X_t, \sum_{s=1}^{n} X_s\right) \\
&= \frac{1}{n^2}[n\gamma_X(0) + (n-1)\gamma_X(1) + (n-2)\gamma_X(2) + \cdots + \gamma_X(n-1) + (n-1)\gamma_X(-1) + \\
&\quad (n-2)\gamma_X(-2) + \cdots + \gamma_X(1-n)) \\
&= \frac{1}{n}\sum_{h=-n}^{n}\left(1 - \frac{|h|}{n}\right)\gamma_X(h)
\end{aligned}
$$

Sampled functions

Definición 7. *The* ***sample autocovariance function*** *is defined as:*

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (X_{t+h} - \bar{X})(X_t - \bar{X})$$

with $\hat{\gamma}(h) = \hat{\gamma}(-h)$ *for* $h = 0, 1, \ldots, n-1$

The sum runs over a restricted range because X_{t+h} is not available for $t + h > n$

Definición 8. *The* ***sample autocorrelation function*** *is defined as:*

$$\hat{\rho} = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}$$

Proposición 9. ***Large-Sample Distribution of the ACF.*** *Under general conditions, if X_t is white noise, then for n large, the sample ACF, $\hat{\rho}_X(h)$, for $h = 1, 2, \ldots, H$, where H is fixed but arbitrary, is approximately normally distributed with zero mean and standard deviation given by:*

$$\sigma_{\hat{\rho}_X} = \frac{1}{\sqrt{n}}$$

Based on the previous result, we obtain a rough method of assessing whether peaks in $\hat{\rho}(h)$ are significant by determining wheter the observed peak is outside the interval $\pm 2/\sqrt{n}$; for a white noise sequence, approximately 95% of the sample ACFs should be within these limits.

The applications of this property develop because many statistical modeling procedures depend on reducing a time series to a white noise series using various kinds of transformations.

Definición 10. *The estimator for the* ***cross-covariance function***, $\gamma_{\mathrm{XY}}(h)$, *is given by*

$$\hat{\gamma}_{\mathrm{XY}}(h) = \frac{1}{n}\sum_{t=1}^{n-h} (X_{t+h} - \bar{X})(Y_t - \bar{Y})$$

Definición 11. *The estimator for the* ***cross-correlation***, $\rho_{\mathrm{XY}}(h)$, *is given by*

$$\hat{\rho}_{\mathrm{XY}}(h) = \frac{\hat{\gamma}_{\mathrm{XY}}}{\sqrt{\hat{\gamma}_X(0)\hat{\gamma}_Y(0)}}$$

Proposición 12. *The large sample distribution of* $\hat{\rho}_{\mathrm{XY}}(h)$ *is normal with mean zero and*

$$\sigma_{\hat{\rho}_{\mathrm{XY}}} = \frac{1}{\sqrt{n}}$$

if at least one of the processes is independent white noise.