Risk Management : 2/5 - Mesure du risque

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How to define a risk measure ?

- $X : \Omega \longrightarrow \mathbb{R}$ is a random **loss**.
- We focus on finite losses, that is the set $\mathcal{X}$ of random variables X such that $\|X\|_\infty < \infty$, where $\|X\|_\infty = \sup_{\omega \in \Omega} |X(\omega)|$.
- $\rho : \mathcal{X} \longrightarrow \mathbb{R}$ will be a capital requirement (or risk measure) associated to the random loss X.
- $F(x) = \mathbb{P}(X \leq x)$ is the cumulative distribution function of X.
- F is not necessarily invertible. Why ? We will use its generalized inverse :

$$F^{-1}(r) = \inf\{x \in \mathbb{R}, F(x) \geq r\}.$$

$F^{-1}(r)$ is the quantile of X of probability r.

Définition

ρ is a **monetary risk measure** *if :*

1. *it is* **monotonic** *:* $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$,
2. *it is* **invariant by translation** *:* $\forall m \in \mathbb{R}$, $\rho(X + m) = \rho(X) + m$.

Volatility is **not** a monetary risk measure.

What are the properties of a risk measure ?

Proposition

If ρ is a monetary risk measure, it is Lipschitz :

$$|\rho(X) - \rho(Y)| \leq \|X - Y\|_\infty.$$

Proof :

- By definition of an infinite norm : $X - Y \leq \|X - Y\|_\infty$.
- By monotony, we get $\rho(X) \leq \rho(Y + \|X - Y\|_\infty)$.
- By invariance by translation, it becomes $\rho(X) \leq \rho(Y) + \|X - Y\|_\infty$.
- So, $\rho(X) - \rho(Y) \leq \|X - Y\|_\infty$ and, by symmetry in X and Y, we also have $\rho(Y) - \rho(X) \leq \|X - Y\|_\infty$.

The financial intuition behind

- The difference of risk measure between two assets is lower than the maximal difference of loss the two assets can simultaneously encounter.
- For example, if stock 1 has a VaR of 1% and stock 2 a VaR of 0.8%, the maximal loss of a portfolio composed of stock 1 (long) minus stock 2 (short) is above 0.2%.
- What about the risk measure of the portfolio, $\rho(X - Y)$, compared to $\rho(X) - \rho(Y)$?

Coherent risk measure

Définition

The risk measure μ is **coherent** *if it satisfies all the following properties :*

- **Normality** *:* $\mu(0) = 0$ *(particular case of positive homogeneity).*
- **Monotonicity** *: if profits $X_1 \leq X_2$ almost surely, then $\mu(X_1) \geq \mu(X_2)$.*
- **Sub-additivity** *(diversification principle) : $\mu(X_1 + X_2) \leq \mu(X_1) + \mu(X_2)$.*
- **Positive homogeneity** *: if $\lambda \geq 0$, then $\mu(\lambda X) = \lambda\mu(X)$.*
- **Invariance by translation** *: If p is a deterministic profit, then $\mu(X + p) = \mu(X) - p$.*

- In other words, a coherent risk measure is a monetary risk measure with sub-additivity and positive homogeneity.
- The notion of coherence may be soften by the introduction of convexity instead of sub-additivity and positive homogeneity : if $\lambda \in [0, 1]$, then $\mu(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda\mu(X_1) + (1 - \lambda)\mu(X_2)$.
- In particular, if μ is monetary and positively homogeneous, then μ convex $\Leftrightarrow$ μ sub-additive.

Spectral risk measure

Rationale : A spectral risk measure is a weighted average of the possible payoffs of a portfolio, with a larger weight for less favourable payoffs. Thereafter, the weight is ϕ.

Définition

Let X be the payoff of a portfolio and F_X its cdf. A spectral measure M_ϕ is a function in $\mathbb{R}$ such that there exists a function $\phi : [0,1] \rightarrow [0,1]$ such that :

$$M_\phi(X) = -\int_0^1 \phi(p) F_X^{-1}(p) dp,$$

where

1. *ϕ is non-increasing,*
2. *ϕ is right-continuous,*
3. *ϕ is integrable and $\int_0^1 \phi(p) dp = 1$*

- We interpret ϕ as a weight due to assumption 3, overweighting bad outcomes due to assumption 1.
- The VaR is not spectral due to assumption 2 (see its distortion measure, in next slides).

Spectral risk measure

- If we consider the distribution of the losses, F_{-X}, which is such that $F_X^{-1}(p) = -F_{-X}^{-1}(1-p)$, then $M_\phi(X) = \int_0^1 \phi(1-p) F_{-X}^{-1}(p) dp$, after a change of variable.
- The expected value is a spectral risk measure, with $\phi = 1$.

Proposition

Every spectral measure is monetary, sub-additive and positively homogeneous. In other words, it is a coherent risk measure.

For discrete observations of price returns, where each outcome is considered as equiprobable, we define a spectral measure as

$$M_\phi(X) = -\frac{1}{N} \sum_{s=1}^{N} \phi_s X_{s:N},$$

where

- $X_{1:N} \leq ... \leq X_{N:N}$ are the order statistics,
- $\forall s,\ \phi_s \geq 0$,
- $\sum_{s=1}^N \phi_s = 1$,
- ϕ_s is non-increasing.

Distortion risk measure

A distortion risk measure is a risk measure defined as a weighted sum of payoffs (or returns) of a portfolio, *whatever the weight*. In particular, a spectral risk measure is a distortion risk measure.

Définition

Let μ be a distortion function : $\mu : [0,1] \to [0,1]$ and μ is non-decreasing and surjective, so that $\mu(0) = 0$ and $\mu(1) = 1$.
The corresponding **distortion risk measure**, *for gains X of cdf F_X is*
$R_\mu(X) = \int_0^1 F_{-X}^{-1}(1-p)d\mu(p) = -\int_0^1 F_X^{-1}(p)d\mu(p)$.

Proposition

Let M_ϕ be a spectral risk measure and R_μ a distortion risk measure, with μ differentiable. Then :

$$M_\phi = R_\mu \Leftrightarrow \phi = \mu'.$$

In such a case and if ϕ is differentiable, then $\mu'' \leq 0$.

Théorème

The distortion μ is concave (overweight of greater losses) $\Leftrightarrow$ R_μ is a coherent risk measure.

Distortion risk measure

Usual risk measures :

- **Value at risk** with confidence α : distortion $\mu(x) = \mathbb{1}_{[1-\alpha,1]}(x)$, not differentiable, not concave : not spectral.
- **Expected shortfall** with confidence α : distortion $\mu(x) = \mathbb{1}_{[1-\alpha,1]}(x) + \mathbb{1}_{[0,1-\alpha)}(x)\frac{x}{1-\alpha}$, not differentiable but concave : coherent, and even spectral.
- **Expected loss** : distortion $\mu(x) = x$, so that $\mu' = 1 > 0$: spectral.

What about the properties of usual risk measures ?

The VaR :

- is monetary,
- is positively homogeneous,
- is **not** necessarily convex : take losses X and Y iid Bernoulli(p) ; then

$$VaR_\alpha\left(\frac{X+Y}{2}\right) = \begin{cases} 1 & \text{if } \alpha \geq 1-p^2 \\ 0 & \text{if } \alpha \leq (1-p)^2 \\ 1/2 & \text{else,} \end{cases}$$

  whereas

$$\frac{1}{2}\left(VaR_\alpha(X) + VaR_\alpha(Y)\right) = \begin{cases} 1 & \text{if } \alpha \geq p \\ 0 & \text{else.} \end{cases}$$

- is therefore neither sub-additive nor coherent.

Expected shortfall

The expected shortfall is coherent and even spectral. That is why the regulator finally prefers this risk measure.

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2. Aspects empiriques

Inverse distribution function

Définition

Let X be a random variable (say a price return) and F be its cumulative distribution function : $F(x) = \mathbb{P}(X \leq x)$. The **generalized inverse** *of F is*

$$F^{-1}(p) = \inf\{x \in \mathbb{R}, F(x) \geq p\}.$$

- The generalized inverse is also called the quantile.
- If F is *continuous* and *strictly increasing*, the generalized inverse is the standard inverse.
- F^{-1} is nondecreasing.
- $F^{-1}(F(x)) \leq x$ and $F(F^{-1}(p)) \geq p$.
- If a random variable U is uniform in $[0, 1]$, then $F^{-1}(U)$ is distributed according to F. This is used to generate pseudo-random numbers in F.

Value at risk

Définition

Let X be a random variable depicting a gain. Then, the value at risk with confidence α is :

$$VaR_\alpha(X) = -F_X^{-1}(1-\alpha).$$

Alternatively, if Y is a loss ($Y = -X$), $VaR_\alpha(Y) = F_Y^{-1}(\alpha)$.

- In other words, the VaR at 99% is the quantile at 99% of the losses or the opposite of the quantile at 1% of the gains.
- The VaR is the maximum expected loss for a given confidence level.
- For successive gains $X_1, ..., X_n$, we can either assume that they are identically distributed (and even independent) or not, for example with conditional cdf $F(X_i | X_{i-1}, ..., X_{i-d})$ or with any model of dynamic.

Estimating Value at risk (1/2)

Several notions of VaR depending on their estimation method.

1. **Historical VaR** : based directly on past values (not used for estimating, but simulation directly in this past dataset). Drawbacks : equal weight to each observation and static (possibility to weight data to have something more dynamic with more weight on recent observations), often iid assumption.
   - **Empirical VaR** : empirical quantile, but need big amount of data (should have a number of observations $>> 1/(1-\alpha)$).
   - **VaR based on empirical distribution** : estimation of the whole distribution, when less data than needed for empirical quantiles (e.g., 100 data for $\alpha = 99.9\%$), and then quantile on the empirical distribution. Empirical distribution could be **parametric** or **non-parametric** (would be better), like a kernel-based distribution.
   - **EVT VaR** : much more accurate for tail behaviour than a Kernel approach, it leverages on the fact that the distribution of extreme events should be close to a known parametrized distribution, whatever the distribution of the returns (restricts the difficulty of the problem).
2. ...

Estimating Value at risk (2/2)

1. ...
2. **Monte-Carlo VaR** : based on a dynamic, a model estimated on historical data.
    - **parametric VaR** : parametric model such as
        - iid Gaussian (or other) returns : sounds arbitrary, not that efficient and does not need Monte-Carlo (closed formulas). But could be used to model factors (and then Monte-Carlo approach useful). Difficulty to identify factors (econometric approach).
        - parametric dynamic, such as GARCH, which is more accurate.
    - **non-parametric VaR** : already proposed non-parametric models, here Monte Carlo is a method and not a model, it can then use any non-parametric model.

Non-parametric distribution

An estimator $\hat{f}$ of a density should check : $\forall x \in \mathbb{R}, \hat{f}(x) \geq 0$, and $\int_{-\infty}^{+\infty} \hat{f}(x)dx = 1$. From a series of observed price returns $X_1, ..., X_n$, the probability density could be estimated by :

- A **histogram** (not continuous, not invariant by translation), with a discretization $a_1 < ... < a_m$: if $x \in [a_i, a_{i+1})$, then

$$\hat{f}(x) = \frac{1}{n(a_{i+1} - a_i)} \sum_{j=1}^{n} \mathbb{1}_{[a_i, a_{i+1})}(X_j).$$

- A **kernel density**, where the Kernel K, a density with cdf $\mathcal{K}$ can be, for example, $K(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$, with a smoothing parameter $h > 0$ indicating the smoothness/robustness :

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right) \text{ and } \hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{K}\left(\frac{x - X_i}{h}\right).$$

  Drawback : fixed-size resolution (h), thus spurious effects for tails and smoothing of essential elements in the main part of the distribution.

- An **adaptive (or variable) kernel density** :

$$\hat{f}(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{hd_{i,k}} K\left(\frac{x - X_i}{hd_{i,k}}\right) \text{ and } \hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{K}\left(\frac{x - X_i}{hd_{i,k}}\right),$$

  where $d_{i,k}$ is the Euclidean distance between X_i and its k-th nearest neighbour among the $n-1$ other observations, $X_1, ..., X_{i-1}, X_{i+1}, ..., X_n$.

Non-parametric distribution

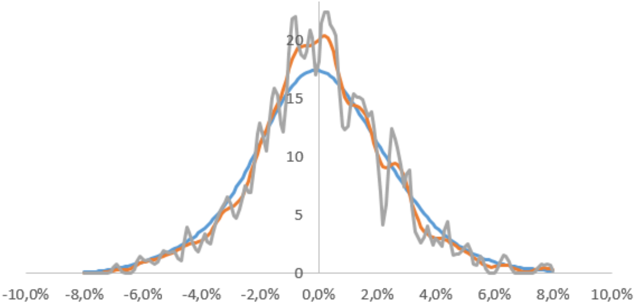

FIGURE – Kernel density of historical daily price returns (Natixis, 2015-2016) for $h = 0.01$ (smooth one), $h = 0.003$, and $h = 0.001$ (most erratic one).

Critiques de la VaR

La VaR est plus robuste que la volatilité et est un standard de marché, mais :

- En général la VaR n'est pas une mesure de risque cohérente car elle n'est pas sous-additive (ne reflète pas bien la diversification). Cela peut pousser une banque à des astuces comptables : supposons que le superviseur d'une banque impose d'immobiliser du capital par trading desk, correspondant à la VaR à 95% de chaque desk. Comment la banque définit-elle ses desks pour minimiser le capital immobilisé sachant qu'il existe des pairs de desks pour lesquelles $VaR_\alpha(Y_1 + Y_2)$ peut être plus grand que $VaR_\alpha(Y_1) + VaR_\alpha(Y_2)$?
- La VaR ne donne aucune indication sur la taille potentielle de la perte au-dessus d'elle (ce n'est qu'une quantile) ; une solution consiste à observer plusieurs quantiles (VaR spectrale).

L'*expected shortfall* n'a pas ces deux inconvénients (avec hypothèse de distribution de pertes absolument continue pour ce qui concerne la cohérence).

Expected shortfall

Let X be the gain of the portfolio, $\alpha \in [0,1]$ a confidence level. We define several risk measures depicting the average of the $1-\alpha$ worst losses :

- the **worst conditional expectation** : $WCE_\alpha(X) = \sup\{\mathbb{E}[-X|A], \mathbb{P}(A) \geq 1-\alpha\}$,
- the **tail conditional expectation** : $TCE_\alpha = \mathbb{E}\left[-X|X \leq -VaR_\alpha(X)\right]$,
- the **average VaR** (AVaR), or **expected shortfall** (ES), or **conditional VaR** : $ES_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_r(X)dr = \frac{1}{1-\alpha}\int_\alpha^1 F_{-X}^{-1}(r)dr$.

Théorème

We have : $AVaR_\alpha(X) \geq WCE_\alpha(X) \geq TCE_\alpha(X) \geq VaR_\alpha(X)$. Moreover, if the distribution of X is without atoms ($\forall\varepsilon > 0$, $\exists(A_i)$, a finite partition of the set Ω such that $\mathbb{P}(A_i) \leq \varepsilon$) : $AVaR_\alpha(X) = WCE_\alpha(X) = TCE_\alpha(X)$.

Théorème

If $(\Omega, \mathcal{F}, \mathbb{P})$ is without atoms, then ES_α is the smallest coherent risk measure, continuous by below (that is for $X_n \nearrow X$, $\rho(X_n) \to \rho(X)$), and invariant by the distribution (ρ is invariant by distribution if $\rho(X) = \rho(Y)$ when X and Y have the same distribution), which dominates VaR_α.

Risk measures based on moments

- Variance and volatility (square root of variance) are not monetary risk measures because they are neither monotonic (X uniform in $[0,1]$, Y constant equal to 2, then $X < Y$ but $\rho(X) > \rho(Y)$) nor invariant by translation ($vol(X+1) = vol(X) \neq vol(X)+1$).
- Nevertheless, massively used to monitor the *risk* of a portfolio.
- Could use other moments too for asymmetry (skewness) or tail behaviour (kurtosis).

About cumulative volatility

We want to know the contribution of one asset to the global volatility, with a linear combination.

- $Var(X+Y) = Var(X) + Var(Y)$ if zero correlation (not independence), but not linear for volatility $Vol(X+Y) = \sqrt{Vol(X)^2 + Vol(Y)^2}$.
- In general, if we note $Vol_X = \sigma_X$ and ρ the correlation, $\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2\rho\sigma_X\sigma_Y$ and thus $\sigma_{X+Y} = \frac{\sigma^2_X + \rho\sigma_X\sigma_Y}{\sigma_{X+Y}} + \frac{\sigma^2_Y + \rho\sigma_X\sigma_Y}{\sigma_{X+Y}}$.
- For n assets, $\sigma^2_{\sum_{i=1}^n X_i} = \sum_{i=1}^n \sigma^2_{X_i} + \sum_{i \neq j} \rho_{i,j}\sigma_{X_i}\sigma_{X_j}$. Therefore the volatility can be written as a linear combination of n volatilities relative to each asset : $\sigma_{\sum_{i=1}^n X_i} = \sum_{i=1}^n \left[\frac{\sigma^2_{X_i} + \sum_{j \neq i} \rho_{i,j}\sigma_{X_i}\sigma_{X_j}}{\sigma_{\sum_{j=1}^n X_j}} \right]$.

Deviation risk measure

A generalization of volatility, quantifies deviation and not downside risk (differences with coherent risk measure, in addition to monetary and monotonicity, which disappears, are in bold) :

1. Normality : $\mu(0) = 0$ (particular case of positive homogeneity).
2. **Positivity** : $\mu(X) > 0$ is X is not constant, $\mu(X) = 0$ else.
3. Sub-additivity (diversification principle) : $\mu(X_1 + X_2) \leq \mu(X_1) + \mu(X_2)$.
4. Positive homogeneity : if $\lambda \geq 0$, then $\mu(\lambda X) = \lambda \mu(X)$.
5. **Invariance by translation** : If p is a deterministic profit, then $\mu(X + p) = \mu(X)$.

Examples :

- volatility (standard deviation) : $\sqrt{\mathbb{E}[(X - \mathbb{E}[X])^2]}$,
- lower and upper semi-deviation : $\sigma_-(X) = \sqrt{E[(X - EX)_-{}^2]}$ and $\sigma_+(X) = \sqrt{E[(X - EX)_+{}^2]}$, where $[X]_- = \max\{0, -X\}$ and $[X]_+ = \max\{0, X\}$,
- mean absolute deviation : $MAD(X) = E(|X - EX|)$.

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2. Aspects empiriques

Defining the entropic risk measure

The exponential or entropic risk measure $e_\gamma(X)$, for a risk aversion $\gamma > 0$ and for a given random variable X representing a profit, say a price return of a financial asset between two dates, is defined by :

$$e_\gamma(X) = \frac{1}{\gamma} \ln \mathbb{E}[e^{-\gamma X}] = \frac{1}{\gamma} \ln M_X(-\gamma), \tag{1}$$

where M_X is the moment-generating function of X. Symmetrically, if a random variable Y represents a risk, say a loss, the corresponding profit X is equal to $-Y$ and the corresponding entropic risk measure is $e_\gamma(-Y)$.

The entropic risk measure $e_\gamma(Y)$ of a random variable Y representing a risk is also the insurance premium against the risk Y in an exponential utility framework. For the risk Y and a utility function u, the premium P is the capital required such that the risk Y is properly hedged, that is to say the expected utility of the risk hedged by its premium is zero :

$$\mathbb{E}[u(P - Y)] = 0.$$

In the case of the entropic risk measure, $e_\gamma(-Y)$ is the premium for an exponential utility function of risk aversion γ :

$$u : x \mapsto \frac{1}{\gamma}\left(1 - e^{-\gamma x}\right).$$

Properties of the entropic risk measure

1. Since the entropic risk measure is related to the moment-generating function of the profits, it is based on all its moments. This is an evident advantage over the simplistic volatility as a risk measure.
2. The entropic risk measure is convex. This implies that diversification does not increase the risk : when two portfolios are merged, the resulting risk is not higher than the sum of the risk of each portfolio. Other popular risk measures, like the value-at-risk (VaR), do not have this important property.
3. Neither the entropic risk measure nor the VaR are coherent risk measures. In particular, the entropic risk measure lacks positive homogeneity. A risk measure ρ is positively homogeneous if $\forall\lambda \geq 0, \rho(\lambda X) = \lambda\rho(X)$. The VaR and the expected shortfall, which is convex as well and thus coherent, are positively homogeneous. The example of a Gaussian profit X of mean μ and variance σ^2 shows that the entropy is not positively homogeneous. Indeed, the entropic risk measure is then equal to $-\mu + \sigma^2\gamma/2$. But if the variable X is doubled, the entropy is not doubled since it is equal to $-2\mu + 2\sigma^2\gamma$. However, positive homogeneity of risk measures is not a realistic property since in practice the risk of a position is often non-linearly related to the held amount, due to liquidity risk.

Coherent risk measure

The risk measure μ is coherent if it satisfies all the following properties :

- Normality : $\mu(0) = 0$.
- Monotonicity : if profits $X_1 \leq X_2$ almost surely, then $\mu(X_1) \geq \mu(X_2)$.
- Sub-additivity (diversification principle) : $\mu(X_1 + X_2) \leq \mu(X_1) + \mu(X_2)$.
- Positive homogeneity : if $\lambda \geq 0$, then $\mu(\lambda X) = \lambda\mu(X)$.
- Invariance by translation : If p is a deterministic profit, then $\mu(X + p) = \mu(X) - p$.

The notion of coherence may be soften by the introduction of convexity instead of sub-additivity and positive homogeneity : if $\lambda \in [0, 1]$, then $\mu(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda\mu(X_1) + (1 - \lambda)\mu(X_2)$.

Scaling rule of the entropic risk measure

- Scaling rule (when horizon multiplied by t) of the VaR for i.i.d. Gaussian returns : $\sqrt{t}$.
- Alternative example of dependent Gaussian : scaling in t^H for a H-fractional Brownian motion.
- Alternative example of i.i.d. non-Gaussian : scaling in t for Cauchy process.

Instead of that, the entropic risk measure has a unique scaling rule for i.i.d. returns, whether they are Gaussian or not : the log-return until time t is the sum of t i.i.d. returns of horizon 1, $X_1, ..., X_t$, and $e_\gamma\left(\sum_{i=1}^t X_i\right) = \sum_{i=1}^t e_\gamma\left(X_i\right) = te_\gamma\left(X_1\right)$, so that the scaling factor is t.

Risk aversion

The risk measure is increasing with the risk aversion γ. However, determining the risk aversion of an agent may seem as arbitrary as choosing a confidence level α for a VaR. In fact, both parameters can be linked. For example, assuming Gaussian price returns X of mean μ and variance σ^2, the corresponding entropic risk measure and VaR are respectively $-\mu + \sigma^2\gamma/2$ and $-\mu + \sigma G^{-1}(1-\alpha)$. Both are equal if $\gamma = 2G^{-1}(1-\alpha)/\sigma$. We can therefore define a risk aversion for a type of asset. For instance, concerning a stock market for which we assume a long-term annual volatility of 20%, values of the VaR parameter α equal to 5%, 1% and 0.01% are respectively associated to a risk aversion γ equal to 16.45, 23.26 and 30.90.

Estimating the risk aversion

- Price of a stock is discounted sum of future dividends : does not take into account variations of preference over different horizons. On the opposite, options exist for different horizons : they are well-suited to estimate risk aversion.
- Estimate mean of a distribution on stocks and all the density on options : $q(S_T) = e^{r(T-t)} \left. \frac{\partial^2 C(t,S_t,K,T)}{\partial K^2} \right|_{K=S_T}$.
- This distribution is estimated at a single date, not on historical data : it is therefore much more responsive to changing market expectations.
- ...

Estimating the risk aversion

- ...
- Unfortunately, the option-implied density is risk-neutral, which may not be consistent with the actual forecast of the most representative investor (the market). The difference between risk-neutral (q) and physical (p) distributions is due to risk aversion (λ for representative utility function U) or equivalently to the **stochastic discount factor** (aka **pricing kernel** : $\zeta(S_T)$) :

$$\frac{p(S_T)}{q(S_T)} = \lambda \frac{U'(S_T)}{U'(S_t)} = \zeta(S_T).$$

- If two among p, q, U are known, the third one is inferred. In practice, only q is known. Solutions : assume stationarity of p or stationarity of U (parametric function, depending on λ, which in this case is constant).
- The risk aversion is then the λ maximizing the forecast ability over historical data.

Stochastic discount factor

- For assets $i \in \{1, ..., n\}$ of pay-off π_i, initial price p_i and return $R_i = \pi_i / p_i$, the stochastic discount factor ζ is such that, $\forall i \in \{1, ..., n\}, \mathbb{E}(\zeta \pi_i) = p_i$, in which $\mathbb{E}$ is expressed in the risk-neutral distribution.
- As a consequence $\forall i, j$, $\mathbb{E}[\zeta(R_i - R_j)] = 0$.
- In the risk-neutral distribution, the risk-free rate R_f is deterministic. Therefore, $\mathbb{E}(\zeta) = 1/R_f$.
- As $\text{cov}(X, Y) = \mathbb{E}[X, Y] - \mathbb{E}[X]\mathbb{E}[Y]$, we have $1 = \text{cov}(\zeta, R_i) + \mathbb{E}(\zeta)\mathbb{E}(R_i)$. In particular :

$$\mathbb{E}(R_i) - R_f = -R_f \text{cov}(\zeta, R_i),$$

which is the risk premium of asset i.

Bibliographie

- Bliss, Panigirtzoglou (2004), *Option-implied risk aversion estimates*