Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2021.

∗ If there is any problem, please contact TA Haolin Zhou.

∗ Name: Zenan Li Student ID: 519021911033 Email: Emiyali@sjtu.edu.cn

1. Prove that for any integer $n > 2$, there is a prime p satisfying $n < p < n!$. (Hint: consider a prime factor p of $n! - 1$ and prove by contradiction)

   Proof. Assume $n > 2$ and there does not exist a prime p that satisfying $n < p < n!$. It means that $p \leq n$ or $p \geq n!$, for any prime number p.

   We have proved in class that $\forall n \in \mathbb{N}$ with $n \geq 2$, it has prime factors. Yet the general, we could take a prime factor of $n! - 1$ for analysis. Since $p \leq n! - 1$, we could tell from the assumption that $p \leq n$.

   However, for any integer $2 \leq k \leq n$, we have:

   $$n! - 1 = 1 \times 2 \times 3 \times ... \times k \times (k+1) \times ... \times n - 1 = k \times M_k - 1$$

   Therefore, we could get that $k \nmid n! - 1$ (for any $2 \leq k \leq n$), which contradicts the assumption that $p \mid n! - 1, and\ p \leq n$ (the prime number p must be larger than 1). □

2. Use the minimal counterexample principle to prove that for any integer $n \geq 7$, there exists integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 2 + j_n \times 3$.

   Proof. If $n = i_n \times 2 + j_n \times 3$ is not true for every $n \geq 7$, then there are values of n which could not be written as $i_n \times 2 + j_n \times 3$, and there must be a smallest such value, say $n = k$.

   Since $7 = 2 \times 2 + 1 \times 3$, we have $k \geq 8$, and $k - 1 \geq 7$.

   Since k is the smallest value for which $n \neq i_k \times 2 + j_k \times 3$, $k - 1 = i_{k-1} \times 2 + j_{k-1} \times 3$. Besides, if we set $j_k = 0$, and i_k to be any possible natural number, we could get that k is odd. So $k - 1$ is even, j_{k-1} could be 0, and $i_{k-1} = \frac{k-1}{2} \geq 4$.

   Hence, we could get that:

   $$\begin{aligned} k &= (k-1) + 1 \\ &= i_{k-1} \times 2 + 1 \\ &= (i_{k-1} - 1) \times 2 + 1 \times 3 \\ &= i_k \times 2 + j_k \times 3 \end{aligned}$$

   The equation above shows that k could be written as $i_k \times 2 + j_k \times 3$. We have derived a contradiction, which allows us to conclude that our original assumption is false. □

3. Suppose the function f be defined on the natural numbers recursively as follows: $f(0) = 0$, $f(1) = 1$, and $f(n) = 5f(n-1) - 6f(n-2)$, for $n \geq 2$. Use the strong principle of mathematical induction to prove that for all $n \in N$, $f(n) = 3^n - 2^n$.

   Proof. Define $P(n)$ be the statement that "$f(n) = 3^n - 2^n$". We will try to prove that $P(n)$ is true for every $n \in \mathbb{N}$.

   Basis step. $P(0)$ is true, since $f(0) = 3^0 - 2^0 = 0$. $P(1)$ is true, since $f(0) = 3^1 - 2^1$ is true.

   Induction hypothesis. For $k \geq 0$ and $0 \leq n \leq k$, $P(n)$ is true.

Proof of induction step. Let's prove $P(k+1)$. We have the following formula:

$$\begin{aligned} f(k+1) &= 5f(k) - 6f(k-1) \\ &= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1}) \\ &= 3 \times 3^k - 2 \times 2^k \\ &= 3^{k+1} - 2^{k+1} \end{aligned}$$

So we get that $f(k+1) = 3^{k+1} - 2^{k+1}$. Thus, $P(k+1)$ is true. $\square$

4. An n-team basketball tournament consists of some set of $n \geq 2$ teams. Team p beats team q iff q does not beat p, for all teams $p \neq q$. A sequence of distinct teams p_1, p_2,..., p_k, such that team p_i beats team p_{i+1} for $1 \leq i < k$ is called a ranking of these teams. If also team p_k beats team p_1, the ranking is called a *k-cycle*.

   Prove by mathematical induction that in every tournament, either there is a "champion" team that beats every other team, or there is a 3-cycle.

Proof. Let $P(n)(n \geq 2)$ be the statement that in any n-team tournament, either there is a "champion" team that beats every other team, or there is a 3-cycle. We prove $P(n)$ is true for $n \geq 2$ by induction.

First of all, it's obvious that any n-team tournament could have a "champion" team. So what we should prove is that "a 3-cycle exists when there is not a champion team in a tournament".

Basis step. As for $n = 2$, there must exist a champion team, so $P(2)$ is true.

Induction hypothesis. Assume $P(k)$ is true for some $k \geq 2$.

Proof of induction step. Let's prove $P(k+1)$.

Using nodes to represent different teams, we could construct a directed graph where "A $\rightarrow$ B" means team A beats team B. Divide the $k+1$ nodes into k original nodes and an additional node and we could get Figure 1. Besides, We could have two typical cases as shown in Figure 2 and Figure 3.

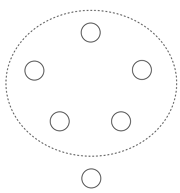

Figure 1: Directed Graph

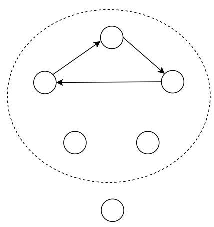

Figure 2: Case 1

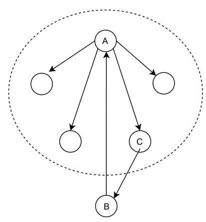

Figure 3: Case 2

Case 1. If there already exists a 3-cycle in the original nodes, $P(k+1)$ is obviously true.

Case 2. If there does not exist a 3-cycle in the original nodes. From the hypothesis, there should be an original "champion" team instead. However, what we should prove is that "a 3-cycle exists when there is not a champion team in a tournament". So in **Case 2**, the additional team must have beaten the original champion, or there will be a "champion" team in the $(k+1)$-team (contrary to the proposition).

We could see in Figure 3 that the additional team B beats the original "champion" A. However, since B could not be the new "champion" either, there must have been a team C in the original nodes that beats it. Hence, we've got a 3-cycle "B $\to$ A $\to$ C $\to$ A" in the figure, which proves that $P(k+1)$ is true.

Conclusion. Since in both cases $P(k+1)$ is true, we've proved the proposition's correctness. $\square$

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