# Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2021.

$*$ If there is any problem, please contact TA Haolin Zhou.

$*$ Name:Yanjie Ze   Student ID:519021910706   Email: zeyanjie@sjtu.edu.cn

1. Prove that for any integer $n > 2$, there is a prime $p$ satisfying $n < p < n!$. (Hint: consider a prime factor $p$ of $n! - 1$ and prove by contradiction)

   **Proof.** Assume for any integer $n > 2$, there is no prime $p$ satisfying $n < p < n!$, which means:

   $$\forall\ integer\ m \in (n, n!), its\ prime\ factor \in [2, n].$$

   However, for the integer $n! - 1$, its prime factor $\notin [2, n]$ because the integer $n!$'s all prime factors $\in [2, n]$.

   Therefore, $n! - 1$ has prime factors $\in\ (n, n!)$, which contradicts the assumption that there is no prime $p$ satisfying $n < p < n!$. □

2. Use the minimal counterexample principle to prove that for any integer $n \geq 7$, there exists integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 2 + j_n \times 3$.

   **Proof.** If there exists a integer $n \geq 7$ which makes us unable to find $i_n \geq 0$ and $j_n \geq 0$ to satisfy $n = i_n \times 2 + j_n \times 3$, assume the minimal interger is $n = k$.

   Since $n = 7 = 2 \times 2 + 1 \times 3$ and $n = 8 = 4 \times 2 + 0 \times 3$, $k \geq 9$.

   Thus the number $k - 2$ satisfies the equation:

   $$k - 2 = i_{k-2} \times 2 + j_{k-2} \times 3$$

   Therefore:

   $$k = (i_{k-2} + 1) \times 2 + j_{k-2} \times 3$$

   which contradicts the assumption and allows us to conclude our original assumption is false.

   □

3. Suppose the function $f$ be defined on the natural numbers recursively as follows: $f(0) = 0$, $f(1) = 1$, and $f(n) = 5f(n-1) - 6f(n-2)$, for $n \geq 2$. Use the strong principle of mathematical induction to prove that for all $n \in N$, $f(n) = 3^n - 2^n$.

   **Proof. Induction hypothesis.** For $k \geq 2$ and $2 \leq n \leq k$, $f(n) = 3^n - 2^n$.

   **Proof of induction step.**

   For $n = k + 1$:

   $$\begin{aligned} f(k+1) &= 5f(k) - 6f(k-1) \\ &= 5 \times (3^k - 2^k) - 6 \times (3^{k-1} - 2^{k-1}) \\ &= 3^{k+1} - 2^{k+1} \end{aligned}$$

   Therefore, for all $n \in N$, $f(n) = 3^n - 2^n$. □

4. An $n$-team basketball tournament consists of some set of $n \geq 2$ teams. Team $p$ beats team $q$ iff $q$ does not beat $p$, for all teams $p \neq q$. A sequence of distinct teams $p_1$, $p_2$,..., $p_k$, such that team $p_i$ beats team $p_{i+1}$ for $1 \leq i < k$ is called a ranking of these teams. If also team $p_k$ beats team $p_1$, the ranking is called a *$k$-cycle*.

   Prove by mathematical induction that in every tournament, either there is a "champion" team that beats every other team, or there is a 3-cycle.

   **Proof.** Define *P(n)* be the statement that "or an $n$-team basketball tournament, either there is a 'champion' team that beats every other team, or there is a 3-cycle.

   **Basic step.** For $n = 2$, given the team $p_1$ and the team $p_2$, if $p_1$ beats $p_2$ then $p_1$ is the "champion". Otherwise, $p_2$ is the "champion". Therefore, *P(2)* is true.

   **Induction hypothesis.** For $n = k$, *P(k)* is true, which means for teams $\{p_1, p_2, ..., p_n\}$,either there is a 'champion' team that beats every other team, or there is a 3-cycle.

   **Proof of induction step.** For $n = k + 1$,there are $k + 1$ teams $\{p_1, p_2, ..., p_n, p_{n+1}\}$ in the tournament and there are generally 2 cases to be considered.

   **Case 1: there exists a 3-cycle in** $\{p_1, p_2, ..., p_n\}$**..** In this case, the new team $p_{n+1}$ will not change the 3-cycle.

   **Case 2: there exists a "champion" in** $\{p_1, p_2, ..., p_n\}$. Assume the champion is $p_1$. For $p_{n+1}$, it either beats the champion or gets defeated by the champion. There are 3 subcases.

   **Case 2.1:** If $p_{n+1}$ beats $p_1$ and gets defeated by another team, $p_j(1 < j \leq n)$, then there would exist a 3-cycle: $p_1, p_j, p_{n+1}$.

   **Case 2.2:** If $p_{n+1}$ beats $p_1$ and all other teams, then $p_{n+1}$ is the new champion.

   **Case 2.3:** If $p_{n+1}$ gets defeated by $p_1$, then we can view the teams $\{p_2, p_3, ..., p_n, p_{n+1}\}$ as a whole, which corresponds *P(n)*, and $p_1$ can be viewed as the new team. Therefore, we transform **Case 2.3** into the cases discussed before(**Case 1, 2.1, 2.2**), which have been proved correct.

   Thus, *P(k+1)* is true.

   $\square$

**Remark:** You need to include your .pdf and .tex files in your uploaded .rar or .zip file.