Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2021.

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1. Prove that for any integer $n > 2$, there is a prime p satisfying $n < p < n!$. (Hint: consider a prime factor p of $n! - 1$ and prove by contradiction)

   Proof. Since $n > 2$, $n!$ must have factors n and 2. Therefore, $n! \geqslant 2n = n + n > n + 1$, then we have $n! - 1 > n > 2$. Additionally, we have a general fact that every integer greater than 1 must have a prime factor, thus $n! - 1$ has a prime factor p with $p \leqslant n! - 1$, that is, $p < n!$.

   Another side, we use the proof of contradiction. If $p \leqslant n$, then due to $p \geqslant 2$, p is a factor of $n!$. Additionally, we have a general fact that there is no integer that is the factor of both n and $n - 1$, where $n \geqslant 2$, except for 1, which is in contradiction to $p \geqslant 2$. So we have $p > n$.

   Then we conclude $n < p < n!$. □

2. Use the minimal counterexample principle to prove that for any integer $n \geq 7$, there exists integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 2 + j_n \times 3$.

   Proof. We first take the statement as $P(n)$.

   So, **Basic step**, $P(n)$: for any integer $n \geqslant 7$, there exist integers $i_n \geqslant 0$ and $j_n \geqslant 0$, such that $n = i_n \times 2 + j_n \times 3$. And we check that $P(7) = 2 \times 2 + 1 \times 3$.

   For **Induction step**, we use minimal counterexample principle, assuming for $7 \leqslant n \leqslant k - 1$, $P(n)$ all holds until $n = k$. So due to the induction hypothesis, we have

$$k - 1 = i_{k-1} \times 2 + j_{k-1} \times 3,$$

   that is,

$$k = i_{k-1} \times 2 + j_{k-1} \times 3 + 1,$$

   where $i_{k-1} \geqslant 0$ and $j_{k-1} \geqslant 0$. However, we can see that i_{k-1} and j_{k-1} cannot both equal to 0 as the following shows. We prove it by contradiction: if so, then $k - 1 = 0$, giving $k = 1$ in contradiction to the condition $k \geqslant 8$. So we conclude that either $i_{k-1} \geqslant 1$ or $j_{k-1} \geqslant 1$, which means either $i_{k-1} - 1 \geqslant 0$ or $j_{k-1} - 1 \geqslant 0$.

   Then we can divide the proof into separate cases.

   (i) If $i_{k-1} - 1 \geqslant 0$, then we can have

$$k = (i_{k-1} - 1) \times 2 + (j_{k-1} + 1) \times 3,$$

   giving $i_k = i_{k-1} - 1 \geqslant 0$ and $j_k = j_{k-1} + 1 \geqslant 1 \geqslant 0$. Here $P(k)$ holds, where we derived a contradiction.

   (ii) And if $j_{k-1} - 1 \geqslant 0$, then we can have

$$k = (i_{k-1} + 2) \times 2 + (j_{k-1} - 1) \times 3,$$

   giving $i_k = i_{k-1} + 2 \geqslant 2 \geqslant 0$ and $j_k = j_{k-1} - 1 \geqslant 0$. Also contradiction!

   Therefore, our original assumption is false. Thus for any integer $n \geqslant 7$, there exist integers $i_n \geqslant 0$ and $j_n \geqslant 0$, such that $n = i_n \times 2 + j_n \times 3$. □

3. Suppose the function f be defined on the natural numbers recursively as follows: $f(0) = 0$, $f(1) = 1$, and $f(n) = 5f(n-1)-6f(n-2)$, for $n \geq 2$. Use the strong principle of mathematical induction to prove that for all $n \in N$, $f(n) = 3^n - 2^n$.

Proof. We first take the statement as $P(n)$.

Basic step. It is true that $f(0) = 0 = 3^0 - 2^0 = 1 - 1$ and $f(1) = 1 = 3^1 - 2^1 = 3 - 2$. It is also true that $f(2) = 5f(1) - 6f(0) = 5 = 3^2 - 2^2 = 9 - 4$.

Induction hypothesis. $k \geqslant 1$, and for every n with $0 \leqslant n \leqslant k$, $f(n) = 3^n - 2^n$.

Statement to be show in induction step. $f(k+1) = 3^{k+1} - 2^{k+1}$.

Proof of induction step. Since $k - 1 \geqslant 0$, we also have definition for $f(k-1)$. Then $f(k+1) = 5f(k) - 6f(k-1)$, using our induction hypothesis, we have

$$f(k+1) = 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1}).$$

So we can get

$$f(k+1) = 5 \times 3^k - 2 \times 3^k - 5 \times 2^k + 3 \times 2^k = 3^{k+1} - 2^{k+1}.$$

Therefore, $P(k+1)$ is true. □

4. An n-team basketball tournament consists of some set of $n \geq 2$ teams. Team p beats team q iff q does not beat p, for all teams $p \neq q$. A sequence of distinct teams p_1, p_2,..., p_k, such that team p_i beats team p_{i+1} for $1 \leq i < k$ is called a ranking of these teams. If also team p_k beats team p_1, the ranking is called a *k-cycle*.

Prove by mathematical induction that in every tournament, either there is a "champion" team that beats every other team, or there is a 3-cycle.

Proof. We also first take the statement as $P(n)$.

Basic step. We first check $P(2)$ which is that there are p and q teams. Then since the hypothesis in the question that team p beats team q iff q does not beat p for all teams $p \neq q$, there must be a "champion" team that beats the other.

We can also check $P(3)$ which is that there are p, q and r three teams. Without loss of generality, we can assume p beats q. Then (i) if p also beats r, then p will be the "champion" team. When it's not the case, then that means r beats p, leaving q and r where (ii) if q beats r then there exists a 3-cycle, (iii) else then r will be the "champion" team. The pictures below show the 3 cases.

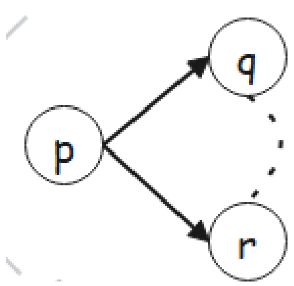

Fig. 1: case i for $P(3)$

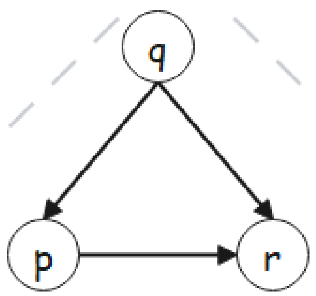

Fig. 2: case ii for $P(3)$

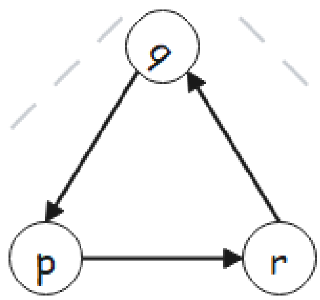

Fig. 3: case iii for $P(3)$

Induction hypothesis. $k \geqslant 2$, and $P(k)$ holds, which means in a tournament with k teams participating, either there is a "champion" team or there is a 3-cycle.

Statement to be show in induction step. $P(k+1)$ holds.

Proof of induction step. For a tournament with $k+1$ teams participating, we can take it as a tournament with k teams p_1, p_2,..., p_k participating and then another team p_{k+1} goes to play with each team additionally. So due to our induction hypothesis, p_1, p_2,..., p_k either have a "champion" team or have a 3-cycle. If the latter, then additional team p_{k+1} participation doesn't influence that 3-cycle, so that 3-cycle still exists in the tournament with $k+1$ teams participating. For the "champion" case, without loss of generality, we assume p_1 was the original "champion". If the p_{k+1} beats all the original k teams, then p_{k+1} becomes the "champion". If not, then when p_{k+1} doesn't beat p_1, p_1 still is the "champion", but when p_{k+1} doesn't beat p_i where $2 \leqslant i \leqslant k$ but beating p_1, then it's easy to see p_1, p_i and p_{k+1} form a 3-cycle. Whichever the case is, either there is a "champion" team or there is a 3-cycle. $\square$

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