Hard Problems in CS499

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Set theory

Problem 6. *A and B are countable sets. Prove that*

1. $A \cup B$ *is countable*
2. $A \times B$ *is countable*

Tall or Width

Problem 4. *Prove the following strengthening of the* ***Erdös-Szekeres Lemma****: Let* κ, ℓ *be natural numbers. Then every sequence of real numbers of length* $\kappa\ell + 1$ *contains an nondecreasing subsequence of length* $\kappa + 1$ *or a decreasing subsequence of length* $\ell + 1$.

Counting

Problem 3. *Calculate (i.e. express by a simple formula not containing a sum)*

1. $\sum_{k=1}^{n} \binom{k}{m} \frac{1}{k}$

2. $\sum_{k=0}^{n} \binom{k}{m} k$

Problem 5. *How many functions* $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ *are there that are* monotone*; that is, for* $i < j$ *we have* $f(i) \leq f(j)$*?*

Problem5可以想像成求非负整数解个数的那个问题。

容斥原理

Problem 4. *Count the permutations with exactly k fixed points. (Remark:* π *is a permutation of the set* $\{1, 2, \ldots, n\}$*. Call an index i with* $\pi(i) = i$ *a* fixed point *of the permutation* π*.)*

Solution. First choose the points that are fixed. It will have $\binom{n}{k}$ possible choices.

The rest is counting the number of permutation without a fixed point, which is $D(n-k)$.

In all, the answer is $\binom{n}{k} \cdot D(n-k)$. $\square$

Problem 6. *How many ways are there to seat n married couples at a round table with $2n$ chairs in such a way that the couples never sit next to each other?*

Solution.(hint)

A_i = {the ways of seating in which the ith couple is adjacent.}

- $|A_i| = (2n-1)! \cdot 2^1/(2n-1)$;
- $|A_i \cap A_j| = (2n-2)! \cdot 2^2/(2n-2)$ for $i \neq j$;
- $\cdots$;
- $|A_{i1} \cap \cdots \cap A_{ik}| = (2n-k)! \cdot 2^k/(2n-k)$ for different A_{ij}s.

The final result should be $(2n)!/(2n) - |A_1 \cup A_2 \cup \cdots \cup A_k|$. Then by PIE....

Generating Function

Problem 3. *Let a_n be the number of ordered triples $\langle i, j, k\rangle$ of integer numbers such that $i \geq 0, j \geq 1, k \geq 1$, and $i + 3j + 3k = n$. Find the generating function of the sequence $(a_0, a_1, a_2, \ldots)$ and calculate a formula for a_n.*

Solution.

$(1 + x + x^2 + x^3 + \cdots)(x^3 + x^6 + x^9 + \cdots)(x3 + x^6 + x^9 + \cdots)$

$= \frac{1}{1-x}\frac{x^3}{1-x^3}\frac{x^3}{1-x^3}$

$= \frac{x^6(1+x+x^2)}{(1-x^3)^3} = x^6(1 + x + x^2)(1 - x^3)^{-3}.$

Then use the generalized binomial theorem. □

Recurrence Function

Problem 1. *Prove that any natural number $n \in \mathbb{N}$ can be written as a sum of mutually distinct Fibonacci numbers.*

函数的渐进式比较

Problem 2. 1. *Find two functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.*

2. *Furthermore, we say a function $h : \mathbb{R} \to \mathbb{R}$ is* monotonically increasing *if it satisfies the property '$x \leq y \implies h(x) \leq h(y)$'.*
   Find two monotonically increasing functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.

(Please give the detailed proof that your functions satisfy the requirements.)

Solution.

1. $\begin{cases} f(x) = \sin(x); \\ g(x) = \cos(x). \end{cases}$

2. $\begin{cases} f(x) = x^{\sin(x)+x}; \\ g(x) = x^{\cos(x)+x}. \end{cases}$

Problem 6.

a) *Show that the product of all primes p with $m < p \leq 2m$ is at most $\binom{2m}{m}$.*

b) *Using a), prove the estimate $\pi(x) = O(\frac{x}{\ln x})$, where $\pi(x)$ denote the number of primes not exceeding the number x.*

这一题很难想啊。第一小问的答案就不放了，看到了自己想想，第二小问的解答：

First proof: Combing a), w.l.o.g. assume n is even and $n = 2m$. It is obvious that

$$B \leq \sum_{i=0}^{2m} \binom{2m}{i} = 4^m$$

With a) we have $\prod_{m<p\leq 2m} p \leq B \leq 4^m$ (p is prime, as above). It follows that

$$\sum_{m<p\leq 2m} \log p \leq m \log 4 = 2m \qquad (\star)$$

Then count the number of primes between m and $2m$, i.e. the number of $p \in (m, 2m]$,

$$\pi(2m)-\pi(m) = \sum_{m<p\leq 2m} 1 \leq \sum_{m<p\leq 2m} \frac{\log p}{\log m} = \frac{1}{\log m}\left(\sum_{m<p\leq 2m} \log p\right) \overset{(\star)}{\leq} \frac{2m}{\log m}.$$

For any given x, there exists $k \geq 1$ such that $x \in (2^{k-1}, 2^k]$.
Finally with the above analysis

$$\pi(x) \leq \pi(2^k) = \sum_{i=1}^{k} \left(\pi(2^i) - \pi(2^{i-1})\right) = O\left(\sum_{i=1}^{k} \frac{2^j}{j}\right) = O\left(\frac{2^k}{k}\right) = O\left(\frac{x}{\ln x}\right).$$

图同构

Problem 3. *How many graphs on the vertex set $\{1, 2, \ldots, 2n\}$ are isomorphic to the graph consisting of n vertex-disjoint edges (i.e. with edge set $\{\{1,2\},\{3,4\},\ldots,\{2n\text{-}1,2n\}\}$?*

Solution. $\frac{(2n\cdot(2n-1))((2n-2)\cdot(2n-3))\cdots(2\cdot 1)}{2^n\cdot n!} = (2n-1)(2n-3)\cdot\cdots\cdot 5\cdot 3.$

思路：

- 先2n选2，再2n-2选2，再...
- 最后要除以n!，因为之前的选法其实做了一个组合。

树

Problem 6. *Given a sequence $(d_1, d_2, \ldots, d_n)$ of positive integers (where $n \geq 1$):*

(i) There exists a tree with score $(d_1, d_2, \ldots, d_n)$.

(ii) $\sum_{i=1}^{n} d_i = 2n - 2$.

Prove that (i) and (ii) are equivalent.

1到2很简单，2到1需要用一下归纳法。注意：归纳法从n+1去掉一个叶节点构造出n的树，而不是从n加一个点变成n+1的树。后者没有证明到generality。

具体过程如下。

Solution.

1. $(i) \Rightarrow (ii)$ is obvious.

2. To prove $(ii) \Rightarrow (i)$:

   By induction on the number n.

   For $n = 1, 2$ the implication holds trivially, so let $n > 2$. Suppose the implication holds for any $n-1$ long positive sequence $(d_1, d_2, \ldots, d_{n-1})$ with $\sum_{i=1}^{n-1} d_i = 2(n-1) - 2$.

   For the induction step, consider an length n positive sequence $\ell = (d_1, d_2, \ldots, d_n)$ with $\sum_{i=1}^{n} d_i = 2n - 2$:

   Since the sum of the d_i is smaller than $2n$, there exists an i with $d_i = 1$. w.l.o.g. we assume $d_1 = 1$. With a similar argument we can also conclude that there must exist some index j such that $d_j \geq 2$. We take $k = \min\{j \mid d_j \geq 2\}$.
   Now the sequence $\ell = (d_1, d_2, \ldots, d_k, \ldots, d_n) = (1, d_2, \ldots, d_k - 1 + 1, \ldots, d_n)$, we can derive a new sequence $\ell' = (d_2, \ldots, d_k - 1, \ldots, d_n)$. Obviously ℓ' is a $n-1$ length sequence (all positive) with the summation to be $2n - 2 - 1 - 1 = 2(n-1) - 2$. Then according to the induction hypothesis, there exists a tree $\mathcal{T}'$ which corresponds to ℓ'.
   Then $\mathcal{T} = (V(\mathcal{T}') \cup \{v_1\}, E(\mathcal{T}') \cup \{v_1, v_k\})$ is the tree which witnesses the validity of the sequence ℓ.

树的同构

TODO：再把这一题看一下

Problem 7. *Let N_k denote the number of spanning trees of K_n in which the vertex n has degree k, $k = 1, 2, \dots, n-1$ (recall that we assume $V(K_n) = \{1, 2, \dots, n\}$).*

i) Prove that $(n-1-k)N_k = k(n-1)N_{k+1}$.

ii) Using i), derive $N_k = \binom{n-2}{k-1}(n-1)^{n-1-k}$.

iii) Prove Cayley's formula from ii).

Solution.

i) Both sides of the equality count the number of pairs spanning trees (T, T^*), where $deg_T(n) = k$, $deg_{T^*}(n) = k+1$, and T^* arises from T by the following operation: pick an edge $\{i, j\} \in E(T)$ with $i \neq n \neq j$, delete it, and add either the edge $\{i, n\}$ or the edge $\{j, n\}$, depending on which of these edges connects the two components of $T - \{i, j\}$.

- From one T we can get $n-1-k$ different T^*: the number of different edges in T which are not connected to n at the beginning;
- And one T^* can be obtained from $k(n-1)$ different T: pick any vertex $v \in \{1, 2, \dots, n-1\}$. If one deletes all edges incident to n in a spanning tree from N_{k+1}, neighbours of n (denoted by $\ell_1, \ell_2, \dots, \ell_{k+1}$) will lie in exactly $k+1$ different components. Suppose v lies in the last component, namely C_{k+1}. Add an edge between v and some ith leaf ℓ_i ($i \in \{1, 2, \dots, k\}$) of n and remove the original edge (n, ℓ_i) simultaneously, one will get a different T. In all, there are $n-1$ ways to pick v and k ways to pick ℓ_i.

概率论

Problem 4. *We have 27 fair coins and one counterfeit coin (28 coins in all), which looks like a fair coin but is a bit heavier. Show that one needs at least 4 weighings to determine the counterfeit coin. We have no calibrated weights, and in one weighing we can only find out which of two groups of some k coins each is heavier, assuming that if both groups consist of fair coins only the result is an equilibrium.*

Solution. Each weighting has 3 possible outcomes, and hence 3 weightings can only distinguish one among 3^3 possibilities.

概率方法

Problem 5. *1. Prove that, for every integer n, there exists a coloring of the edges of the complete graph K_n by two colors so that the total number of monochromatic copies of K_4 is at most $\binom{n}{4}2^{-5}$.*

2. Give a randomized algorithm for finding a coloring with at most $\binom{n}{4}2^{-5}$ monochromatic (i.e. single-color) copies of K_4 that runs in expected time polynomial in n.

思路:

- 第一问用概率方法做
- 第二问类似于las vegas 算法。
- 第二问的解答:

2. Color each edge independently and uniformly. Let $p = Pr\left(X \le \binom{n}{4}2^{-5}\right)$ where X is the number of chromatic K_4.

$$\begin{aligned} \binom{n}{4}2^{-5} &= \mathbf{E}(X) \\ &= \sum_{i \le \binom{n}{4}2^{-5}} i \cdot Pr(X = i) + \sum_{i > \binom{n}{4}2^{-5}} i \cdot Pr(X = i) \\ &\ge p + (1-p)\left(\binom{n}{4}2^{-5} + 1\right) \end{aligned}$$

which implies $p \ge \frac{32}{\binom{n}{4}}$. The expected number of sampling before finding a suitable coloring is $1/p = \frac{\binom{n}{4}}{32}$. For each sampling, the time needs to count the number of chromatic K_4 is bounded by $\binom{n}{4}$ which is also polynomial. Thus the expected running time of this algorithm is polynomial.

Problem 6. *Use the Lovasz local lemma to show that if*

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1$$

then it is possible to color the edges of K_n with two colors so that it has no monochromatic (i.e. single color) K_k subgraph.

Solution. E_i: the $i-$th K_k is monochromatic. $Pr(E_i) = 2^{1-\binom{k}{2}}$. Consider the dependency graph, for any different E_i and E_j, they are adjacent if the corresponding K_k share at least one edge. Thus the degree of the dependency graph is bounded by $\binom{k}{2}\binom{n}{k-2}$.

According to the Lovasz local lemma, it is possible that none of the E_i happens under the given inequality. □

这一题套一下lovasz local lemma就行，关键在于怎么设计p和d。

随机图

Problem 7. *What is the expected number of trees with k vertices in $G \in \mathcal{G}(n, p)$?*

Solution. By *Cayley's formula* and the linearity of expectation, it is $\binom{n}{k}k^{k-2}p^{k-1}$ □

Problem 8. *Show that if almost all $G \in \mathcal{G}(n, p)$ have a graph property $\mathcal{P}_1$ and almost all $G \in \mathcal{G}(n, p)$ have a graph property $\mathcal{P}_2$, then almost all $G \in \mathcal{G}(n, p)$ have both properties.*

Problem 1. *Show that, for constant $p \in (0, 1)$, almost no graph in $\mathcal{G}(n, p)$ has a separating complete subgraph.*

[Hint]

1. Recall the property $P_{i,j}$ from the slides

2. You may need to recall the definitions:

Problem 2. *Consider* $\mathbf{G}(n, p)$ *with* $p = \frac{1}{3n}$.

Use the second moment method to show that with high probability there exists a simple path of length 10.

这一题求期望的时候要分类讨论一下。