Reinforcement Learning

Lecture 6: RL algorithms 2.0

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Objectives of this lecture

Present and analyse two online algorithms based on the "optimism in front of uncertainty" principle, and compare their regret to algorithms with random exploration

- UCB-VI for episodic RL problems
- UCRL2 for ergodic RL problems

Lecture 6: Outline

1. Minimal exploration in RL
2. UCB-VI
3. UCRL2

Lecture 6: Outline

1. **Minimal exploration in RL**
2. UCB-VI
3. UCRL2

Towards minimal exploration

The MDP model is unknown and has to be learnt. Solutions for on-policy algorithms:

1. Estimate the model then optimise: poor regret and premature exploitation
2. ϵ greedy exploration: undirected exploration (explores too much (state, action) pairs with low values)
3. **Bandit-like optimal exploration-exploitation trade-off**

But how much should a (state,action) pair be explored?

Regret lower bounds

In the case of ergodic RL problems:

- Problem-specific lower bound (Burnetas - Katehakis 1997)

$$\lim \inf_{T \to \infty} \frac{\mathbb{E}[N_{(s,a)}(T)]}{\log(T)} \geq \frac{1}{\mathcal{K}_M(s,a)}$$

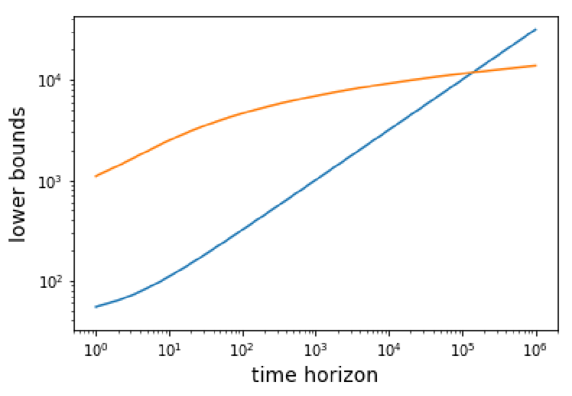
  Leading to an **asymptotic** regret lower bound scaling as $SA\log(T)$
- Minimax lower bound $\Theta(\sqrt{SAT})$

We don't know when the asymptotic problem-specific regret lower bound is representative, often for very large T!

Read for bandit optimisation: "Explore First, Exploit Next: The True Shape of Regret in Bandit Problems", Garivier et al. , https://arxiv.org/abs/1602.07182

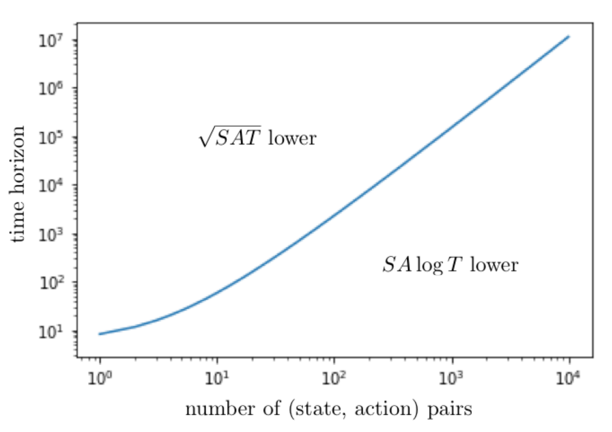
Which regret lower bound should we target?

Example: $SA = 1000$, comparison of $\sqrt{SAT}$ and $SA\log(T)$



Which regret lower bound should we target?

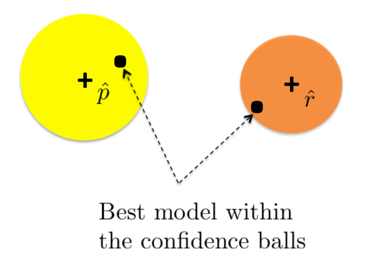
Boundary: $SA = \frac{T}{\log(T)^2}$



Optimism in front of uncertainty

Estimate the unknown system parameters (here $p(\cdot|\cdot,\cdot)$ and $r(\cdot,\cdot)$) and build an optimistic reward estimate to trigger exploration.

Estimate: find confidence balls containing the true model w.h.p.
Optimistic reward estimate: find the model within the confidence balls leading to the highest value.



Optimism in front of uncertainty: generic algorithm

Algorithm. (for Infinite horizon RL problems)
Initialise $\hat{p}$, $\hat{r}$, and $N(s,a)$ For $t = 1, 2, \ldots$

1. Build an optimistic reward model $(\bar{Q}(s,a))_{s,a}$ from $\hat{p}$, $\hat{r}$, and $N(s,a)$
2. Select action $a(t)$ maximising $\bar{Q}(s(t),a)$ over $\mathcal{A}_{s(t)}$
3. Observe the transition to $s(t+1)$ and collect reward $r(s(t),a(t))$
4. Update $\hat{p}$, $\hat{r}$, and $N(s,a)$

Examples

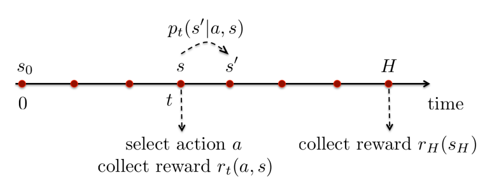
UCB-VI: directly build a confidence ball for the Q function based on the empirical estimates of the model.

UCRL2: first build confidence balls for the reward and transition probabilities, and then identify $\bar{Q}$.

Lecture 6: Outline

1. Minimal exploration in RL
2. **UCB-VI**
3. UCRL2

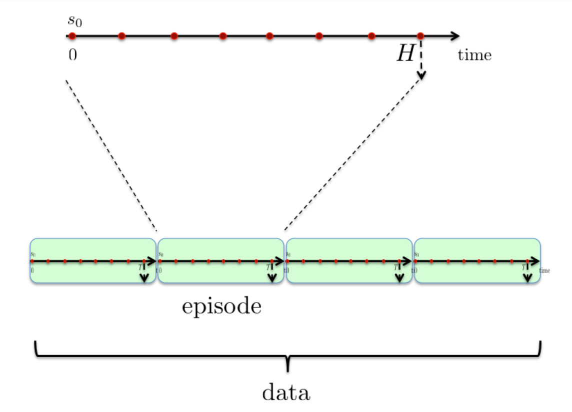
Finite-horizon MDP to episodic RL problems



- Initial state s_0 (could be a r.v.)
- Transition probabilities at time t: $p(s'|s, a)$
- Reward at time t: $r(s, a)$ and at time H: $r_H(s)$
- **Unknown** transition probabilities and reward function
- Objective: *quickly* learn a policy $\pi^\star$ maximising over $\pi_0 \in MD$

$$V_H^{\pi_0} := \mathbb{E}\left[\sum_{u=0}^{H-1} r(s_u^{\pi_0}, a_u^{\pi_0}) + r_H(s_H^{\pi_0})\right].$$

Finite-horizon MDP to episodic RL problems



- Data: K episodes of length H (actions, states, rewards)
- Learning algorithm π : data $\mapsto \pi_K \in MD$
- Performance of π: how close π_K is from the optimal policy $\pi^\star$

UCB-VI

UCBVI is an extension of Value Iteration, guaranteeing that the resulting value function is a (high-probability) upper confidence bound (UCB) on the optimal value function.

At the beginning of episode k, it computes state-action values using empirical transition kernel and reward function. In step h of backward induction (to update $Q_{k,h}(s,a)$ for any (s,a)), it adds a bonus $b_{k,h}(s,a)$ to the value, and ensures that $Q_{k,h}$ never exceeds $Q_{k,h-1}$.

Two variants of UCBVI, depending on the choice of bonus $b_{k,h}$:

- UCBVI-CH
- UCBVI-FB

UCB-VI algorithm

Variables to be maintained by the algorithm: for known reward function

- $\hat{p} = (\hat{p}(s'|s,a), s, s' \in \mathcal{S}, a \in \mathcal{A}_s)$: estimated transition probabilities
- $Q = (Q_h(s,a), h \leq H, s \in \mathcal{S}, a \in \mathcal{A}_s)$: estimated Q-function
- $b = (b_h(s,a), h \leq H, s \in \mathcal{S}, a \in \mathcal{A}_s)$: Q-value bonus
- $N = (N(s,a), s \in \mathcal{S}, a \in \mathcal{A}_s)$: number of visits to (s,a) so far
- $N' = (N_h(s,a), h \leq H, s \in \mathcal{S}, a \in \mathcal{A}_s)$: number of visits in the h-step of episodes to (s,a) so far

UCB-VI algorithm

Algorithm. UCB-VI

Input: Initial state distribution ν_0, precision δ

Initialise the variables $\hat{p}$, N, and N'

For episode $k = 1, 2, \ldots$

1. Optimistic reward:
   a. Compute the bonus: $b \leftarrow \text{bonus}(N, N', \hat{p}, Q, \delta)$
   b. Estimate the Q-function: $Q \leftarrow \text{bellmanOpt}(Q, b, \hat{p})$
2. Initialise the state $s(0) \sim \nu_0$
3. for $h = 1, \ldots, H$, select action
   $a \in \arg\max_{a' \in \mathcal{A}_{s(h-1)}} Q_h(s(h-1), a')$
4. Observe the transition and update $\hat{p}$, N, and N'

UCB-VI algorithm: bonus

UCBVI-CH:

$$b_h(s,a) = \frac{7H}{\sqrt{N(s,a)}} \log(5SAT/\delta)$$

UCBVI-BF:

$$\begin{aligned} b_h(s,a) = & \sqrt{\frac{8L}{N(s,a)} \mathrm{Var}_{\widehat{p}(\cdot|s,a)}(V_{h+1}(Y))} + \frac{14HL}{3N(s,a)} \\ & + \sqrt{\frac{8}{N(s,a)} \sum_y \widehat{p}(y|s,a) \min\left\{ \frac{10^4 H^3 S^2 A L^2}{N'_{h+1}(y)}, H^2 \right\}} \end{aligned}$$

where $L = \log(5SAT/\delta)$.

UCB-VI algorithm: Optimistic Bellman operator

bellmanOpt$(Q, b, \hat{p})$ applies Dynamic Programming with a bonus.

Initialisation: $Q_H(s,a) = r_H(s)$ for all (s,a)
For step $h = H-1, \dots, 1$**:** for all (s,a) visited at least once so far:
$Q_h(s,a) \leftarrow \min\left(Q_h(s,a), H, r(s,a) + \sum_y \hat{p}(y|s,a) V_{h+1}(s) + b_h(s,a)\right)$

UCB-VI: Regret guarantees

Regret up to time $T = KH$:
$R^{UCBVI}(T) = \sum_{k=1}^{K}(V^\star(x_{k,1}) - V^{\pi_k}(x_{k,1}))$.

Theorem *For any $\delta > 0$, the regret of UCB-VI-CH(δ) is bounded w.p. at least $1-\delta$ by:*

$$R^{UCBVI-CH}(T) \leq 20HL\sqrt{SAT} + 250H^2S^2AL^2,$$

with $L = \log(5HSAT/\delta)$.

For $T \geq HS^3A$ and $SA \geq H$, the regret upper bound scales as $\tilde{\mathcal{O}}(H\sqrt{SAT})$ (!?)

Sketch of proof

Notations:

- π_k is the policy applied by UCBVI in the k-th episode
- $V_{k,h}$ is the optimistic value function computed by UCBVI in the h-step of the k-th episode
- V_h^π is the value function from step h under π
- $P^\pi = (p(s'|s,\pi(s)))_{s,s'}$
- $\hat{P}_k^\pi = (\hat{p}_k(s'|s,\pi(s)))_{s,s'}$ where $\hat{p}_k$ is the estimated transitions in episode k

Claim 1: by construction with high probability, $V_{k,h} \geq V_h^\star$. Then:

$$R^{UCBVI}(T) \leq \tilde{R}(T) = \sum_{k=1}^{K} (V_{k,1}(x_{k,1}) - V^{\pi_k}(x_{k,1}))$$

Sketch of proof

Let $\tilde{\Delta}_{k,h} = V_{k,h} - V_h^{\pi_k}$, so that $\tilde{R}(T) = \sum_{k=1}^{K} \tilde{\Delta}_{k,1}(x_{k,1})$.
Backward induction on h to bound $\tilde{\Delta}_{k,1}$: introduce $\tilde{\delta}_{k,h} = \tilde{\Delta}_{k,h}(x_{k,h})$, then

$$\tilde{\delta}_{k,h} \leq (\hat{P}_k^{\pi_k} - P^{\pi_k})\tilde{\Delta}_{k,h+1}(x_{k,h}) + \tilde{\delta}_{k,h+1} + \epsilon_{k,h} + b_{k,h} + e_{k,h}$$

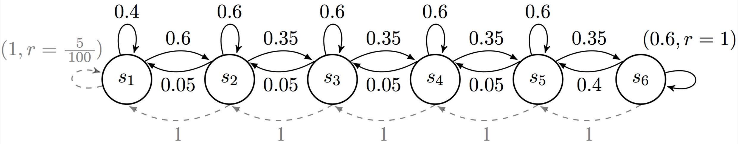
where

$$\begin{cases} \epsilon_{k,h} = P^{\pi_k}\tilde{\Delta}_{k,h+1}(x_{k,h}) - \tilde{\Delta}_{k,h+1}(x_{k,h+1}) \\ e_{k,h} = (\hat{P}_k^{\pi_k} - P^{\pi_k})V_{h+1}^{\star}(x_{k,h}) \end{cases}$$

Concentration + Martingale (Azuma) + bounding bonus

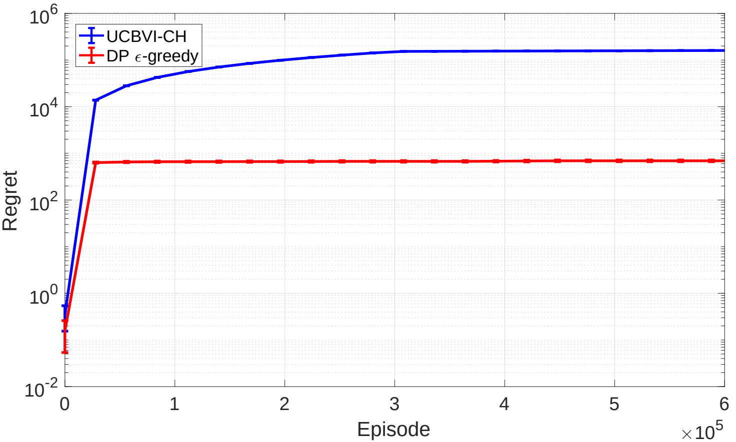
Numerical experiments

The river-swim example ...



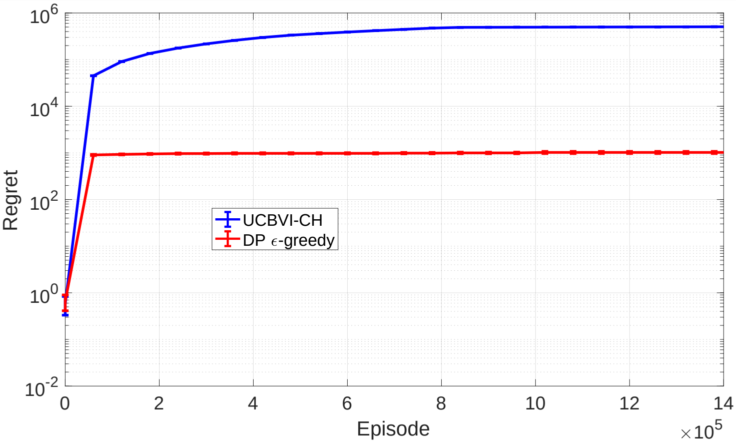
Regret

4 states, $H = 2$, $\delta = 0.05$ (for UCBVI), ϵ-greedy: $\epsilon_t = \min(1, 1000/t)$



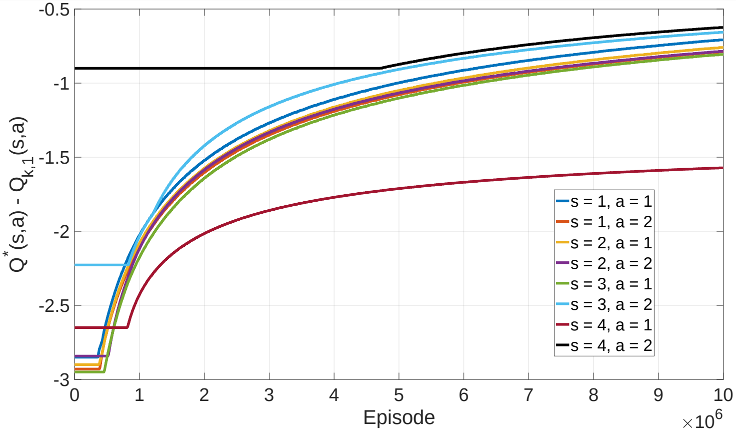
Regret

4 states, $H = 3$, $\delta = 0.05$ (for UCBVI), ϵ-greedy: $\epsilon_t = \min(1, 1000/t)$



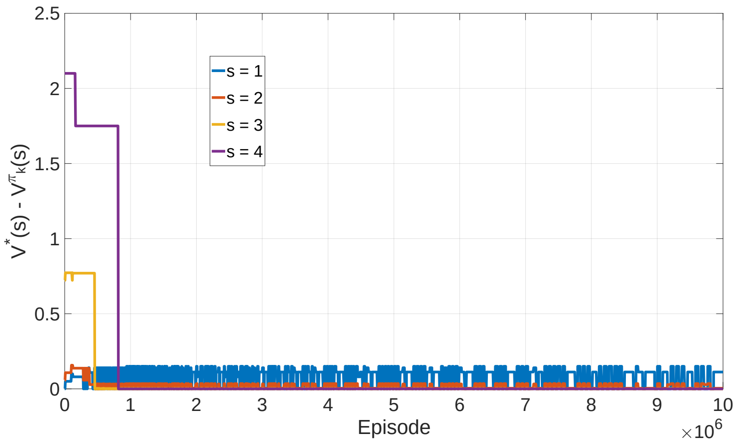
Optimistic Q-values

4 states, $H = 3$, $\delta = 0.05$ (for UCBVI)



Value function convergence under UCBVI

4 states, $H = 3$, $\delta = 0.05$ (for UCBVI)



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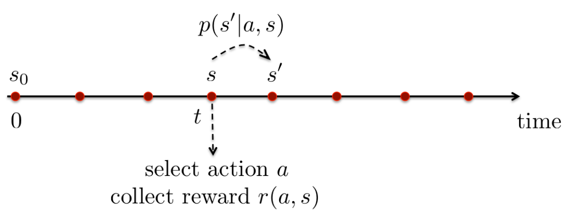
Expected average reward MDP to ergodic RL problems



- Stationary transition probabilities $p(s'|s,a)$ and rewards $r(s,a)$, uniformly bounded: $\forall a, s,\ |r(s,a)| \leq 1$
- Objective: learn from data a policy $\pi \in MD$ maximising (over all possible policies)

$$g^\pi = V^\pi(s_0) := \lim \inf_{T\to\infty} \frac{1}{T}\mathbb{E}_{s_0}\left[\sum_{u=0}^{T-1} r(s_u^\pi, a_u^\pi,)\right]$$

Ergodic RL problems: Preliminaries

Optimal policy

Recall Bellman's equation

$$g^\star + h^\star(s) = \max_{a\in\mathcal{A}}\Big(r(s,a) + h^\top p(\cdot|s,a)\Big), \quad \forall s$$

where $g^\star$ is the maximal gain, and $h^\star$ is the *bias* function ($h^\star$ is uniquely determined up to an additive constant). Note: $g^\star$ does not depend on the initial state for communicating MDPs.

Let $a^\star(s)$ denote any optimal action for state s (i.e., a maximizer in the above). Define the gap for sub-optimal action a at state s:

$$\phi(s,a) := \big(r(s,a^\star(s)) - r(s,a)\big) + {h^\star}^\top\big(p(\cdot|s,a^\star(s)) - p(\cdot|s,a)\big)$$

Ergodic RL problems: Preliminaries

Diameter D: defined as

$$D := \max_{s \neq s'} \min_{\pi} \mathbb{E}[T^{\pi}_{s,s'}]$$

where $T^{\pi}_{s,s'}$ denotes the first time step in which s' is reached under π staring from initial state s.

Remark: all communicating MDPs have a finite diameter.

Important parameters impacting performance

- Diameter D
- Gap $\Phi := \min_{s,a \neq a^\star(s)} \phi(s,a)$
- Gap $\Delta := \min_{\pi}(g^\star - g^{\pi})$

Ergodic RL problems: Regret lower bounds

- **Problem-specific regret lower bound:** (Burnetas-Katehakis)
  For any algorithm π,

$$\liminf_{T\to\infty} \frac{R^\pi(T)}{\log(T)} \geq c_{\text{bk}} := \sum_{s,a} \frac{\phi(s,a)}{\inf\{\mathrm{KL}(p(\cdot|s,a), q) : q \in \Theta_{s,a}\}}$$

  where $\Theta_{s,a}$ is the set of distributions q s.t. replacing (only) $p(\cdot|s,a)$ by q makes a the unique optimal action in state s.
  - asymptotic (valid as $T \to \infty$)
  - valid for any ergodic MDP
  - scales as $\Omega(\frac{DSA}{\Phi}\log(T))$ for specific MDPs

- **Minimax regret lower bound:** $\Omega(\sqrt{DSAT})$
  - non-asymptotic (valid for all $T \geq DSA$)
  - derived for a specific family of *hard-to-learn* communicating MDPs

Ergodic RL problems: State-of-the-art

Two types of algorithms targeting different regret guarantees:

- Problem-specific guarantees
  - MDP-specific regret bound scaling as $\mathcal{O}(\log(T))$
  - Algorithms: B-K (Burnetas & Katehakis, 1997), OLP (Tewari & Bartlett, 2007), UCRL2 (Jaksch et al. 2009), KL-UCRL (Filippi et al. 2010)
- Minimax guarantees
  - Valid for a class of MDPs with S states and A actions, and (typically) diameter D
  - Scaling as $\widetilde{\Omega}(\sqrt{T})$
  - Algorithms: UCRL2 (Jaksch et al. 2009), KL-UCRL (Filippi et al. 2010), REGAL (Bartlett & Tewari, 2009), A-J (Agrawal & Jia, 2010)

Ergodic RL problems: State-of-the-art

Algorithm	Setup	Regret
B-K	ergodic MDPs, known rewards	$\mathcal{O}\left(c_{\text{bk}}\log(T)\right)$ – asympt.
OLP	ergodic MDPs, known rewards	$\mathcal{O}\left(\frac{D^2SA}{\Phi}\log(T)\right)$ – asympt.
UCRL	unichain MDPs	$\mathcal{O}\left(\frac{S^5A}{\Delta^2}\log(T)\right)$
UCRL2, KL-UCRL	communicating MDPs	$\mathcal{O}\left(\frac{D^2S^2A}{\Delta}\log(T)\right)$
Lower Bound	ergodic MDPs, known rewards	$\Omega\left(c_{\text{bk}}\log(T)\right),\ \Omega\left(\frac{DSA}{\Phi}\log(T)\right)$

Algorithm	Setup	Regret
UCRL2	communicating MDPs	$\widetilde{\mathcal{O}}\left(DS\sqrt{AT}\right)$
KL-UCRL	communicating MDPs	$\widetilde{\mathcal{O}}\left(DS\sqrt{AT}\right)$
REGAL	weakly comm. MDPs, known rewards	$\widetilde{\mathcal{O}}\left(BS\sqrt{AT}\right)^*$
A-J	communicating MDPs, known rewards	$\widetilde{\mathcal{O}}\left(D\sqrt{SAT}\right),\ T \geq S^5A$
Lower Bound	known rewards	$\Omega\left(\sqrt{DSAT}\right),\ T \geq DSA$

*B denotes the span of bias function of true MDP, and $B \leq D$

UCRL2

UCRL2 is an optimistic algorithm that works in episodes of increasing lengths.

- At the beginning of each episode k, it maintains a set of plausible MDPs $\mathcal{M}_k$ (which contains the true MDP w.h.p.)
- It then computes an optimal policy π_k, which has the largest gain over all MDPs in $\mathcal{M}_k$ ($\pi_k \in \mathrm{argmax}_{M' \in \mathcal{M}_k, \pi}\, g^\pi(M')$).
  - For computational efficiency, UCRL2 computes an $\frac{1}{\sqrt{t_k}}$-optimal policy, where t_k is the starting step of episode k
  - To find a near-optimal policy, UCRL2 uses Extended Value Iteration
- It then follows π_k within episode k until the number of visits for some pair (s, a) is doubled (and so, a new episode starts).

UCRL2

Notations:

- $k \in \mathbb{N}$: index of an episode
- $N_k(s,a)$: total no. visits of pairs (s,a) before episode k
- $\hat{p}_k(\cdot|s,a)$: empirical transition probability of (s,a) made by observations up to episode k
- $\hat{r}_k(s,a)$: empirical reward distribution of (s,a) made by observations up to episode k
- π_k: policy followed in episode k
- $\mathcal{M}_k$: set of models for episode k (defined next)
- $\nu_k(s,a)$: no. of visits of pairs (s,a) seen so far in episode k

UCRL2: Main ingredients

- **The set of plausible MDPs** $\mathcal{M}_k$: for confidence parameter δ, define

$$\mathcal{M}_k = \Bigg\{ M' = (\mathcal{S}, \mathcal{A}, \tilde{r}, \tilde{p}) : \forall (s,a), |\tilde{r}(s,a) - \hat{r}_k(s,a)| \leq \sqrt{\frac{3.5 \log(2SAt/\delta)}{N_k(s,a)^+}}$$
$$\|\tilde{p}(\cdot|s,a) - \hat{p}_k(\cdot|s,a)\|_1 \leq \sqrt{\frac{14S \log(2At/\delta)}{N_k(s,a)^+}} \Bigg\}$$

- **Optimistic gain**: find in $\mathcal{M}_k$ the MDP that leads to the highest gain. We need to solve for episode k:

$$\text{maximise over } (M, \pi) \quad g^\pi(M)$$
$$\text{subject to } M \in \mathcal{M}_k$$

UCRL2 pseudo-code

Algorithm. UCRL2

Input: Initial state s_0, precision δ, $t = 1$

For each episode $k \geq 1$:

1. Initialisation. $t_k = t$ (start time of the episode)
   Update $N_k(s,a)$, $\hat{r}_k(s,a)$, and $\hat{p}_k(s,a)$ for all (s,a)
2. Compute the set of possible MDPs $\mathcal{M}_k$ (using δ)
3. Compute the policy
   $\pi_k \leftarrow$ ExtendedValueIteration$(\mathcal{M}_k, 1/\sqrt{t_k})$
4. Execute π_k and end the episode:
   While $[\nu_k(s_t, \pi_k(s_t)) < \max(1, N_k(s_t, \pi_k(s_t)))]$
    - Play $\pi_k(s_t)$, observe the reward and the next state
    - Update $\nu_k(s_t, \pi_k(s_t)) \leftarrow \nu_k(s_t, \pi_k(s_t)) + 1$ and $t \leftarrow t + 1$

Extended value iteration

Set of plausible MDPs $\mathcal{M}_k$:

$$\mathcal{M}_k = \Bigg\{ M' = (\mathcal{S}, \mathcal{A}, \tilde{r}, \tilde{p}) : \forall (s,a), |\tilde{r}(s,a) - \hat{r}_k(s,a)| \leq d(s,a)$$

$$\|\tilde{p}(\cdot|s,a) - \hat{p}_k(\cdot|s,a)\|_1 \leq d'(s,a) \Bigg\}$$

We wish to find $M' \in \mathcal{M}_k$ and a policy π_k maximising $g^{\pi}(M')$ over all possible $M' \in \mathcal{M}_k$ and policy π.

Ideas:

a. we can fix the reward to its maximum: $\bar{r}(s,a) = \hat{r}(s,a) + d(s,a)$

b. solve a large MDP whose set of actions is $\mathcal{A}'_s$ where $(a,q) \in \mathcal{A}'_s$ if and only if $q \in \mathcal{P}_k(s,a)$ with:

$$\mathcal{P}_k(s,a) = \{q : \|q(\cdot) - \hat{p}_k(\cdot|s,a)\|_1 \leq d'(s,a)\}$$

Extended value iteration

Solution: apply one of the known algorithms to find an optimal policy in MDPs, i.e., value iteration algorithm.

Extended Value Iteration: For all $s \in \mathcal{S}$, starting from $u_0(s) = 0$:

$$u_{i+1}(s) = \max_{a \in \mathcal{A}} \Big\{ \bar{r}(s,a) + \max_{q \in \mathcal{P}_k(s,a)} u_i^\top q \Big\}$$

- $\mathcal{P}_k(s,a)$ is a polytope, and the inner maximisation can be done in $\mathcal{O}(S)$ operations.
- To obtain an ε-optimal policy, the update is stopped when $\max_s(u_{i+1}(s) - u_i(s)) - \min_s(u_{i+1}(s) - u_i(s)) \leq \varepsilon$

UCRL2: Regret guarantees

Let $\pi =$UCRL2 Regret up to time T: $\mathcal{R}^\pi(T) = Tg^\star - \sum_{t=1}^T r(s_t^\pi, a_t^\pi)$, a random variable capturing the learning cost and the mixing time problems.

Theorem *W.p. at least $1-\delta$, the regret of UCRL2 satisfies, for any initial state, for any $T > 1$,*

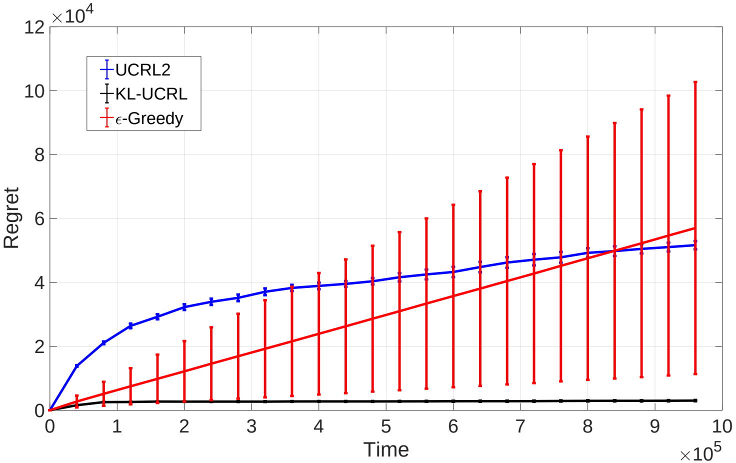
$$\mathcal{R}^\pi(T) \leq 34DS\sqrt{AT\log(\frac{T}{\delta})}.$$

For any initial state, and any $T \geq 1$, we have w.p. at least $1-3\delta$,

$$\mathcal{R}^\pi(T) \leq 34^2\frac{D^2S^2A\log(\frac{T}{\delta})}{\epsilon} + \epsilon T.$$

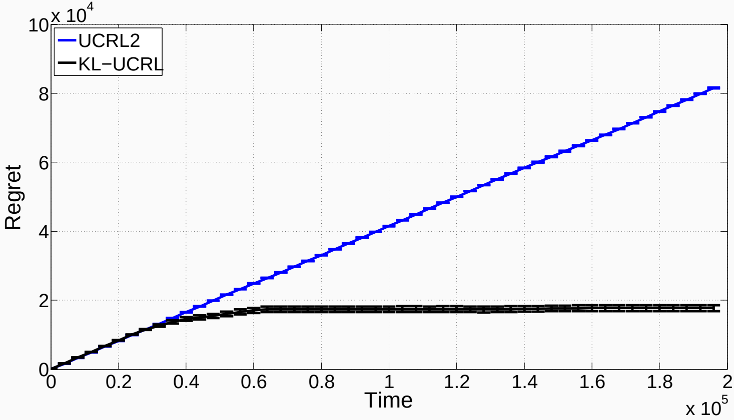
Regret

6 states, $\delta = 0.05$ (for UCRL2), ϵ-greedy: $\epsilon_t = \min(1, 1000/t)$



Regret

12 states, $\delta = 0.05$ (for UCRL2)



References

Episodic RL

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Ergodic RL

- **UCRL algorithm:**
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- **UCRL2 algorithm and minimax LB:**
  P. Auer, T. Jaksch, and R. Ortner, "Near-optimal regret bounds for reinforcement learning," *J. Machine Learning Research*, 2010.