MATH 3310 LECTURE NOTES

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Lecture 1.19

1. The real numbers

1.1. The irrationality of $\sqrt{2}$.

Theorem 1.1.1. There if no rational number whose square is 2.

Proof. We prove this by contradiction. Suppose there exists rational number p such that $p^2 = 2$. Then we could write $p = m/n$ where $m, n \in \mathbb{Z}$ share no common divisor. Then we have

$$\frac{m^2}{n^2} = p^2 = 2,$$

so

$$m^2 = 2n^2.$$

This shows that m^2 is even, so we may write $m = 2r$ where $r \in \mathbb{Z}$. Then we have

$$(2r)^2 = 2n^2,$$

which we can simplify to

$$2r^2 = n^2.$$

Therefore n is also even. This leads to the conclusion that both m, n are even and share a common divisor. $\square$

1.2. Some preliminaries (see textbook).

1.3. The Axiom of Completeness.

An initial definition for $\mathbb{R}$.

(1) $\mathbb{R}$ is a set that contains $\mathbb{Q}$.
(2) $\mathbb{R}$ is a field. $\mathbb{R}$ inherits the operations of addition and multiplication from $\mathbb{Q}$, such that every element in $\mathbb{R}$ has an additive inverse and every non-zero element in $\mathbb{R}$ has a multiplicative inverse. Addition and multiplication in $\mathbb{R}$ are commutative, associative, and distributive.
(3) Ordering of $\mathbb{Q}$ extends to $\mathbb{R}$. (A total order)
(4) The Axiom of Completeness.

Def 1.3.1 (Bounded above)**.** A set A is bounded above if there exists a $p \in \mathbb{R}$ such that $\forall a \in A$, $a \leq p$. Then we call p a upper bound of A.

Def 1.3.2 (Least upper bound)**.** A real number s is a least upper bound or supremum of A, denoted as $\sup(A)$ or $\sup A$, if s is an upper bound and for every upper bound p of A, $s \leq p$.

Axiom 1.3.1 (The Axiom of Completeness)**.** Every non-empty set of real numbers that is bounded above has a least upper bound (i.e. supremum)

Lecture 1.24

1.4. **Consequences of Completeness.**

Theorem 1.4.1 (Thm 1.4.2 in textbook, known as the archimedean property)**.** (1) Given any $x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ such that $n \geq x$.
(2) Given any $y > 0$, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} \leq y$

Proof of Thm 1.4.1 (1):

Proof. We proceed by proof by contradiction. Suppose $\exists x \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$, $n < x$. Then by definition of upper bound (def 1.3.1), x is an upper bound for $\mathbb{N}$. Then by the axiom of completeness, there exists a least upper bound for $\mathbb{N}$, call it s. Then $s - 1$ must exist, and $s - 1 < s$. Therefore $s - 1$ is not a upper bound for $\mathbb{N}$, thus $\exists k \in \mathbb{N}$ such that $k > s - 1$. We add 1 to both sides and we have

$$k + 1 > s$$

where $k + 1$ is a natural number. Then s is not a upper bound of $\mathbb{N}$ and we reach a contradiction.

$\square$

Proof of Thm 1.4.1 (2):

Proof. Let $y > 0$, then $1/y < 0$ (its correctness, though obvious, is not discussed yet). Then by part(1), we may find $n \in \mathbb{N}$ such that $n \geq 1/y$.

$$\begin{aligned} n &\geq 1/y \\ ny &\geq 1 \\ y &\geq 1/n \end{aligned}$$

$\square$

Theorem 1.4.2 (Thm 1.4.5 in textbook)**.** There exists $\alpha \in \mathbb{R}$ such that $\alpha^2 = 2$.

We try to avoid mentioning $\sqrt{2}$, since we don't know whether it exists or not. The proof idea: we show that $\sqrt{2}$ is the supremum of the set A, which is $\{r \in \mathbb{R} : r > 0 \text{ and } r^2 < 2\}$. First, we need to show that A satisfies two properties (otherwise we cannot talk about supremum): non-empty and bounded above.
Non-empty is straight-forward that $1 \in \mathbb{R}$ is in A. Bounded-above property is easy to see, that 100 is a upper bound of A, but the proof needs some work. We proceed by proof by contradiction. Suppose $\exists x \in A$ and $x \geq 100$, then

$$\begin{aligned} 100x &\geq 10000 \\ 100x^2 &\geq 10000x \\ x^2 &\geq 100x \\ x^2 &\geq 10000 \end{aligned}$$

Which contradicts assumption that $x^2 < 2$. Thus A is bounded above.
Then we may invoke the axiom of completeness that $\exists \sup(A)$, call this number s. The rest of the proof is divided into two parts: first we show that $s^2 \geq 2$, then we show that $s^2 \leq 2$. If done, we can safely conclude that $s^2 = 2$ and $s = \sqrt{2}$ exists.

Proof of $s^2 \geq 2$: We proceed by proof by contradiction. Suppose $s^2 < 2$, notice that the following inequality holds when $n > 0$:

$$\begin{aligned}(s+\frac{1}{n})^2 &= s^2 + \frac{2s}{n} + \frac{1}{n^2} \\ &\leq s^2 + \frac{2s}{n} + \frac{1}{n} \\ &\leq s^2 + \frac{2s+1}{n}\end{aligned}$$

Choose n such that $\frac{1}{n} < \frac{2-s^2}{2s+1}$ (Thm 1.4.1). Then we have

$$(s+\frac{1}{n})^2 \leq s^2 + (2s+1)\cdot\frac{1}{n} = s^2 + 2 - s^2 = 2$$

Then $s+\frac{1}{n} \in A$ and $s+\frac{1}{n} > s$. This leads to a contradiction that s is an upper bound of A. Hence the original claim is true.

Proof of $s^2 \leq 2$: *Exercise*

Lecture 1.26

Why the Axiom of Choice is reasonable for us to accept?(This part is from professor)

It can be shown that no contradictions arise from assuming the axiom of completeness. Therefore, in order to accept the axiom of completeness as an axiom about the real numbers, all we need is to be subjectively convinced that it's a reasonable axiom to assume about whatever we mean by *number*.

Now length or distance is a pretty reasonable notion of number. For every positive number x, there should be two points on a line that are distance x apart from each other, and conversely, the distance between any two points on a line should be a positive number. This, for instance, is why I initially felt compelled to accept that the square root of 2 is a real number. Now if you take a line that's infinite in both directions and mark a point as 0 and another point as 1, then every point on that line ought to represent a real number, and every real number ought to be represented by a point on the line. This is, I think, what we intuitively mean when we talk about real numbers. So we can think of the *set* of real numbers as the set of points on the line. Now the argument goes as follows:

(1) Intuitively, it makes sense that the real number line should "look the same everywhere."
(2) If you cut the line into two pieces at 0 (or any rational number), then there's a point that must go either to the left or to the right. You can split the real numbers up as $(-\infty, 0) \cup [0, \infty)$ or as $(-\infty, 0] \cup (0, \infty)$, but not as $(-\infty, 0) \cup (0, \infty)$, because then there's a number you're missing. At the point 0, you can't split the real numbers up as the union of two disjoint intervals that have no least element and no greatest element.
(3) If we accept point 1, then there should be no place where you can split the real numbers up as the union of two disjoint intervals that have no least element and no greatest element.
(4) Now for a set A, consider the set of upper bounds for A and the set of numbers that are not upper bounds for A. Certainly if any number is an upper bound, then anything bigger is an upper bound, and if a number is not an upper bound, then anything smaller is also not an upper bound. So we end up with two disjoint intervals. By point 3, either the set on the left (the set of numbers that are not upper bounds) must have a greatest element, or the set on the right (the set of upper bounds) must have a least element.
(5) It is impossible for the set of numbers that are not upper bounds to have a greatest element, because this element would have to be an upper bound.Therefore, the set of upper bounds for A must have a least element.

All this boils down to an argument that the axiom of completeness follows from the idea that the real number line should look the same everywhere. If there's a set that doesn't have a least upper bound, then there's a gap in the number line, and it's not a single connected continuum (the words "connected" and "continuum" have mathematical meanings, but I'm trying to use them in a more colloquial sense).

Theorem 1.4.3. For any $a, b \in \mathbb{R}$, with $a < b$, there exists a $x \in \mathbb{Q}$ such that $a < x < b$ and there exists $y \in \mathbb{R} - \mathbb{Q}$ such that $a < y < b$.

Proof. Let $0 \leq a < b$, then $b - a > 0$. By Archimedean property (b), $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$. Similarly, $\exists n_0$ such that $\frac{1}{n_0} < \sqrt{2}(b-a)$, namely, $\frac{1}{\sqrt{2}n_0} < b - a$.

Then choose m to be the largest integer that $\frac{m}{n} \leq a$. We can find such a m since the set $\{m \in \mathbb{N} : \frac{m}{n} \leq a\}$ is non-empty and finite. Since m is the largest number possible, then

$$\frac{1+m}{n} > a$$

Since $\frac{1}{n} < b - a$, then by adding a to both sides we have $\frac{1}{n} + a < b - a + a$ and

$$\frac{1}{n} + \frac{m}{n} < \frac{1}{n} + a < b$$

Therefore we find such a number that $a < \frac{m+1}{n} < b$.

Similarly, we may choose m_0 to be the largest integer that $\frac{m_0}{\sqrt{2}n_0} \leq a$ and it follows that $a < \frac{m_0+1}{\sqrt{2}n_0} < b$.

(Note that in class we only prove the case where $0 \leq a < b$. Then for the case where $a < 0 < b$, using the Archimedean property, we can find $n \in \mathbb{N}$ such that $a < \frac{1}{n} < b$ and also $a < \frac{1}{\sqrt{2}n_0} < b$. For the case where $a < b \leq 0$, we may flip the sign and use the proof for the case $0 \leq a < b$.) $\square$

For this property, we say that $\mathbb{Q}$ is **dense** in $\mathbb{R}$ and $\mathbb{R} - \mathbb{Q}$ is **dense** in $\mathbb{R}$.

We briefly went over a informal proof for the Theorem 1.6.1 in textbook that states: There is no bijection between $\mathbb{N}$ and $[0, 1]$. Consequently $[0, 1]$ is uncountable. The proof idea is by **diagonalization**. We assume that there exists a function $f : \mathbb{N} \to [0, 1]$. More concretely, say

$$\begin{aligned} f(1) &= 0.12345678... \\ f(2) &= 0.23456789... \\ f(3) &= 0.34567890... \\ &\vdots \end{aligned}$$

Then we may construct a number in $[0, 1]$ such that no natural number can be mapped to. A way to find this number: consider $f(n)$, for its n^{th} decimal place, if it is a 1, we change it to 5, and if it is not a 1, change it to 1. Using the example above, a carefully-constructed number n_0 will start with 0.511. Since $\forall n \in \mathbb{N}$, n_0 is different from $f(n)$ in n^{th} decimal place. Then we find such a number n_0 in $[0, 1]$ that no natural number is mapped to, and f cannot be a bijection.

Lecture 1.31

Theorem 1.4.4. (Nested interval property, equivalent to Axiom of Completeness) Suppose we have an infinite sequence of nested closed intervals $I_n = [a_n, b_n]$. Suppose further that $I_n \supseteq I_{n+1}(i.e. I_1 \supseteq I_2 \supseteq I_3 \dots)$, then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

(This theorem is not true of $\mathbb{Q}$, consider $I_n = [\sqrt{2} + \frac{1}{n}, \sqrt{2} - \frac{1}{n}]$)

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$. We first show that A is non-empty, since $a_1 \in A$, it is bounded above since $b_1 \geq a_n$ for all n (or any b_n). Thus by A.C, there is a supremum s. Then $sgeqa_n$ for all n, and all b_n is a upper bound of A and then $b_n \geq s$. Hence $a_n \leq s \leq b_n$ for all n and hence $s \in \bigcap_{n=1}^{\infty} I_n$ and thus the intersection is not empty. □

To show some usage of our new theorem:

Theorem 1.4.5. $\mathbb{R}$ is uncountable, that is there is no bijection between $\mathbb{N}$ and $\mathbb{R}$.

General strategy: for all n, construct interval such that $f(n) \notin I_n$

Proof. Let $f : \mathbb{N} \to \mathbb{R}$. Let $I_1 = [f(1) + 1, f(1) + 2] = [a_1, b_1]$. Now we construct I_2: if $f(a) \notin I_1$, then $I_2 = I_1$. If $f(2) \in I_1$, then let

$$b_2 = \frac{f(1) + 1 + f(2)}{2}$$

which is the midpoint of $f(2)$ and a_1 however, if $f(2)$ happens to be a_1, then change

$$a_2 = \frac{a_1 + b_1}{2}$$

which is the midpoint of I_1.

More formally, suppose I_n is defined, then if $f(n+1) \notin I_n$, then let $I_{n+1} = I_n$; if $f(n+1) \in (a_n, b_n]$, then let

$$b_{n+1} = \frac{a_n + f(n+1)}{2}, a_{n+1} = a_n$$

if $f(n+1) = a_n$, then let

$$a_{n+1} = \frac{b_n + a_n}{2}, b_{n+1} = b_n$$

Now we verify that $f(n) \notin I_n$ (by proof by cases). Lastly, by NIP, there exists $x \in \bigcap_{n=1}^{\infty} I_n$, so $f(n) \neq x$ for all n, thus f is not surjective. □

Side note: there is bijection between $(0, 1)$ and $\mathbb{R}^2$! How to construct a bijection of $(0, 1)$ and $\mathbb{R}^2$? One way is that you can encode all the odd digit into the first coordinate, and the even digit into the second coordinate and vice versa. For example we may have $f(0.1212121212) = (11111, 22222)$

Def 1.4.1. Let X be a set, then $P(X)$ is the set of all subsets of X.

Theorem 1.4.6. There is no surjection from any set to its power set.

Proof. Let $f : X \to P(X)$. Let $A = \{x \in X : x \notin f(x)\}$. Then we show that A cannot be mapped. Suppose $\exists a \in X$ such that $f(a) = A$, then is $a \in f(a)$? If $a \in f(a)$, then $a \notin f(a)$ by definition of A. The other way is similar. So no such f exists. $\square$

It is a similar argument comparing to the diagonalization argument. (Write everything in binary, then you assign 1 if $f(n)(n) = 0$ (the n^{th} position of $f(n)$), and 0 if $f(n)(n) = 1$)

Lecture 2.2

2. Sequences and series

2.1. Discussion: Rearrangement of infinite series.

2.2. Limit of a sequence.

Def 2.2.1. A *sequence* is a function from the natural numbers to real numbers.

Write it as (a_n) or $(a_n)n$ where $a_n \in \mathbb{R}$. We may define a sequence explicitly: $a_n = 2^n$ or recursively: $a_1 = 2$ and $a_n = \frac{a_n+1}{2}$

Def 2.2.2. A sequence (a_n) converges to a *limit* a if $\forall \epsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n \geq N, |a_n - a| < \epsilon$.

This is a concise way to write, it reads out as: for all epsilon greater than 0, there exists a natural number N such that for all numbers greater than N, the distance of a_n and a is less than epsilon.

How to prove convergence of a sequence?

(1) let $\epsilon > 0$.
(2) come up with a N (can be done by the Archimedean property), and verify that $f(N) < \epsilon$.
(3) show that when $n \geq N$, $|a_n - a| \leq \epsilon$.
(4) conclude that $\lim_{n\to\infty}(a_n) = a$

Def 2.2.3. If a sequence does not converge to any number, then it *diverges*.

Def 2.2.4. A sequence (x_n) is *bounded*, if there exists $M > 0$, such that $|x_n| < M$ for all n.

Theorem 2.2.1. Every convergent sequence is bounded.

Proof. Let $\epsilon = 15$, then since (x_n) is convergent, $\exists N \in \mathbb{N}$ such that for $n > N$, $|x_n - x| \leq 15$. Then $|x_n| \leq |x| + 15$. Then let $M = \max\{|x| + 15, |a_1|, \ldots, |a_N|\}$. Then $|x_n| < M$. $\square$

Lecture 2.7

2.3. **The Algebraic and Order Limit Theorem.**

Theorem 2.3.1. (Algebraic Limit Theorem) Suppose (a_n) is a sequence that converges to a, and (b_n) is a sequence that converges to b. Then

(1) $\lim_{n\to\infty} c(a_n) = ca$ for $c \in \mathbb{R}$
(2) $\lim_{n\to\infty}(a_n + b_n) = a + b$
(3) $\lim_{n\to\infty}(a_n b_n) = ab$
(4) $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{a}{b}$ if $b \neq 0$

Proof for Thm 2.2.1(2):

Proof. Let $\epsilon > 0$, then we want to show $|(a_n + b_n) - a - b| < \epsilon$. Note that by the triangular inequality,

$$|(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$$

Then there exists $N_a \in \mathbb{N}$ such that $\forall n > N_a$ we have $|a_n - a| < \frac{\epsilon}{2}$, and there exists $N_b \in \mathbb{N}$ such that $\forall n > N_b$ we have $|b_n - b| < \frac{\epsilon}{2}$. Then pick $N = \max(N_a, N_b)$, then $\forall n > N$ we have

$$|(a_n + b_n) - a - b| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Then $\lim_{n\to\infty}(a_n + b_n) = a + b$. □

Theorem 2.3.2. (Order limit theorem) Suppose (a_n) is a sequence that converges to a, and (b_n) is a sequence that converges to b.

(1) If $a_n \geq 0$ for $n \in \mathbb{N}$, then $a \geq 0$.
(2) If $a_n \leq b_n$, then $a \leq b$.
(3) If $\exists c \in \mathbb{R}$ such that $a_n \leq c$, then $a \leq c$.

Proof for Thm 2.2.3(1):

Proof. Suppose $a < 0$, $a_n \geq 0$ for all n, and (a_n) has limit a. Let $\epsilon = |a|$, then for large enough n, $|a_n - a| \leq \epsilon$. Then $a_n \in (a - \epsilon, a + \epsilon) = (2a, 0)$, then we reached a contradiction that $a_n < 0$. Hence $a \geq 0$. □

Proof for Thm 2.2.3(2):

Proof. Assume $b_n \geq a_n$, then consider the sequence $(b_n - a_n)$, then $b_n - a_n \geq 0$, then from Algebraic Limit Theorem we have $(b_n - a_n)$ has a limit $(b - a)$, and by part (1), we have $b - a \geq 0$, then $b \geq a$. □

Proof for Thm 2.2.3(3):

Proof. Take $b_n = c$, then (b_n) has limit c, we have $a_n \leq c = b_n$, then by (2) we have $a < c$. □

2.4. **The Monotone Convergence Theorem.**

Def 2.4.1. A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all n. *Strictly increasing* if $a_n < a_{n+1}$ for all n.
Similarly, a sequence (a_n) is *decreasing* if $a_n \geq a_{n+1}$ for all n. *Strictly decreasing* if $a_n > a_{n+1}$ for all n.
Finally, a sequence is *monotone* if it is either increasing or decreasing.

Theorem 2.4.1. (Monotone Convergence Theorem) If a sequence is monotone and bounded, it converges.

Side note: this theorem is not generally true of $\mathbb{Q}$, consider the decimal expansion of $\sqrt{2}$, it is approaching $\sqrt{2}$ but it does not have a limit in $\mathbb{Q}$. This would suggest us using some property that $\mathbb{R}$ has but $\mathbb{Q}$ does not: Axiom of Completeness.

Proof. Without loss of generality, let (a_n) be increasing. Let $A = \{a_n : n \in \mathbb{N}\}$. Then the set is non-empty and bounded by assumption. Let $s = \sup(A)$. Then let $\epsilon > 0$, we want to show that the sequence is in $(s - \epsilon, s + \epsilon)$ eventually. Since $s = \sup(A)$, and $s - \epsilon < s$, then $\exists k$ such that $a_k > s - \epsilon$ (otherwise $s - \epsilon$ is an upperbound, which is a contradiction). Let $N = k$, if $n \geq N$, we have

$$s + \epsilon > s \geq a_n \geq a_N \geq s - \epsilon$$

□

Def 2.4.2. Let (b_n) be a sequence, a *infinite series* is the formal expression

$$\sum_{k=1}^{\infty} b_k$$

It converges to B, if the sequence of partial sums

$$S_n = \sum_{k=1}^{n} b_k$$

converges to B.

An example of a convergent infinite series: Let $(b_n) = \frac{1}{2^n}$, then $S_n = \sum_{k=1}^{n} b_k$ converges to 2.

Lecture 2.9
In order to show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges:

Let $S_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$, then

$$0 \leq S_n \leq 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} \dots$$

Then all we need is to show that the right hand side is bounded. Note that

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

And we rearrange the terms:

$$0 \leq S_n \leq 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4} + \dots) = 2 - \frac{1}{n^2}$$

Thus by the Monotone Convergence Theorem that if a sequence is monotone and bounded, then it converges.
In fact $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Not proved yet)

Now what about $\sum_{n=1}^{\infty} \frac{1}{n}$?

Then let

$$\begin{aligned} S_n &= \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \\ &> 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + \dots \\ S_{2^n} &> \frac{1}{2} \cdot n + 1 \end{aligned}$$

Then as n goes to infinity, $S_n = \infty$.

2.5. **Subsequences and the Bolzano–Weierstrass Theorem.**

Def 2.5.1. Let (a_n) be a sequence $\in \mathbb{R}$, let $n_1 < n_2 < n_3 < \dots$ be an increasing sequence in $\mathbb{N}$. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, \dots)$$

is a *subsequence* of (a_n), denoted as (a_{n_k}), where $k \in \mathbb{N}$ indexes the sequence.

Remark: $n_k \geq k$.

Theorem 2.5.1. Subsequences of a convergent sequence converge to the same limit as that sequence.

Proof. We want to show $(a_n \to a)$. Then let $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that $\forall n > N$, $|a_n - a| < \epsilon$. Let $k \geq N$, then $n_k \geq k \geq N$ so $|a_{n_k} - a| < \epsilon$. □

The contrapositive of the theorem is also useful: if two subsequences do not converge to the same value, then the original sequence does not converge.

Theorem 2.5.2. (Bolzano-Weierstrass Theorem) Every bounded sequence contains a convergent subsequence.

This theorem is not generally true of $\mathbb{Q}$, and we need Axiom of Completeness or Nested Interval Property for the proof.

Proof. Let (a_n) be a bounded sequence, so $\exists M > 0$ such that $a_n \in [-M, M]$ for all n. Then name our first interval as

$$I_0 = [-M, M]$$

Then either $[-M, 0]$ or $[0, M]$ gets hit infinitely many times. Then without loss of generality assume $[0, M]$ gets hit infinitely many times and let

$$I_1 = [0, M]$$

Inductively, assume we have I_n and I_n gets hit infinitely many times, assume $I_n = [a, b]$, then either $[a, \frac{a+b}{2}]$ or $[\frac{a+b}{2}, b]$ gets hit infinitely many times, then call that infinite interval I_{n+1} (if both infinite, choose arbitrarily). Then by construction we have

$$I_1 \supseteq I_2 \supseteq I_3 \ldots$$

By the Nested Interval Property, there exists $x \in \bigcap_{i=1}^{\infty} I_n$.

Now we want to show that there is a sequence that has limit x. Let $n_0 = 0$ and $a_{n_0} \in I_0$, let $n_1 > n_0$ such that $a_{n_1} \in I_1$. We can find such n_1 since I_1 has infinitely many points and we may pick one that is bigger than n_1. Inductively, having chosen n_k, let $n_{k+1} > n_k$ such that $a_{n_{k+1}} \in I_{k+1}$, then every $a_{n_k} \in I_k$.
Now let $\epsilon > k$, then choose N such that $\frac{M}{2^{N-2}} < \epsilon$. If $k \geq N$, then $|I_k| < |I_N| < \epsilon$ ($||$ here denotes the length of the interval), then since $x \in I_k$, and we have $a_{n_k} \in I_k$, which is equivalent to $a_{n_k} \in [x - \epsilon, x + \epsilon]$. Then $|a_{n_k} - x| < \epsilon$. □

So far, our proof begins with finding a candidate for the limit and prove that the sequence converges to that limit. Now we introduce another criterion for determining whether the sequence converges.

2.6. **The Cauchy Criterion.**

Def 2.6.1. (x_n) is *Cauchy* if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$,

$$|x_n - x_m| < \epsilon$$

We now show that this definition is equivalent to the definition of convergence.We first show the forward direction, that is, if a sequence converges, it is Cauchy.

Proof. Suppose $(x_n) \to x$. Let $\epsilon > 0$, choose N such that for all $n \geq N$,

$$|x_n - x| < \frac{\epsilon}{2}$$

Let $n, m \geq N$,

$$|x_n - x_m| = |(x_n - x) + (x - x_m)| \leq |x_n - x| + |x_m - x| < \epsilon$$

□

Lecture 2.14

We show the backward direction of Cauchy Criterion: If a sequence is Cauchy, then it converges.

Proof idea: since this one is also not true of $\mathbb{Q}$, we recall some big theorems related to the Axiom of Completeness so far:

(1) NIP
(2) Monotone Convergence theorem
(3) Bolzano-Weierstrass Theorem

This time we use the most recent theorem, Bolzano-Weierstrass Theorem.

Proof. Assume (a_n) is Cauchy, then the sequence is bounded (a similar proof to why convergent sequence is bounded). By Bolzano-Weierstrass Theorem, there exists a convergent subsequence of (a_n), call it (a_{n_k}), and call its limit a.
Let $\epsilon > 0$, then choose N_1 such that $\forall n, m > N_1$ we have $|a_n - a_m| < \frac{\epsilon}{2}$, then choose N_2 such that $\forall k > N_2$ we have $|a_{n_k} - a| < \frac{\epsilon}{2}$. Then let $N = \max(N_1, N_2)$, then for any $n > N$, choose $k > n$, then $n_k \geq k \geq n > N$, so

$$|a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

We switch our mind off the sequence and we consider Cauchy criterion for series:

Theorem 2.6.1. $\sum_{k=1}^{\infty} a_k$ converges if and only if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n, m > N$ with $m > n$, it follows $|a_{n+1} + \cdots + a_m| < \epsilon$.

Proof. Apply the Cauchy Criterion to the sequence of the partial sums. If $S_n = \sum_{k=1}^{n}$, then S_n converges if and only if it is Cauchy. Then note that the Cauchy Criterion states that $\exists N \in \mathbb{N}$ such that $\forall m, n$, we have $|s_m - s_n| < \epsilon$ which is equivalent to $|a_{n+1} + \cdots + a_m| < \epsilon$ (assume without loss of generality $m > n$). □

2.7. **Properties of Infinite Series.**

Theorem 2.7.1. If $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.

Proof. Let $m = n + 1$ in proof of Theorem 2.6.1 (My numbering is a bit different from the professor's) □

Theorem 2.7.2. (Comparison test) Suppose (a_k) and (b_k) are sequences with $0 \leq a_n \leq b_n$, if $\sum_{k=1}^{\infty} b_k$ converges, so does $\sum_{k=1}^{\infty} a_k$.

Proof. Use Theorem 2.6.1, noting $|a_{n+1} + \cdots + a_m| \leq |b_{n+1} + \cdots + b_m|$. □

Theorem 2.7.3. (Absolute value test) If $\sum_{k=1}^{\infty} |a_k|$ converges, so does $\sum_{k=1}^{\infty} a_k$.

Proof. Use Theorem 2.6.1, noting $|a_{n+1} + \cdots + a_m| < |a_{n+1}| + \cdots + |a_m|$ by triangular inequality. □

Remark: The converse is not necessarily true, a counterexample is the alternating harmonic series: $\sum_{k=1}^{\infty} a_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$.

We now show that

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \ldots$$

converges. Note that

$$\frac{S}{2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} \dots$$

And we add them up we have

$$\frac{3S}{2} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} \dots$$

To show that S actually converges, we would first introduce a new theorem:

Theorem 2.7.4. Suppose $\sum_{k=1}^{\infty} a_k$ converges to A and $\sum_{k=1}^{\infty} b_k$ converges to B, then

(1) $\sum_{k=1}^{\infty}(a_k + b_k)$ converges to $A + B$.
(2) for all $c \in \mathbb{R}$, $\sum_{k=1}^{\infty} c \cdot (a_k) = cA$.

Proof for (1):

Proof. Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$. Let $r_n = \sum_{k=1}^{n}(a_k + b_k)$ and $r_n = s_n + t_n$, and we may apply Algebraic Limit Theorem. The proof is similar for (2), note $c \cdot s_n = \sum_{k=1}^{n} ca_k$. $\square$

Theorem 2.7.5. (Alternating Series Test) If $(a_n) \to 0$, and $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$, then $\sum_{k=1}^{\infty}(-1)^{k+1} a_k$ converges.

Proof. Let $S_n = \sum_{k=1}^{\infty}(-1)^{k+1} a_k$. We want to show that S_{2n+1} is decreasing. We need to show $S_{2k+1} \leq S_{2k+3}$. Then

$$S_{2k+3} - S_{2k+1} = -a_{2k+2} + a_{2k+3} \leq 0$$

$\square$

Lecture 2.16
We continue our discussion of Alternating Series Test

Proof. (continue) Now we need to show $S_{2k} \geq S_{2k}$, then

$$S_{2k+2} - S_{2k} = a_{k+1} - a_{k+2} \geq 0$$

Now we want to show that S_n is Cauchy. Let $\epsilon > 0$, choose even N such that $a_N \leq \epsilon$ (since $(a_n) \to 0$), let $n \geq N+2$. If n is even, then $S_n \leq S_N$, then $S_n \leq S_{n-1}$ since the even terms of (a_n) takes on the negative value. We have

$$S_n \leq S_{n-1} \leq S_{N+1}$$

Hence, if n is odd, then $S_n \geq S_{n-1} \geq S_N$, and $S_n \leq S_{N+1}$. Either way,

$$S_N \leq S_n \leq S_{N+1}$$

Then, if $m, n \geq N+2$, then $m, n \in [S_N, S_{N+1}]$. So $|S_m - S_n| \leq S_{N+1} - S_N = a_{N+1} \leq a_N < \epsilon$. $\square$

We've seen from the last lecture that we can rearrange the terms of a series. We rigorously formulate the notion of rearrangement.

Def 2.7.1. $\sum_{k=1}^{\infty} b_k$ is a *rearrangement* of $\sum_{k=1}^{\infty} a_k$ if $\exists$ a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all k.

Theorem 2.7.6. If $\sum_{k=1}^{\infty} a_k$ converges absolutely and $\sum_{k=1}^{\infty} b_k$ is a rearrangement of $\sum_{k=1}^{\infty} a_k$, then $\sum_{k=1}^{\infty} b_k$ converges to the same limit.

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation such that $b_k = a_{f(k)}$. Suppose $s_n = \sum_{k=1}^{\infty} a_k \to a$, and $t_n = \sum_{k=1}^{\infty} b_k$. Now we want to show that $(t_m) \to a$.

Let $\epsilon > 0$, choose N_1 such that $|s_n - a| \leq \frac{\epsilon}{2}$ for all $n \geq N_1$. (Since $\sum_{k=1}^{\infty} |a_k| < a' < \infty$) Choose N_2 such that $\sum_{k=N_2+1}^{\infty} |a_k| \leq \frac{\epsilon}{2}$. Let $N = \max(N_1, N_2)$, let $M = \max\{f(k) : 1 \leq k \leq N\}$. Let $m \geq M$,

$$|t_m - s_N| = |\sum_{k \leq m, f(k) > N}^{\infty} b_k|$$

$$\leq \sum_{k \leq m, f(k) > N}^{\infty} |b_k| \leq \sum_{f(k) > N}^{\infty} |b_k| = \sum_{k=N+1}^{\infty} |a_k| \leq \frac{\epsilon}{2}$$

Then

$$|t_m - a| = |t_m - s_N + s_N - a| \leq |t_m - s_N| + |s_N - a| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\square$

(One comment: there is a subtle issue with this proof that when we use $f(k)$, sometimes it means $f'(k)$, the inverse function, but I think this does not matter quite much if we understand the meaning of each $f(k)$ here).

3. Basic Topology of $\mathbb{R}$

3.1. Discussion: The Cantor set. Consider

$$C_0 = [0, 1]$$
$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$
$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$

Then define the *Cantor set* to be the intersection

$$C = \bigcap_{i=0}^{\infty} C_n$$

Some claims: C is of length zero, and C is uncountable. An intuition may be if we write each number between $[0, 1]$ in binary expansion, then 0 and 1 indicates different direction that we go, so we can locate each number in C. Furthermore, no two elements can go to the same closed interval since we can always find a closed interval that is of shorter length than that interval. Thus we can build a bijection between C and $[0, 1]$.

Lecture 2.21

A way to think about **dimension**:
If you stretch a point, you still have a point, so the dimension of a point is of dimension 0.
If you stretch an interval by three, then you have an interval with 3 times of the original length, so an interval would have dimension 1.
If you stretch a cube by 3 times then you end up with a cube 3^3 times of its original volume, so the cube has dimension 3.
When you stretch a Cantor set by 3 times, you end up having 2 copies of the Cantor set then the dimension of C is $\log_3(2)$.

3.2. **Open and closed sets.**

Def 3.2.1. A set O is *open* if $\forall x \in O$, $\exists \epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \in O$.

It is a good definition since you can generalize this notion to any metric space.

Examples of open sets:
1.We use our definition to show that (a,b) is open. Let $x \in (a,b)$, then let $\epsilon = \min(x-a, b-x)$, then $(x-\epsilon, x+\epsilon) \subset (a,b)$.
2.Using the same logic, we can show that $(a,b) \cup (c,d)$ is also open.
3.$\mathbb{R}$ is open
4.$\emptyset$ is also open since there is no element in the empty set and everything is trivially true.

Theorem 3.2.1. (a) The union of any collection of open set is open.
(b) The intersection of any *finite* collection of open sets is open. (The counterexample is $\bigcap_{i=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$)

Proof. Let $\{O_\lambda : \lambda \in \Lambda\}$ be a collection of open sets, and let $O = \bigcup_{\lambda\in\Lambda} O_\lambda$. Let $x \in O$, then $x \in O_\lambda$ for some $\lambda \in \Lambda$, then by assumption that O_λ is open, there exists a $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset O_\lambda \subset O$. Thus concludes the proof for part (a). (Side note: $(x-\epsilon, x+\epsilon)$ is called as epsilon neighborhood.)

Proof for part (b): Let $(O_i)_{i=1}^n$ be a finite collection of open sets. Let $x \in O = \bigcap_{i=1}^n O_i$. Then $x \in O_i$ for all i, then for each $1 \leq i \leq n$, $\exists \epsilon_i$ such that $(x-\epsilon_i, x+\epsilon_i) \subseteq O_i$. Then let $\epsilon = \min(\epsilon_i : 1 \leq i \leq n)$. Then $\epsilon > 0$ (here's where it breaks for uncountably many open sets, you cannot find such ϵ.), and $(x-\epsilon, x+\epsilon) \subseteq (x-\epsilon_i, x+\epsilon_i)$ for all i, hence $(x-\epsilon, x+\epsilon) \subseteq \bigcap_{i=1}^n O_i$

□

Def 3.2.2. Let C be a set. We say C is *closed* if for any convergent sequence (x_n) with $x_n \in C$, $\lim_{n\to\infty} x_n = x$, we have $x \in C$.

Examples:
1.$[a,b]$ let $x_n \to x$, with $a \leq x_n \leq b$, then by the Order Limit Theorem, $a \leq x \leq b$, so $x \in [a,b]$.
2.Similarly, $[a,b] \cup [c,d]$ is also closed.
3.$\{x\}$ is closed

4.$\mathbb{R}$ is closed
5.$\emptyset$ is closed

Theorem 3.2.2. A set is closed if and only if its complement is open.

Proof. We first show the backward direction. Suppose O is open. Let (x_n) be a convergent sequence in O^c, say $(x_n) \to x$, then we need to show $x \in O^c$. Assume toward contradiction that $x \in O$, then $\exists \epsilon > 0$, such that $(x-\epsilon, x+\epsilon) \subset O$. Then since (x_n) is convergent, then $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|x_n - x| \in (x-\epsilon, x+\epsilon)$, however, $x_n \in O^c$ and we reached a contradiction.

Now suppose C is closed then we need to show that C^c is open. Let $x \in C^c$, assume toward contradiction that $\forall \epsilon > 0$, $(x-\epsilon, x+\epsilon) \cap C \neq \emptyset$. Then $\forall n \in \mathbb{N}$, $\exists$ a point $x_n \in (x-\frac{1}{n}, x+\frac{1}{n}) \cap C$, then we want to show that $(x_n) \to x$. ($\forall \epsilon > 0, \exists N$ such that $\frac{1}{N} \leq \epsilon$, then for $n > N$, $|x_n - x| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$) Since C is closed so $x \in C$, which yields a contradiction. $\square$

Lecture 2.23

Theorem 3.2.3. The finite union of closed sets in closed, the arbitrary intersection of closed sets is closed.

The counterexample for arbitrary union not being closed: $\bigcap_{n=1}^{\infty}[\frac{1}{n}, 1-\frac{1}{n}]$, then as n goes to infinity, then $\bigcap_{n=1}^{\infty}[\frac{1}{n}, 1-\frac{1}{n}] \to (0,1)$, which is open.

Proof. We use the De Morgan's Law here: consider $\{C_\lambda : \lambda \in \Lambda\}$, we have $(\bigcap_{\lambda\in\Lambda} C_\lambda)^c = \bigcup_{\lambda\in\Lambda} C_\lambda^c$ is open, and $(\bigcup_{i=1}^{n} C_i)^c = \bigcap_{i=1}^{n} C_i^c$ is closed. □

Side note: Is Cantor set open or closed? The complement of an Cantor set is a union of open sets, then Cantor set (as its complement) must be closed.

Three parallel definitions of *closure* of a set:

Def 3.2.3. For a set A, $\overline{A}$, the closure of A, is the smallest closed set containing A.

Def 3.2.4. Let L be the set of limits of sequences in A, then $\overline{A} = A \cup L$.

Def 3.2.5. Let

$$\overline{A} = \bigcap_{\text{closed sets C s.t.} A\subseteq C} C$$

That ends temporarily our discussion of closure.

3.3. **Compact set.**

Def 3.3.1. A set K is *compact* if every sequence in K has a convergent subsequence with a limit in K.

It is equivalent to say K is compact and K is closed and bounded, which is equivalent to every open cover of K has a finite subcover.

Theorem 3.3.1. $K \subseteq \mathbb{R}$ is compact if and only if K is closed and bounded.

Proof. ($\Leftarrow$) Let K be a closed and bounded set. Let (k_n) be a sequence in K. By Bolzano–Weierstrass theorem, $\exists$ convergent subsequence, since K is closed, its limit is in K, thus K is compact.

($\Rightarrow$) Assume K is compact. We first show that K is closed. Let (k_n) be an convergent sequence, say $(k_n) \to k$. Since K is compact, $\exists$ a subsequence (k_{n_r}) such that $k_{n_r} \to k' \in K$. Since every subsequence of a convergent sequence converges to the same limit, so $k' = k$, therefore $k \in K$.

Now we show that K is bounded, assume K is bounded, then $\forall M > 0$, $\exists k_m \notin [-M, M]$, then $(k_n)_{n=1}^{\infty}$ is unbounded, and any subsequence is also unbounded, since for any subsequence (k_{n_r}), we have $k_{n_r} \notin [-n_r, n_r] \supset [-r, r]$. Thus K is bounded. □

Def 3.3.2. An *open cover* of a set A is a collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ where

$$A \subseteq \bigcup_{\lambda\in\Lambda} O_\lambda$$

And a *finite subcover* is a collection $\{O_\lambda : \lambda \in \Lambda^*\}$ where $A \subseteq \bigcup_{\lambda \in \Lambda^*} O\lambda$ and $\Lambda^* \subseteq \Lambda$ and Λ^* is finite.

Theorem 3.3.2. If K is closed and bounded then every open cover of K has a finite subcover.

Lecture 2.28

We continue the proof of Theorem 3.2.2. We start with an example of $[0,1]$:

Proof. Let $K=[0,1]$, let $O=\{O_\lambda : \lambda\in\Lambda\}$ be an open cover of $[0,1]$. Note that $\exists[0,x]$ such that a finite subset of O covers $[0,x]$ with $x>0$.
Why? Because $0\in O_\lambda$ for some λ, since O_λ is open, then $\exists\epsilon$ such that $(-\epsilon,\epsilon)\subseteq O_\lambda$, so

$$[0,\frac{\epsilon}{2}]\subseteq O_\lambda$$

so O_λ covers $[0,\frac{\epsilon}{2}]$. Let y be the supremum of $\{x : x\in[0,1]\}$ such that $[0,x]$ is covered by a finite subset of O. I know $y>0$, and $y\in O_\lambda$ for some $\lambda\in\Lambda$, so $\exists\epsilon$ such that

$$(y-\epsilon,y+\epsilon)\subseteq O_\lambda$$

By the definition of y, there exists a finite subset $\{O_\lambda : \lambda\in\Lambda^*\}$ with $\Lambda^*\subseteq\Lambda$ of O that covers $[0,y-\frac{\epsilon}{2}]$ where $\epsilon<y$. Now $\{O_\lambda : \lambda\in\Lambda^*\}\cup O_\lambda$ covers $[0,y+\frac{\epsilon}{2}]$. If $y<1$, then this contradicts the definition of y, if $y=1$, we have a finite subcover of $[0,1]$. □

Here comes the official proof:

Proof. Let $K\subseteq[-M,M]$ and let K be closed. Then let $O=\{O_\lambda : \lambda\in\Lambda\}$ be an open cover of K. Let y be the supremum of $\{x : x\in[-M,M]\}$ such that $[-M,x]\cap K$ is covered by a finite subset of O. There are some cases to worry about:

(1) Case 1: $y\in K$, then $y\in K^c$ and K^c is open, so $\exists\epsilon$ such that

$$(y-\epsilon,y+\epsilon)\subseteq K^c$$

Then $K\cap[-M,y-\frac{\epsilon}{2}]$ is covered by a finite subset of O. Then

$$K\cap[-M,y+\frac{\epsilon}{2}]=[-M,y-\frac{\epsilon}{2}]$$

Thus $[-M,y+\frac{\epsilon}{2}]$ is covered by a finite subset of O. Then if $y<M$ this is a contradiction, so $y=M$ and this means K is covered by a finite subset fo O.

(2) Case 2: $y\in K$, $\exists O_\lambda$ such that $y\in O_\lambda$, so $\exists\epsilon>0$ such that

$$(y-\epsilon,y+\epsilon)\subseteq O_\lambda$$

Then $K\cap[-M,y-\frac{\epsilon}{2}]$ is covered by $\{O_\lambda : \lambda\in\Lambda^*\}$ for $\Lambda^*\subseteq\Lambda$, then $K\cap[-M,y+\frac{\epsilon}{2}]$ is covered by $\{O_\lambda : \lambda\in\Lambda^*\}\cup O_\lambda$. If $y<M$ then there is a contradiction, else of $y=M$ then K is covered by a finite subset of O.

□

Theorem 3.3.3. (Nested compact sets, a generalization of NIP) If $K_1\supseteq K_2\supseteq K_3\supseteq\dots$ with each K_i being non-empty and compact, then

$$\bigcap_{n=1}^{\infty}K_n$$

is non-empty.

Proof. Consider the sequence (x_n) such that $x_n\in K_n$ for all n, then each $x_n\in K_1$, then (x_n) has a convergent subsequence converging to a point $x\in K_1$, denoted by $(x_n)\to x$. Then each subsequence $(x_n,x_{n+1},x_{n+2},\dots)$ is in K_n, and $(x_n,x_{n+1},x_{n+2},\dots)\to x$. (Every subsequence of a convergent sequence converges to the same limit). Since K_n is compact, then $x \in K_n$. So

$$x \in \bigcap_{1}^{\infty} K_n$$

$\square$

Def 3.3.3. Let A be a set. A point $x \in \mathbb{R}$ is a *cluster point* of A if $\exists$ a sequence (a_n) in $A \setminus \{x\}$ such that $(a_n) \to x$.

Def 3.3.4. A point $a \in A$ is *isolated* if it is not a cluster point. Note: A set A is closed if and only if it contains all of its cluster points.

Def 3.3.5. If A is a set, let A' be its set of cluster points. A is *perfect* if $A = A'$, which is equivalent to A is closed and A has no isolated points.

An example of a perfect set: $\mathbb{R}$, $\emptyset$, any closed interval, the Cantor set.

Theorem 3.3.4. Every non-empty perfect set is uncountable.

Lecture 3.2
Definition of continuous on a point:

Def 4.0.1. Let $A \subseteq \mathbb{R}$. A function $f : A \to \mathbb{R}$ is *continuous* at a point $x \in A$ if for any sequences (a_n) in A converging to x, we have

$$f(a_n) \to f(x)$$

Note: A function is continuous at any isolated point of its domain.

Now we view another version of continuity:

Def 4.0.2. Let $A \subseteq \mathbb{R}$, a function $f : A \to \mathbb{R}$ is continuous at $x \in A$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then

$$|f(y) - f(x)| < \epsilon$$

This is particular useful when we want to prove some function is continuous.

FACT: two definitions are equivalent.

Proof. We first show that $(2 \Rightarrow 1)$. Let $f : A \to \mathbb{R}$ be continuous at a point x satisfying the second definition. Let (a_n) be a sequence in A converging to x.
Let $\epsilon > 0$, so by assumption, $\exists \delta > 0$, if $|y - x| \leq \delta$, then $|f(y) - f(x)| < \epsilon$. Since (a_n) converges to x, $\exists$ some $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - x| < \delta$, so $|f(a_n) - f(x)| < \epsilon$.

We now show that $(1 \Rightarrow 2)$. We proceed by contradiction. Let $f : A \to \mathbb{R}$ be continuous satisfying the first definition at x, suppose towards a contradiction that f is not continuous using the definition 2 at x. Then $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists y \in A$ with $|y - x| < \delta$ such that $|f(y) - f(x)| \geq \epsilon$. Then for each n, let $|a_n - x| < \frac{1}{n}$ and $|f(a_n) - f(x)| \geq \epsilon$. Then $(a_n) \to x$, by assumption, $f(a_n)$ converges to $f(x)$, but all elements in (a_n) are some distance away from $f(x)$, hence we reached a contradiction. $\square$

Def 4.0.3. A function $f : A \to \mathbb{R}$ is continuous if it is continuous at every point A.

We now investigate a concrete function, a modified version Dirichlet function:

$$D(x) = \begin{cases} 0 \text{ if } x \in \mathbb{R} - \mathbb{Q} \\ x \text{ if } x \in \mathbb{Q} \end{cases}$$

We will show that it is not continuous.

Proof. Let $x \in \mathbb{R}$, if $x \neq 0 \in \mathbb{Q}$, then take a sequence of irrational numbers approaching x and it breaks. Else if $x \in \mathbb{R} - \mathbb{Q}$, then take a rational sequence approaching x and the argument breaks. However, this function is continuous at 0:
Let $\epsilon > 0$, let $\delta = \epsilon$, if $|y - 0| < \delta$, then either $y \in \mathbb{Q}$, so $|f(y) - f(0)| = |y - 0| < \epsilon$ or $y \notin \mathbb{Q}$, so $|f(y) - f(0)| = |0 - 0| < \epsilon$. $\square$

Another example : Thomae's function

$$T(x) = \begin{cases} 0 \text{ if } x \in \mathbb{R} - \mathbb{Q} \\ \frac{1}{q} \text{ if } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

This function is continuous at irrational points if you take an ϵ neighborhood around irrational numbers. It is still not continuous at rational numbers.