Homework 1

Material derivative and error propagation

due January 25, 23:59

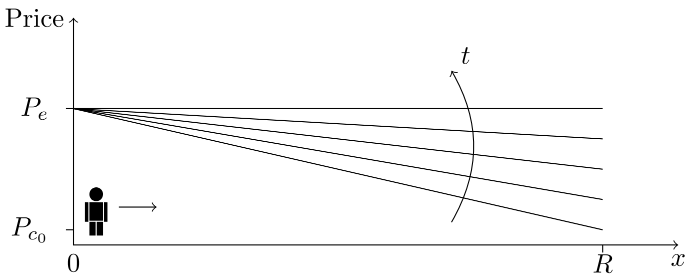
Guidelines:

- Use only a **single pdf**. Preferably scan your homework, if you take a photo, make sure it is sharp and bright;
- **Include all plots in the pdf** in the right place;
- Do not present plots without commenting them: always write a (short) description of what the plot tells you and what you can conclude from it;
- You can work in groups up to 3;
- If you work in groups:
  - write the names of all group members at the beginning of the report;
  - you can work and discuss together but each group member is required to submit an individual report written with his/her own words; **copy-paste reports will not be accepted**
- Please use the following naming convention: **"surname_hwX.pdf"** and include all Matlab files as a single **separate** archive: **"surname_hwX.zip"**. X is the number of the homework.

Task 1: Material derivative

Task 1

A man goes to a large, round market to buy fish. He does not have much money so he wants to spend as little as possible. His problem is that the fish stands at the edge of the market have higher prices than those at the centre: the price decreases linearly from the edge to the centre. To complicate his life, the price at the centre of the market rises linearly with time, and so does the price in the rest of the market, always changing linearly from edge to centre.



Given that:

- the market is a circle with radius $R = 5$ km;
- the man can walk at a maximum speed of 2.5 km/h;
- the price at the edge of the market is fixed to $P_e = 350$ kr/kg;

- at the start, $t = 0$, the price of fish at the centre of the market is $P_{c_0} = 40$ kr/kg;
- in one hour the price at the centre has risen such that the price is the same in the whole market: $P_f = 350$ kr/kg.

At what time and distance from the edge of the market will the man be able to buy the cheapest fish? How many kilograms of fish will he be able to buy if he has 130 kr with him?

(*Hint:* determine the price as a function of space and time, then use the definition of the material derivative: the price changes because the man is moving in a non-uniform 'price field' and because time passes)

Task 2: Machine precision

The following code can be used in MATLAB to determine the machine accuracy ε (or Machine epsilon):

```
numprec=double(1.0); % Define 1.0 with double precision
numprec=single(1.0); % Define 1.0 with single precision
while(1 < 1 + numprec)
numprec=numprec*0.5;
end
numprec=numprec*2
```

Alternatively, the following Python code can be used for the same purpose:

```
import numpy as np
numprec=1.0
while 1<np.float32(1+numprec):   # Check sum with single precision
# while 1<1+numprec:             # Check sum with double precision
    numprec*=0.5

numprec*=2
```

1. Determine ε using the above code, both for single and double precision. A double precision number uses 8 bytes of storage, whereas a single precision number only occupies 4 bytes.
2. Give a definition of the machine accuracy based on the code above (use words and not mathematical expression).

Task 3: Round-off Error

In this exercise, the errors involved in numerically calculating derivatives are examined.

1. The propagation error through a general function g of variables $a_1, a_2, ...$ with relative uncertainties $\varepsilon_1, \varepsilon_2, ...$ is defined as:

$$\xi_p^2 = \sum_{j=1}^{n} \left(\frac{a_j}{g} \frac{\partial g}{\partial a_j} \right)^2 \varepsilon_j^2 \tag{1}$$

   By applying this formula, show that the propagation error, ξ_p, of the sum of two numbers X_1 and X_2 is given by:

$$\xi_p^2 = \left(\frac{X_1}{X_1 + X_2} \varepsilon_1 \right)^2 + \left(\frac{X_2}{X_1 + X_2} \varepsilon_2 \right)^2 \tag{2}$$

   where ε_1 and ε_2 are the relative uncertainties.

2. In order to analyze the discretization error we consider the derivative of a function $f(x)$, approximated with the centered difference scheme

$$f'_{\mathrm{n}}(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}. \tag{3}$$

Your task is to compute, numerically, the relative discretization error of the derivative of the function

$$f(x) = \frac{1}{2+x} + x^2. \tag{4}$$

The estimate of the relative discretization error is given by:

$$\varepsilon_d = \left| \frac{f'(x) - f'_{\mathrm{n}}(x)}{f'(x)} \right| \tag{5}$$

where $f'(x)$ is the analytic derivative of $f(x)$. Compute ε_d for $x = 2$ and use step sizes $\Delta x = 10^{-20}, ..., 10^0$. Use both single and double precision for the calculation and present the results in a double logarithmic plot (ε_d vs. Δx). You can use the function *loglog* in MATLAB. Note that all the variables used here should be defined as double or single precision.

3. Show that, when using the proposed central differences schemes, that the propagation error is given by:

$$\xi_p^2 = \left(\frac{f(x)\varepsilon}{f'(x)\sqrt{2}\Delta x} \right)^2 \tag{6}$$

where ε is the machine accuracy (hint: f'_{n} is the sum of two terms and Δx is small, which helps in evaluating $f(x \pm \Delta x)$).

Furthermore, show that the relative discretization error is given by:

$$\xi_d = \frac{\Delta x^2 |f'''(x)|}{6|f'(x)|} \tag{7}$$

(use the Taylor expansion to do so)

Find, analytically, the value of Δx that minimises the total error:

$$\xi_g^2 = \xi_p^2 + \xi_d^2 \tag{8}$$

Plot ξ_d, ξ_p and ξ_g in a double logarithmic plot (vs. Δx).