Fluid Dynamics:
Theory and Computation

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Contents

Preface

These lecture notes has evolved from a CFD course (5C1212) and a Fluid Mechanics course (5C1214) at the department of Mechanics and the department of Numerical Analysis and Computer Science (NADA) at KTH. Erik Stålberg and Ori Levin has typed most of the LaTeXformulas and has created the electronic versions of most figures.

Stockholm, August 2004
Dan Henningson
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In the latest version of the lecture notes study questions for the CFD course 5C1212 and recitation material for the Fluid Mechanics course 5C1214 has been added.

Stockholm, August 2005
Dan Henningson

Chapter 1

Derivation of the Navier-Stokes equations

1.1   Notation

The Navier-Stokes equations in vector notation has the following form

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} &= -\dfrac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \\[2mm] \nabla \cdot \mathbf{u} &= 0 \end{cases}$$

where the velocity components are defined

$$\mathbf{u} = (u, v, w) = (u_1, u_2, u_3)$$

the nabla operator is defined as

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$$

the Laplace operator is written as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and the following definitions are used

$$\begin{array}{rcl} \nu & - & \text{kinematic viscosity} \\ \rho & - & \text{density} \\ p & - & \text{pressure} \end{array}$$

see figure 1.1 for a definition of the coordinate system and the velocity components.
The Cartesian tensor form of the equations can be written

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + u_j \dfrac{\partial u_i}{\partial x_j} &= -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_i} + \nu \dfrac{\partial^2 u_i}{\partial x_j \partial x_j} \\[2mm] \dfrac{\partial u_i}{\partial x_i} &= 0 \end{cases}$$

where the summation convention is used. This implies that a repeated index is summed over, from 1 to 3, as follows

$$u_i u_i = u_1 u_1 + u_2 u_2 + u_3 u_3$$

Thus the first component of the vector equation can be written out as
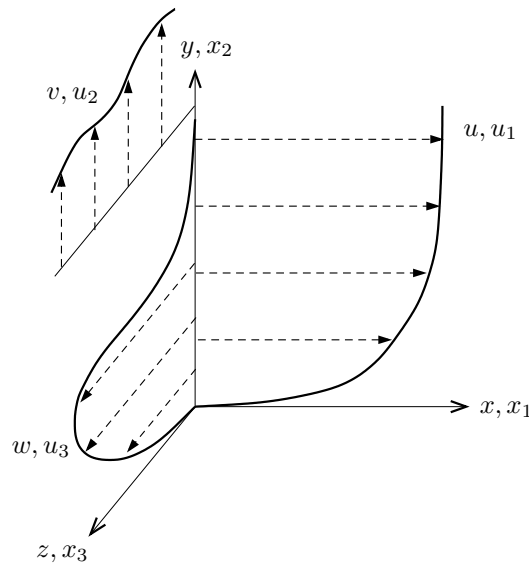
Figure 1.1: Definition of coordinate system and velocity components
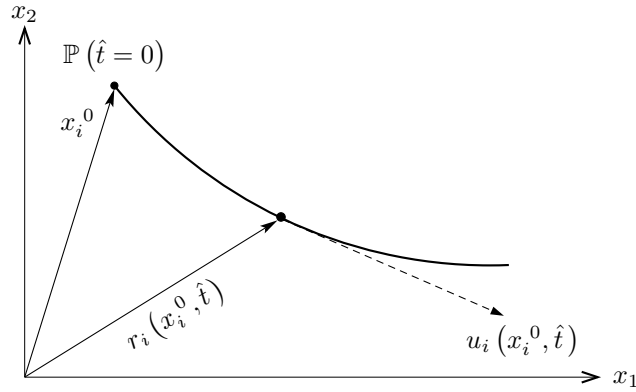
Figure 1.2: Particle path.

$$i = 1 \quad \frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + u_3\frac{\partial u_1}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu\left(\frac{\partial^2 u_1}{\partial x_1{}^2} + \frac{\partial^2 u_1}{\partial x_2{}^2} + \frac{\partial^2 u_1}{\partial x_3{}^2}\right)$$

$$\text{or} \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{\nabla^2 u}$$

1.2   Kinematics

Lagrangian and Euler coordinates

Kinematics is the description of motion without regard to forces. We begin by considering the motion of a fluid particle in Lagrangian coordinates, the coordinates familiar from classical mechanics.

Lagrange coordinates: every particle is marked and followed in flow. The independent variables are

$$x_i{}^0 \quad - \quad \text{initial position of fluid particle}$$

$$\hat{t} \quad - \quad \text{time}$$

where the particle path of $\mathbb{P}$, see figure 1.2, is

$$r_i = r_i\left(x_i{}^0, \hat{t}\right)$$

and the velocity of the particle is the rate of change of the particle position, i.e.

$$u_i = \frac{\partial r_i}{\partial \hat{t}}$$

Note here that when $x_i{}^0$ changes we consider new particles. Instead of marking every fluid particle it is most of the time more convenient to use Euler coordinates.

Euler coordinates: consider fixed point in space, fluid flows past point. The independent variables are

$$x_i \quad - \quad \text{space coordinates}$$

$$t \quad - \quad \text{time}$$

Thus the fluid velocity $u_i = u_i(x_i, t)$ is now considered as a function of the coordinate x_i and time t.

The relation between Lagrangian and Euler coordinates, i.e. $(x_i{}^0, \hat{t})$ and (x_i, t), is easily found by noting that the particle position is expressed in fixed space coordinates x_i, i.e.

$$\begin{cases} x_i &= r_i\left(x_i{}^0, \hat{t}\right) \qquad \text{at the time} \\ t &= \hat{t} \end{cases}$$

Material derivative

Although it is usually most convenient to use Euler coordinates, we still need to consider the rate of change of quantities following a fluid particle. This leads to the following definition.

Material derivative: rate of change in time following fluid particle expressed in Euler coordinates.

Consider the quantity F following fluid particle, where

$$F = F_{\mathrm{L}}\left(x_i{}^0, \hat{t}\right) = F_{\mathrm{E}}(x_i, t) = F_{\mathrm{E}}\left(r_i\left(x_i, \hat{t}\right), t\right)$$

The rate of change of F following a fluid particle can then be written

$$\frac{\partial F}{\partial \hat{t}} = \frac{\partial F_{\mathrm{E}}}{\partial x_i} \cdot \frac{\partial r_i}{\partial \hat{t}} + \frac{\partial F_{\mathrm{E}}}{\partial t} \cdot \frac{\partial t}{\partial \hat{t}} = \frac{\partial F_{\mathrm{E}}}{\partial t} + u_i \frac{\partial F_{\mathrm{E}}}{\partial x_i}$$

Based on this expression we define the material derivative $\dfrac{D}{Dt}$ as

$$\frac{\partial}{\partial \hat{t}} \equiv \frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$$

In the material or substantial derivative the first term measures the local rate of change and the second measures the change due to the motion with velocity u_i.

As an example we consider the acceleration of a fluid particle in a steady converging river, see figure 1.3. The acceleration is defined

$$a_j = \frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}$$

which can be simplified in 1D for stationary case to

$$a = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}$$

Note that the acceleration $\neq 0$ even if velocity at fixed x does not change. This has been experienced by everyone in a raft in a converging river. The raft which is following the fluid is accelerating although the flow field is steady.

Description of deformation

Evolution of a line element

Consider the two nearby particles in figure 1.4 during time $d\hat{t}$. The position of $\mathbb{P}_2$ can by Taylor expansion be expressed as

$$\begin{aligned} \mathbb{P}_2 \quad : \quad & r_i\left(d\hat{t}\right) + dr_i\left(d\hat{t}\right) = \\ & = r_i(0) + \frac{\partial r_i}{\partial t}(0)\, d\hat{t} + dr_i(0) + \frac{\partial}{\partial t}(dr_i)\Big|_0 d\hat{t} \\ & = x_i{}^0 + dx_i{}^0 + u_i(0)\, d\hat{t} + du_i(0)\, d\hat{t} \end{aligned}$$

x, x_1
y, x_2
z, x_3
u, u_1
v, u_2
w, u_3
x_1
x_2
$x_i{}^0$
$r_i\left(x_i{}^0, \hat{t}\right)$
$u_i\left(x_i{}^0, \hat{t}\right)$
$\mathbb{P}\left(\hat{t}=0\right)$

x

u_1

$u_2 > u_1$
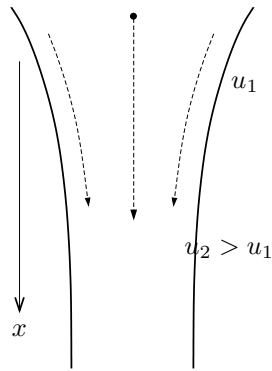
Figure 1.3: Acceleration of fluid particles in converging river.

x, x_1
y, x_2
z, x_3
u, u_1
v, u_2
w, u_3
x_1
x_2
$x_i{}^0$
$r_i\left(x_i{}^0, \hat{t}\right)$
$u_i\left(x_i{}^0, \hat{t}\right)$
$\mathbb{P}\left(\hat{t}=0\right)$

x
u_1
$u_2 > u_1$

x_2

$\mathrm{d}x_i{}^0$

$\mathrm{d}u_i\,\mathrm{d}\hat{t}$

$\mathbb{P}_2$

$u_i\,\mathrm{d}\hat{t}$

$\mathrm{d}r_i\left(\mathrm{d}\hat{t}\right)$

$\mathbb{P}_1$

$x_i{}^0$

$r_i\left(\mathrm{d}\hat{t}\right)$

x_1

x_3
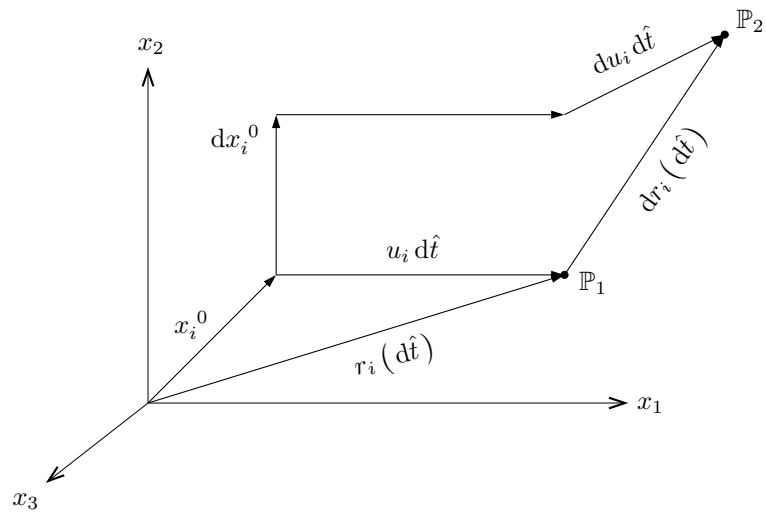
Figure 1.4: Relative motion of two nearly particles.

where we have used

$$
\begin{aligned}
\frac{\partial}{\partial \hat{t}} \left(\mathrm{d}r_i \right) &= \frac{\partial}{\partial \hat{t}} \left(\frac{\partial r_i}{\partial x_j{}^0} \right) \mathrm{d}x_j{}^0 = \frac{\partial}{\partial x_j{}^0} \left(\frac{\partial r_i}{\partial \hat{t}} \right) \mathrm{d}x_j{}^0 \\
&= \frac{\partial u_i}{\partial x_j{}^0} \mathrm{d}x_j{}^0 = \mathrm{d}u_i
\end{aligned}
$$

and where

$$
\begin{array}{cl}
\dfrac{\partial u_i}{\partial x_j{}^0} & - \quad \text{change of } u_i \text{ with initial pos.} \\
\mathrm{d}x_j{}^0 & - \quad \text{difference in initial pos.} \\
\mathrm{d}u_i & - \quad \text{difference in velocity}
\end{array}
$$

We can transform the expression $\dfrac{\partial}{\partial \hat{t}} \left(\mathrm{d}r_i \right) = \mathrm{d}u_i$ in Lagrange coordinates to an equation for a material line element in Euler coordinates

$$
\begin{aligned}
\frac{D}{Dt} \left(\mathrm{d}r_i \right) &= \mathrm{d}u_i \\
&= \{\text{expand in Euler coordinates}\} \\
&= \frac{\partial u_i}{\partial x_j} \mathrm{d}r_j
\end{aligned}
$$

where

$$
\begin{array}{cl}
\dfrac{\partial u_i}{\partial x_j} & - \quad \text{change in velocity with spatial position} \\
\mathrm{d}r_j & - \quad \text{difference in spatial pos. of particles}
\end{array}
$$

Relative motion associated with invariant parts

We consider the relative motion $\mathrm{d}u_i = \dfrac{\partial u_i}{\partial x_j} \mathrm{d}r_j$ by dividing $\dfrac{\partial u_i}{\partial x_j}$ in its invariant parts, i.e.

$$
\frac{\partial u_i}{\partial x_j} = \xi_{ij} + \underbrace{\overline{e}_{ij} + \overline{\overline{e}}_{ij}}_{e_{ij}}
$$

where e_{ij} is the deformation rate tensor and

$$
\begin{aligned}
\xi_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial u_j} - \frac{\partial u_j}{\partial x_i} \right) && \text{anti-symmetric part} \\
\overline{e}_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_r}{\partial u_r} \delta_{ij} \right) && \text{traceless part} \\
\overline{\overline{e}}_{ij} &= \frac{1}{3} \frac{\partial u_r}{\partial u_r} \delta_{ij} && \text{isotropic part}
\end{aligned}
$$

The symmetric part of $\dfrac{\partial u_i}{\partial x_j}$, e_{ij}, describes the deformation and is considered in detail below, whereas the anti-symmetric part can be written in terms of the vorticity ω_k and is associated with solid body rotation, i.e. no deformation.

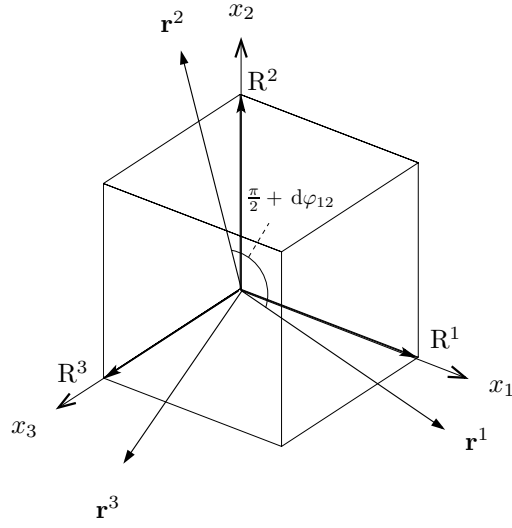The anti-symmetric part can be written

Figure 1.5: Deformation of a cube.

$$[\xi_{ij}] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) & -\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) & 0 \end{bmatrix}$$

$$= \{\text{use } \boldsymbol{\omega} = \nabla \times \mathbf{u}, \ \omega_k = \epsilon_{kji}\frac{\partial}{\partial x_j}u_i\}$$

$$= \begin{bmatrix} 0 & -\frac{1}{2}\omega_3 & \frac{1}{2}\omega_2 \\ \frac{1}{2}\omega_3 & 0 & -\frac{1}{2}\omega_1 \\ \frac{1}{2}\omega_2 & \frac{1}{2}\omega_1 & 0 \end{bmatrix}$$

Deformation of a small cube

Consider the deformation of the small cube in figure 1.5, where we define

$$\mathrm{R}_k^l = \mathrm{R}\delta_{kl} \qquad \text{component } k \text{ of side } l$$

$$r_k^l = \mathrm{R}_k^l + \underbrace{\frac{\partial u_k}{\partial x_j}\mathrm{R}_j^l \, dt}_{du_k^l} \qquad \text{relative motion of side } l$$

$$= \mathrm{R}\left(\delta_{kl} + \frac{\partial u_k}{\partial x_l}\, dt\right) \qquad \text{deformed cube}$$

First, we consider the deformation on side 1, which can be expressed as

$$dR^1 = \left|\mathbf{r}^1\right| - \mathrm{R} = \sqrt{r_k^1 r_k^1} - \mathrm{R}$$

The inner product can be expanded as

$$r_k^1 r_k^1 = \left[\left(1 + \frac{\partial u_1}{\partial x_1}\, dt\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\, dt\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\, dt\right)^2\right]\mathrm{R}^2$$

$$= \{\text{drop quadratic terms}\}$$

$$= \mathrm{R}^2\left(1 + 2\frac{\partial u_1}{\partial x_1}\, dt\right)$$

We have dropped the quadratic terms since we are assuming that dt is small. dR^1 becomes

$$
\begin{aligned}
dR^1 &= R\sqrt{1 + 2\frac{\partial u_1}{\partial x_1}\,dt} - R = R\left(1 + \frac{\partial u_1}{\partial x_1}\,dt + ...\right) - R \\
&= R\frac{\partial u_1}{\partial x_1}\,dt = Re_{11}\,dt
\end{aligned}
$$

which implies that

$$
\frac{dR^1}{dt} = Re_{11}
$$

Thus the deformation rate of side 1 depends on e_{11}, both traceless and isotropic part of $\frac{\partial u_i}{\partial x_j}$.

Second, we consider the deformation of the angle between side 1 and side 2. This can be expressed as

$$
\begin{aligned}
\cos\left(\frac{\pi}{2} + d\phi_{12}\right) &= \frac{\mathbf{r}^1 \cdot \mathbf{r}^2}{|\mathbf{r}^1|\,|\mathbf{r}^2|} = r_k^1 r_k^2 \cdot \left(r_m^1 r_m^1 \cdot r_n^2 r_n^2\right)^{-1/2} \\
&= \left(\delta_{k1} + \frac{\partial u_k}{\partial x_1}\,dt\right)\left(\delta_{k2} + \frac{\partial u_k}{\partial x_2}\,dt\right) \cdot \left(1 + 2\frac{\partial u_1}{\partial x_1}\,dt\right)^{-1/2}\left(1 + 2\frac{\partial u_2}{\partial x_2}\,dt\right)^{-1/2} \\
&= \left(\frac{\partial u_1}{\partial x_2}\,dt + \frac{\partial u_2}{\partial x_1}\,dt\right)\left(1 - \frac{\partial u_1}{\partial x_1}\,dt\right)\left(1 - \frac{\partial u_2}{\partial x_2}\,dt\right) \\
&= \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)dt = 2\bar{e}_{12}\,dt
\end{aligned}
$$

where we have dropped quadratic terms. We use the trigonometric identity

$$
\cos\left(\frac{\pi}{2} + d\varphi_{12}\right) = \cos\frac{\pi}{2} \cdot \cos d\varphi_{12} - \sin\frac{\pi}{2} \cdot \sin d\varphi_{12} \approx - d\varphi_{12}
$$

which allow us to obtain the finial expression

$$
\frac{d\varphi_{12}}{dt} = -2\bar{e}_{12}
$$

Thus the deformation rate of angle between side 1 and side 2 depends only on traceless part of $\frac{\partial u_i}{\partial x_j}$.

Third, we consider the deformation of the volume of the cube. This can be expressed as

$$
\begin{aligned}
dV &= \left|\mathbf{r}^1\ \mathbf{r}^2\ \mathbf{r}^3\right| - R^3 \\
&= R^3\begin{vmatrix} 1 + \frac{\partial u_1}{\partial x_1}\,dt & \frac{\partial u_1}{\partial x_2}\,dt & \frac{\partial u_1}{\partial x_3}\,dt \\ \frac{\partial u_2}{\partial x_1}\,dt & 1 + \frac{\partial u_2}{\partial x_2}\,dt & \frac{\partial u_2}{\partial x_3}\,dt \\ \frac{\partial u_3}{\partial x_1}\,dt & \frac{\partial u_3}{\partial x_2}\,dt & 1 + \frac{\partial u_3}{\partial x_3}\,dt \end{vmatrix} - R^3 \\
&= R^3\left(1 + \frac{\partial u_1}{\partial x_1}\,dt\right)\left(1 + \frac{\partial u_2}{\partial x_2}\,dt\right)\left(1 + \frac{\partial u_3}{\partial x_3}\,dt\right) - R^3 \\
&= R^3\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)dt \\
&= R^3\frac{\partial u_k}{\partial x_k}\,dt
\end{aligned}
$$

where we have again omitted quadratic terms. Thus we have

$$
\frac{dV}{dt} = R^3\bar{\bar{e}}_{rr}
$$

and the deformation rate of volume of cube (or expansion rate) depends on isotropic part of $\frac{\partial u_i}{\partial x_j}$.

In summary, the motion of a fluid particle with velocity u_i can be divided into the following invariant parts
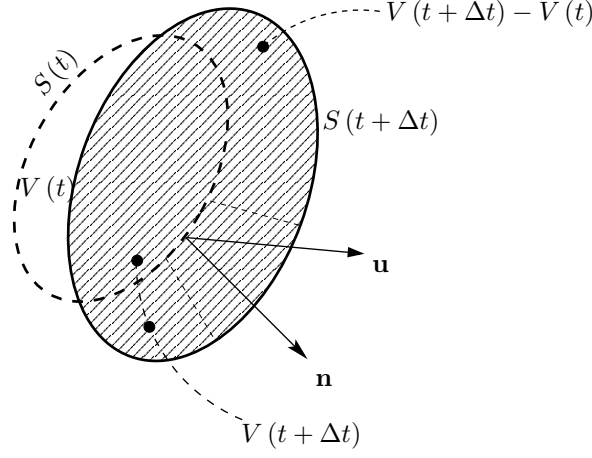
Figure 1.6: Volume moving with the fluid.

i) u_i solid body translation

ii) $\xi_{ij} = \frac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} - \dfrac{\partial u_j}{\partial x_i}\right) = -\frac{1}{2}\epsilon_{kij}\omega_k$ solid body rotation

iii) $\overline{e}_{ij} = \frac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} - \dfrac{2}{3}\dfrac{\partial u_r}{\partial u_r}\delta_{ij}\right)$ volume constant deformation

iv) $\overline{\overline{e}}_{ij} = \frac{1}{3}\dfrac{\partial u_r}{\partial u_r}\delta_{ij}$ volume expansion rate

1.3 Reynolds transport theorem

Volume integral following with the fluid

Consider the time derivative of a <u>material</u> volume integral, i.e. a volume integral where the volume is moving with the fluid. We obtain the following expressions

$$
\begin{aligned}
\frac{D}{Dt}\int_{V(t)} T_{ij}\,\mathrm{d}V &= \lim_{\Delta t \to 0}\left\{\frac{1}{\Delta t}\left[\int_{V(t+\Delta t)} T_{ij}(t+\Delta t)\,\mathrm{d}V - \int_{V(t)} T_{ij}(t)\,\mathrm{d}V\right]\right\} \\
&= \lim_{\Delta t \to 0}\left\{\frac{1}{\Delta t}\left[\int_{V(t+\Delta t)} T_{ij}(t+\Delta t)\,\mathrm{d}V - \int_{V(t)} T_{ij}(t+\Delta t)\,\mathrm{d}V\right]\right. \\
&\qquad\qquad \left.+ \frac{1}{\Delta t}\left[\int_{V(t)} T_{ij}(t+\Delta t)\,\mathrm{d}V - \int_{V(t)} T_{ij}(t)\,\mathrm{d}V\right]\right\} \\
&= \lim_{\Delta t \to 0}\left\{\frac{1}{\Delta t}\int_{V(t+\Delta t)-V(t)} T_{ij}(t+\Delta t)\,\mathrm{d}V\right\} + \int_{V(t)}\frac{\partial T_{ij}}{\partial t}\,\mathrm{d}V
\end{aligned}
$$

The volume in the first integral on the last line is represented in figure 1.6, where a volume element describing the change in volume between V at time t and $t+\Delta t$ can be written as

$$
\mathbf{u}\cdot\mathbf{n}\Delta t = u_k n_k \Delta t \Rightarrow \mathrm{d}V = u_k n_k \Delta t\,\mathrm{d}S
$$

This implies that the volume integral can be converted to a surface integral. This surface integral can in turn be changed back to a volume integral by the use of Gauss (or Greens) theorem. We have

$$\frac{D}{Dt}\int_{V(t)} T_{ij}\,\mathrm{d}V = \lim_{\Delta t \to 0}\left[\oint_{S(t)} T_{ij}\,(t+\Delta t)\,u_k n_k\,\mathrm{d}S\right] + \int_{V(t)}\frac{\partial T_{ij}}{\partial t}\,\mathrm{d}V$$

$$= \oint_{S(t)} T_{ij}u_k n_k\,\mathrm{d}S + \int_{V(t)}\frac{\partial T_{ij}}{\partial t}\,\mathrm{d}V = \{\text{Gauss/Green's theorem}\}$$

$$= \int_{V(t)}\left[\frac{\partial T_{ij}}{\partial t} + \frac{\partial}{\partial x_k}\left(u_k T_{ij}\right)\right]\mathrm{d}V$$

which is the Reynolds transport theorem.

Conservation of mass

By the substitution $T_{ij} \to \rho$ and the use of the Reynolds transport theorem above we can derive the equation for the conservation of mass. We have

$$\frac{D}{Dt}\int_{V(t)} \rho\,\mathrm{d}V = \int_{V(t)}\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}\left(u_k \rho\right)\right] = 0$$

Since the volume is arbitrary, the following must hold for the integrand

$$0 = \underbrace{\frac{\partial \rho}{\partial t}}_{①} + \underbrace{\frac{\partial}{\partial x_k}\left(u_k \rho\right)}_{②} = \frac{\partial \rho}{\partial t} + u_k\frac{\partial \rho}{\partial x_k} + \rho\frac{\partial u_k}{\partial x_k} = \underbrace{\frac{D\rho}{Dt}}_{③} + \underbrace{\rho\frac{\partial u_k}{\partial x_k}}_{④}$$

where we have used the definition of the material derivative in order to simplify the expression. The terms in the expression can be given the following interpretations:

①	:	accumulation of mass in fixed element
②	:	net flow rate of mass out of element
③	:	rate of density change of material element
④	:	volume expansion rate of material element

By considering the transport of a quantity given per unit mass, i.e. $T_{ij} = \rho t_{ij}$, we can simplify Reynolds transport theorem. The integrand in the theorem can then be written

$$\frac{\partial}{\partial t}\left(\rho t_{ij}\right) + \frac{\partial}{\partial x_k}\left(u_k \rho\, t_{ij}\right) = t_{ij}\overbrace{\frac{\partial \rho}{\partial t}}^{0\,(\text{cont. eq.})} + \rho\frac{\partial t_{ij}}{\partial t} + t_{ij}\overbrace{\frac{\partial}{\partial x_k}\left(u_k \rho\right)}^{0} + \rho u_k\frac{\partial t_{ij}}{\partial x_k} = \rho\frac{Dt_{ij}}{Dt}$$

which implies that Reynolds transport theorem becomes

$$\frac{D}{Dt}\int_{V(t)} \rho\, t_{ij}\,\mathrm{d}V = \int_{V(t)} \rho\frac{Dt_{ij}}{Dt}\,\mathrm{d}V$$

where $V(t)$ again is a material volume.

1.4   Momentum equation

Conservation of momentum

The momentum equation is based on the principle of conservation of momentum, i.e. that the time rate of change of momentum in a material region = sum of the forces on that region. The quantities involved are:

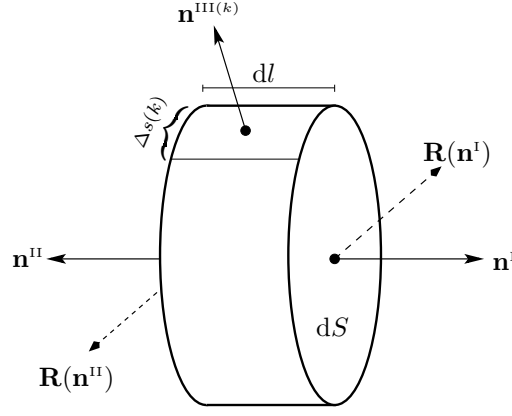|		
F_i	-	body forces per unit mass
R_i	-	surface forces per unit area
ρu_i	-	momentum per unit volume

Figure 1.7: Momentum balance for fluid element

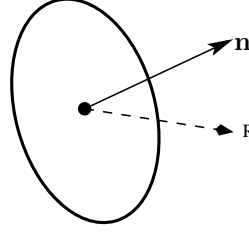Figure 1.8: Surface force and unit normal.

We can put the momentum conservation in integral form as follows

$$\frac{D}{Dt}\int\limits_{V(t)}\rho u_i\,\mathrm{d}V = \int\limits_{V(t)}\rho F_i\,\mathrm{d}V + \int\limits_{S(t)}R_i\,\mathrm{d}S$$

Using Reynolds transport theorem this can be written

$$\int\limits_{V(t)}\rho\frac{Du_i}{Dt}\,\mathrm{d}V = \int\limits_{V(t)}\rho F_i\,\mathrm{d}V + \int\limits_{S(t)}R_i\,\mathrm{d}S$$

which is Newtons second law written for a volume of fluid: mass · acceleration = sum of forces. To proceed R_i must be investigated so that the surface integral can be transformed to a volume integral. In order to do that we have to define the stress tensor.

The stress tensor

Remove a fluid element and replace outside fluid by surface forces as in figure 1.7. Here $\mathbf{R}(\mathbf{n})$ is the surface force per unit area on surface $\mathrm{d}S$ with normal $\mathbf{n}$, see figure 1.8. Momentum conservation for the small fluid particle leads to

$$\rho\frac{Du_i}{Dt}\,\mathrm{d}S\,\mathrm{d}l = \rho F_i\,\mathrm{d}S\,\mathrm{d}l + R_i\left(n_j^{\mathrm{I}}\right)\,\mathrm{d}S + R_i\left(n_j^{\mathrm{II}}\right)\,\mathrm{d}S + \sum_k R_i\left(n_j^{\mathrm{III}(k)}\right)\Delta s^{(k)}\,\mathrm{d}l$$

Letting $\mathrm{d}l \to 0$ gives

$$0 = R_i\left(n_j^{\mathrm{I}}\right)\,\mathrm{d}S + R_i\left(n_j^{\mathrm{II}}\right)\,\mathrm{d}S$$

Now $n_j = n_j^{\mathrm{I}} = -n_j^{\mathrm{II}}$ which leads to

$$R_i\left(n_j\right) = -R_i\left(-n_j\right)$$

implying that a surface force on one side of a surface is balanced by an equal an opposite surface force at the other side of that surface. Note that it is a general principle that the terms proportional to the volume of a small fluid particle approaches zero faster than the terms proportional to the surface area of the particle. Thus the surface forces acting on a small fluid particle has to balance, irrespective of volume forces or acceleration terms.

$$\mathbf{R(e_1)} = (T_{11}, T_{21}, T_{31})$$

T_{21}

T_{11}

x_1

T_{31}
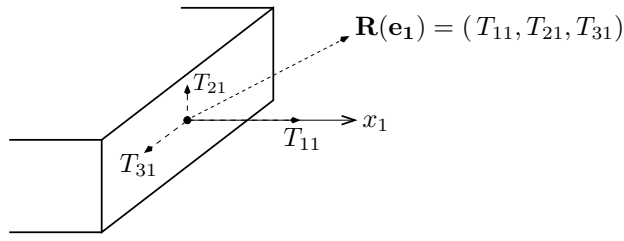
Figure 1.9: Definition of surface force components on a surface with a normal in the 1-direction.

x_2

$$\mathbf{R(e_2)} = (T_{12}, T_{22}, T_{32})$$

T_{22}

T_{12}

T_{32}
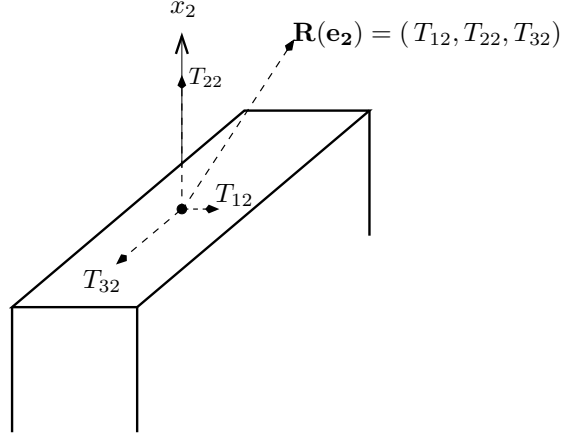
Figure 1.10: Definition of surface force components on a surface with a normal in the 2-direction.

We now divide the surface forces into components along the coordinate directions, as in figures 1.9 and 1.10, with corresponding definitions for the force components on a surface with a normal in the 3-direction. Thus, T_{ij} is the i-component of the surface force on a surface dS with a normal in the j-direction.

Consider a fluid particle with surfaces along the coordinate directions cut by a slanted surface, as in figure 1.11. The areas of the surface elements are related by

$$dS_j = \mathbf{e}_j \cdot \mathbf{n}\, dS = n_j\, dS$$

where dS is the area of the slanted surface. Momentum balance require the surface forces to balance in the element, we have

$$0 = R_1\, dS - T_{11}n_1\, dS - T_{12}n_2\, dS - T_{13}n_3\, dS$$

which implies that the total surface force R_i can be written in terms of the components of the stress tensor T_{ij} as

$$R_i = T_{i1}n_1 + T_{i2}n_2 + T_{i3}n_3 = T_{ij}n_j$$

$\mathbf{e_2}$

$\mathbf{R}$

$\mathbf{n}$

R_1

dS_3

dS_1

dS_2

$\mathbf{e_1}$

$\mathbf{e_3}$

Figure 1.11: Surface force balance on a fluid particle with a slanted surface.

Here the diagonal components are <u>normal stresses</u> off-diagonal components are <u>shear</u> stresses.

Further consideration of the moment balance around a fluid particle show that T_{ij} is a symmetric tensor, i.e.

$$T_{ij} = T_{ji}$$

Momentum equation

Using the definition of the surface force in terms of the stress tensor, the momentum equation can now be written

$$\int_{V(t)} \rho \frac{Du_i}{Dt}\, \mathrm{d}V = \int_{V(t)} \rho F_i\, \mathrm{d}V + \int_{S(t)} T_{ij} n_j\, \mathrm{d}S = \int_{V(t)} \left[\rho F_i + \frac{\partial T_{ij}}{\partial x_j} \right] \mathrm{d}V$$
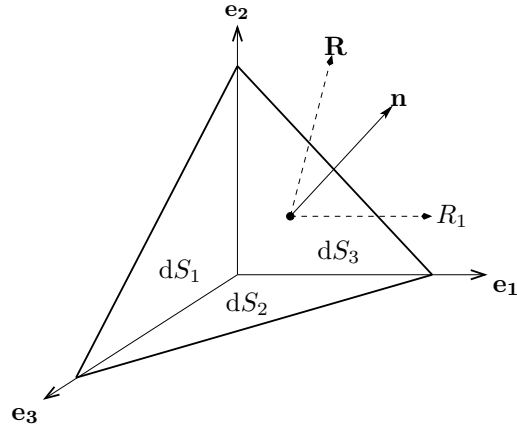
The volume is again arbitrary implying that the integrand must itself equal zero. We have

$$\rho \frac{Du_i}{Dt} = \rho F_i + \frac{\partial T_{ij}}{\partial x_j}$$

Pressure and viscous stress tensor

For fluid at rest only normal stresses present otherwise fluid element would deform. We thus divide the stress tensor into an isotropic part, the hydrodynamic pressure p, and a part depending on the motion of the fluid. We have

$$T_{ij} = -p\delta_{ij} + \tau_{ij}$$

p -   hydrodynamic pressure, directed inward
τ_{ij} -   viscous stress tensor, depends on fluid motion
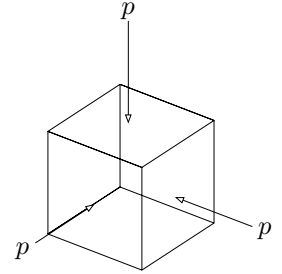
Newtonian fluid

It is natural to assume that the viscous stresses are functions of deformation rate e_{ij} or strain. Recall that the invariant symmetric parts are

$$e_{ij} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_r}{\partial x_r}\delta_{ij} \right)}_{\overline{e}_{ij}} + \underbrace{\frac{1}{3}\frac{\partial u_r}{\partial x_r}\delta_{ij}}_{\overline{\overline{e}}_{ij}}$$

where $\overline{e}_{ij}$ is the volume constant deformation rate and $\overline{\overline{e}}_{ij}$ is the uniform rate of expansion. For an isotropic fluid, the viscous stress tensor is a linear function of the invariant parts of e_{ij}, i.e.

$$\tau_{ij} = \lambda \overline{\overline{e}}_{ij} + 2\mu \overline{e}_{ij}$$

where we have defined the two viscosities as

$\mu(T)$:   dynamic viscosity (here T is the temperature)
$\lambda(T)$:   second viscosity, often$=0$

For a Newtonian fluid, we thus have the following relationship between the viscous stress and the strain (deformation rate)

$$\tau_{ij} = 2\mu \overline{e}_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_r}{\partial x_r}\delta_{ij} \right)$$

which leads to the momentum equation

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_r}{\partial x_r}\delta_{ij} \right) \right] + \rho F_i$$

1.5   Energy equation

The energy equation is a mathematical statement which is based on the physical law that the rate of change of energy in material particle = rate that energy is received by heat and work transfers by that particle. We have the following definitions

$\rho \left[e + \frac{1}{2} u_i u_i \right] \, dV$ energy of particle, with e the internal energy

$\rho u_i F_i \, dV$ $\underbrace{\text{work rate}}_{\text{force} \cdot \text{velocity}}$ of F_i on particle

$u_j R_j \, dS$ work rate of R_j on particle

$n_i q_i \, dS$ heat loss from surface, with q_i the heat flux vector, directed outward

Using Reynolds transport theorem we can put the energy conservation in integral form as

$$\frac{D}{Dt} \int_{V(t)} \rho \left[e + \frac{1}{2} u_i u_i \right] \, dV = \int_{V(t)} \rho F_i u_i \, dV + \int_{S(t)} [n_i T_{ij} u_j - n_i q_i] \, dS = \int_{V(t)} \left[\rho F_i u_i + \frac{\partial}{\partial x_i} (T_{ij} u_j - q_i) \right] \, dV$$

Compare the expression in classical mechanics, where the momentum equation is $m\dot{\mathbf{u}} = \mathbf{F}$ and the associated kinetic energy equation is

$\begin{array}{lcl} \frac{m}{2} \frac{d}{dt} (\mathbf{u} \cdot \mathbf{u}) & = & \mathbf{F} \cdot \mathbf{u} \\ \text{work rate} & = & \text{force} \cdot \text{velocity} \\ (\text{ work} & = & \text{force} \cdot \text{dist. }) \end{array}$

From the integral energy equation we obtain the total energy equation by the observation that the volume is arbitrary and thus that the integrand itself has to be zero. We have

$$\rho \frac{D}{Dt} \left(e + \frac{1}{2} u_i u_i \right) = \rho F_i u_i - \frac{\partial}{\partial x_i} (p u_i) + \frac{\partial}{\partial x_i} (\tau_{ij} u_j) - \frac{\partial q_i}{\partial x_i}$$

The mechanical energy equation is found by taking the dot product between the momentum equation and $\mathbf{u}$. We obtain

$$\rho \frac{D}{Dt} \left(\frac{1}{2} u_i u_i \right) = \rho F_i u_i - u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial \tau_{ij}}{\partial x_j}$$

Thermal energy equation is then found by subtracting the mechanical energy equation from the total energy equation, i.e.

$$\rho \frac{De}{Dt} = -p \frac{\partial u_i}{\partial x_i} + \tau_{ij} \frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

The work of the surface forces divides into viscous and pressure work as follows

$$-\frac{\partial}{\partial x_i} (p u_i) \quad = \quad -p \frac{\partial u_i}{\partial x_i} \quad - \quad u_i \frac{\partial p}{\partial x_i}$$

$$\qquad\qquad\qquad\qquad ① \qquad\qquad ②$$

$$\frac{\partial}{\partial x_i} (\tau_{ij} u_j) \quad = \quad \tau_{ij} \frac{\partial u_i}{\partial x_j} \quad + \quad u_j \frac{\partial \tau_{ij}}{\partial x_i}$$

where following interpretations can be given to the thermal and the mechanical terms

① :   thermal terms
     (force $\cdot$ deformation): heat generated by compression and viscous dissipation
② :   mechanical terms
     (velocity $\cdot$ force gradients): gradients accelerate fluid and increase kinetic energy

The heat flux need to be related to the temperature gradients with Fouriers law

$$q_i = -\kappa \frac{\partial T}{\partial x_i}$$

where $\kappa = \kappa(T)$ is the thermal conductivity. This allows us to write the thermal energy equation as

$$\rho \frac{De}{Dt} = -p \frac{\partial u_i}{\partial x_i} + \Phi + \frac{\partial}{\partial x_i}\left(\kappa \frac{\partial T}{\partial x_i}\right)$$

where the positive definite dissipation function Φ is defined as

$$\begin{aligned}
\Phi &= \tau_{ij}\frac{\partial u_i}{\partial x_j} = 2\mu\left[e_{ij}e_{ij} - \frac{1}{3}\left(\frac{\partial u_k}{\partial x_k}\right)^2\right] \\
&= 2\mu\left(e_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)^2 > 0
\end{aligned}$$

Alternative form of the thermal energy equation can be derived using the definition of the enthalpy

$$h = e + \frac{p}{\rho}$$

We have

$$\frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} - \frac{p}{\rho^2}\underbrace{\frac{D\rho}{Dt}}_{-\left(\rho\frac{\partial u_i}{\partial x_i}\right)}$$

which gives the final result

$$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt} + \Phi + \frac{\partial}{\partial x_i}\left(\kappa\frac{\partial T}{\partial x_i}\right)$$

To close the system of equations we need a

  i)   thermodynamic equation $e = e(T, p)$ simplest case: $e = c_v T$ or $h = c_p T$

  ii)   equation of state $p = \rho R T$

where c_v and c_p are the specific heats at constant volume and temperature, respectively. At this time we also define the ratio of the specific heats as

$$\gamma = \frac{c_p}{c_v}$$

1.6   Navier-Stokes equations

The derivation is now completed and we are left with the Navier-Stokes equations. They are the equation describing the conservation of mass, the equation describing the conservation of momentum and the equation describing the conservation of energy. We have

$$\frac{D\rho}{Dt} + \rho\frac{\partial u_k}{\partial x_k} = 0$$

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho F_i$$

$$\rho\frac{De}{Dt} = -p\frac{\partial u_i}{\partial x_i} + \Phi + \frac{\partial}{\partial x_i}\left(\kappa\frac{\partial T}{\partial x_i}\right)$$

where

$$\begin{cases} \Phi &= \tau_{ij}\dfrac{\partial u_i}{\partial x_j} \\ \tau_{ij} &= \mu\left(\dfrac{\partial u_i}{\partial x_j}+\dfrac{\partial u_j}{\partial x_i}-\dfrac{2}{3}\dfrac{\partial u_r}{\partial x_r}\delta_{ij}\right) \end{cases}$$

and the thermodynamic relation and the equation of state for a gas

$$e = e\left(T,p\right)$$

$$p = \rho R T$$

We have 7 equations and 7 unknowns and therefore the necessary requirements to obtain a solution of the system of equations.

unknown		equations	
ρ	1	continuity	1
u_i	3	momentum	3
p	1	energy	1
e	1	thermodyn.	1
T	1	gas law	1
	$\sum 7$		$\sum 7$

Equations in conservative form

A slightly different version of the equations can be found by using the identity

$$\rho\frac{Dt_{ij}}{Dt} = \frac{\partial}{\partial t}\left(\rho t_{ij}\right)+\frac{\partial}{\partial x_k}\left(u_k\rho t_{ij}\right)$$

to obtain the system

$$\begin{cases} \dfrac{\partial\rho}{\partial t}+\dfrac{\partial}{\partial x_k}\left(u_k\rho\right)=0 \\[2mm] \dfrac{\partial}{\partial t}\left(\rho u_i\right)+\dfrac{\partial}{\partial x_k}\left(\rho u_k u_i\right)=-\dfrac{\partial p}{\partial x_i}+\dfrac{\partial\tau_{ij}}{\partial x_j}+\rho F_i \\[2mm] \dfrac{\partial}{\partial t}\left(e+\dfrac{1}{2}u_i u_i\right)+\dfrac{\partial}{\partial x_k}\left[\rho u_k\left(e+\dfrac{1}{2}u_i u_i\right)\right]=\rho F_i u_i-\dfrac{\partial}{\partial x_i}\left(p u_i\right)+\dfrac{\partial}{\partial x_i}\left(\tau_{ij}u_j\right)+\dfrac{\partial}{\partial x_i}\left(\kappa\dfrac{\partial T}{\partial x_i}\right) \end{cases}$$

These are all of the form

$$\frac{\partial\mathbf{U}}{\partial t}+\frac{\partial\mathbf{G}^{(i)}}{\partial x_i}=\mathbf{J}\qquad\text{or}\qquad\frac{\partial\mathbf{U}}{\partial t}+\frac{\partial\mathbf{G}}{\partial x}+\frac{\partial\mathbf{G}}{\partial y}+\frac{\partial\mathbf{G}}{\partial z}=\mathbf{J}$$

where

$$\mathbf{U}=\left(\rho,\rho u_1,\rho u_2,\rho u_3,\rho\left(e+u_i u_i/2\right)\right)^T$$

is the vector of unknowns,

$$\mathbf{J}=\left(0,\rho F_1,\rho F_2,\rho F_3,\rho u_i F_i\right)^T$$

is the vector of the right hand sides and

$$\mathbf{G}^{(i)}=\begin{bmatrix} \rho u_i \\ \rho u_1 u_i+p\delta_{1i}-\tau_{1i} \\ \rho u_2 u_i+p\delta_{2i}-\tau_{2i} \\ \rho u_3 u_i+p\delta_{3i}-\tau_{3i} \\ \rho\left(e+u_i u_i/2\right)u_i+p u_i+\kappa\frac{\partial T}{\partial x_i}-u_j\tau_{ij} \end{bmatrix}$$

is the flux of mass, momentum and energy, respectively. This form of the equation is usually termed the conservative form of the Navier-Stokes equations.

1.7   Incompressible Navier-Stokes equations

The conservation of mass and momentum can be written

$$
\begin{cases}
\dfrac{D\rho}{Dt} + \rho \dfrac{\partial u_i}{\partial x_i} = 0 \\[2mm]
\rho \dfrac{Du_i}{Dt} = -\dfrac{\partial p}{\partial x_i} + \dfrac{\partial \tau_{ij}}{\partial x_j} + \rho F_i \\[2mm]
\tau_{ij} = \mu \left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} - \dfrac{2}{3}\dfrac{\partial u_k}{\partial u_k}\delta_{ij} \right)
\end{cases}
$$

for incompressible flow $\rho = $ constant, which from the conservation of mass equation implies

$$
\frac{\partial u_i}{\partial x_i} = 0
$$

implying that a fluid particle experiences no change in volume. Thus the conservation of mass and momentum reduce to

$$
\begin{cases}
\dfrac{\partial u_i}{\partial t} + u_j \dfrac{\partial u_i}{\partial x_j} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + F_i \\[2mm]
\dfrac{\partial u_i}{\partial x_i} = 0
\end{cases}
$$

since the components of the viscous stress tensor can now be written

$$
\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij} \right) = \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \nabla^2 u_i
$$

We have also introduced the kinematic viscosity, ν, above, defined as

$$
\nu = \mu/\rho
$$

The incompressible version of the conservation of mass and momentum equations are usually referred to as the incompressible Navier-Stokes equations, or just the Navier-Stokes equations in case it is evident that the incompressible limit is assumed. The reason that the energy equation is not included in the incompressible Navier-Stokes equations is that it decouples from the momentum and conservation of mass equations, as we will see below. In addition it can be worth noting that the conservation of mass equation is sometimes referred to as the continuity equation.

The incompressible Navier-Stokes equations need boundary and initial conditions in order for a solution to be possible. Boundary conditions (BC) on solid surfaces are $u_i = 0$. They are for obvious reasons usually referred to as the no slip conditions. As initial condition (IC) one needs to specify the velocity field at the initial time $u_i\,(t=0) = u_i^0$. The pressure field does not need to be specified as it can be obtained once the velocity field is specified, as will be discussed below.

Integral form of the Navier-Stokes eq.

We have derived the differential form of the Navier-Stokes equations. Sometimes, for example when a finite-volume discretization is derived, it is convenient to use an integral form of the equations.

An integral from of the continuity equation is found by integrating the divergence constraint over a fixed volume V_F and using the Gauss theorem. We have

$$
\int_V \frac{\partial}{\partial x_i}(u_i)\,\mathrm{d}V_F = \int_S u_i n_i\,\mathrm{d}S_F = 0
$$

An integral form of the momentum equation is found by taking the time derivative of a fixed volume integral of the velocity, substituting the differential form of the Navier-Stokes equation, using the continuity equation and Gauss theorem. We have

$$
\frac{\partial}{\partial t}\int_{V_F} u_i\,\mathrm{d}V = \int_{V_F}\frac{\partial u_i}{\partial t}\,\mathrm{d}V = -\int_{V_F}\left[\frac{\partial}{\partial x_j}(u_j u_i) + \frac{1}{\rho}\frac{\partial p}{\partial x_i} - \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \right]\mathrm{d}V =
$$

$$
-\int_{V_F}\frac{\partial}{\partial x_j}\left[u_j u_i + \frac{1}{\rho}p\delta_{ij} - \nu\frac{\partial u_i}{\partial x_j} \right]\mathrm{d}V = -\int_{S_F}\left[u_i u_j n_j + \frac{p}{\rho}n_i - \nu\frac{\partial u_i}{\partial x_j}n_j \right]\mathrm{d}S
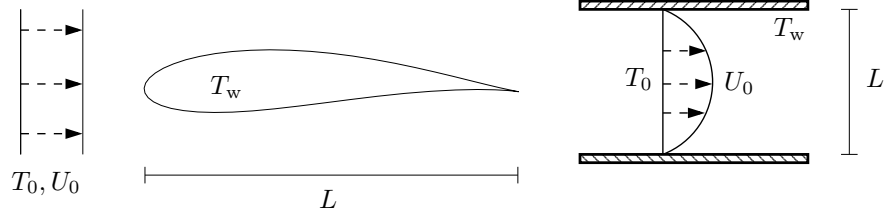$$

Figure 1.12: Examples of length, velocity and temperature scales.

Note that the integral from of the continuity equation is a compatibility condition for BC in incompressible flow.

Dimensionless form

It is often convenient to work with a non-dimensional form of the Navier-Stokes equations. We use the following scales and non-dimensional variables

$$x_i^* = \frac{x_i}{L} \qquad t^* = \frac{tU_0}{L} \qquad u_i^* = \frac{u_i}{U_0} \qquad p^* = \frac{p}{p_0} \qquad \frac{\partial}{\partial x_i} = \frac{1}{L}\frac{\partial}{\partial x_i^*} \qquad \frac{\partial}{\partial t} = \frac{U_0}{L}\frac{\partial}{\partial t^*} \qquad \Rightarrow$$

where $*$ signifies a non-dimensional variable. The length and velocity scales have to be chosen appropriately from the problem under investigation, so that they represent typical lengths and velocities present. See figure 1.12 for examples.

If we introduce the non-dimensional variables in the Navier-Stokes equations we find

$$\begin{cases} \dfrac{U_0^2}{L}\cdot\dfrac{\partial u_i^*}{\partial t^*} + \dfrac{U_0^2}{L}u_j^*\dfrac{\partial u_i^*}{\partial x_j^*} = -\dfrac{p_0}{\rho L}\dfrac{\partial p^*}{\partial x_i^*} + \dfrac{\nu U_0}{L^2}\dfrac{\partial^2 u_i^*}{\partial x_j^*\partial x_j^*} \\ \dfrac{U_0}{L}\cdot\dfrac{\partial u_i^*}{\partial x_i^*} = 0 \end{cases}$$

Now drop $*$ as a sign of non-dimensional variables and divide through with U_0^2/L, we find

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + u_j\dfrac{\partial u_i}{\partial x_j} = -\dfrac{p_0}{\rho U_0^2}\dfrac{\partial p}{\partial x_i} + \dfrac{\nu}{U_0 L}\nabla^2 u_i \\ \dfrac{\partial u_i}{\partial x_i} = 0 \end{cases}$$

We have defined the Reynolds number Re as

$$Re = \frac{U_0 L}{\nu} = \frac{U_0^2/L}{\nu U_0/L^2} \qquad \frac{\text{``inertial forces''}}{\text{``viscous forces''}}$$

which can be interpreted as a measure of the inertial forces divided by the viscous forces. The Reynolds number is by far the single most important non-dimensional number in fluid mechanics. We also define the pressure scale as

$$p_0 = \rho U_0^2$$

which leaves us with the final form of the non-dimensional incompressible Navier-Stokes equations as

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + u_j\dfrac{\partial u_i}{\partial x_j} = -\dfrac{\partial p}{\partial x_i} + \dfrac{1}{Re}\nabla^2 u_i \\ \dfrac{\partial u_i}{\partial x_i} = 0 \end{cases}$$

Non-dimensional energy equation

We stated above that the energy equation decouple from the rest of the Navier-Stokes equations for incompressible flow. This can be seen from a non-dimensionalization of the energy equation. We use the definition of the enthalpy ($h = c_p T$) to write the thermal energy equation as

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \Phi + \kappa \nabla^2 T$$

and use the following non-dimensional forms of the temperature and dissipation function

$$T^* = \frac{T - T_0}{T_w - T_0} \qquad \Phi^* = \frac{L^2}{U_0^2 \mu} \Phi$$

This leads to

$$
\begin{aligned}
\frac{DT^*}{Dt^*} &= \frac{\mu}{\rho U_0 L} \cdot \frac{\kappa}{\mu c_p} \cdot \nabla^{*2} T^* + \frac{U_0^2}{c_p (T_w - T_0)} \left[\frac{Dp^*}{Dt^*} + \frac{\mu}{\rho U_0 L} \phi^* \right] \\
&= \frac{1}{Re} \cdot \frac{1}{Pr} \cdot \nabla^{*2} T^* + \underbrace{\frac{U_0^2}{c_p (T_w - T_0)}}_{Ma^2 \frac{a_0^2}{c_p (T_w - T_0)}} \left(\frac{Dp^*}{Dt^*} + \frac{1}{R} \Phi^* \right)
\end{aligned}
$$

where a_0 speed of sound in free-stream and $Ma = \dfrac{U_0}{a_0}$ is Mach number. We now drop * and let $Ma \to 0$ to obtain

$$\frac{DT}{Dt} = \frac{1}{Re} \cdot \frac{1}{Pr} \cdot \nabla^2 T$$

where the Prandtl number $Pr = \frac{\mu c_p}{\kappa}$, which takes on a value of ≈ 0.7 for air.

As the Mach number approaches zero, the fluid behaves as an incompressible medium, which can be seen from the definition of the speed of sound. We have

$$a_0 = \frac{\gamma}{\rho \alpha}, \qquad \alpha = \frac{1}{\rho} \frac{\partial \rho}{\partial p} \bigg|_T$$

where α is the isothermal compressibility coefficient. If the density is constant, not depending on p, we have that $\alpha \to 0$ and $a_0 \to \infty$, which implies that $Ma \to 0$.

1.8   Role of the pressure in incompressible flow

The role of the pressure in incompressible flow is special due to the absence of a time derivative in the continuity equation. We will illustrate this in two different ways.

Artificial compressibility

First we discuss the artificial compressibility version of the equations, used sometimes for solution of the steady equations. We define a simplified continuity equation and equation of state as

$$\frac{\partial \rho}{\partial t} + \frac{\partial u_i}{\partial x_i} = 0 \qquad \text{and} \qquad p = \beta \rho$$

This allows us to write the Navier-Stokes equations as

$$
\begin{cases}
\dfrac{\partial u_i}{\partial t} + \dfrac{\partial}{\partial x_j} (u_j u_i) &= -\dfrac{\partial p}{\partial x_i} + \dfrac{1}{Re} \nabla^2 u_i \\
\dfrac{\partial p}{\partial t} + \beta \dfrac{\partial u_i}{\partial x_i} &= 0
\end{cases}
$$

Let

$$
\mathbf{u} = \begin{bmatrix} p \\ u \\ v \\ w \end{bmatrix} \qquad
\mathbf{e} = \begin{bmatrix} \beta u \\ p + u^2 \\ uv \\ uw \end{bmatrix} \qquad
\mathbf{f} = \begin{bmatrix} \beta v \\ uv \\ p + v^2 \\ vw \end{bmatrix} \qquad
\mathbf{g} = \begin{bmatrix} \beta w \\ uw \\ vw \\ p + \beta^2 \end{bmatrix}
$$

be the vector of unknowns and their fluxes. The Navier-Stokes equations can then be written

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{e}}{\partial x} + \frac{\partial \mathbf{f}}{\partial y} + \frac{\partial \mathbf{g}}{\partial z} = \frac{1}{Re}\nabla^2 \mathbf{Du}$$

or $$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \mathbf{B}\frac{\partial \mathbf{u}}{\partial \mathbf{y}} + \mathbf{C}\frac{\partial \mathbf{u}}{\partial \mathbf{z}} = \frac{1}{R}\nabla^2 \mathbf{Du}$$

where the matrices above are defined

$$\mathbf{A} = \frac{\partial \mathbf{e}}{\partial \mathbf{u}} = \begin{bmatrix} 0 & \beta & 0 & 0 \\ 1 & 2u & 0 & 0 \\ 0 & v & u & 0 \\ 0 & w & 0 & u \end{bmatrix}, \quad \mathbf{B} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{bmatrix} 0 & 0 & \beta & 0 \\ 0 & v & u & 0 \\ 1 & 0 & 2v & 0 \\ 0 & 0 & w & v \end{bmatrix}$$

$$\mathbf{C} = \frac{\partial \mathbf{g}}{\partial \mathbf{u}} = \begin{bmatrix} 0 & 0 & 0 & \beta \\ 0 & v & u & 0 \\ 1 & 0 & 2v & 0 \\ 0 & 0 & w & v \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the eigenvalues of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are the wave speeds of plane waves in the respective coordinate directions, they are

$$\left(u, u, u \pm \sqrt{u^2 + \beta}\right), \qquad \left(v, v, v \pm \sqrt{v^2 + \beta}\right) \text{ and } \qquad \left(w, w, w \pm \sqrt{w^2 + \beta}\right)$$

Note that the effective acoustic or pressure wave speed $\sim \sqrt{\beta} \to \infty$ for incompressible flow.

Projection on a divergence free space

Second we discuss the projection of the velocity field on a divergence free space. We begin by the following theorem.

Theorem 1. Any w_i in Ω can uniquely be decomposed into

$$w_i = u_i + \frac{\partial p}{\partial x_i}$$

$$where \qquad \frac{\partial u_i}{\partial x_i} = 0, \quad u_i \cdot n_i = 0 \qquad on \qquad \partial \Omega.$$

i.e. into a function u_i that is divergence free and parallel to the boundary and the gradient of a function, here called p.

Proof. We start by showing that u_i and $\frac{\partial p}{\partial x_i}$ is orthogonal in the L_2 inner product.

$$\left(u_i, \frac{\partial p}{\partial x_i}\right) = \int_\Omega u_i \frac{\partial p}{\partial x_i} \, dV = \int_\Omega \frac{\partial}{\partial x_i}(u_i p) \, dV = \oint_{\partial \Omega} p u_i n_i \, dS = 0$$

The uniqueness of the decomposition can be seen by assuming that we have two different decompositions and showing that they have to be equivalent. Let

$$w_i = u_i^{(1)} + \frac{\partial p^{(1)}}{\partial x_i} = u_i^{(2)} + \frac{\partial p^{(2)}}{\partial x_i}$$

The inner product between

$$u_i^{(1)} - u_i^{(2)} + \frac{\partial}{\partial x_i}\left(p^{(1)} - p^{(2)}\right) \qquad \text{and} \quad u_i^{(1)} - u_i^{(2)}$$

gives

$$0 = \int_\Omega \left[\left(u_i^{(1)} - u_i^{(2)}\right)^2 + \left(u_i^{(1)} - u_i^{(2)}\right)\frac{\partial}{\partial x_i}\left(p^{(1)} - p^{(2)}\right)\right] dV = \int_\Omega \left(u_i^{(1)} - u_i^{(2)}\right)^2 dV$$

Thus we have

$$u_i^{(1)} = u_i^{(2)}, \qquad p^{(1)} = p^{(2)} + C$$

$\frac{\pi}{2} + \mathrm{d}\varphi_{12}$

$V(t)$

$S(t)$

$V(t + \Delta t) - V(t)$

$S(t + \Delta t)$

u

n

$V(t + \Delta t)$

p

T_0, U_0

T_{w}

L

T_0

U_0

Gradient fields

w_i

$\frac{\partial p}{\partial x_i}$

u_i

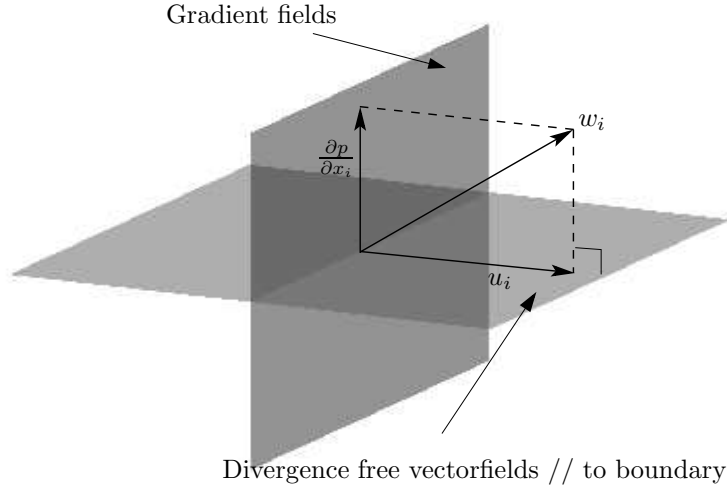Divergence free vectorfields // to boundary

Figure 1.13: Projection of a function on a divergence free space.

We can find an equation for a p with above properties by noting that

$$\nabla^2 p = \frac{\partial w_i}{\partial x_i} \qquad \text{in } \Omega \text{ with} \qquad \frac{\partial p}{\partial n} = w_i n_i \text{ on } \partial\Omega$$

has a unique solution. Now, if

$$u_i = w_i - \frac{\partial p}{\partial x_i} \Rightarrow$$

we have

$$0 = \frac{\partial u_i}{\partial x_i} = \frac{\partial w_i}{\partial x_i} - \frac{\partial^2 p}{\partial x_i \partial x_i}; \quad \text{and} \qquad 0 = u_i n_i = w_i n_i - \frac{\partial p}{\partial n}$$

□

To sum up, to project w_i on divergence free space let $w_i = u_i + \dfrac{\partial p}{\partial x_i}$

1) solve for p: $\dfrac{\partial w_i}{\partial x_i} = \nabla^2 p, \qquad \dfrac{\partial p}{\partial n} = 0$

2) let $u_i = w_i - \dfrac{\partial p}{\partial x_i}$

This is schematically shown in figure 1.13. Note also that $(w_i - u_i) \perp u_i$.

Apply to Navier-Stokes equations

Let $\mathbb{P}$ be orthogonal projector which maps w_i on divergence free part u_i, i.e. $\mathbb{P} w_i = u_i$. We then have the following relations

$$w_i = \mathbb{P} w_i + \frac{\partial p}{\partial x_i}$$

$$\mathbb{P} u_i = u_i, \qquad \mathbb{P} \frac{\partial p}{\partial x_i} = 0$$

Applying the orthogonal projector to the Navier-Stokes equations, we have

$$\mathbb{P}\left(\frac{\partial u_i}{\partial t} + \frac{\partial p}{\partial x_i}\right) = \mathbb{P}\left(-u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{Re}\nabla^2 u_i\right)$$

Now, u_i is divergence free and parallel to boundary, thus

$$\mathbb{P}\frac{\partial u_i}{\partial t} = \frac{\partial u_i}{\partial t}$$

However,

$$\mathbb{P}\nabla^2 u_i \neq \nabla^2 u_i$$

since $\nabla^2 u_i$ need not be parallel to the boundary. Thus we find an evolution equation without the pressure, we have

$$\frac{\partial u_i}{\partial t} = \mathbb{P}\left(\underbrace{-u_j\frac{\partial u_i}{\partial x_j} + \frac{1}{Re}\nabla^2 u_i}_{w_i}\right)$$

The pressure term in the Naiver-Stokes equations ensures that the right hand side w_i is divergence free. We can find a Poisson equation for the pressure by taking the divergence of w_i, according to the result above. We find

$$\nabla^2 p = \frac{\partial}{\partial x_i}\left(\underbrace{-u_j\frac{\partial u_i}{\partial x_j} + \frac{1}{Re}\nabla^2 u_i}_{w_i}\right) = -\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}$$

thus the pressure satisfies elliptic Poisson equation, which links the velocity field in the whole domain instantaneously. This can be interpreted such that the information in incompressible flow spreads infinitely fast, i.e. we have an infinite wave speed for pressure waves, something seen in the analysis of the artificial compressibility equations.

Evolution equation for the divergence

It is common in several numerical solution algorithms for the Naiver-Stokes equations to discard the divergence constraint and instead use the pressure Poisson equation derived above as the second equation in the incompressible Naiver-Stokes equations. If this is done we may encounter solutions that are not divergence free.

Consider the equation for the evolution of the divergence, found by taking the divergence of the momentum equations and substituting the Laplacian of the pressure with the right hand side of the Poisson equation. We have

$$\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_i}\right) = \frac{1}{Re}\nabla^2\left(\frac{\partial u_i}{\partial x_i}\right)$$

Thus the divergence is not automatically zero, but satisfies a heat equation. For the stationary case the solution obeys the maximum principle. This states that the maximum of a harmonic function, a function satisfying the Laplace equation, has its maximum on the boundary. Thus, the divergence is zero inside the domain, if and only if it is zero everywhere on the boundary.

Since it is the pressure term in the Navier-Stokes equations ensures that the velocity is divergence free, this has implications for the boundary conditions for the pressure Poisson equation. A priori we have none specified, but they have to be chosen such that the divergence of the velocity field is zero on the boundary. This may be a difficult constraint to satisfy in a numerical solution algorithm.

z, x_3
u, u_1
v, u_2
w, u_3
x_1
x_2
x_i^0

Chapter 2

$u_i\left(\mathbf{x}_i, \hat{t}\right)$
$u_i\left(x_i{}^0, \hat{t}\right)$
$\mathbb{P}\left(\hat{t} = 0\right)$

Flow physics

x
u_1
$u_2 > u_1$
x_1
x_2
x_3
x_i

2.1   Exact solutions

$u_i \, \mathrm{d}\hat{t}$

Plane Pouseuille flow - exact solution for channel flow

$\frac{\mathrm{d}u_i \, \mathrm{d}\hat{t}}{\mathrm{d}t}$

The flow inside the two-dimensional channel, see figure 2.1, is driven by a pressure difference $p_0 - p_1$ between the inlet and the outlet of the channel. We will assume two-dimensional, steady flow, i.e.

$r_i\left(\mathrm{d}\hat{t}\right)$
$\mathrm{d}r_i\left(\mathrm{d}t\right)$

$\mathbb{P}_1$
$\mathbb{P}_2$

$$\frac{\partial}{\partial t} = 0, \qquad u_3 = 0$$

x_1
x_2

and that the flow is fully developed, meaning that effects of inlet conditions have disappeared. We have

x_3
R^3
R^2
R

$$u_1 = u_1\left(x_2\right), \quad u_2 = u_2\left(x_2\right)$$

The boundary conditions are $u_1 = u_2 = 0$ at $x_2 = \pm h$. The continuity equation reads

$\mathbf{r}^3$
$\mathbf{r}^2$
$\mathbf{r}^1$
$\frac{\pi}{2} + \mathrm{d}\varphi_{12}$
$V(t)$
$S(t)$
$V(t + \Delta t) - V(t)$
$S(t + \Delta t)$
$\mathbf{u}$
$\mathbf{n}$
$V(t + \Delta t)$
p
T_0, U_0
T_w

$$\frac{\partial \cancel{u_1}}{\cancel{\partial x_1}}^{\,0} + \frac{\partial u_2}{\partial x_2} + \frac{\partial \cancel{u_3}}{\cancel{\partial x_3}}^{\,0} = 0$$

where the first and the last term disappears due to the assumption of two-dimensional flow and that the flow is fully developed. This implies that $u_2 = C = 0$, where the constant is seen to be zero from the boundary conditions.

The steady momentum equations read

$$\begin{cases} \rho u_1 \dfrac{\partial u_1}{\partial x_1} + \rho u_2 \dfrac{\partial u_1}{\partial x_2} = -\dfrac{\partial p}{\partial x_1} + \mu \nabla^2 u_1 \\[2mm] \rho u_1 \dfrac{\partial u_2}{\partial x_1} + \rho u_2 \dfrac{\partial u_2}{\partial x_2} = -\dfrac{\partial p}{\partial x_2} + \mu \nabla^2 u_2 - \rho g \end{cases}$$

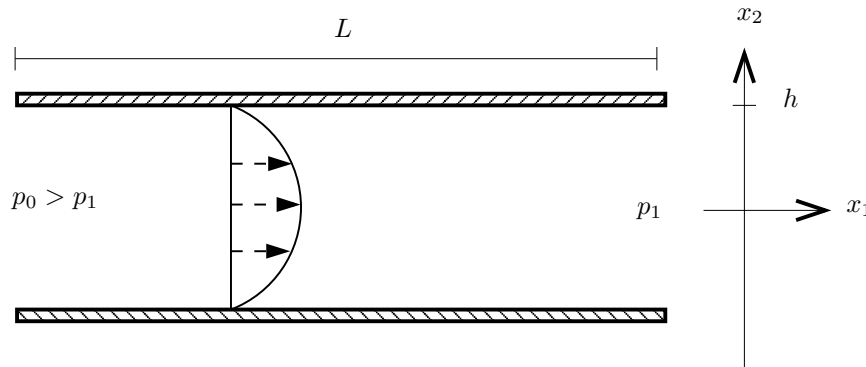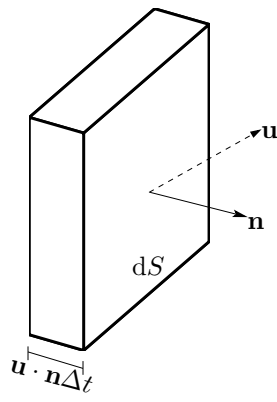which, by applying the assumptions above, reduce to

L
T_0
U_0

Gradient fields

w_i
$\frac{\partial p}{\partial x_i}$
u_i

ree vectorfields // to boundary

$p_0 > p_1$

Figure 2.1: Plane channel geometry.

Figure 2.2: Volume of fluid flowing through surface dS in time Δt

$$0 = -\frac{\partial p}{\partial x_1} + \mu \frac{\mathrm{d}^2 u_1}{\mathrm{d}x_2^2}$$

$$0 = -\frac{\partial p}{\partial x_2} - \rho g$$

The momentum equation in the x_2-direction (or normal direction) implies

$$p = -\rho g x_2 + P(x_1) \qquad \Rightarrow \qquad \frac{\partial p}{\partial x_1} = \frac{\mathrm{d}P}{\mathrm{d}x_1}$$

showing that the pressure gradient in the x_1-direction (or streamwise direction) is only a function of x_1. The momentum equation in the streamwise direction implies

$$0 = -\frac{\mathrm{d}P}{\mathrm{d}x_1} + \mu \frac{\mathrm{d}^2 u_1}{\mathrm{d}x_2^2}$$

Since $P(x_1)$ and $u_1(x_2)$ we have

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}x_2^2} = \frac{1}{\mu} \frac{\mathrm{d}P}{\mathrm{d}x_1} = \text{const.} \ (\text{independent of } x_1, x_2)$$

We can integrate u_1 in the normal direction to obtain

$$u_1 = \frac{1}{2\mu} \frac{\mathrm{d}P}{\mathrm{d}x_1} x_2^2 + C_1 x_2 + C_2$$

The constants can be evaluated from the boundary conditions $u_1(\pm h) = 0$, giving the parabolic velocity profile

$$u_1 = \frac{h^2}{2\mu} \cdot \frac{\mathrm{d}P}{\mathrm{d}x_1} \left[1 - \left(\frac{x_2}{h}\right)^2\right]$$

Evaluating the maximum velocity at the center of the channel we have

$$u_{\text{max}} = -\frac{h^2}{2\mu} \cdot \frac{\mathrm{d}P}{\mathrm{d}x_1} > 0$$

if flow is in the positive x_1-direction. The velocity becomes

$$\frac{u_1}{u_{\text{max}}} = 1 - \left(\frac{x_2}{h}\right)^2$$

Flow rate

From figure 2.2 we can evaluate the flow rate as $\mathrm{d}Q = \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S$. Integrating over the channel we find

$$Q = \int_S u_i n_i \, \mathrm{d}S = \{\text{channel}\} = \int_{-h}^{h} u_1 \, \mathrm{d}x_2 = -\frac{2h^3}{3\mu} \cdot \frac{\mathrm{d}P}{\mathrm{d}x_1} = \frac{4}{3} h u_{\text{max}}$$

$$L$$
$$h$$
$$x_1$$
$$x_2$$
$$p_0 > p_1$$
$$p_1$$
$$\mathbf{u}$$
$$\mathbf{n}$$
$$\mathrm{d}S$$
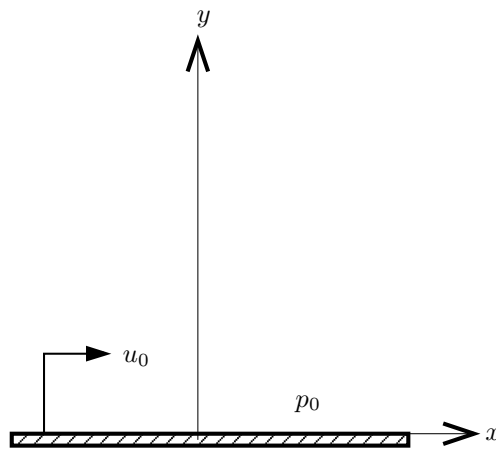$$\mathbf{u} \cdot \mathbf{n}\Delta t$$
$$u_0$$
$$p_0$$

Figure 2.3: Stokes instantaneously plate.

Wall shear stress

The wall shear stress can be evaluated from the velocity gradient at the wall. We have

$$\tau_{12} = \tau_{21} = \mu \frac{\mathrm{d}u_1}{\mathrm{d}x_2} = -\frac{2\mu \cdot u_{\max} \cdot x_2}{h^2}$$

implying that

$$\tau_{\mathrm{wall}} = \tau\big|_{x_2 = -h} = \frac{2\mu \, u_{\max}}{h}$$

Vorticity

The vorticity is zero in the streamwise and normal directions, i.e. $\omega_1 = \omega_2 = 0$ and has the following expression in the x_3-direction (or spanwise direction)

$$\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = -\frac{\mathrm{d}u_1}{\mathrm{d}x_2} = \frac{2u_{\max} \cdot x_2}{h^2}$$

Stokes 1:st problem: instantaneously started plate

Consider the instantaneously starting plate of infinite horizontal extent shown in figure 2.3. This is a time dependent problem where plate velocity is set to $u = u_0$ at time $t = 0$. We assume that the velocity can be written $u = u(y, t)$ which using continuity and the boundary conditions imply that normal velocity $v = 0$. The momentum equations reduce to

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g$$

The normal momentum equation implies that $p = -\rho g y + p_0$, where p_0 is the atmospheric pressure at the plate surface. This gives us the following diffusion equation, initial and boundary conditions for u

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$
$$u(y, 0) = 0 \quad 0 \leq y \leq \infty$$
$$u(0, t) = u_0 \quad t > 0$$
$$u(\infty, t) = 0 \quad t > 0$$

We look for a similarity solution, i.e. we introduce a new dependent variable which is a combination of y and t such that the partial differential equation reduces to an ordinary differential equation. Let

$$f(\eta) = \frac{u}{u_0} \qquad \eta = C \cdot y \cdot t^b$$

where C and b are constants to be chosen appropriately. We transform the time and space derivatives according to

$$\frac{\partial}{\partial t} = \frac{\partial \eta}{\partial t}\frac{d}{d\eta} = Cbyt^{b-1}\frac{d}{d\eta}$$

$$\frac{\partial}{\partial y} = \frac{\partial \eta}{\partial y}\frac{d}{d\eta} = Ct^b\frac{d}{d\eta}$$

$$\frac{\partial^2}{\partial y^2} = C^2 t^{2b}\frac{d^2}{d\eta^2}$$

If we substitute this into the equation for u and use the definition of η, we find an equation for f

$$bt^{-1}\eta\frac{df}{d\eta} = \nu C^2 t^{2b}\frac{d^2 f}{d\eta^2}$$

where no explicit dependence on y and t can remain if a similarity solution is to exist. Thus, the coefficient of the two η-dependent terms have to be proportional to each other, we have

$$bt^{-1} = \kappa \nu C^2 t^{2b}$$

where κ is a proportionality constant. The exponents of these expressions, as well as the coefficients in front of the t-terms have to be equal. This gives

$$b = -\frac{1}{2} \qquad C = \frac{1}{\sqrt{-2\kappa\nu}}$$

We choose $\kappa = -2$ and obtain the following

$$f'' + 2\eta f' = 0 \qquad \eta = \frac{y}{2\sqrt{\nu t}}$$

where $' = \frac{d}{d\eta}$. The boundary and initial conditions also have to be compatible with the similarity assumption. We have

$$\frac{u(y,0)}{u_0} = f(\infty) = 0 \qquad \frac{u(0,t)}{u_0} = f(0) = 1 \qquad \frac{u(\infty,t)}{u_0} = f(\infty) = 0$$

This equation can be integrated twice to obtain

$$f = C_1 \int_0^\eta e^{\xi^2} d\xi + C_2$$

Using the boundary conditions and the definition of the error-function, we have

$$f = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{\xi^2} d\xi = 1 - \mathrm{erf}(\eta)$$

This solution is shown in figure 2.4, both as a function of the similarity variable η and the unscaled normal coordinate y. Note that the time dependent solution is diffusing upward.

From the velocity we can calculate the spanwise vorticity as a function of the similarity variable η and the unscaled normal coordinate y. Here the maximum of the vorticity is also changing in time, from the infinite value at $t = 0$ it is diffusing outward in time.

$$\omega_3 = \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{u_0}{\sqrt{\pi\nu t}}e^{-\eta^2}$$

The vorticity is shown in figure 2.5, both as a function of the similarity variable η and the unscaled normal coordinate y. Note again that the time dependent solution is diffusing upward.

2.2 Vorticity and streamfunction

Vorticity is an important concept in fluid dynamics. It is related to the average angular momentum of a fluid particle and the swirl present in the flow. However, a flow with circular streamlines may have zero vorticity and a flow with straight streamlines may have a non-zero vorticity.
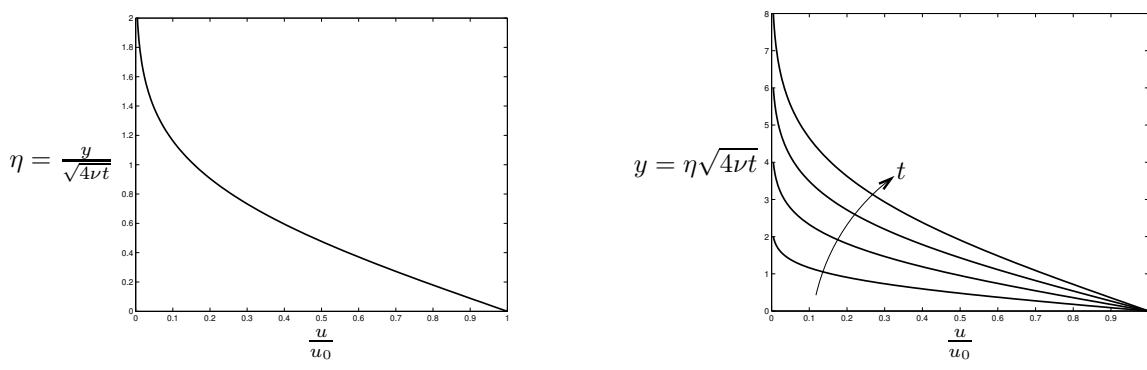
Figure 2.4: Velocity above the instantaneously started plate. a) U as a function of the similarity variable η. b) $U(y,t)$ for various times.
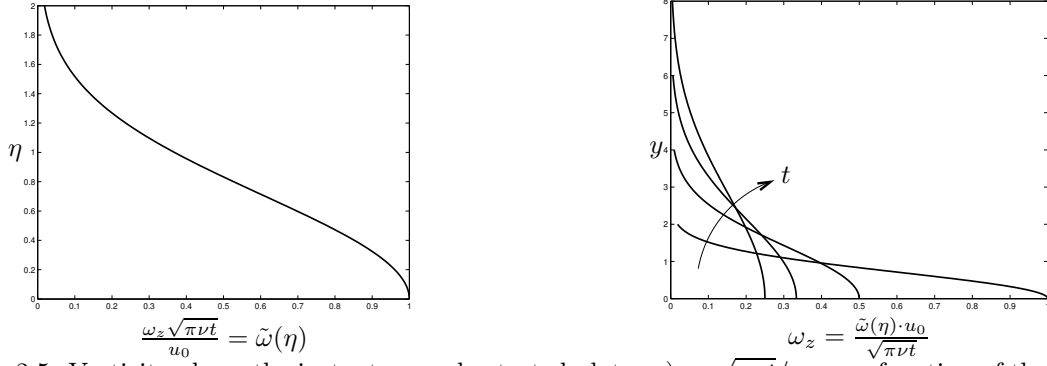

Figure 2.5: Vorticity above the instantaneously started plate. a) $\omega_z\sqrt{\pi\nu t}/u_0$ as a function of the similarity variable η. b) $\omega_z(y,t)$ for various times.

Vorticity and circulation

The vorticity is defined mathematically as the curl of the velocity field as

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

In tensor notation this expression becomes

$$\omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}$$

Another concept closely related to the vorticity is the circulation, which is defined as

$$\Gamma = \oint_C u_i\,\mathrm{d}x_i = \oint_C u_i t_i\,\mathrm{d}s$$

In figure 2.6 we show a closed curve C. Using Stokes theorem we can transform that integral to one over the area S as

$$\Gamma = \int_S \epsilon_{ijk}n_i\frac{\partial u_k}{\partial x_j}\,\mathrm{d}S = \int_S \omega_i n_i\,\mathrm{d}S$$

This allows us to find the following relationship between the circulation and vorticity $\mathrm{d}\Gamma = \omega_i n_i\,\mathrm{d}S$, which can also be written

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}S} = \omega_i n_i$$

Thus, one interpretation is that the vorticity is the circulation per unit area for a surface perpendicular to the vorticity vector.

Example 1. The circulation of an ideal vortex. Let the azimuthal velocity be given as $u_\theta = C/r$. We can calculate the circulation of this flow as

$$\Gamma = \oint_C u_\theta r\,\mathrm{d}\theta = C\int_0^{2\pi}\,\mathrm{d}\theta = 2\pi C$$

Thus we have the following relationship for the ideal vortex
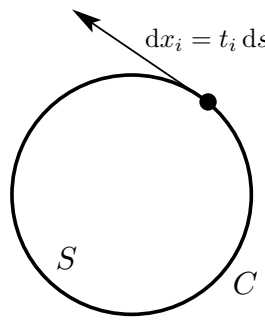
$$u_\theta = \frac{\Gamma}{2\pi r}$$

Figure 2.6: Integral along a closed curve C with enclosed area S.

The ideal vortex has its name from the fact that its vorticity is zero everywhere, except for an infinite value at the center of the vortex.

Derivation of the Vorticity Equation

We will now derive an equation for the vorticity. We start with the dimensionless momentum equations. We have

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i$$

We slightly modify this equation by introducing the following alternative form of the non-linear term

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{1}{2}u_j u_j\right) + \epsilon_{ijk}\omega_j u_k$$

this can readily be derived using tensor manipulation. We take the curl ($\epsilon_{pqi}\frac{\partial}{\partial x_q}$) of the momentum equations and find

$$\frac{\partial}{\partial t}\left(\epsilon_{pqi}\frac{\partial u_i}{\partial x_q}\right) + \epsilon_{pqi}\frac{\partial^2}{\partial x_q \partial x_i}\left(\frac{1}{2}u_j u_j\right) + \epsilon_{pqi}\frac{\partial}{\partial x_q}(\epsilon_{ijk}\omega_j u_k) = -\epsilon_{pqi}\frac{\partial^2 p}{\partial x_q \partial x_i} + \frac{1}{Re}\epsilon_{pqi}\nabla^2\frac{\partial u_i}{\partial x_q}$$

The second and the fourth terms are zero, since ϵ_{pqi} is anti-symmetric in i, q and $\partial x_q \partial x_i$ is symmetric in i, q

$$\epsilon_{pqi}\frac{\partial^2}{\partial x_q \partial x_i} = 0$$

The first term and the last term can be written as derivatives of the vorticity, and we now have the following from of the vorticity equation

$$\frac{\partial \omega_p}{\partial t} + \epsilon_{pqi}\frac{\partial}{\partial x_q}(\epsilon_{ijk}\omega_j u_k) = \frac{1}{Re}\nabla^2\omega_p$$

The second term in this equation can be written as

$$\epsilon_{ipq}\epsilon_{ijk}\frac{\partial}{\partial x_q}(\omega_j u_k) = (\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj})\frac{\partial}{\partial x_q}(\omega_j u_k) = \frac{\partial}{\partial x_k}(\omega_p u_k) - \frac{\partial}{\partial x_j}(\omega_j u_p) =$$

$$u_k\frac{\partial \omega_p}{\partial x_k} + \omega_p\frac{\partial u_k}{\partial x_k} - u_p\frac{\partial \omega_j}{\partial x_j} - \omega_j\frac{\partial u_p}{\partial x_j}$$

Due to continuity $\frac{\partial \omega_j}{\partial x_j} = \epsilon_{jpq}\frac{\partial^2 u_q}{\partial x_j \partial x_p} = 0$ and we are left with the relation

$$\epsilon_{ipq}\epsilon_{ijk}\frac{\partial}{\partial x_q}(\omega_j u_k) = u_k\frac{\partial \omega_p}{\partial x_k} - \omega_j\frac{\partial u_p}{\partial x_j}$$

Thus, the vorticity equation becomes

$$\frac{\partial \omega_p}{\partial t} + u_k\frac{\partial \omega_p}{\partial x_k} = \omega_j\frac{\partial u_p}{\partial x_j} + \frac{1}{Re}\nabla^2\omega_p$$

or written in vector-notation

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \frac{1}{Re}\nabla^2\boldsymbol{\omega}$$

Components in Cartesian coordinates

In cartesian coordinates the components of the vorticity vector in three-dimensions become

$$\boldsymbol{\omega} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \mathbf{e}_x + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \mathbf{e}_y + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathbf{e}_z$$

For two-dimensional flow it is easy to see that this reduces to a non-zero vorticity in the spanwise direction, but zero in the other two directions. We have

$$\omega_x = 0, \qquad \omega_y = 0, \qquad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

This simplifies the interpretation and use of the concept of vorticity greatly in two-dimensional flow.

Vorticity and viscocity

There exists a subtle relationship between flows with vorticity and flows on which viscous forces play a role. It is possible to show that

$$\nabla \times \boldsymbol{\omega} \neq 0 \iff \text{viscous forces} \neq 0$$

This means that without viscous forces there cannot be a vorticity field in the flow which varies with the coordinate directions. This is expressed by the following relationship

$$\frac{\partial \tau_{ij}}{\partial x_j} = \mu \nabla^2 u_i = -\mu \epsilon_{ijk} \frac{\partial \omega_k}{\partial x_j}$$

This can be derived in the following manner. We have

$$\begin{aligned} -\mu \epsilon_{ijk} \frac{\partial \omega_k}{\partial x_j} &= -\mu \epsilon_{ijk} \epsilon_{kmn} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_m} u_n \\ &= -\mu (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \frac{\partial^2 u_n}{\partial x_j \partial x_m} \\ &= \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \\ &= \mu \nabla^2 u_i \end{aligned}$$

Terms in the two-dimensional vorticity equation

In two dimensions the equation for the only non-zero vorticity component can be written

$$\frac{\partial \omega_3}{\partial t} + u_1 \frac{\partial \omega_3}{\partial x_1} + u_2 \frac{\partial \omega_3}{\partial x_2} = \nu \nabla^2 \omega_3$$

We will take two examples elucidating the meaning of the terms in the vorticity equation. First the diffusion of vorticity and second the advection or transport of vorticity.

Example 2. Diffusion of vorticity: the Stokes 1st problem revisited.

From the solution of Stokes 1st problem above, we find that the vorticity has the following form

$$\omega_3 = \omega_z = \omega(y, t)$$

and that it thus is a solution of the following diffusion equation

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2}$$

The time evolution of the vorticiy can be seen in figure 2.7, and is given by

$$\omega = \frac{u_0}{\sqrt{\pi \nu t}} e^{-\eta^2}, \qquad \text{where } \eta = \frac{y}{2\sqrt{\nu t}}$$

where the factor $\sqrt{\nu t}$ can be identified as the diffusion length scale, where we have used the following dimensional relations

$$[\nu] = \frac{L^2}{T}, \qquad [t] = T \qquad \Rightarrow \qquad [\sqrt{\nu t}] = L$$

ree vectorfields // to boundary

L

h

x_1

x_2

$p_0 > p_1$

p_1

$\mathbf{u}$

$\mathbf{n}$

$\mathrm{d}S$

$\mathbf{u} \cdot \mathbf{n} \Delta t$

x

y

u_0

p_0

$\eta = \frac{y}{\sqrt{4\nu t}}$

$\frac{u}{u_0}$

$y = \eta\sqrt{4\nu t}$

t

η

y

$\omega_z = \frac{\tilde{\omega}(\eta) \cdot u_0}{\sqrt{\pi\nu t}}$
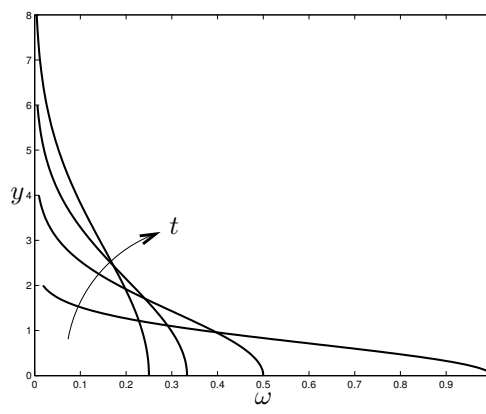
t

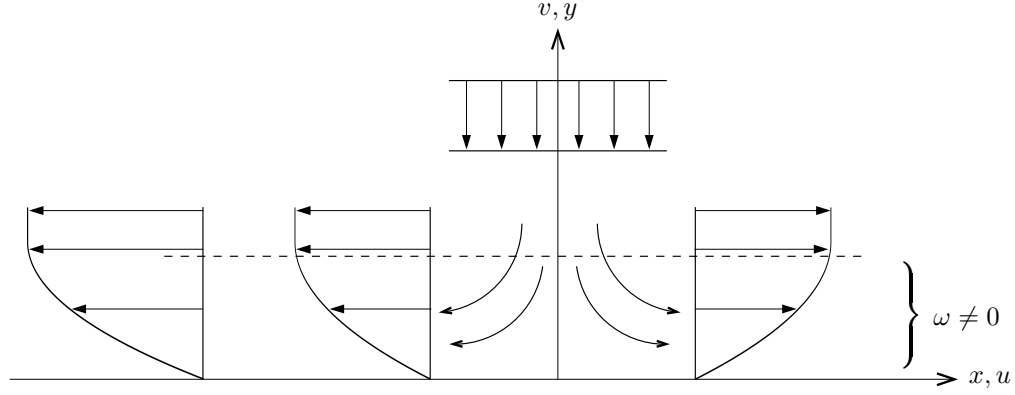Figure 2.7: A sketch of the evolution of vorticity in Stokes 1st problem.

Figure 2.8: Stagnation point flow with a viscous layer close to the wall.

In this flow the streamlines are straight lines and therefore the non-linear term, or the transport/advection term, is zero. We now turn to an example where the transport of vorticity is important.

Example 3. Transport/advection of vorticity: Stagnation point flow.

The stagnation point flow is depicted in figure 2.8, and is a flow where a uniform velocity approaches a plate where it is showed down and turned to a flow parallel two the plates in both the positive and negative x-direction. At the origin we have a stagnation point.

The inviscid stagnation point flow is given as follows

$$\text{inviscid:} \begin{cases} u = cx \\ v = -cy \end{cases} \Rightarrow \quad \omega_z = \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

Note that this flow has zero vorticity and that it does not satisfy the no-slip condition ($u = v = 0, y = 0$). In order to satisfy this condition we introduce viscosity, or equally, vorticity. Thus close to the plate we have to fulfill the equation

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

The two first terms represent advection or transport of vorticity. We will try a simple modification of the inviscid flow close to the boundary, which is written as

$$\begin{cases} u = xf'(y) \\ v = -f(y) \end{cases}$$

This satisfies the continuity equation, and will be zero on the boundary and approach the inviscid flow far above the plate if we apply the following boundary conditions

$$\begin{cases} u = v = 0, & y = 0 \\ u \propto cx, \ v \propto -cy, & y \to \infty \end{cases}$$

The vorticity can be written $\omega = -xf''(y)$. We introduce this and the above definitions of the velocity components into the vorticity equation, and find
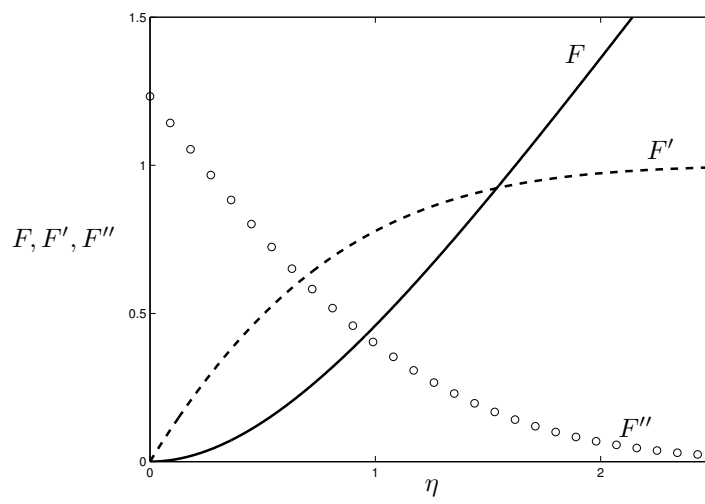
$$-f'f'' + ff''' + \nu f'''' = 0$$

$$\eta = \frac{y}{\sqrt{4\nu t}}$$

$$y = \eta\sqrt{4\nu t}$$

$$\omega_z = \frac{F''}{\sqrt{\pi\nu t}}$$

Figure 2.9: Solution to the non-dimensional equation for stagnation point flow. F is proportional to the normal velocity, F' to the streamwise velocity and F'' to the vorticity.

The boundary conditions become

$$\begin{cases} f = f' = 0 & y = 0 \\ f \propto cy, \ f' \propto c, & y \to \infty \end{cases}$$

An integral to the equation exists, which can be written

$$f'^2 - ff'' - \nu f''' = K = \{\text{evaluate } y \to \infty\} = c^2$$

This is as far as we can come analytically and to make further numerical treatment simpler we make the equation non-dimensional with ν and c. Note that they have the dimensions

$$[\nu] = \frac{L^2}{T}, \quad [c] = \frac{1}{T}$$

and that we can use the length scale $\sqrt{\nu/c}$ and the velocity scale $\sqrt{\nu c}$ to scale the independent and the dependent variables as

$$\eta = \frac{y}{\sqrt{\nu/c}}$$

$$F(\eta) = \frac{f(y)}{\sqrt{\nu c}}$$

This results in the following non-dimensional equation

$$F'^2 - FF'' - F''' = 1$$

with the boundary conditions

$$\begin{cases} F = F' = 0, & \eta = 0 \\ F \propto \eta, & \eta \to \infty \end{cases}$$

In figure 2.9 a solution of the equation is shown. It can be seen that <u>diffusion</u> of vorticity from the wall is balanced by <u>transport</u> downward by v and outward by u. The thin layer of vortical flow close to the wall has the thickness $\sim \sqrt{\nu/c}$.

Stretching and tilting of vortex lines

In three-dimensions the vorticity is not restricted to a single non-zero component and the three-dimensional vorticity equation governing the vorticity vector becomes

$$\frac{\partial \omega_i}{\partial t} + \underbrace{u_j \frac{\partial \omega_i}{\partial x_j}}_{\text{advection of vorticity}} = \omega_j \frac{\partial u_i}{\partial x_j} + \underbrace{\frac{1}{Re}\nabla^2 \omega_i}_{\text{diffusion of vorticity}}$$
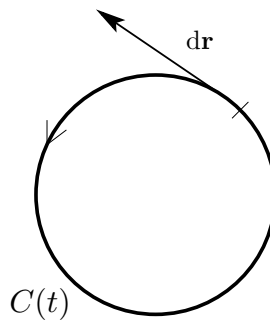
Figure 2.10: A closed material curve.

The complex motion of the vorticity vector for a full three-dimensional flow can be understood by an analogy to the equation governing the evolution of a material line, discussed in the first chapter. The equation for a material line is

$$\frac{D}{Dt}\,dr_i = dr_j\frac{\partial u_i}{\partial x_j}$$

Note that if we disregard the diffusion term, i.e. the last term of the vorticity equation, the two equations are identical. Thus we can draw the following conclusions about the development of the vortex vector

i) Stretching of vortex lines (line $\parallel \boldsymbol{\omega}$) produce ω_i like stretching of dr_i produces length

ii) Tilting vortex lines produce ω_i in one direction at expense of ω_i in other direction

Helmholz and Kelvin's theorems

The analogy with the equation for the material line directly give us a proof of Helmholz theorem. We have

Theorem 1. Helmholz theorem for inviscid flow: Vortex lines are material lines in inviscid flow.

A second theorem valid in inviscid flow is Kelvin's theorem.

Theorem 2. Kelvin's theorem for inviscid flow: Circulation around a material curve is constant in inviscid flow.

Proof. We show this by considering the circulation for a closed material curve, see figure 2.10,

$$\Gamma_m = \oint_{C(t)} u_i\,dr_i$$

We can evaluate the material time derivative of this expression with the use of Lagrangian coordinates as follows

$$
\begin{aligned}
\frac{D\Gamma_m}{Dt} &= \frac{D}{Dt}\oint_{C(t)} u_i\,dr_i \\
&= \left\{
\begin{array}{l}
\text{Lagrange coordinates} \\
u_i = u_i(x_i^0,\hat{t}) = u_i(x_i^0(m),\hat{t}) \\
r_i = r_i(x_i^0,\hat{t}) = r_i(x_i^0(m),\hat{t})
\end{array}
\right. \\
&= \frac{d}{d\hat{t}}\oint_m u_i\frac{\partial r_i}{\partial m}\,dm \\
&= \oint_m \frac{\partial u_i}{\partial\hat{t}}\frac{\partial r_i}{\partial m}\,dm + \oint_m u_i\frac{\partial}{\partial\hat{t}}\left(\frac{\partial r_i}{\partial m}\right)\,dm
\end{aligned}
$$

The last term of this expression will vanish since

$$\oint_m u_i\frac{\partial}{\partial\hat{t}}\left(\frac{\partial r_i}{\partial m}\right)\,dm = \oint_m u_i\frac{\partial u_i}{\partial m}\,dm = \oint_m \frac{\partial}{\partial m}\left(\frac{1}{2}u_i u_i\right)\,dm = 0$$
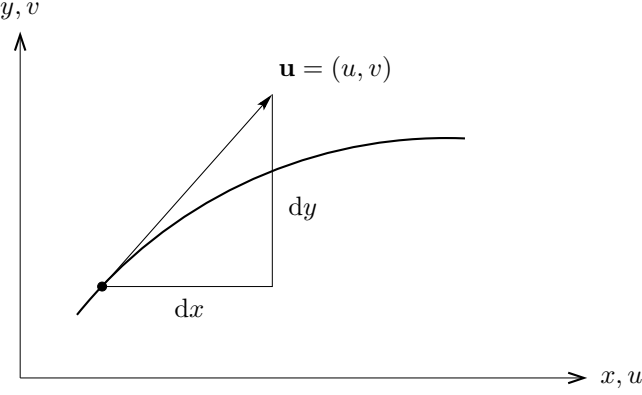
Figure 2.11: A streamline which is tangent to the velocity vector.

Thus we find the following resulting expression

$$
\begin{aligned}
\frac{D\Gamma_m}{Dt} &= \oint_{C(t)} \frac{Du_i}{Dt}\,\mathrm{d}r_i \\
&= \oint_{C(t)} \left[-\frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i \right]\,\mathrm{d}r_i \\
&= \frac{1}{Re} \oint_{C(t)} \nabla^2 u_i\,\mathrm{d}r_i \to 0, \qquad Re \to \infty
\end{aligned}
$$

Which shows that the circulation for a material curve changes only if the viscous force $\neq 0$ $\square$

Streamfunction

We end this section with the definition and some properties of the streamfunction. To define the stream-function we need the streamlines which are tangent to the velocity vector. In two dimensions, see figure 2.11, we have

$$
\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{u}{v} \quad \Longleftrightarrow \quad \frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} \quad \Longleftrightarrow \quad u\,\mathrm{d}y - v\,\mathrm{d}x = 0
$$

and in three dimensions we can write

$$
\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w}
$$

For two-dimensional flow we can find the streamlines using a streamfunction. We can introduce a streamfunction $\psi = \psi(x,y)$ with the property that the streamfunction is constant along streamlines. We can make the following derivation

$$
\begin{aligned}
\mathrm{d}\psi &= \frac{\partial \psi}{\partial x}\,\mathrm{d}x + \frac{\partial \psi}{\partial y}\,\mathrm{d}y \\
&= \left\{ \text{assume } u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x} \text{ and check for consistency} \right\} \\
&= u\,\mathrm{d}y - v\,\mathrm{d}x \\
&= 0, \qquad \text{on streamlines}
\end{aligned}
$$

We can also quickly check that the definition given above satisfies continuity. We have

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0
$$

Another property of the streamfunction is that the volume flux between two streamlines can be found from the difference by their respective streamfunctions. We use figure 2.12 and figure 2.13 in the following derivations of the volume flux:
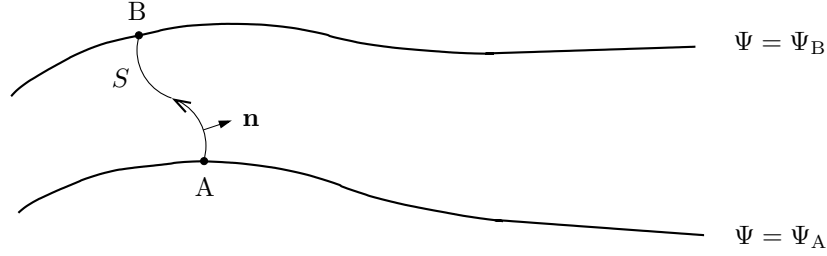
$$\omega_z = \frac{}{\sqrt{\pi\nu t}}\mathbf{u}$$

B

$$\Psi = \Psi_B$$

$\mathbf{n}$

$\mathrm{d}S$ S

$\mathbf{u}\cdot\mathbf{n}\Delta t$ $\mathbf{n}$

x

y A

u_0

p_0 $\Psi = \Psi_A$

$$\eta = \frac{y}{\sqrt{4\nu t}}$$
$$\frac{u}{u_0}$$

$$y = \eta\sqrt{4\nu t}$$

t

η

y

$$\omega_z = \frac{\tilde\omega(\eta)\cdot u_0}{\sqrt{\pi\nu t}}$$

$t\mathrm{d}y$

Figure 2.12: Integration paths between two streamlines.

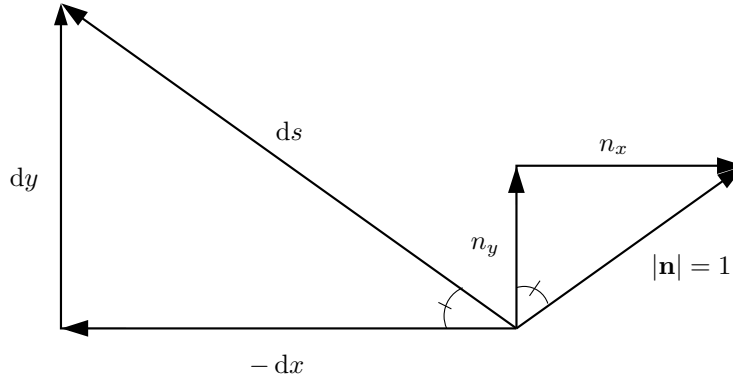$\mathrm{d}s$ n_x n_y $|\mathbf{n}| = 1$

$-\,\mathrm{d}x$

Figure 2.13: The normal vector to a line element.

$$
\begin{aligned}
Q_{AB} &= \int_A^B u_i n_i \,\mathrm{d}s \\
&= \int_A^B (n_x u + n_y v)\,\mathrm{d}s \\
&= \left\{ \frac{\mathrm{d}s}{\mathrm{d}y} = \frac{1}{n_x}, \quad -\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{1}{n_y} \right\} \\
&= \int_A^B (u\,\mathrm{d}y - v\,\mathrm{d}x) = \int_A^B \mathrm{d}\psi = \psi_B - \psi_A
\end{aligned}
$$

2.3   Potential flow

Flows which can be found from the gradient of a scalar function, a so called potential, will be dealt with in this section. Working with the velocity potential, for example, will greatly simplify calculations, but also restrict the validity of the solutions. We start with a definition of the velocity potential, then discuss simplifications of the momentum equations when such potentials exist, and finally we deal with two-dimensional potential flows, where analytic function theory can be used.

Velocity potential

Assume that the velocity can be found from a potential. In three dimensions we have

$$u_i = \frac{\partial\phi}{\partial x_i} \quad \text{or} \quad \mathbf{u} = \nabla\phi$$

with the corresponding relations in two dimensions

$$
\begin{cases}
u &= \dfrac{\partial\phi}{\partial x} \\[2mm]
v &= \dfrac{\partial\phi}{\partial y}
\end{cases}
$$

There are several consequences of this definition of the velocity field.

i) Potential flow have no vorticity, which can be seen by taking the curl of the gradient of the velocity potential, i.e.

$$\omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \epsilon_{ijk}\frac{\partial^2 \phi}{\partial x_j \partial x_k} = 0$$

ii) Potential flow is not influenced by viscous forces, which is seen by the relationship between viscous forces and vorticity, derived earlier. We have

$$\frac{\partial \tau_{ij}}{\partial x_j} = \mu \nabla^2 u_i = -\mu \epsilon_{ijk}\frac{\partial \omega_k}{\partial x_j}$$

iii) Potential flow satisfies Laplace equation, which is a consequence of the divergence constraint applied to the gradient of the velocity potential, i.e.

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial^2 \phi}{\partial x_i \partial x_i} = \nabla^2 \phi = 0$$

Bernoulli's equation

For potential flow the velocity field can thus be found by solving Laplace equation for the velocity potential. The momentum equations can then be used to find the pressure from the so called Bernoulli's equation. We start by rewriting the momentum equation

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i - g\delta_{i3}$$

using the alternative form of the non-linear terms, i.e.

$$u_j\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{1}{2}u_k u_k\right) - \epsilon_{ijk}u_j\omega_k$$

and obtain the following equation

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\left(\frac{1}{2}u_k u_k + \frac{p}{\rho} + gx_3\right) = \epsilon_{ijk}u_j\omega_k + \nu\epsilon_{ijk}\frac{\partial \omega_k}{\partial x_j}$$

This equation can be integrated in for two different cases: potential flow (without vorticity) and stationary inviscid flow with vorticity.

i) Potential flow. Introduce the definition of the velocity potential $u_i = \frac{\partial \phi}{\partial x_i}$ into the momentum equation above, we have

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}u_k u_k + \frac{p}{\rho} + gx_3\right) = 0 \quad \Rightarrow$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(u_1^2 + u_2^2 + u_3^2) + \frac{p}{\rho} + gx_3 = f(t)$$

ii) Stationary, inviscid flow with vorticity. Integrate the momentum equations along a streamline, see figure 2.14. We have

$$\int t_i \frac{\partial}{\partial x_i}\left(\frac{1}{2}u_k u_k + \frac{p}{\rho} + gx_3\right) \mathrm{d}s = \int \epsilon_{ijk}\frac{u_i}{u}u_j\omega_k \,\mathrm{d}s = 0 \quad \Rightarrow$$

$$\frac{1}{2}(u_1^2 + u_2^2 + u_3^2) + \frac{p}{\rho} + gx_3 = \text{constant along streamlines}$$

Thus, Bernoulli's equation can be used to calculate the pressure when the velocity is known, for two different flow situations.

Example 4. Ideal vortex: Swirling stationary flow without vorticity. The velocity potential $\mathbf{u} = \nabla\phi$ in polar coordinates can be written

$$\begin{cases} u_r &= \dfrac{\partial \phi}{\partial r} \\[2mm] u_\theta &= \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} \end{cases}$$
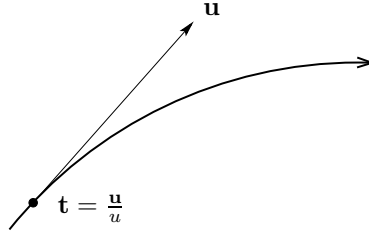
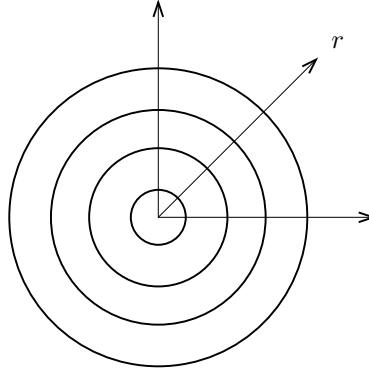Figure 2.14: Integration of the momentum equation along a streamline.

Figure 2.15: Streamlines for the ideal vortex.

The velocity potential satisfies Laplace equation, which in polar coordinates becomes

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

The first term vanishes since $\frac{\partial \phi}{\partial r} = u_r = 0$. Integrate the resulting equation twice and we find

$$\phi = C\theta + D$$

The constant D is arbitrary so we can set it to zero. Using the definition of the circulation we find

$$u_\theta = \frac{C}{r}, \quad C = \frac{\Gamma}{2\pi}$$

We can easily find the streamfunction for this flow. In polar coordinates the streamfunction is defined

$$\begin{cases} u_r &= \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} \\[2mm] u_\theta &= -\dfrac{\partial \psi}{\partial r} \end{cases}$$

Note that the streamfunction satisfies continuity, i.e. $\nabla \cdot \mathbf{u} = 0$. We use the solution from the velocity potential to find the streamfunction, we have

$$\frac{\partial \psi}{\partial r} = -\frac{\Gamma}{2\pi r} \quad \Rightarrow \quad \psi = -\frac{\Gamma}{2\pi}\ln r$$

The streamfunction is constant along streamlines resulting in circular streamlines, see figure 2.15

We can now use Bernoulli's equation to calculate the pressure distribution for the ideal vortex as

$$\frac{1}{2}u_\theta^2 + p = p_\infty \quad \Rightarrow \quad p = p_\infty - \frac{\Gamma^2}{8\pi^2 r^2}$$

Note that the solution has a singularity at the origin.

Two-dimensional potential flow using analytic functions

For two-dimensional flows we can use analytic function theory, i.e. complex valued functions, to calculate the velocity potential. In fact, as we will show in the following. Any analytic function represents a two-dimensional potential flow.

We start by showing that both the velocity potential and the streamfunciton satisfy Laplace equation. Velocity potential ϕ for two-dimensional flow is defined

$$\begin{cases} u & = & \dfrac{\partial \phi}{\partial x} \\[3mm] v & = & \dfrac{\partial \phi}{\partial y} \end{cases}$$

Using the continuity equation gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla^2 \phi = 0$$

The streamfunction ψ is defined

$$\begin{cases} u & = & \dfrac{\partial \psi}{\partial y} \\[3mm] v & = & -\dfrac{\partial \psi}{\partial x} \end{cases}$$

By noting that potential flow has no vorticity we find

$$-\omega = -\frac{\partial v}{\partial x} + \frac{\partial y}{\partial y} = \nabla^2 \psi = 0$$

Thus, both the velocity potential ϕ and the streamfunction ψ satisfy Laplace equation.

We can utilize analytic function theory by introducing the complex function

$$F(z) = \phi(x, y) + i\psi(x, y)$$

where

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

which will be denoted the complex potential.

We now recall some useful results from analytic function theory.

$$F'(z) \text{ exists and is unique} \quad \Longleftrightarrow \quad F(z) \text{ is an analytic function}$$

The fact that a derivative of a complex function should be unique rests on the validity of the Cauchy–Riemann equations. We can derive those equations by requiring that a derivative of a complex function should be the same independent of the direction we take the limit in the complex plane, see figure 2.16. We have

$$\begin{aligned} F'(z) & = & \lim_{\Delta z \to 0} \frac{1}{\Delta z}[F(z + \Delta z) - F(z)] \\[2mm] & = & \lim_{\Delta x \to 0} \frac{1}{\Delta x}[\phi(x + \Delta x, y) - \phi(x, y) + i\psi(x + \Delta x, y) - i\psi(x, y)] \\[2mm] & = & \lim_{i\Delta y \to 0} \frac{1}{i\Delta y}[\phi(x, y + \Delta y) - \phi(x, y) + i\psi(x, y + \Delta y) - i\psi(x, y)] \end{aligned}$$

$$\Rightarrow \begin{cases} \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y} & \text{real part} \\[3mm] i\dfrac{\partial \psi}{\partial x} = \dfrac{1}{i}\dfrac{\partial \phi}{\partial y} \Rightarrow \dfrac{\partial \psi}{\partial x} = -\dfrac{\partial \phi}{\partial y} & \text{imaginary part} \end{cases}$$

Thus we have the Cauchy–Riemann equations as a necessary requirement for a complex function to be analytic, i.e.

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Using the Cauchy-Riemann equations we can show that the real ϕ and imaginary part ψ of the complex potential satisfy Laplace equations and thus that they are candidates for identification with the velocity potential and streamfunction of a two-dimensional potential flow. We have

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \psi}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \psi}{\partial x}\right) = -\dfrac{\partial^2 \phi}{\partial y^2} \\[4mm] \dfrac{\partial^2 \psi}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \phi}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \phi}{\partial y}\right) = -\dfrac{\partial^2 \psi}{\partial x^2} \end{cases}$$

u_0

p_0

$\eta = \dfrac{y}{\sqrt{4\nu t}}$

$\dfrac{u}{u_0}$

$y = \eta\sqrt{4\nu t}$

t

η

y

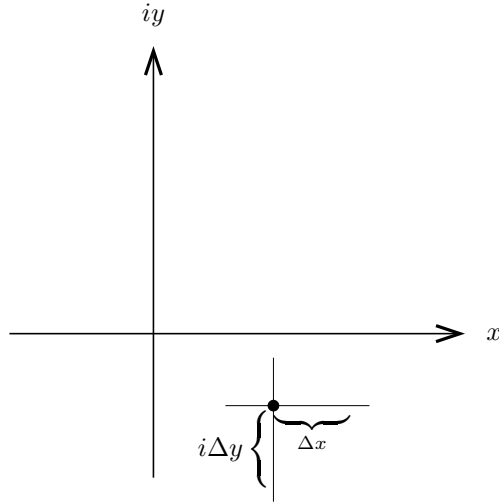$\omega_z = \dfrac{\tilde{\omega}(\eta)\cdot u_0}{\sqrt{\pi\nu t}}$

t

Figure 2.16: The complex plane and the definitions of analytic functions.

In addition to satisfying the Laplace equations, we also have to show that the Cauchy-Riemann equations are consistent with the definition of the velocity potential and the streamfunciton. The Cauchy–Riemann equations are

$$\begin{cases} u = \dfrac{\partial\phi}{\partial x} = \dfrac{\partial\psi}{\partial y} \\[3mm] v = \dfrac{\partial\phi}{\partial y} = -\dfrac{\partial\psi}{\partial x} \end{cases}$$

which recovers the definitions of the velocity potential and the streamfunction. Thus any analytic complex function can be identified with a potential flow, where the real part is the velocity potential and the imaginary part is the streamfunction.

We can now define the complex velocity as the complex conjugate of the velocity vector since

$$W(z) = \frac{\mathrm{d}F}{\mathrm{d}z} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = u - iv$$

It will be useful to have a similar relation for polar coordinates, see figure 2.17. In this case we have

$$\begin{cases} u = u_r\cos\theta - u_\theta\sin\theta \\ v = u_r\sin\theta + u_\theta\cos\theta \end{cases}$$

$$W(z) = u - iv = [u_r(r,\theta) - iu_\theta(r,\theta)]\underbrace{e^{-i\theta}}_{\cos\theta - i\sin\theta}$$

We will now take several examples of how potential velocity fields can be found from analytic functions.

Example 5. Calculate the velocity field from the complex potential $F = Ue^{-i\alpha}z$. The complex velocity becomes

$$W(z) = F' = Ue^{-i\alpha} = U(\cos\alpha - i\sin\alpha)$$

which gives us the solution for the velocity field in physical variables, see figure 2.18, as

$$\begin{cases} u = U\cos\alpha \\ v = U\sin\alpha \end{cases}$$

Example 6. Calculate the velocity field from the complex potential $F = Az^n$. In polar coordinates the complex potential becomes

$$Ar^n e^{in\theta} = Ar^n\cos(n\theta) + iAr^n\sin(n\theta)$$

Thus the velocity potential and the streamfunction can be identified as

$$\begin{cases} \phi = Ar^n\cos(n\theta) \\ \psi = Ar^n\sin(n\theta) \end{cases}$$

$\mathbf{q} = m\mathbf{P}_2^\cdot$
$\hat{x}$
x_1
$\hat{y}$
x_2
$\hat{u}_0$
x_3
$\hat{p}_0$
$\eta = \dfrac{\mathbf{R}^3}{\sqrt{4\nu t}}$
$\mathbf{R}^2$
$\dfrac{u_0}{\mathbf{R}} y$
$y = \eta\sqrt{4\nu t}\, y$
$\mathbf{r}^3$
t
$\mathbf{r}^2$
η
$\mathbf{r}^1$
y
$\dfrac{\pi}{2} + \mathrm{d}\varphi_{13}$
$\omega_z = \dfrac{\bar{\omega}(\eta)\cdot u_0}{\sqrt{\pi\nu t}}$
$S(t)$
$V(t + \Delta t) - V(t)$
$S(t + \Delta t)$
$\mathbf{u}$
$\mathbf{n}$
$V(t + \Delta t)$
p
T_0, U_0
T_{w}
L
T_0
U_0
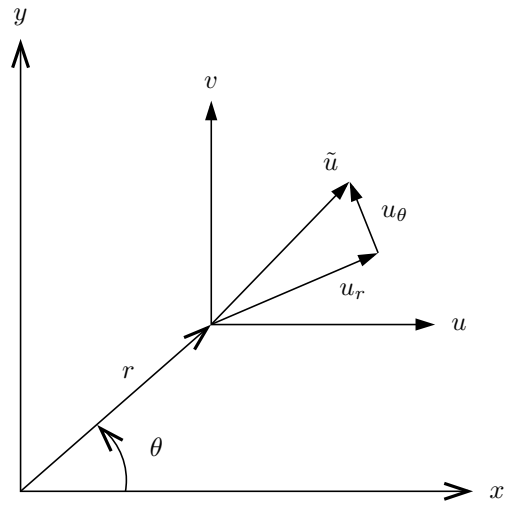
Figure 2.17: The definition of polar coordinates.

Gradient fields
w_i
$\dfrac{\partial p}{\partial x_i}$
u_i
Divergence free vectorfields // to boundary
L
h
x_1
x_2
$p_0 > p_1$
p_1
$\mathbf{u}$
$\mathbf{n}$
$\mathrm{d}S$
$\mathbf{u}\cdot\mathbf{n}\Delta t$
x
y
u_0
p_0
$\eta = \dfrac{y}{\sqrt{4\nu t}}$
$\dfrac{u}{u_0}$
$y = \eta\sqrt{4\nu t}$
t
η
y
$\omega_z = \dfrac{\bar{\omega}(\eta)\cdot u_0}{\sqrt{\pi\nu t}}$
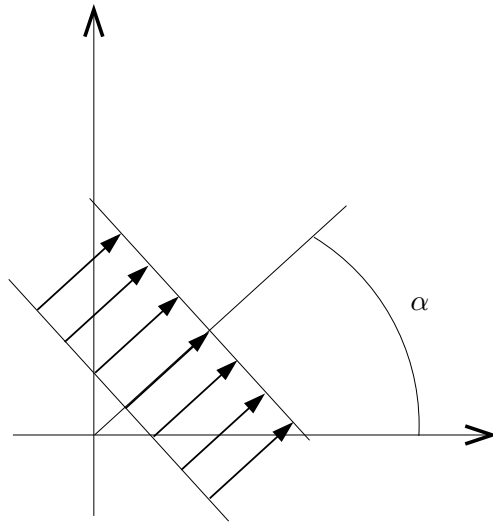t

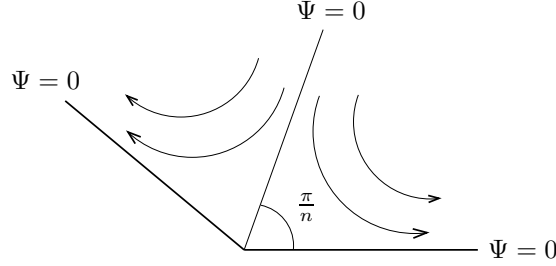Figure 2.18: Constant velocity at an angle α.
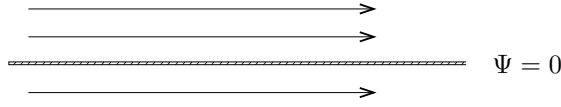
Figure 2.19: Flow towards a corner.

Figure 2.20: Uniform flow over a flat plate.

and we can calculate the complex velocity as

$$
\begin{aligned}
W(z) &= F' = nAz^{n-1} \\
&= nAr^{n-1}e^{in\theta}e^{-i\theta} \\
&= nAr^{n-1}[\cos(n\theta) + i\sin(n\theta)]e^{-i\theta}
\end{aligned}
$$

This gives us the solution for the velocity in polar coordinates as

$$
\left\{
\begin{aligned}
u_r &= nAr^{n-1}\cos(n\theta) \\
u_\theta &= -nAr^{n-1}\sin(n\theta)
\end{aligned}
\right.
$$

This is a flow which approaches a corner, see figure 2.19. The streamfunction is zero on the walls and on the stagnation point streamline. By solving for the zero streamlines we can calculate the angle between a wall aligned with the x-axis and the stagnation point streamline. We have

$$
\psi = Ar^n \sin(n\theta) = 0 \quad \Rightarrow \quad \theta = \frac{\pi k}{n}
$$

Figure 2.20 shows the uniform flow over a flat plate which results when $n = 1$.

Figure 2.21 shows the stagnation point flow when $n = 2$. Here the analysis can be done in Cartesian coordinates

$$
\begin{aligned}
F &= Az^2 \\
W(z) &= F' = 2Az \\
&= 2A(x + iy) \\
\Rightarrow \quad &\left\{
\begin{aligned}
u &= 2Ax \\
v &= -2Ay
\end{aligned}
\right.
\end{aligned}
$$

Figure 2.22 shows the flow towards a wedge when $1 \le n \le 2$. The velocity along the wedge edge (x axis in figure 2.22) is

$$
u_r = nAr^{n-1}\cos(n\theta) \Rightarrow \{\theta = 0\} \Rightarrow u = u_r = nAr^{n-1} = Cx^{n-1}
$$

The wedge angle is
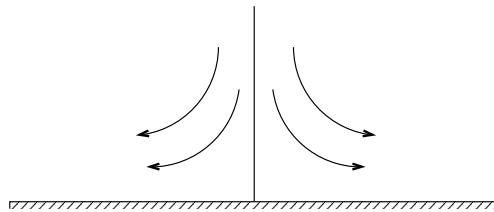
$$
\varphi = 2\pi - \frac{2\pi}{n} = 2\pi\frac{n-1}{n}
$$

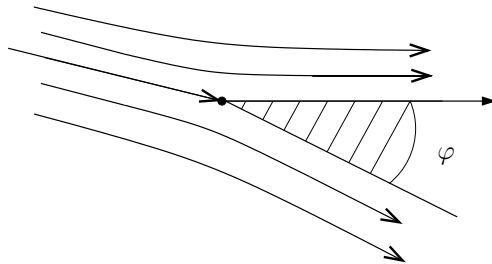Figure 2.21: Stagnation point flow when $n = 2$.

$\frac{u}{u_0}$

$y = \eta\sqrt{4\nu t}$ L

h

t x_1

η x_2

$p_0 > p_1$

$\omega_z = \frac{\tilde{\omega}(\eta)\cdot u_0}{\sqrt{\pi\nu t}}$ p_1

t **u**

n

dS

φ

Figure 2.22: The flow over a wedge when $1 \le n \le 2$.

u · **n** Δt

x

y

u_0

p_0

$\eta = \frac{y}{\sqrt{4\nu t}}$

$\frac{u}{u_0}$

$y = \eta\sqrt{4\nu t}$

t

η

y

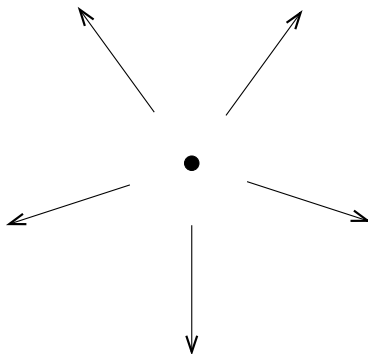$\omega_z = \frac{\tilde{\omega}(\eta)\cdot u_0}{\sqrt{\pi\nu t}}$

t

Figure 2.23: Line source.

We now list several other examples of potential flow fields obtained from analytic functions.

Example 7. Line source.

$$\begin{aligned} F(z) &= \frac{m}{2\pi}\ln z \\ &= \frac{m}{2\pi}\ln(re^{i\theta}) \\ &= \frac{m}{2\pi}[\ln r + i\theta] \end{aligned}$$

The velocity potential and the streamfunction become

$$\begin{cases} \phi = \dfrac{m}{2\pi}\ln r \\[2mm] \psi = \dfrac{m}{2\pi}\theta \end{cases}$$

The complex velocity can be written as

$$W(z) = F' = \frac{m}{2\pi z} = \frac{m}{2\pi r}e^{-i\theta}$$

which gives the solution in polar coordinates, see figure 2.23, as

$$\begin{cases} u_r = \dfrac{m}{2\pi r} \\[2mm] u_\theta = 0 \end{cases}$$

We can now calculate the volume flux from the source as

$$Q = \int_0^{2\pi} \frac{m}{2\pi r}r\,d\theta = m$$

Example 8. Line vortex.

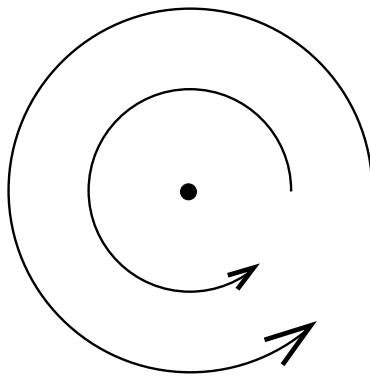$$\begin{aligned} F(z) &= -i\frac{\Gamma}{2\pi}\ln z \\ &= \frac{\Gamma}{2\pi}[\theta - i\ln r] \end{aligned}$$

x

y

u_0

p_0

$\eta = \dfrac{y}{\sqrt{4\nu t}}$

$\dfrac{u}{u_0}$

$y = \eta\sqrt{4\nu t}$

t

η

y

$\omega_z = \dfrac{\tilde{\omega}(\eta)\cdot u_0}{\sqrt{\pi\nu t}}$

t

Figure 2.24: Ideal vortex.

The velocity potential and the streamfunction become

$$\begin{cases} \phi = \dfrac{\Gamma}{2\pi}\theta \\[2ex] \psi = -\dfrac{\Gamma}{2\pi}\ln r \end{cases}$$

The complex velocity can be written as

$$W(z) = F' = -i\dfrac{\Gamma}{2\pi r}e^{-i\theta}$$

which gives the solution in polar coordinates, see figure 2.24, as

$$\begin{cases} u_r = 0 \\[2ex] u_\theta = \dfrac{\Gamma}{2\pi r} \end{cases}$$

Example 9. The dipole has the complex potential

$$F(z) = \dfrac{m\epsilon}{\pi z}$$

Example 10. The potential flow around a cylinder has the complex potential

$$F(z) = Uz + U\dfrac{r_0^2}{z}$$

Example 11. The potential flow around a cylinder with circulation has the complex potential

$$F(z) = Uz + U\dfrac{r_0^2}{z} - i\dfrac{\Gamma}{2\pi}\ln\dfrac{z}{r_0}$$

Example 12. Airfoils. The inviscid flow around some airfoils can be calculated with conformal mappings of solutions from cylinder with circulation.

2.4 Boundary layers

For flows with high Reynolds number Re, where the viscous effects are small, most of the flow can be considered inviscid and a simpler set of equations, the Euler equations, can be solved. The Euler equations are obtained if $Re \to \infty$ in the Navier-Stokes equations. However, the highest derivatives are present in the viscous terms, thus a smaller number of boundary conditions can be satisfied when solving the Euler equations. In particular, the no flow condition can be satisfied at a solid wall but not the no slip condition. In order to satisfy the no slip condition we need to add the viscous term. Thus, there will be a layer close to the solid walls where the viscous terms are important even for very high Reynolds number flow. This layer will be very thin and the flow in that region is called boundary layer flow.
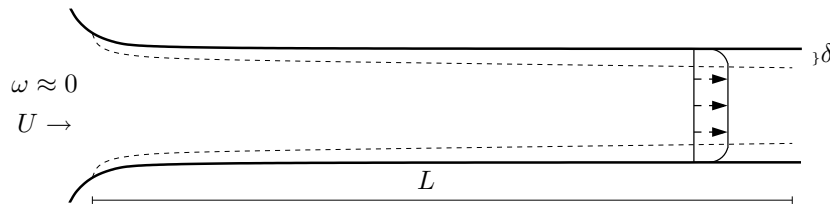
Figure 2.25: Estimate of the viscous region for an inviscid vorticity free inflow in a plane channel.
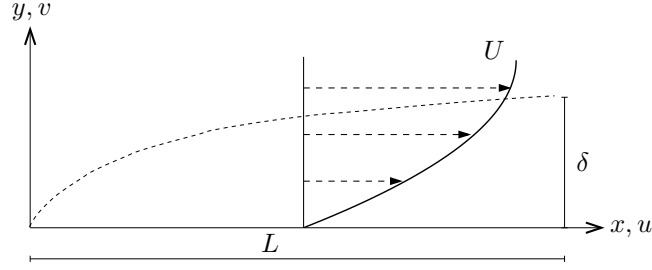

Figure 2.26: Boundary layer profile close to a solid wall.

We begin this section by estimating the thickness of the viscous region in channel with inviscid inflow (zero vorticity), see figure 2.25.

By analogy to the Stokes solution for the instantaneously starting plate, we can estimate the distance the vorticity has diffused from the wall toward the channel centerline as $\delta \sim \sqrt{\nu t_s}$. Where t_s is time of a fluid particle with velocity U has travelled length L, i.e. $t_s = \dfrac{L}{U}$. Thus we can estimate δ/L as

$$\Rightarrow \frac{\delta}{L} = \frac{\sqrt{\nu t_s}}{L} = \frac{1}{L}\left(\frac{\nu L}{U}\right)^{1/2} = \left(\frac{\nu}{UL}\right)^{1/2} = \frac{1}{\sqrt{Re}} \to 0 \text{ as } Re \to \infty$$

We will now divide the flow into a central/outer inviscid part and a boundary layer part close to walls, where the no slip condition is fulfilled. Since we expect this region to be thin for high Reynolds numbers, we expect large velocity gradients near walls.

We choose different length scales inside boundary layer and in inviscid region as indicated in figure 2.26, as

Inviscid : U velocity scale for u, v
: L length scale for x, y

BL : U velocity scale for u
: α velocity scale for v
: L length scale for x
: δ length scale for y

The pressure scale is assumed to be ρU^2 for both the inner and the outer region. The unknown velocity scale for the normal velocity α is determined using continuity

$$\underbrace{\frac{\partial u}{\partial x}}_{\mathcal{O}\left[\frac{U}{L}\right]} + \underbrace{\frac{\partial v}{\partial y}}_{\mathcal{O}\left[\frac{\alpha}{\delta}\right]} = 0$$

which implies that

$$\frac{U}{L} = \frac{\alpha}{\delta} \quad \Rightarrow \quad \alpha = \frac{\delta}{L}U$$

In this manner both terms in the continuity equation have the same size.

Let us use these scalings in the steady momentum equations and disregard small terms. We obtain the following estimates for the size of the terms in the normal momentum equation

$$\uparrow y: \quad u\frac{\partial v}{\partial x} \quad + \quad v\frac{\partial v}{\partial y} \quad = \quad -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad + \quad \nu\frac{\partial^2 v}{\partial x^2} \quad + \quad \nu\frac{\partial^2 v}{\partial y^2}$$

$$U\cdot\frac{1}{L}\frac{\delta U}{L} \qquad \frac{\delta U}{L}\cdot\frac{1}{\delta}\cdot\frac{\delta U}{L} \qquad \frac{U^2}{\delta} \qquad \frac{\nu}{L^2}\cdot\frac{\delta U}{L} \qquad \frac{\nu}{\delta^2}\frac{\delta U}{L}$$

$$\left(\frac{\delta}{L}\right)^2 \qquad\qquad \left(\frac{\delta}{L}\right)^2 \qquad\qquad 1 \qquad\qquad \frac{1}{Re}\left(\frac{\delta}{L}\right)^2 \qquad\qquad \frac{1}{Re}$$

where the last line was obtained by dividing all the terms on the second line by U^2/δ. When $R \to \infty$ and δ/L is small, only pressure term $\mathcal{O}(1)$, which implies that

$$\frac{\partial p}{\partial y} = 0 \qquad \Rightarrow \qquad p = p(x)$$

Thus, the pressure is constant through the boundary layer, given by the inviscid outer solution.
Next we estimate the size of the terms in the streamwise momentum equation as

$$\to x: \quad u\frac{\partial u}{\partial x} \quad + \quad v\frac{\partial u}{\partial y} \quad = \quad -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad + \quad \nu\frac{\partial^2 u}{\partial x^2} \quad + \quad \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{U^2}{L} \qquad\qquad \frac{\delta U}{L}\cdot\frac{U}{\delta} \qquad\qquad \frac{U^2}{L} \qquad\qquad \frac{\nu}{L^2}\cdot U \qquad\qquad \frac{\nu}{\delta^2}U$$

$$1 \qquad\qquad 1 \qquad\qquad 1 \qquad\qquad \frac{1}{Re} \qquad\qquad \frac{1}{Re}\cdot\left(\frac{L}{\delta}\right)^2$$

The first three terms are order one and the fourth term negligible when the Reynolds number become large. In order to keep the last term, the only choice that gives a consistent approximation, we have to assume that

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{Re}}$$

which is the same estimate as we found for the extent of the boundary layer in the channel inflow case. Thus we are left with the parabolic equation in x since the second derivative term in that direction is neglected. We have

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

In summary, the steady boundary layer equations for two-dimensional flow is

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

with the initial (IC) and boundary conditions (BC)

$$\text{IC}: \quad u(x=x_0, y) \quad = \quad u_{\text{in}}(y)$$

$$\text{BC}: \quad u(x, y=0) \quad = \quad 0$$
$$v(x, y=0) \quad = \quad 0$$
$$u(x, y\to\infty) \quad = \quad u_e(x) \text{ outer inviscid flow}$$

where the initial condition is at $x = x_0$ and the equations are marched in the downstream direction. There are several things to note about these equations.

First, v_{in} cannot be given as its own initial condition. By using continuity one can show that once the u_0 is given the momentum equation can be written in the form

$$-\frac{\partial v}{\partial y} + f(y)v = g(y)$$

which can be integrated in the y-direction to give the initial normal velocity.

Second, the system of equations is a third order system in y and thus only three boundary conditions can be enforced. We have the no flow and the no slip conditions at the wall and the $u = u_e$ in the free stream. Thus no boundary condition for v in the free stream can be given.
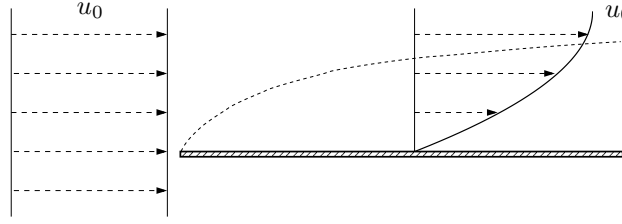
Figure 2.27: Boundary layer flow over a flat plate with zero pressure gradient.

Third, $u_e(x) = u_{inv}(x, y = 0)$, i.e. the boundary layer solution in the free stream approaches the inviscid solution evaluated at the wall, since the thickness of the boundary layer is zero from the inviscid point of view.

Fourth, the pressure gradient term in the momentum equation can be written in terms of the edge velocity u_e using the Bernoulli equation. We have

$$p(x) + \frac{1}{2}\rho u_e^2(x) = \text{const}$$

which implies that

$$-\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = u_e\frac{\mathrm{d}u_e}{\mathrm{d}x}$$

Blasius flow over flat plate

Consider the boundary layer equations for a flow over a flat plate with zero pressure gradient, i.e. the outer inviscid flow is $u_e(x) = u_0$. The boundary layer equations become

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \nu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

$$\begin{aligned} \text{IC}: \quad u(x = x_0, y) \quad &= \quad u_0 \\ \text{BC}: \quad u(x, y = 0) \quad &= \quad 0 \\ v(x, y = 0) \quad &= \quad 0 \\ u(x, y \to \infty) \quad &= \quad u_0 \end{aligned}$$

We introduce the two-dimensional stream function Ψ

$$u = \frac{\partial\Psi}{\partial y}, \qquad v = -\frac{\partial\Psi}{\partial x}$$

which implicitly satisfies continuity. The boundary layer equations become

$$\frac{\partial\Psi}{\partial y}\cdot\frac{\partial^2\Psi}{\partial x\partial y} - \frac{\partial\Psi}{\partial x}\cdot\frac{\partial^2\Psi}{\partial y^2} = \nu\frac{\partial^3\Psi}{\partial y^3}$$

with appropriate initial and boundary conditions.

Similarity solution

We will now look for a similarity solution. Let

$$u = \frac{\partial\Psi}{\partial y} = u_0 f'(\eta), \qquad \eta = \frac{y}{\delta(x)}$$

which by integration in the y-direction implies

$$\Psi(x, y) = u_0\delta(x)f(\eta)$$

We can now evaluate the various terms in the momentum equation. We give two below

$$v = -\frac{\partial\Psi}{\partial x} = -u_0\left(\delta'f + \delta f'\frac{\partial\eta}{\partial x}\right) = -u_0\left(\delta'f - \frac{y}{\delta}f'\delta'\right) = u_0(\eta f' - f)\delta'$$
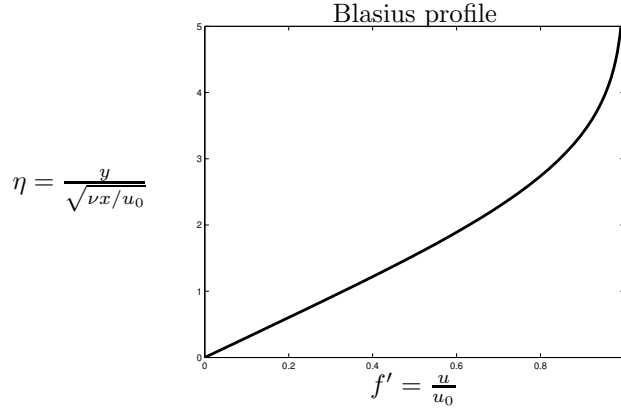
Figure 2.28: Similarity solution giving the Blasius boundary layer profile.

$$\frac{\partial^2 \Psi}{\partial x \partial y} = -u_0 f'' \frac{\eta}{\delta} \delta'$$

If these and the other corresponding terms are introduced into the momentum equation we have the following equation for f

$$-u_0^2 f' f'' \frac{\eta}{\delta} \delta' + u_0^2 \left(\eta f' - f\right) \frac{f'' \delta'}{\delta} = \nu u_0 \frac{f'''}{\delta^2}$$

which can be simplified to

$$f''' + \frac{u_0 \delta \delta'}{\nu} f f'' = 0$$

For a similarity solution to be possible, the coefficient in front of the $f f''$ term has to be constant. This implies

$$\frac{u_0 \delta \delta'}{\nu} = \frac{1}{2} \qquad \Rightarrow \qquad \int \delta \delta' \, dx = \frac{1}{2} \int \frac{\nu \, dx}{u_0} \qquad \Rightarrow \qquad \frac{1}{2} \delta^2 = \frac{1}{2} \frac{\nu x}{u_0}$$

which implies that $\delta(x) = \sqrt{\frac{\nu x}{u_0}}$, where we have chosen the constant of integration such that $\delta = 0$ when $x = 0$. We are left with the Blasius equation and boundary conditions

$$f''' + \frac{1}{2} f f'' = 0$$

$$\text{BC} : \begin{cases} f(0) & = & f'(0) = 0 \\ f'(\infty) & = & 1 \end{cases}$$

The numerical solution to the Blasius equation gives the self-similar velocity profile seen in figure 2.28.

The boundary layer thickness can be evaluated as the height where the velocity has reached 99% of the free stream value. We have

$$f'(\eta_{99}) = 0.99 \qquad \Rightarrow \qquad \eta_{99} = \frac{\delta_{99}}{\sqrt{\frac{\nu x}{u_0}}} = 4.9 \qquad \Rightarrow \qquad \delta_{99} = 4.9 \sqrt{\frac{\nu x}{u_0}}$$

where it can be seen that the boundary layer thickness grows as $\sqrt{x}$.

Displacement thickness and skin friction coefficient

Another measure of the boundary layer thickness is the displacement δ^* of the wall needed in order to have the same volume flux of the inviscid flow as for the flow in the boundary layer. With the help of figure 2.29 we can evaluate this as

$$\int_0^{y_0} u \, dy \equiv \int_{\delta^*}^{y} u_0 \, dy = \int_0^{y} u_0 \, dy - \delta^* u_0$$
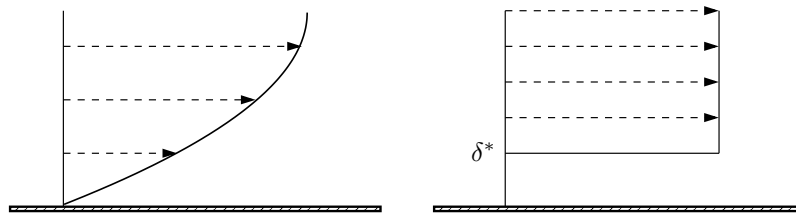
which implies that

$\omega_z = \dfrac{}{\sqrt{\pi \nu t}}$ U

t

δ

u_0

$\eta \dfrac{L}{\delta} \dfrac{y}{\sqrt{\nu x / u_0}}$

$\omega \approx 0$ $f' = \dfrac{u}{u_0}$

Blasius profile

y, v

x, u

L

U

δ

u_0

$\eta = \dfrac{y}{\sqrt{\nu x / u_0}}$

$f' = \dfrac{u}{u_0}$

Blasius profile

δ^*

Figure 2.29: The displacement thickness δ^*

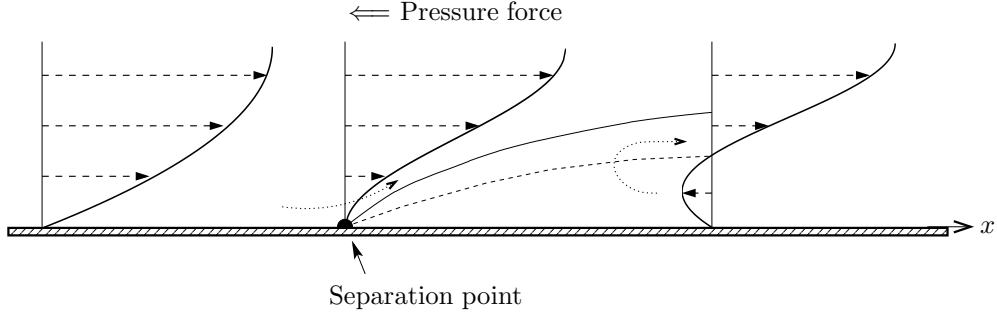$\Longleftarrow$ Pressure force

Separation point

Figure 2.30: Boundary layer separation caused by a pressure force directed upstream, a so called adverse pressure gradient.

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_0}\right) \mathrm{d}y = \sqrt{\frac{\nu x}{u_0}} \int_0^\infty \left[1 - f'(\eta)\right] \mathrm{d}\eta$$

If we evaluate this expression for the Blasius boundary layer we have $\delta^* = 1.72 \sqrt{\frac{\nu x}{u_0}}$.

A quantity which is of large importance in practical applications is the skin friction coefficient. The skin friction at the wall is

$$\tau_\mathrm{w} = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{\mu u_0}{\delta} f''(0) = \frac{\rho u_0^2}{\sqrt{\frac{u_0 x}{\nu}}} f''(0) = \frac{0.332 \rho u_0^2}{\sqrt{Re_x}}$$

where the last term is the expression evaluated for Blasius flow. This can be used to evaluate the skin friction coefficient as

$$C_f = \frac{\tau_\mathrm{w}}{\frac{1}{2}\rho u_0^2} = \frac{0.664}{\sqrt{Re_x}}$$

where the last terms again is the Blasius value.

Boundary layer separation

A large adverse pressure gradient $\left(\frac{\partial p}{\partial x} > 0\right)$ may cause a back flow region close to wall, see figure 2.30.

The boundary layer equations cannot be integrated past a point of separation, since they become singular at that point. This can be seen from the following manipulations of the boundary layer equations. If we take the momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \nu\frac{\partial^2 u}{\partial y^2}$$

take the normal derivative and use continuity we have

$$u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} = \nu\frac{\partial^3 u}{\partial y^3}$$

If we again take the normal derivative and use continuity we can write this as

$$\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3} = \nu\frac{\partial^4 u}{\partial y^4}$$

This expression evaluated at the wall, with the use of the boundary conditions, become

$$\frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y}\bigg|_{y=0} \right)^2 = \underbrace{\nu \frac{\partial^4 u}{\partial y^4}\bigg|_{y=0}}_{\pm \kappa^2}$$

which can be integrated to yield

$$\left(\frac{\partial u}{\partial y}\bigg|_{y=0} \right)^2 = \pm \kappa^2 x + C$$

At the point of separation $x = x_s$, i.e. where $\left(\frac{\partial u}{\partial y}\bigg|_{y=0} = 0 \right)$ this expression is

$$\frac{\partial u}{\partial y}\bigg|_{y=0} = \kappa \sqrt{x_s - x}$$

We can see that the boundary layer equations are not valid beyond point of separation since we have a square root singularity at that point. In addition, the point of separation is the limiting point after which there is back flow close to the wall. This means that there is information propagating upstream, invalidating the downstream parabolic nature of the equations assumed by the downstream integration of the equations.

In practical situations it is very important to be able to predict the point of separation, for example when an airfoil stalls and experience a severe loss of lift.

2.5   Turbulent flow

Reynolds average equations

Turbulent flow is inherently time dependent and chaotic. However, in applications one is not usually interested in knowing full the details of this flow, but rather satisfied with the influence of the turbulence on the averaged flow. For this purpose we define an ensemble average as

$$U_i = \overline{u}_i = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} u_i^{(n)}$$

where each member of the ensemble $u_i^{(n)}$ is regarded as an independent realization of the flow.

We are now going to derive an equation governing the mean flow U_i. Divide the total flow into an average and a fluctuating component u_i' as

$$u_i = \overline{u}_i + u_i' \qquad p = \overline{p} + p'$$

and introduce into the Navier-Stokes equations. We find

$$\frac{\partial u_i'}{\partial t} + \frac{\partial \overline{u}_i}{\partial t} + u_j' \frac{\partial \overline{u}_i}{\partial x_j} + \overline{u}_j \frac{\partial u_i'}{\partial x_j} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} + u_j' \frac{\partial u_i'}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p'}{\partial x_i} - \frac{1}{\rho} \frac{\partial \overline{p}}{\partial x_i} + \nu \nabla^2 u_i' + \nu \nabla^2 \overline{u}_i$$

We take the average of this equation, useing

$$\overline{u}_i' = 0 \qquad \overline{u_j' \overline{u}_i} = \overline{u}_i \cdot \overline{u}_j' = 0$$

which gives

$$\begin{cases} \dfrac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \dfrac{\partial \overline{u}_i}{\partial x_j} = -\dfrac{1}{\rho} \dfrac{\partial \overline{p}}{\partial x_i} + \nu \nabla^2 \overline{u}_i - \dfrac{\partial}{\partial x_j} \left(\overline{u_i' u_j'} \right) \\ \dfrac{\partial \overline{u}_i}{\partial x_i} = 0 \end{cases}$$

where the average of the continuity equation also has been added. Now, let $U_i = \overline{u}_i$, as above, and drop the $'$. We find the Reynolds average equations

$$
\begin{cases}
\dfrac{\partial U_i}{\partial t} + U_j \dfrac{\partial U_i}{\partial x_j} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial x_i} + \dfrac{\partial}{\partial x_j}\left(\tau_{ij} - \overline{u_i u_j}\right) \\[3mm]
\dfrac{\partial U_i}{\partial x_i}
\end{cases}
$$

where $\overline{u_i u_j}$ is the Reynolds stress. Thus the effect of the turbulence on the mean is through an additional stress R_{ij} which explicitly written out is

$$
[R_{ij}] = [\overline{u_i u_j}] =
\begin{bmatrix}
\overline{u^2} & \overline{uv} & \overline{uw} \\
\overline{uv} & \overline{v^2} & \overline{vw} \\
\overline{uw} & \overline{vw} & \overline{w^2}
\end{bmatrix}
$$

The diagonal component of this stress is the kinetic energy of the turbulent fluctuations

$$
\overline{k} = R_{ii}/2 = \overline{u_i u_i}/2 = \frac{1}{2}\left(\overline{u^2} + \overline{v^2} + \overline{w^2}\right)
$$

Turbulence modelling

One may derive equations for the components of the Reynolds stress tensor. However, they would involve additional new unknowns, and the equations for those would in turn involve even more unknowns. This closure problem implies that if one cannot calculate the complete turbulent flow, but is only interested in the mean, the effect of the Reynolds stress components on in the Reynolds average equations have to be modelled.

The simplest consistent model is to introduce a turbulent viscosity as a proportionality constant between the Reynolds stress components and the strain or deformation rate tensor, in analogy with the stress-strain relationship in for Newtonian fluids. We have

$$
R_{ij} = \frac{2}{3} k \delta_{ij} - 2\nu_T \overline{E}_{ij}
$$

The first term on the right hand side is introduced to give the correct value of the trace of R_{ij}. Here the deformation rate tensor is calculated based on the mean flow, i.e.

$$
\overline{E}_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{2}{3}\frac{\partial U_r}{\partial x_r}\delta ij\right)
$$

where we have assumed incompressible flow. Introducing this into the Reynolds average equations gives

$$
\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left[-\left(\frac{P}{\rho} + \frac{2}{3}k\right)\delta_{ij} + 2\left(\nu + \nu_T\right)\overline{E}_{ij}\right] = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \left(\nu + \nu_T\right)\nabla^2 U_i
$$

where the last equality assumes that ν_T is constant, an approximation only true for very simple turbulent flow. It is introduced here only to point out the analogy between the molecular viscosity and the turbulent viscosity.

The turbulent viscosity ν_T must be modelled and most numerical codes which solved the Navier-Stokes equations for practical applications uses some kind of model for ν_T.

For simple shear flows, i.e. velocity fields of the form $U(y)$, the main Reynolds stress component becomes

$$
\overline{uv} = -\nu_T \frac{\partial U}{\partial y}
$$

where, in analogy with kinetic gas theory, ν_T

$$
\nu_T = u_T l
$$

Here, u_T is a characteristic turbulent velocity scale and l is a characteristic turbulent length scale. In kinetic gas theory l would be the mean free path of the molecules and u_T the characteristic velocity of the molecules.
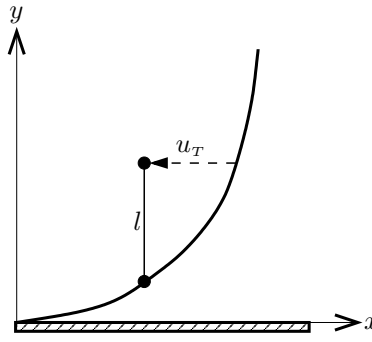
Figure 2.31: Prandtl's mixing length l is the length for which a fluid particle displaced in the normal direction can be expected to retain its horizontal momentum.

Zero-equation models

The simplest model for u_T and l is to model them algebraically.

In simple free shear flows like jets, wakes and shear layers we can take ν_T as constant in normal direction or y-direction and to vary in a self-similar manner in x.

In wall bounded shear flow we can take l as Prandtl's mixing length (1920's). This is a normal distance over which the particle retains its momentum and is depicted in figure 2.31.

We can thus evaluate u_T and find ν_T

$$u_T = l \left| \frac{\mathrm{d}U}{\mathrm{d}y} \right| \qquad \Rightarrow \qquad \nu_T = l^2 \left| \frac{\mathrm{d}U}{\mathrm{d}y} \right|$$

Define τ_w as the shear stress at the wall and $u_\tau = \sqrt{\tau_w/\rho}$ as the skin friction velocity. With

$$u_T = u_\tau, \qquad l = \kappa \cdot y$$

as Prandtl's estimation of mixing length we get

$$\frac{\mathrm{d}U}{\mathrm{d}y} = \frac{u_T}{l} = \frac{u_\tau}{\kappa y} \qquad \Rightarrow \qquad \frac{U}{u_\tau} = \frac{1}{\kappa} \ln y + C.$$

κ is the von Karman constant, approximately equal to 0.41. Thus the mixing length is proportional to the distant from the wall and the region close to the wall in a turbulent flow has a logarithmic velocity profile, called the log-region.

One-equation models

Instead of modelling the characteristic turbulent velocity algebraically, the next level of modelling uses the assumption that

$$u_T = \sqrt{k} = \sqrt{\frac{1}{2}\overline{u_i u_i}}$$

and a differential equation is used to calculate the turbulent kinetic energy. We can derive such an equation by taking the previously derived equation for $U_i + u_i$ and subtracting the equation for U, the Reynolds average equation. We find

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} = -u_j \frac{\partial U_i}{\partial x_j} - u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(-\frac{p}{\rho}\delta_{ij} + \nu \frac{\partial u_i}{\partial x_j} + \overline{u_i u_j} \right)$$

We now take the average of the inner product between the turbulent fluctuations u_i and the above equation, and find

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i u_i}\right) + U_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i u_i}\right) = -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(-\frac{\overline{p u_j}}{\rho} - \frac{1}{2}\overline{u_i u_i u_j} + \frac{\nu}{2}\frac{\partial}{\partial x_j}\overline{u_i u_i} \right) - \underbrace{\nu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}}_{\epsilon}$$

where ϵ is the mean dissipation rate of the turbulent kinetic energy. We have now introduced a number of new unknowns which have to be modelled.

First, we model the Reynolds stress as previously indicated, i.e. $\overline{u_i u_j} = \frac{2}{3}\overline{k}\delta_{ij} - \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$.

Second, the turbulent dissipation is modelled purely based on dimension using $\overline{k}$ and l. We have

$$\epsilon = c_D \frac{\overline{k}^{3/2}}{l}$$

since the dimension of ϵ is $[\epsilon] = \frac{L^2}{T}$. Here c_D is a modelling constant.

Third, a gradient diffusion model is used for the transport terms, i.e.

$$-\frac{1}{2}\overline{u_i u_i u_j} - \frac{1}{\rho}\overline{p u_j} = \frac{\nu_T}{Pr_k}\frac{\partial \overline{k}}{\partial x_j}$$

where Pr_k is the turbulent Prandtl number.

For simplicity we write the resulting equation for the turbulent kinetic energy for constant ν_T. We find

$$\underset{\text{rate of increase}}{\frac{D\overline{k}}{Dt}} = \underset{\text{diffusion rate}}{(\nu + \nu_T)\nabla^2 \overline{k}} - \underset{\text{generation rate}}{\overline{u_i u_j}\frac{\partial U_i}{\partial x_j}} - \underset{\text{dissipation rate}}{\epsilon}$$

Two-equation models

The most popular model in use in engineering applications is a two equation model, the so called $\overline{k}$-ϵ-model. This is based on the above model for $\overline{k}$ but instead of modelling ϵ a partial differential equation is derived governing the turbulent dissipation. This equation has the form as the equation for $\overline{k}$, with a diffusion term, a generation term and a dissipation term. Many new model constants need to be introduced in order to close the ϵ equation.

Once $\overline{k}$ and ϵ is calculated the turbulent viscosity is evaluated as a quotient of the two, such that the dimension is correct. We have

$$\Rightarrow \nu_T = C_\mu \cdot \frac{\overline{k}^2}{\epsilon}$$

where C_μ is a modelling constant.

Chapter 3

Finite volume methods for incompressible flow

3.1 Finite Volume method on arbitrary grids

Equations with 1st order derivatives: the continuity equation

In finite volume methods one makes approximations of the differential equations in integral form. We start with an equation with only first order derivatives, the continuity equation. Recall that the integral from of this equation can be written

$$\int_V \frac{\partial}{\partial x_k} \left(u_k \right) \, \mathrm{d}V = \int_S u_k n_k \, \mathrm{d}S = 0$$

Evaluating the normal vector n_i to the curve l for two-dimensional flow, see figure 3.1, we have the normal with length dl as $\mathbf{n} \, \mathrm{d}l = (\, \mathrm{d}y, - \, \mathrm{d}x)$. This implies that

$$\int_S u_k n_k \, \mathrm{d}l = \int_S \left(u \, \mathrm{d}y - v \, \mathrm{d}x \right) = 0$$

We now apply this to an arbitrary two-dimensional grid defined as in figure 3.2.

We make use of the following definitions and approximations

$$u_{ab} = u_{i+\frac{1}{2},j} \approx \left(u_{i+1,j} + u_{i,j} \right) / 2 \qquad \Delta x_{ab} = x_b - x_a$$

$$v_{ab} = v_{i+\frac{1}{2},j} \approx \left(v_{i+1,j} + v_{i,j} \right) / 2 \qquad \Delta y_{ab} = y_b - y_a$$
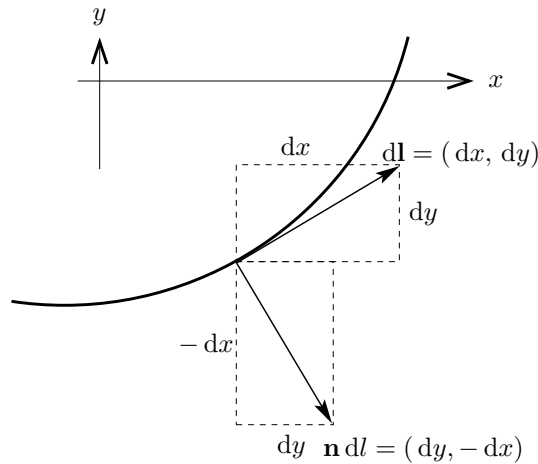

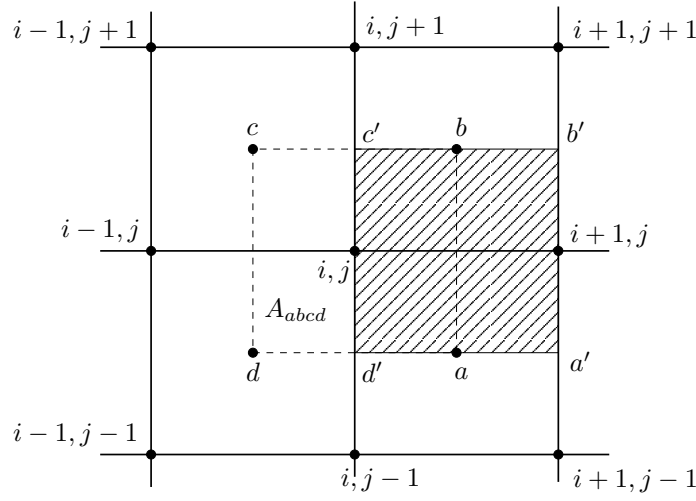Figure 3.1: Normal vector to the curve l.

Figure 3.2: Arbitrary grid. Note that the grid lines need not be parallel to the coordinate directions.

with analogous expressions for the quantities on the bc, cd and da faces. We can now evaluate the integral associated with the two-dimensional version of the continuity equation for the surface defined by the points $abcd$. We have

$$\int\limits_S (u\,dy - v\,dx) \approx u_{i+\frac{1}{2},j}\Delta y_{ab} - v_{i+\frac{1}{2},j}\Delta x_{ab} + u_{i,j+\frac{1}{2}}\Delta y_{bc} - v_{i,j+\frac{1}{2}}\Delta x_{bc} +$$

$$u_{i-\frac{1}{2},j}\Delta y_{cd} - v_{i-\frac{1}{2},j}\Delta x_{cd} + u_{i,j-\frac{1}{2}}\Delta y_{da} - v_{i,j-\frac{1}{2}}\Delta x_{da}$$

On a cartesian grid we have the relations

$$\Delta x_{ab} = \Delta x_{cd} = \Delta y_{bc} = \Delta y_{da} = 0 \text{ and } \Delta y_{cd} = -\Delta y_{ab}, \Delta x_{bc} = -\Delta x_{da}$$

which implies that the approximation of the continuity equation can be written

$$\int\limits_S (u\,dy - v\,dx) \approx \left(u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}\right)\Delta y_{ab} + \left(v_{i,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}}\right)\Delta x_{da} =$$

$$\frac{1}{2}\left(u_{i+1,j} - u_{i-1,j}\right)\Delta y_{ab} + \frac{1}{2}\left(v_{i,j+1} - v_{i,j-1}\right)\Delta x_{da}$$

Note that dividing with $\Delta y_{ab} \cdot \Delta x_{da}$ gives a standard central difference discretization of the continuity equation, i.e.

$$\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x_{da}} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y_{ab}} = 0$$

Equations with 2nd derivatives: the Laplace equation

We use the Gauss or Greens theorem to derive the integral from of the two-dimensional Laplace equation. We find

$$0 = \int\limits_V \frac{\partial^2 \Phi}{\partial x_j \partial x_j}\,dV = \int\limits_S \frac{\partial \phi}{\partial x_j}n_j\,dS = \{2D\} = \int\limits_S \left(\frac{\partial \Phi}{\partial x}\,dy - \frac{\partial \Phi}{\partial y}\,dx\right)$$

If we approximate this integral in the same manner as for the continuity equation we have

$$\left[\frac{\partial \Phi}{\partial x}\right]_{i+\frac{1}{2},j}\Delta y_{ab} - \left[\frac{\partial \Phi}{\partial y}\right]_{i+\frac{1}{2},j}\Delta x_{ab} + \left[\frac{\partial \Phi}{\partial x}\right]_{i,j+\frac{1}{2}}\Delta y_{bc} - \left[\frac{\partial \Phi}{\partial y}\right]_{i,j+\frac{1}{2}}\Delta x_{bc} +$$

$$\left[\frac{\partial \Phi}{\partial x}\right]_{i-\frac{1}{2},j}\Delta y_{cd} - \left[\frac{\partial \Phi}{\partial y}\right]_{i-\frac{1}{2},j}\Delta x_{cd} + \left[\frac{\partial \Phi}{\partial x}\right]_{i,j-\frac{1}{2}}\Delta y_{da} - \left[\frac{\partial \Phi}{\partial y}\right]_{i,j-\frac{1}{2}}\Delta x_{da}$$

First derivatives are evaluated as mean value over adjacent control volumes (areas in the two-dimensional case). We use Gauss theorem over the area defined by the points $a'b'c'd'$, see figure 3.2. For the three-dimensional case we have

$$\frac{1}{V}\int_V \frac{\partial \Phi}{\partial x_j}\, dV = \frac{1}{V}\int_S \Phi n_j\, dS \Rightarrow$$

which in the two-dimensional case is approximated by

$$\left(\left[\frac{\partial \Phi}{\partial x}\right]_{i+\frac{1}{2},j}, \left[\frac{\partial \Phi}{\partial y}\right]_{i+\frac{1}{2},j}\right) \approx \frac{1}{A_{a'b'c'd'}}\int_{a'b'c'd'} (\Phi\, dy, -\Phi\, dx)$$

where

$$\int_{a'b'c'd'} \Phi\, dy \approx \Phi_{i+1,j}\Delta y_{a'b'} + \Phi_b\Delta y_{b'c'} + \Phi_{i,j}\Delta y_{c'd'} + \Phi\Delta y_{d'a'}$$

and the area evaluated as half magnitude of cross products of diagonals, i.e.

$$A_{ab} \equiv A_{a'b'c'd'} = \frac{1}{2}\left|\Delta x_{d'b'}\cdot \Delta y_{a'c'} - \Delta y_{d'b'}\cdot \Delta x_{a'c'}\right|$$

and the value Φ_a is taken as the average

$$\Phi_a = \frac{1}{4}\left(\Phi_{i,j} + \Phi_{i+1,j} + \Phi_{i,j-1} + \Phi_{i+1,j-1}\right)$$

Evaluating the derivatives $\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}$ along the other cell faces and substituting back we obtain a 9 point formula which couples all of the nine points around the point i, j, of the form

$$
\begin{array}{ccccccccc}
A_{i,j}\Phi_{i+1,j+1} & + & B_{i,j}\Phi_{i+1,j} & + & C_{i,j}\Phi_{i+1,j-1} & + & D_{i,j}\Phi_{i,j+1} & + & E_{i,j}\Phi_{i,j} & + \\
F_{i,j}\Phi_{i,j-1} & + & G_{i,j}\Phi_{i-1,j+1} & + & H_{i,j}\Phi_{i-1,j} & + & I_{i,j}\Phi_{i-1,j-1} & = & 0 &
\end{array}
$$

If we have a total of $N = m \times n$ interior nodes, we obtain N equations which in matrix form can be written

$$A\mathbf{x} = \mathbf{b}$$

Here $\mathbf{x}^T = (\Phi_{11}, \Phi_{12}, \Phi_{13}, \ldots, \Phi_{mn})$ and $\mathbf{b}$ contains the boundary conditions. The matrix A is a $N \times N$ with the terms A_{ij}, B_{ij}, etc. as coefficients.

Rather than showing the general case in more detail, we instead work out the expression for cartesian grids in two-dimensions. We find

$$\int \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) dx\, dy = \int \left(\frac{\partial \Phi}{\partial x}\, dy - \frac{\partial \Phi}{\partial y}\, dx\right) \approx$$

$$\left(\left[\frac{\partial \Phi}{\partial x}\right]_{i+\frac{1}{2},j+} - \left[\frac{\partial \Phi}{\partial x}\right]_{i-\frac{1}{2},j}\right)\Delta y + \left(\left[\frac{\partial \Phi}{\partial y}\right]_{i,j+\frac{1}{2}} - \left[\frac{\partial \Phi}{\partial y}\right]_{i,j-\frac{1}{2}}\right)\Delta x$$

We can evaluate the derivatives as

$$\left[\frac{\partial \Phi}{\partial x}\right]_{i+\frac{1}{2},j} = (\Phi_{i+1,j} - \Phi_{i,j})\,\Delta y \cdot \frac{1}{\Delta x \cdot \Delta y}$$

and so on for the rest of the terms, which implies that

$$\int \left(\frac{\partial \Phi}{\partial x}\, dy - \frac{\partial \Phi}{\partial y}\, dx\right) \approx (\Phi_{i+1,j} - 2\Phi_{i,j} + \Phi_{i-1,j})\frac{\Delta y}{\Delta x} + (\Phi_{i,j+1} - 2\Phi_{i,j} + \Phi_{i,j-1})\frac{\Delta x}{\Delta y}$$

Dividing with the area $\Delta x \cdot \Delta y$ we obtain the usual 5-point Laplace formula, i. e.

$$\frac{\Phi_{i+1,j} - 2\phi_{i,j} + \Phi_{i-1,j}}{\Delta x^2} + \frac{\Phi_{i,j+1} - 2\phi_{i,j} + \Phi_{i,j-1}}{\Delta y^2} = 0$$

which of course also can be written in the matrix form shown for the general case above.
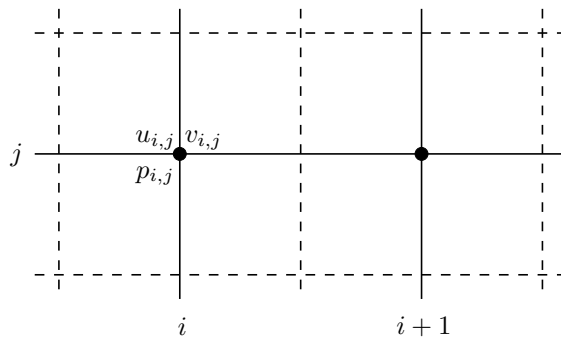
Figure 3.3: Cartesian finite-volume grid where the velocity components and the pressure are co-located.

3.2 Finite-volume discretizations of 2D NS

Co-located Cartesian grid

We will apply the finite volume discretizations to the Navier-Stokes equations in integral form, which in two-dimensions can be written

$$\int \left(u \, \mathrm{d}y - v \, \mathrm{d}x \right) = 0$$

$$\iint \frac{\partial u}{\partial t} \, \mathrm{d}S = - \int \left[u^2 \, \mathrm{d}y - uv \, \mathrm{d}x + p \, \mathrm{d}y - \frac{1}{Re} \left(\frac{\partial u}{\partial x} \, \mathrm{d}y - \frac{\partial u}{\partial y} \, \mathrm{d}x \right) \right]$$

$$\iint \frac{\partial v}{\partial t} \, \mathrm{d}S = - \int \left[uv \, \mathrm{d}y - v^2 \, \mathrm{d}x - p \, \mathrm{d}x - \frac{1}{Re} \left(\frac{\partial v}{\partial x} \, \mathrm{d}y - \frac{\partial v}{\partial y} \, \mathrm{d}x \right) \right]$$

where we have used the earlier derived result $\mathbf{n} = (\, \mathrm{d}y, - \, \mathrm{d}x)$. We start with a co-located grid which means that the finite volume used for both velocity components and the pressure is the same, see figure 3.3. In addition we choose a Cartesian grid in order to simplify the expressions. Recall that the finite volume discretization of the continuity equation becomes

$$\frac{1}{2} \left(u_{i+1,j} - u_{i-1,j} \right) \Delta y + \frac{1}{2} \left(v_{i,j+1} - v_{i,j-1} \right) \Delta x = 0$$

which is equivalent to the finite difference discretization

$$\text{or} \quad \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} = 0$$

The discretization of the streamwise momentum equation needs both first and second derivatives. We have seen that the finite volume discretizations of both are equal to standard second order finite difference approximations. Thus we obtain the following result for the u-component
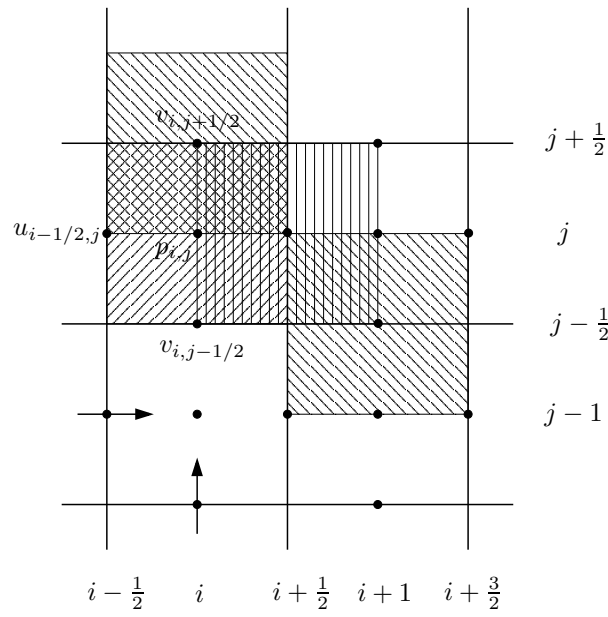
$$\frac{\partial u_{i,j}}{\partial t} + \frac{u_{i+1,j}^2 - u_{i-1,j}^2}{2\Delta x} + \frac{(uv)_{i,j+1} - (uv)_{i,j-1}}{2\Delta y} + \frac{p_{i+1,j} - p_{i-1,j}}{2\Delta x}$$

$$- \frac{1}{Re} \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} \right) = 0$$

In a similar manner the discretization for the v-component becomes

$$\frac{\partial v_{i,j}}{\partial t} + \frac{(uv)_{i+1,j} - (uv)_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1}^2 - v_{i,j-1}^2}{2\Delta y} + \frac{p_{i,j+1} - p_{i,j-1}}{2\Delta y}$$

$$- \frac{1}{Re} \left(\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{\Delta x^2} + \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta y^2} \right) = 0$$

This simple discretization cannot be used in practice without some kind of modification. The reason is that one solution to these equations are in the form of spurious checkerboard modes. Consider the following solution to the discretized equations

$$u_{i,j} = v_{i,j} = (-1)^{i+j} \cdot g \left(t \right), \qquad p = (-1)^{i+j}, \qquad \Delta x = \Delta y$$

$$i, j$$

$i+1, j=1$

$i+1, j=1$

$i, j-1$

$i+1, j+1$

$u_{i,j}$

$i+v_{i,j}$

$u_{i,j}$

i

$v_{i,j+1/2}$

$v_{i,j}$

$u_{i,j+1/2}$

$v_{i,j-1/2}$

$u_{i+1/2,j}$

$v_{i,j-1/2}$

$u_{i-1/2,j}$

j
i
$i+1$
$u_{i,j}$
$v_{i,j}$
$p_{i,j}$

$u_{i,j+1/2}$

$j+\frac{1}{2}$

$u_{i-1/2,j}$

$p_{i,j}$

j

$j-\frac{1}{2}$

$v_{i,j-1/2}$

$j-1$

$i-\frac{1}{2} \quad i \quad i+\frac{1}{2} \quad i+1 \quad i+\frac{3}{2}$

control volume for divergence condition.

control volume for streamwise momentum equation (u).

control volume for normal momentum equation (v).

Figure 3.4: Cartesian finite-volume grid where the velocity components and the pressure uses staggered grids.

which satisfies the divergence constraint. If it is input into the equation for u and v we find the expression

$$\frac{\mathrm{d}g}{\mathrm{d}t} + \frac{8}{Re \cdot \Delta x^2} g = 0 \qquad \Rightarrow \qquad g = e^{-\frac{8t}{Re \cdot \Delta x^2}}$$

Thus we have a spurious solution consisting of checkerboard modes which are damped slowly for velocity and constant in time for the pressure. This mode will corrupt any true numerical solution and yields this discretization unusable as is. These modes are a result of the very weak coupling between nearby finite volumes or grid points, and can be avoided if they are coupled by one sided differences or if they are damped with some type of artificial viscosity.

Staggered cartesian grid

To eliminate the problem with spurious checkerboard modes, we can use a staggered grid as in figure 3.4, where the control volumes for the streamwise, spanwise and pressure are different.

The control volume for the continuity equation is centered abound the pressure point and the discretization becomes

$$\left(u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}\right)\Delta y + \left(v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}}\right)\Delta x = 0$$

$$\text{or} \qquad \frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{\Delta x} + \frac{v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}}}{\Delta y} = 0$$

The control volume for the streamwise velocity is centered around the streamwise velocity point and the discretization becomes

$$\frac{\partial u_{i+1/2,j}}{\partial t} + \frac{u_{i+1,j}^2 - u_{i,j}^2}{\Delta x} + \frac{(uv)_{i+1/2,j+1/2} - (uv)_{i+1/2,j-1/2}}{\Delta y} + \frac{p_{i+1,j} - p_{i,j}}{\Delta x} -$$

$$\frac{1}{Re}\frac{u_{i+3/2,j} - 2u_{i+1/2,j} + u_{i-1/2,j}}{\Delta x^2} - \frac{1}{Re}\frac{u_{i+1/2,j+1} - 2u_{i+1/2,j} + u_{i+1/2,j-1}}{\Delta y^2} = 0$$
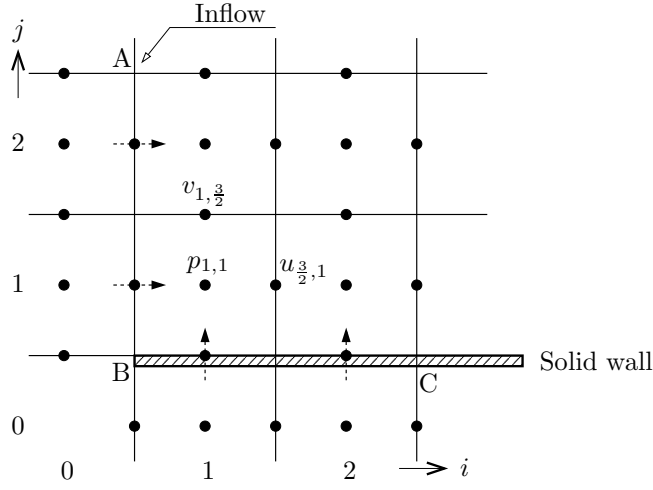
Figure 3.5: Staggered grid near the boundary.

where the $u_{i+1,j}^2, u_{i,j}^2, (uv)_{i+1/2,j+1/2}, (uv)_{i+1/2,j-1/2}$ terms need to be interpolated from the points where the corresponding velocities are defined.

The control volume for the normal velocity is centered around the normal velocity point and the discretization becomes

$$\frac{\partial v_{i,j+1/2}}{\partial t} + \frac{(uv)_{i+1/2,j+1/2} - (uv)_{i-1/2,j+1/2}}{\Delta x} + \frac{v_{i,j+1}^2 - v_{i,j}^2}{\Delta y} + \frac{p_{i,j+1} - p_{i,j}}{\Delta y} -$$

$$\frac{1}{Re} \frac{v_{i+1,j+1/2} - 2v_{i,j+1/2} + v_{i-1,j+1/2}}{\Delta x^2} - \frac{1}{Re} \frac{v_{i,j+3/2} - 2v_{i,j+1/2} + v_{i,j-1/2}}{\Delta y^2} = 0$$

where the $(uv)_{i+1/2,j+1/2}, (uv)_{i-1/2,j+1/2}, u_{i,j+1}^2, u_{i,j}^2$ terms need to be interpolated from the points where the corresponding velocities are defined.

For this discretization no checkerboard modes possible, since there is no decoupling in the divergence constraint, the pressure or the convective terms.

Boundary conditions

To close the system we need to augment the discretized equation with discrete versions of the boundary conditions. See figure 3.5 for a definition of the points near a boundary.

The boundary conditions on the solid wall, BC in figure 3.5, consists of the no flow and the no slip conditions, i.e. $u = v = 0$. The normal points are located on the boundary so that the no flow condition simply become

$$v_{1,\frac{1}{2}} = v_{2,\frac{1}{2}} = \cdots = 0$$

The evaluation of the u-equation at the point $(\frac{3}{2}, 1)$, for example, requires $u_{\frac{3}{2},0}$. We can find this value as follows

$$0 = u_{\frac{3}{2},\frac{1}{2}} = \frac{1}{2}\left(u_{\frac{3}{2},1} + u_{\frac{3}{2},0}\right) \qquad \Rightarrow \qquad u_{\frac{3}{2},0} = -u_{\frac{3}{2},1}$$

The boundary conditions on the inflow, AB in figure 3.5, consist given values of u and v, i.e. so called Dirichlet conditions. The streamwise points are located on the boundary so that the boundary condition is simply consist of the known values

$$u_{\frac{1}{2},1}, u_{\frac{1}{2},2}, \ldots$$

The evaluation of the v-equation at the point $(1, \frac{3}{2})$, for example, requires $v_{0,\frac{3}{2}}$. We can find this value as follows

$$v_{\frac{1}{2},\frac{3}{2}} = \frac{1}{2}\left(v_{1,\frac{3}{2}} + v_{0,\frac{3}{2}}\right) \qquad \Rightarrow \qquad v_{0,\frac{3}{2}} = 2v_{\frac{1}{2},\frac{3}{2}} - v_{1,\frac{3}{2}}$$

If AB in figure 3.5 is an outflow boundary, the boundary conditions may consist of $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$, i.e. so called Neumann conditions. These conditions should be evaluated at $i = \frac{1}{2}$. For the streamwise and normal velocity we find

$$u_{-\frac{1}{2},2} = u_{\frac{3}{2},2}, \qquad v_{0,\frac{3}{2}} = v_{1,\frac{3}{2}}$$

Note that no reference is made to the pressure points outside the domain, so that in this formulation no boundary conditions for p is needed so far. However, depending on the particular discretization of the time derivative and other the details of the solution method one may need to augment the the above with boundary conditions for the pressure. If this is the case particular care has to be taken so that those conditions do not upset the divergence free condition.

3.3 Summary of the equations

The discretized equations derived for the staggered grid can be written in the following form

$$
\begin{cases}
\dfrac{\partial u_{i+\frac{1}{2},j}}{\partial t} + A_{i+\frac{1}{2},j} + \dfrac{p_{i+1,j} - p_{i,j}}{\Delta x} = 0 \\
\dfrac{\partial v_{i,j+\frac{1}{2}}}{\partial t} + B_{i,j+\frac{1}{2}} + \dfrac{p_{i,j+1} - p_{i,j}}{\Delta y} = 0 \\
D_{i,j} = 0
\end{cases}
$$

where the expression for the $A_{i+\frac{1}{2},j}$ can be written

$$A_{i+\frac{1}{2},j} =$$

$$
= \frac{u_{i+1,j}^2 - u_{i,j}^2}{\Delta x} + \frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y}
$$

$$
- \frac{1}{Re} \frac{u_{i+\frac{3}{2},j} - 2u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}}{\Delta x^2} - \frac{1}{Re} \frac{u_{i+\frac{1}{2},j+1} - 2u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j-1}}{\Delta y^2}
$$

$$= \{\text{expand and interpolate using grid values}\}$$

$$
= \left[\frac{1}{2\Delta x}(u_{i+\frac{3}{2},j} - u_{i-\frac{1}{2},j}) + \frac{1}{4\Delta y}(v_{i+1,j+\frac{1}{2}} - v_{i,j+\frac{1}{2}} - v_{i+1,j-\frac{1}{2}} - v_{i,j-\frac{1}{2}}) + \frac{2}{Re}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \right] u_{i+\frac{1}{2},j}
$$

$$
+ \left(\frac{1}{4\Delta x}u_{i+\frac{3}{2},j} - \frac{1}{Re\Delta x^2} \right) u_{i+\frac{3}{2},j} + \left(\frac{1}{4\Delta x}u_{i-\frac{1}{2},j} - \frac{1}{Re\Delta x^2} \right) u_{i-\frac{1}{2},j}
$$

$$
+ \left(\frac{1}{4\Delta y}(v_{i+1,j+\frac{1}{2}} + v_{i,j+\frac{1}{2}}) - \frac{1}{Re\Delta y^2} \right) u_{i+\frac{1}{2},j+1} + \left(\frac{1}{4\Delta y}(v_{i+1,j-\frac{1}{2}} + v_{i,j-\frac{1}{2}}) - \frac{1}{Re\Delta y^2} \right) u_{i+\frac{1}{2},j-1}
$$

$$
= a(u,v)_{i+\frac{1}{2},j} u_{i+\frac{1}{2},j} + \sum_{nb} a(u,v)_{nb} u_{nb}
$$

The last line summarizes the expressions, where the sum over nb indicates a sum over the nearby nodes. The expression for $B_{i,j+\frac{1}{2}}$ can in a similar way be written

$$B_{i,j+\frac{1}{2}} =$$

$$
= \frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} + \frac{v_{i,j+1}^2 - v_{i,j}^2}{\Delta y}
$$

$$
- \frac{1}{R} \frac{v_{i+1,j+\frac{1}{2}} - 2v_{i,j+\frac{1}{2}} + v_{i-1,j+\frac{1}{2}}}{\Delta x^2} - \frac{1}{R} \frac{v_{i,j+\frac{3}{2}} - 2v_{i,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}}}{\Delta y^2}
$$

$$= \{\text{expand and interpolate using grid values}\}$$

$$
= b(u,v)_{i,j+\frac{1}{2}} v_{i,j+\frac{1}{2}} + \sum_{nb} b(u,v)_{nb} v_{nb}
$$

where the expressions for $b(u,v)_{i,j+\frac{1}{2}}$ and $b(u,v)_{nb}$ are analogous to $a(u,v)_{i+\frac{1}{2},j}$ and $a(u,v)_{nb}$, and therefore not explicitly written out. The expression originating from the continuity equation is

$$D_{i,j} = \frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{\Delta x} + \frac{v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}}}{\Delta y}$$

These equations, including the boundary conditions, can be assembled into matrix form. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \\ 0 \end{pmatrix} + \begin{pmatrix} A(u,v) & 0 & G_x \\ 0 & B(u,v) & G_y \\ D_x & D_y & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} f_u \\ f_v \\ 0 \end{pmatrix}$$

Here we we have used the definitions

u	-	vector of unknown streamwise velocities
v	-	vector of unknown normal velocities
p	-	vector of pressure unknowns
$A(u,v)$	-	non-linear algebraic operator from advective and viscous terms of u-eq.
$B(u,v)$	-	non-linear algebraic operator from advective and viscous terms of v-eq.
G_x	-	linear algebraic operator from stremwise pressure gradient
G_y	-	linear algebraic operator from normal pressure gradient
D_x	-	linear algebraic operator from x-part of divergence constraint
D_y	-	linear algebraic operator from y-part of divergence constraint
f	-	vector of source terms from BC

With obvious changes in notation, we can write the system of equations in an even more compact form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ 0 \end{pmatrix} + \begin{pmatrix} N(u) & G \\ D & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

If this is compared to the finite element discretization in the next chapter, it is seen that the form of the discretized matrix equations are the same.

3.4 Time dependent flows

The projection method

For time-dependent flow a common way to solve the discrete equations is the projection or pressure correction method. Sometimes this method and its generalizations are also called splitting methods. We illustrate the method using the system written in the compact notation defined above.

We start to discretize the time derivative with a first order explicit method

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} + N(u^n)\,u^n + G\,p^{n+1} = f(t^n) \\ \\ D u^{n+1} = 0 \end{cases}$$

Then we make a prediction of the velocity field at the next time step u^* not satisfying continuity, and then a correction involving p such that $Du^{n+1} = 0$. We have

$$\begin{cases} \dfrac{u^* - u^n}{\Delta t} + N(u^n)\,u^n = f(t^n) \\ \\ \dfrac{u^{n+1} - u^*}{\Delta t} + G\,p^{n+1} = 0 \\ \\ D u^{n+1} = 0 \end{cases}$$

Next, apply the divergence operator D to the correction step

$$DG\,p^{n+1} = \frac{1}{\Delta t}D\,u^*$$

Here DG represents $divgrad = $ Laplacian, i.e. discrete version of pressure Poisson equation. The velocity at the next time level thus becomes

$$u^{n+1} = u^* - \Delta t\, G\, p^{n+1}$$

Note here that u^{n+1} is projection of u^* on divergence free space. This is a discrete version of the projection of a function on a divergence free space discussed in the section about the role of the pressure in incompressible flow.

In order to be able to discuss the boundary condition on the pressure Poisson equation we write the prediction and correction step more explicitly. Using the staggered grid discretization the prediction step can be written

$$\begin{cases} \dfrac{u^*_{i+\frac{1}{2},j} - u^n_{i+\frac{1}{2},j}}{\Delta t} + A^n_{i+\frac{1}{2},j} = 0 \\[2mm] \dfrac{v^*_{i,j+\frac{1}{2}} - v^n_{i,j+\frac{1}{2}}}{\Delta t} + B^n_{i,j+\frac{1}{2}} = 0 \end{cases}$$

and the projection step as

$$\begin{cases} \dfrac{u^{n+1}_{i+\frac{1}{2},j} - u^*_{i+\frac{1}{2},j}}{\Delta t} + \dfrac{p^{n+1}_{i+1,j} - p^{n+1}_{i,j}}{\Delta x} = 0 \\[2mm] \dfrac{v^{n+1}_{i,j+\frac{1}{2}} - v^*_{i,j+\frac{1}{2}}}{\Delta t} + \dfrac{p^{n+1}_{i,j+1} - p^{n+1}_{i,j}}{\Delta y} = 0 \end{cases}$$

By solving for $u^{n+1}_{i+\frac{1}{2},j}$ and $v^{n+1}_{i,j+\frac{1}{2}}$ above and using this, and the corresponding expressions for $u^{n+1}_{i-\frac{1}{2},j}$ and $v^{n+1}_{i,j-\frac{1}{2}}$, in the expression for the divergence constraint $D_{i,j}$ we find the following explicit expression for the discrete pressure Poisson equation

$$\frac{p^{n+1}_{i+1,j} - 2p^{n+1}_{i,j} + p^{n+1}_{i-1,j}}{\Delta x^2} + \frac{p^{n+1}_{i,j+1} - 2p^{n+1}_{i,j} + p^{n+1}_{i,j-1}}{\Delta y^2} = \frac{D^*_{i,j}}{\Delta t}$$

where $D^*_{i,j}$ is $D_{i,j}$ evaluated using the discrete velocities at the prediction step, u^* and v^*.

When the discrete pressure Poisson equation is solved we need values of the unknowns which are outside of the domain, i.e. we need additional boundary conditions. If we take the vector equation for the correction or projection step and project it normal to the solid boundary in figure 3.5 we have

$$\frac{p^{n+1}_{2,1} - p^{n+1}_{2,0}}{\Delta y} = -\frac{1}{\Delta t}\left(v^{n+1}_{2,\frac{1}{2}} - v^*_\Gamma\right)$$

where $v^*_\Gamma = v^*_{2,\frac{1}{2}}$ is an unknown value of the velocity in the prediction step at the boundary. In general it is important to choose this boundary condition such that discrete velocity after the correction step is divergence at the boundary, see the discussion in the end of the section on the role of the pressure. However, for this particular discretization it turns out that we can choose v^*_Γ arbitrarily since it completely decouples from the rest of the discrete problem. To see this we write the pressure Poisson equation, centered abound the point (2,1), close to the boundary. We find

$$\frac{p^{n+1}_{3,1} - 2p^{n+1}_{2,1} + p^{n+1}_{1,1}}{\Delta x^2} + \frac{p^{n+1}_{2,2} - 2p^{n+1}_{2,1} + p^{n+1}_{2,0}}{\Delta y^2} = \frac{1}{\Delta t}\left(\frac{u^*_{\frac{5}{2},1} - u^*_{\frac{3}{2},1}}{\Delta x} + \frac{v^*_{2,\frac{3}{2}} - v^*_\Gamma}{\Delta y}\right)$$

if we substitute the value of $p^{n+1}_{2,1} - p^{n+1}_{2,0}$ from the boundary condition obtained for the pressure into the above equation we can se that v^*_Γ cancels in the right hand side of the pressure Poisson equation. Since the discretization of the time derivative is explicit there is no other place where the values of v^* or u^* on the boundaries enter. Thus we find that we can choose the value of v^*_Γ arbitrarily. In particular, we can choose $v^*_\Gamma = v^{n+1}_{2,\frac{1}{2}}$, in which case we obtain a Neumann boundary condition for the pressure. Note, however, that this is a numerical artifact and does not imply that the real pressure gradient is zero at the boundary.

A number of other splitting methods have been devised for schemes with higher order accuracy for the time discretization. In general it is preferred to make the time discretization first for the full Navier-Stokes equations and afterward split the discrete equation into one predictor and one corrector step. See Peyret and Taylor for details.

Time step restriction

For stability it is necessary that prediction step is stable, numerical evidence indicates that this is sufficient. In the predictor case the momentum equations decouple if we linearize them abound the solution U_0, V_0. It thus suffices to consider the equation

$$\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} + V_0 \frac{\partial u}{\partial y} = \frac{1}{Re} \nabla^2 u$$

For simplicity we will here consider one-dimensional version

$$\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} = \frac{1}{Re} \frac{\partial^2 u}{\partial x^2}$$

which in discretized form can be written

$$u_j^{n+1} = u_j^n + \Delta t \left[-U_0 \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{1}{Re} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \right]$$

where we have chosen to call the discrete points where the u-velocity is evaluate for x_j.
We introduce

$$u_j^n = Q^n \cdot e^{i \cdot \omega x_j}$$

into the discrete equation, together with the definitions

$$\lambda = \frac{U_0 \Delta t}{\Delta x}, \qquad \alpha = \frac{2\Delta t}{Re\, \Delta x^2}, \qquad \xi = \omega \cdot \Delta x$$

This implies that the amplification factor Q can be written

$$Q = 1 - i\lambda \sin(\xi) - 2\alpha \sin^2 \left(\frac{\xi}{2} \right)$$

We find that the absolute value of Q satisfies

$$
\begin{aligned}
|Q| &= 1 - 4\left(\lambda^2 - \alpha^2\right)s^2 + 4\left(\lambda^2 - s^2\right)s \\
&= 1 - 4s\left[\left(\lambda^2 - \alpha^2\right)s - \lambda^2 + \alpha\right] \le 1
\end{aligned}
$$

where $s = \sin^2\left(\frac{\xi}{2}\right)$. This implies that

$$\Phi(s) = -\left(\lambda^2 - \alpha^2\right)s + \lambda^2 - \alpha \le 0 \qquad \forall s : 0 \le s \le 1$$

$\Phi(s)$ is linear function of s and thus the the limiting condition on $\Phi(s)$ is given by its values at the two end points of the s-interval. We find

$$\Phi(0) = \lambda^2 - \alpha \le 0, \qquad \Phi(1) = \alpha^2 - \alpha \le 0$$

which implies that

$$\Rightarrow \qquad \lambda^2 \le \alpha \le 1$$

substituting back the definitions of λ and α we find the following stability conditions

$$
\begin{cases}
\dfrac{1}{2} U_0^2 \Delta t\, Re & \le \quad 1 \\[2mm]
\dfrac{2\Delta t}{Re\Delta x^2} & \le \quad 1
\end{cases}
$$

For the two dimensional version of the predictor step Peyret and Taylor [8] find the following stability limits

$$
\begin{cases}
\dfrac{1}{4}\left(U_0^2 + V_0^2\right)\Delta t \cdot Re & \le \quad 1 \\[2mm]
\dfrac{4\Delta t}{Re\Delta x^2} & \le \quad 1
\end{cases}
$$

Note that this is a stronger condition than in the one-dimensional case.

3.5 General iteration methods for steady flows

Distributive iteration

For steady flows we can still use the methods introduced for unsteady flows, and iterate the solution in time to obtain a steady state. However, in many cases it is more effective to use other iteration methods. We will here introduce a general distributive iteration method and apply it to the discretized steady Navier-Stokes equations.

For the steady case the discretized equations can be written in the form

$$
\begin{pmatrix} N(u) & G \\ D & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}
$$

which is of the form $A\,y = b$. Now, let $y = B\,\hat{y}$ which implies that

$$
AB\,\hat{y} = b
$$

and use the iteration scheme on this modified equation. We divide the matrix AB in the following manner

$$
AB = M - T \qquad \Rightarrow \qquad M\,\hat{y} = T\hat{y} + b
$$

and use this to define the iteration scheme

$$
MB^{-1}\,y^{k+1} = TB^{-1}\,y^{k} + b = MB^{-1}\,y^{k} + b - A\,y^{k}
$$

which can be simplified to

$$
y^{k+1} = y^{k} + BM^{-1}\underbrace{\left(b - A\,y^{k}\right)}_{r^{k}}
$$

where r^{k} is the residual error in the k:th step of the iteration procedure.

If $B = I$ we obtain a regular iteration method where residual r^{k} is used to update y^{k} by inverting M, a simplified part of A, as in the Gauss-Seidell method for example. For this method we have

$$
\begin{aligned}
A\,x &= b \\
A &= L - U \\
(L - U)\,x &= b \\
L\,x^{n+1} &= U\,x^{n} + b
\end{aligned}
$$

Application to the steady Navier-Stokes equations

When the distributive iteration method is applied to the steady Naiver-Stokes we have

$$
A = \begin{pmatrix} N(u) & G \\ D & 0 \end{pmatrix}
$$

we can obtain AB in the following block triangular form

$$
AB = \begin{pmatrix} N(u) & 0 \\ D & -DN(u)^{-1}G \end{pmatrix}
$$

if the matrix B has the form

$$
B = \begin{pmatrix} I & -N(u)^{-1}G \\ 0 & I \end{pmatrix} \approx \begin{pmatrix} I & -\tilde{N}(u)^{-1}G \\ 0 & I \end{pmatrix}
$$

where $\tilde{N}(u)^{-1}$ is a simple approximation of the inverse of the discrete Navier-Stokes operator. We now split $AB = M - T$ where

$$
M = \begin{pmatrix} Q & 0 \\ D & R \end{pmatrix}
$$

and where Q is an approximation of $N(u)$ and R is an approximation of $-DN^{-1}G$. Note that R is a discrete Laplace like operator.

The SIMPLE (Semi-Implicit Method for Pressure Linked Equations) by Pantankar and Spalding (1972) can be thought of as a distributive iteration method. Wesseling [10] indicates that by choosing

$$
\begin{aligned}
\tilde{N}(u)^{-1} &= [\text{diag}N(u^k)]^{-1} \\
Q &= N(u^k) \\
R &= -D[\text{diag}N(u^k)]^{-1}G
\end{aligned}
$$

we have a method which is essentially the original version of the SIMPLE method. Thus we have the following iteration steps

1. First, we calculate the residual

$$
b - A\,y^k = \begin{pmatrix} f \\ 0 \end{pmatrix} - \begin{pmatrix} N(u^k) & G \\ D & 0 \end{pmatrix} \begin{pmatrix} u^k \\ p^k \end{pmatrix} = \begin{pmatrix} r_1^k \\ r_2^k \end{pmatrix}
$$

2. Second, we calculate $M^{-1}r^k$ which implies

$$
Q\,\delta\tilde{u} = r_1^k \quad \Rightarrow \quad \delta\tilde{u} = Q^{-1}\,r_1^k
$$

$$
R\,\delta\tilde{p} = r_2^k - D\,\delta\tilde{u}
$$

3. Third, we have the distribution step $BM^{-1}\,r^k$ which becomes

$$
\begin{pmatrix} \delta u \\ \delta p \end{pmatrix} = B \begin{pmatrix} \delta\tilde{u} \\ \delta\tilde{p} \end{pmatrix} = \begin{pmatrix} \delta\tilde{u} - \tilde{N}^{-1}G\,\delta\tilde{p} \\ \delta\tilde{p} \end{pmatrix}
$$

4. Fourth, there is an under relaxation step, where the velocities and the pressure is updated

$$
\begin{cases}
u^{k+1} = u^k + \omega_u \delta u \\
p^{k+1} = p^k + \omega_p \delta p
\end{cases}
$$

The under relaxation is needed for the method to converge, thus ω_u and ω_p is usually substantially lower that one.

dx
dy
− dx
dl = (dx, dy)
n dl = (dy, − dx)
i − 1, j + 1
i, j + 1
i + 1, j + 1
c
c′
b
b′
i − 1, j
i + 1, j
i, j
A_abcd
d
d′

Chapter 4

Finite element methods for incompressible flow

The finite-element method (FEM) approximates an integral form of the governing equations. However, the integral form—called the *variational form*—is usually different from the one used for the finite-volume method. Moreover, different integral forms may be used for the same equation depending on circumstances such as boundary conditions. We introduce FEM first for a scalar advection–diffusion problem before discussing the Navier–Stokes equations.

4.1 FEM for an advection–diffusion problem

We consider the steady flow of an incompressible fluid in a bounded domain Ω, and we assume that its velocity field U_j is known (from a previous numerical solution, for instance). We assume that the boundary Γ of the domain can be decomposed into two portions Γ_0 and Γ_1, as in figure 4.1 for instance. We are interested to compute the temperature field u in the domain, modeled by the advection–diffusion problem

$$U_j \frac{\partial u}{\partial x_j} - \nu \nabla^2 u = f \qquad \text{in } \Omega, \tag{4.1a}$$

$$u = 0 \qquad \text{on } \Gamma_0, \tag{4.1b}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \Gamma_1. \tag{4.1c}$$

The coefficient $\nu > 0$ is the thermal diffusivity. Incompressibility yields that $\frac{\partial U_j}{\partial x_j} = 0$. We assume that Γ_1 is a solid wall, so that $n_j U_j = 0$ on Γ_1. This simplifies the analysis but is not essential. The *Dirichlet* boundary condition $u = 0$ on Γ_0 is called an *isothermal condition*, whereas the *Neumann condition* $\frac{\partial u}{\partial n} = 0$ on Γ_1 is an *adiabatic condition*.

Deriving the integral form that is used to form the finite-element discretization requires some vector calculus. The product rule of differentiation yields

$$\frac{\partial}{\partial x_i}\left(v \frac{\partial u}{\partial x_i} \right) = \frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i} + v \nabla^2 u. \tag{4.2}$$
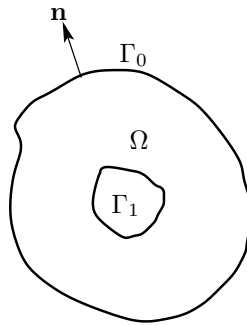
Figure 4.1: A typical domain associated with the advection–diffusion problem (4.1).

Recall the *divergence theorem*

$$\int_\Omega \frac{\partial w_i}{\partial x_i} \, \mathrm{d}\Omega = \int_\Gamma n_i w_i \, \mathrm{d}\Gamma. \tag{4.3}$$

Integrating expression (4.2) and applying the divergence theorem on the left side yields

$$\int_\Gamma v \frac{\partial u}{\partial n} \, \mathrm{d}\Gamma = \int_\Omega \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i} \, \mathrm{d}\Omega + \int_\Omega v \nabla^2 u \, \mathrm{d}\Omega. \tag{4.4}$$

(Note that

$$\frac{\partial u}{\partial n} = n_i \frac{\partial u}{\partial x_i}.)$$

Let v be any smooth function such that $v = 0$ on Γ_0. Multiplying equation (4.1a) with v, integrating over Ω, and utilizing expression (4.4) yields

$$\int_\Omega f v \, \mathrm{d}\Omega = \int_\Omega v U_j \frac{\partial u}{\partial x_j} \, \mathrm{d}\Omega - \nu \int_\Omega v \nabla^2 u \, \mathrm{d}\Omega = \int_\Omega v U_j \frac{\partial u}{\partial x_j} \, \mathrm{d}\Omega + \nu \int_\Omega \frac{\partial v}{\partial x_j} \frac{\partial u}{\partial x_j} \, \mathrm{d}\Omega - \nu \int_\Gamma v \overset{0}{\cancel{\frac{\partial u}{\partial n}}} \, \mathrm{d}\Gamma,$$

where the last term vanishes on Γ_0 due to the requirements on v, and on Γ_1 due to boundary condition (4.1c)

We have thus shown that a smooth (twice continuously differentiable) solution to problem (4.1) (also called a *classical* solution) satisfies the *varational form*

$$\int_\Omega v U_j \frac{\partial u}{\partial x_j} \, \mathrm{d}\Omega + \nu \int_\Omega \frac{\partial v}{\partial x_j} \frac{\partial u}{\partial x_j} \, \mathrm{d}\Omega = \int_\Omega f v \, \mathrm{d}\Omega \tag{4.5}$$

for each smooth function v vanishing on Γ_0

We will now "forget" about the differential equation (4.1), and directly consider the variational form (4.5). For this, we need to introduce the *function space*

$$V = \left\{ v : \int_\Omega \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_i} \, \mathrm{d}\Omega < +\infty, v = 0 \text{ on } \Gamma_0 \right\}.$$

The space V is a *linear space*, that is, $u, v \in V \Rightarrow \alpha u + \beta v \in V \; \forall \alpha, \beta \in \mathbf{R}$. The requirement that the gradient-square of each function in V is bounded should be intepreted as a requirement that the *energy* is bounded.

The variational problem will thus be

Find $u \in V$ such that

$$\int_\Omega v U_j \frac{\partial u}{\partial x_j} \, \mathrm{d}\Omega + \nu \int_\Omega \frac{\partial v}{\partial x_j} \frac{\partial u}{\partial x_j} \, \mathrm{d}\Omega = \int_\Omega f v \, \mathrm{d}\Omega \qquad \forall v \in V. \tag{4.6}$$

A solution to variational problem (4.6) is called a *weak solution* to advection–diffusion problem (4.1). The derivation leading up to expression (4.5) shows that a classical solution is a weak solution. However, a weak solution may not be a classical solution since functions in V may fail to be twice continuously differentiable.

Note that the boundary conditions are incorporated into the variational problem. The Dirichlet boundary condition on Γ_0 is explicitly included in the definition of V and is therefore referred to as an *essential boundary condition*. The Neumann condition on Γ_1 is defined implicitly through the variational form and is therefore called a *natural boundary condition*.

4.1.1 Finite element approximation

Let us *triangulate* the domain Ω by dividing it up into triangles, as in figure 4.2. Let M be the number of triangles and N the number of nodes in $\Omega \cup \Gamma_1$, that is, the nodes on Γ_0 are excluded in the count. Let h be the largest side of any triangle.

Let V_h be the space of all functions that are
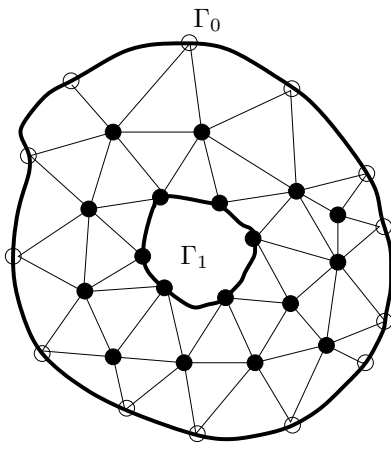
- continuous on $\overline{\Omega}$,

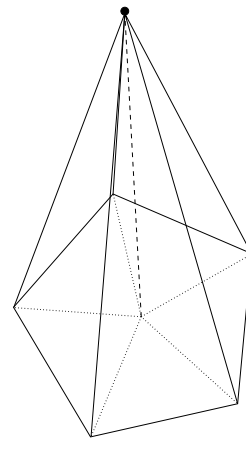Figure 4.2: A triangulation. Here, the number N are the black nodes.

Figure 4.3: The "tent or "hat basis function associated with continuous, piece-wise linear functions.

- linear on each triangle,

- vanishing on Γ_0.

Note that $V_h \subset V$. We get a *finite-element approximation* of our advection-diffusion problem by simply replacing V by V_h in variational problem (4.6), that is:

Find $u_h \in V_h$ such that

$$\int_\Omega v_h U_j \frac{\partial u_h}{\partial x_j}\, \mathrm{d}\Omega + \nu \int_\Omega \frac{\partial v_h}{\partial x_j}\frac{\partial u_h}{\partial x_j}\, \mathrm{d}\Omega = \int_\Omega f v_h\, \mathrm{d}\Omega \qquad \forall v_h \in V_h. \tag{4.7}$$

Restricting the function space in a variational form to a subspace is called a *Galerkin* approximation. The finite-element method is a Galerkin approximation in which the subspace is given by piecewise polynomials.

4.1.2   The algebraic problem. Assembly.

Once the value of a function u_h in V_h is known at all node points, it is easy to reconstruct it by linear interpolation. This interpolation can be written as a sum of the "tent" basis functions $\Phi^{(n)}$ (figure 4.3), which are functions in V_h that are zero at each node point except node n, where it is unity. Thus,

$$u_h\left(x_i\right) = \sum_{n=1}^N u^{(n)} \Phi^{(n)}\left(x_i\right), \tag{4.8}$$

where $u^{(n)}$ denotes the value of u_h at node n. Note that expansion (4.8) forces $u_h = 0$ on Γ_0 since it excludes the nodes on Γ_0 in the sum.

Substituting expansion (4.8) into FEM approximation (4.7) yields

$$\sum_{n=1}^N u^{(n)} \int_\Omega v_h U_j \frac{\partial \Phi^{(n)}}{\partial x_j}\, \mathrm{d}\Omega + \nu \sum_{n=1}^N u^{(n)} \int_\Omega \frac{\partial v_h}{\partial x_j}\frac{\partial \Phi^{(n)}}{\partial x_j}\, \mathrm{d}\Omega = \int_\Omega f v_h\, \mathrm{d}\Omega \qquad \forall\, v_h \in V_h. \tag{4.9}$$

Since each basis function is in V_h, we may choose $v_h = \Phi^{(m)} \in V_h$ for $m = 1, \dots, N$ in equation (4.9), which then becomes

$$\sum_{n=1}^N u^{(n)} \int_\Omega \Phi^{(m)} U_j \frac{\partial \Phi^{(n)}}{\partial x_j}\, \mathrm{d}\Omega + \nu \sum_{n=1}^N u^{(n)} \int_\Omega \frac{\partial \Phi^{(m)}}{\partial x_j}\frac{\partial \Phi^{(n)}}{\partial x_j}\, \mathrm{d}\Omega = \int_\Omega f \Phi^{(m)}\, \mathrm{d}\Omega, m = 1, \dots, N. \tag{4.10}$$

This is a linear system system $Au = b$, where $A = C + \nu K$ and

$$K_{mn} = \int_\Omega \frac{\partial \Phi^{(m)}}{\partial x_j}\frac{\partial \Phi^{(n)}}{\partial x_j}\, \mathrm{d}\Omega, \qquad C_{mn} = \int_\Omega \Phi^{(m)} U_j \frac{\partial \Phi^{(n)}}{\partial x_j}\, \mathrm{d}\Omega, \tag{4.11}$$
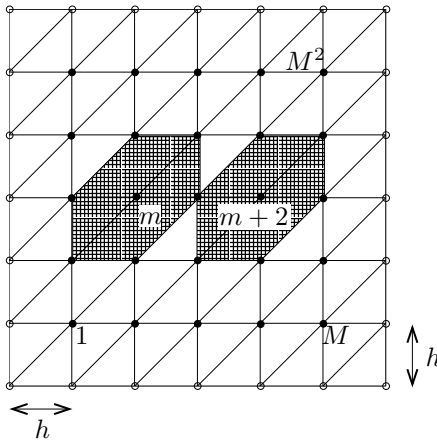
Figure 4.4: An example mesh that gives a particular simple stiffness matrix K

$$u = \begin{pmatrix} u^{(1)} \\ \vdots \\ u^{(n)} \end{pmatrix} \qquad b = \begin{pmatrix} \int\limits_{\Omega} f\Phi^{(1)} \, d\Omega \\ \vdots \\ \int\limits_{\Omega} f\Phi^{(N)} \, d\Omega \end{pmatrix}$$

The union of all triangles constitutes the domain Ω and that the triangles do not overlap. Thus, any integral over Ω can be written as a sum of integrals over each triangle,

$$K_{mn} = \int\limits_{\Omega} \frac{\partial \Phi^{(m)}}{\partial x_j} \frac{\partial \Phi^{(n)}}{\partial x_j} \, d\Omega = \sum_{p=1}^{M} \int\limits_{T_p} \frac{\partial \Phi^{(m)}}{\partial x_j} \frac{\partial \Phi^{(n)}}{\partial x_j} \, d\Omega = \sum_{p=1}^{M} K_{mn}^{(p)}.$$

Note that $K_{mn}^{(p)}$ vanishes as soon as m or n is not a corner node of triangle p. Thus $K_{mn}^{(p)}$ contributes to the sum only when m and n both are corner nodes of triangle p. Matrix K can thus be computed by looping all M triangles, calculating the *element contribution* $K_{mn}^{(p)}$ for m, n being the three nodes of triangle p and adding these 9 contribution to matrix K. Note that the derivatives of $\Phi^{(n)}$ are constant on each triangle, so $K_{mn}^{(p)}$ is easy to compute exactly. Matrix C and vector b can also be computed by element assembly, but numerical quadrature is usually needed, except if U_j and f happen to be particularly "easy" functions (such as constants).

4.1.3   An example

As an example, we will compute the elements of K, the "stiffness matrix" (a term borrowed from structural mechanics), for the particular case of the structured mesh depicted in figure 4.4.

Let M be the number of internal nodes in the horizontal as well as the vertical direction (the solid nodes in figure 4.4.) The width of each "panel" is then $h = 1/(M+1)$, and the area of each triangle is $|T_k| = h^2/2$. The components of K are

$$K_{mn} = \int\limits_{\Omega} \frac{\partial \Phi^{(m)}}{\partial x_j} \frac{\partial \Phi^{(n)}}{\partial x_j} \, d\Omega \tag{4.12}$$

Note that K_{mn} vanishes for most m and n. For instance, $K_{m,m+2} = 0$, since basis functions $\Phi^{(m)}$ and $\Phi^{(m+2)}$ do not overlap (figure 4.4). In fact, $K_{mn} \neq 0$ *only* when m and n are associated with nodes that are nearest neighbors.

To compute matrix element (4.12), we need expressions for the gradients of basis functions $\Phi^{(m)}$ and $\Phi^{(m+1)}$. Since the functions are piecewise linear, the gradient is constant on each triangles and may thus easily be computed by finite-differences in the coordinate direction (figure 4.5). By the expressions in
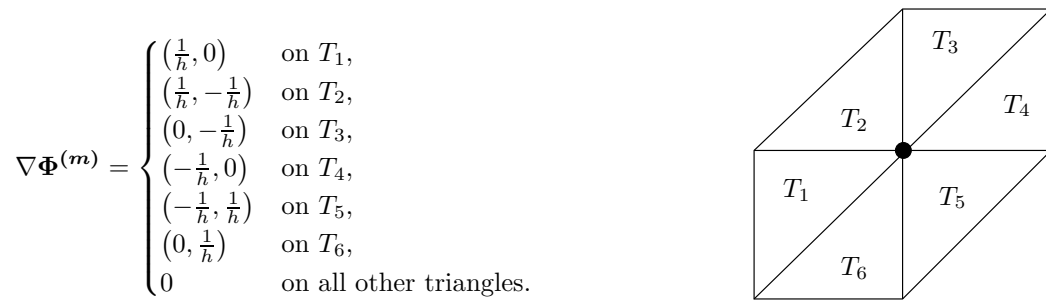
$$
\nabla \boldsymbol{\Phi}^{(\boldsymbol{m})} = \begin{cases} \left(\frac{1}{h}, 0\right) & \text{on } T_1, \\ \left(\frac{1}{h}, -\frac{1}{h}\right) & \text{on } T_2, \\ \left(0, -\frac{1}{h}\right) & \text{on } T_3, \\ \left(-\frac{1}{h}, 0\right) & \text{on } T_4, \\ \left(-\frac{1}{h}, \frac{1}{h}\right) & \text{on } T_5, \\ \left(0, \frac{1}{h}\right) & \text{on } T_6, \\ 0 & \text{on all other triangles.} \end{cases}
$$

Figure 4.5: The gradient of test function $\Phi^{(m)}$ for m being the (black) middle node.

$$
\nabla \Phi^{(m)} = \begin{cases} \left(-\frac{1}{h}, \frac{1}{h}\right) & \text{on } T_1, \\ \left(-\frac{1}{h}, 0\right) & \text{on } T_2, \\ 0 & \text{on all other triangles} \end{cases}
$$

$$
\nabla \Phi^{(m+1)} = \begin{cases} \left(\frac{1}{h}, 0\right) & \text{on } T_1, \\ \left(\frac{1}{h}, -\frac{1}{h}\right) & \text{on } T_2. \end{cases}
$$

Figure 4.6: The support for basis functions Φ^m and Φ^{m+1} are marked with vertical and horizontal stripes respectively.

figure 4.5, we have

$$
K_{mm} = \int_\Omega \frac{\partial \Phi^{(m)}}{\partial x_j} \frac{\partial \Phi^{(m)}}{\partial x_j} \, \mathrm{d}\Omega = \sum_{k=1}^{6} \int_{T_k} \left[\left(\frac{\partial \Phi^{(m)}}{\partial x}\right)^2 + \left(\frac{\partial \Phi^{(m)}}{\partial y}\right)^2 \right] \mathrm{d}\Omega
$$

$$
= \frac{1}{h^2} |T_1| + 2\frac{1}{h^2} |T_2| + \frac{1}{h^2} |T_3| + \frac{1}{h^4} |T_4| + 2\frac{1}{h^2} |T_5| + \frac{1}{h^2} |T_6|
$$

$$
= 8\frac{1}{h^2} \frac{h^2}{2} = 4
$$

The common *support* for basis functions Φ^m and Φ^{m+1} is only the two triangles marked in figure 4.6; that is, it is only on these two triangles that both functions are nonzero at the same time. Utilizing the gradient expressions given in figure 4.6 yields

$$
K_{m,m+1} = \int_\Omega \nabla \Phi^{(m)} \cdot \nabla \Phi^{(m+1)} \, \mathrm{d}\Omega = \sum_{k=1}^{2} \int_{T_k} \nabla \Phi^{(m)} \cdot \nabla \Phi^{(m+1)} \, \mathrm{d}\Omega = -\frac{1}{h^2} |T_1| - \frac{1}{h^2} |T_2| = -\frac{2}{h^2} \frac{h^2}{2} = -1,
$$

and in the same manner,

$$
K_{m,m-1} = K_{m,m+M} = K_{m,m-M} = -1.
$$

Altogether, K becomes the block-tridiagonal M^2-by-M^2 matrix

$$
K = \begin{pmatrix} T & -I & & & \\ -I & T & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & T & -I \\ & & & -I & T \end{pmatrix},
$$

where I is the M-by-M identity matrix, and T the M-by-M matrix

$$
T = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix}.
$$

Thus, a typical row in the matrix–vector product Ku becomes

$$
4u_m - u_{m+1} - u_{m-1} - u_{m+M} - u_{m-M}, \tag{4.13}
$$

that is, after dividing expression (4.13) with h^2, we recognize the classical *five-point stencil* for the negative Laplacian $-\nabla^2$.

Thus our finite-element discretization is for this particular mesh equivalent to a finite-volume or finite-difference discretization. However, for a general unstructured mesh, the finite-element discretization will not give rise to any obvious finite-difference stencil.

4.1.4 Matrix properties and solvability

Matrix A is *sparse*: the elements of martix A are mostly zeros. At row m, matrix element $A_{mn} \neq 0$ only in the columns where n represents a nearest neighbor to node m. The number of nonzero elements on each row does *not* grow (on average) when the mesh is refined (since the number of nearest neighbors in a mesh does not grow). Thus, the matrices become sparser and sparser as the mesh is refined. A common technique in finite-element codes is therefore to use *sparse representation*, that is, to store only the nonzero elements and pointers to matrix locations.

It is immediate from expression (4.11) that $K_{mn} = K_{nm}$, that is, K is a *symmetric matrix* ($K^T = K$). Integration by parts also shows that matric C is *skew symmetric* ($C^T = -C$).

Theorem 1. K *is positive definite, that is,* $v^T K v > 0$ *for each* $v \neq 0$.

Proof. Let $v_h \in V_h$. Then $v_h = \sum_{n=1}^{N} v^{(N)} \Phi^{(n)}$. Denote $v = (v^{(1)}, \ldots, v^{(N)})^T$.

$$v^T K v = \sum_{m=1}^{N} \sum_{n=1}^{N} v^{(n)} \int_{\Omega} \frac{\partial \Phi^{(n)}}{\partial x_i} \frac{\partial \Phi^{(m)}}{\partial x_i} \, d\Omega \, v^{(m)}$$

$$= \int_{\Omega} \sum_{m=1}^{N} \frac{\partial}{\partial x_i} \left(v^{(m)} \Phi^{(m)} \right) \sum_{n=1}^{N} \frac{\partial}{\partial x_i} \left(v^{(m)} \Phi^{(n)} \right) \, d\Omega = \int_{\Omega} \frac{\partial v_h}{\partial x_i} \frac{\partial v_h}{\partial x_i} \, d\Omega \geq 0,$$

with equality if and only if $\frac{\partial v_h}{\partial x_i} = 0$, that is, if $v_h = \text{Const}$. But since $v_h|_{\Gamma_0} = 0$, $v_h \equiv 0$. $\quad\square$

From Theorem 1 also follows that A is positive definite. To see this, note that the skew-symmetry of C yields that

$$v^T C v = \left(v^T C v \right)^T = v^T C^T v = -v^T C v$$

that is, $v^T C v = 0$ for each v. Thus

$$v^T A v = v^T \left(K + C \right) v = v^T K v + v^T C v = v^T K v > 0$$

for each $v \neq 0$, that is, A is positive definite. However, A is *not* symmetric whenever $U_j \neq 0$.

Recall that a linear system $Au = b$ has a unique solution for each b if the matrix is nonsingular. A positive-definite matrix is a particular example of a nonsingular matrix. We thus conclude that the linear system resulting from the finite-element discretization (4.7) has a unique solution.

4.1.5 Stability and accuracy

We have defined the finite-element approximation and showed that the finite-element approximation yields a linear system $Au = b$ that has a unique solution. Now the question is whether the approximation is any good. We will study the *stability* and *accuracy* of the numerical solution. FEM usually uses *integral norms* for analysis of stability and accuracy. We start by reviewing some mathematical concepts that will be used in the analysis.

Basic definitions and inequalities

The most basic integral norm is

$$\|v\|_{L^2(\Omega)} = \left(\int_{\Omega} v^2 \, d\Omega \right)^{1/2}$$

For the current problem, a more fundamental quantity is the *energy norm*

$$\|v\|_V = \left(\int_{\Omega} |\nabla v|^2 \, d\Omega \right)^{1/2} = \left(\int_{\Omega} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_i} \, d\Omega \right)^{1/2}$$

and associated inner product

$$(u, v)_V = \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega = \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, d\Omega. \tag{4.14}$$

The *Cauchy–Schwarz* inequality for vectors $a = (a_1, \ldots, a_n)$ is

$$|a \cdot b| \leq |a||b| = |a \cdot a|^{1/2}|b \cdot b|^{1/2},$$

whereas for functions it reads

$$\left| \int_\Omega fg \, \mathrm{d}\Omega \right| \leq \left(\int_\Omega f^2 \, \mathrm{d}\Omega \right)^{1/2} \left(\int_\Omega g^2 \, \mathrm{d}\Omega \right)^{1/2}, \tag{4.15}$$

which can be denoted

$$(f, g)_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

The Cauchy–Schwarz inequality (4.15) implies that, for each nonzero, square-integrable g,

$$\left(\int_\Omega f^2 \, \mathrm{d}\Omega \right)^{1/2} \geq \frac{\left| \int_\Omega fg \, \mathrm{d}\Omega \right|}{\left(\int_\Omega g^2 \, \mathrm{d}\Omega \right)^{1/2}}.$$

Choosing $g = f$ yields equality in above expression. Thus

$$\|f\|_{L^2(\Omega)} = \left(\int_\Omega f^2 \, \mathrm{d}\Omega \right)^{1/2} = \max_{g \neq 0} \frac{\left| \int_\Omega fg \, \mathrm{d}\Omega \right|}{\left(\int_\Omega g^2 \, \mathrm{d}\Omega \right)^{1/2}} = \max_{g \neq 0} \frac{\left| \int_\Omega fg \, \mathrm{d}\Omega \right|}{\|g\|_{L^2(\Omega)}}$$

The rightmost expression is a "variational characterization" of the $L^2(\Omega)$ norm, that is, it expresses the norm in terms of something that is optimized. By inserting derivatives in the nominator and/or denominator of the variational characterization, various exotic norms can be defined. A norm, different from the $L^2(\Omega)$ norm and of importance for the Navier–Stokes equations is

$$\|p\|_{\text{exotic}} = \max_{w_i} \frac{\left| \int_\Omega p \left(\frac{\partial w_i}{\partial x_i} \right) \mathrm{d}\Omega \right|}{\left(\int_\Omega |\nabla w_i|^2 \, \mathrm{d}\Omega \right)^{1/2}}. \tag{4.16}$$

This norm will be of crucial importance when we analyze the stability of FEM form the Navier–Stokes equations.

The *Poincaré inequality* relates the energy and $L^2(\Omega)$ norms:

Theorem 2 (Poincaré inequality). *There is a $C > 0$ such that*

$$\int_\Omega v^2 \, \mathrm{d}\Omega \leq C \int_\Omega \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_i} \, \mathrm{d}\Omega,$$

that is,

$$\|v\|^2_{L^2(\Omega)} \leq C \|v\|^2_V$$

for each $v \in V$.

Remark 1. Theorem 2 says that the energy norm is *stronger* than then $L^2(\Omega)$ norm. The smallest possible value of C grows with the size of Ω. The Poincaré inequality holds only if Ω is bounded in at least one direction. Also, the inequality does not hold if $\int_{\Gamma_0} \mathrm{d}\Gamma = 0$. However, the inequality

$$\int_\Omega v^2 \, \mathrm{d}\Omega \leq C \left(\int_\Omega v^2 \, \mathrm{d}\Omega + \int_\Omega \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_i} \, \mathrm{d}\Omega \right)$$

holds also when $\int_{\Gamma_0} \mathrm{d}\Gamma = 0$.

Stability

Choosing $v_h = u_h$ in equation (4.7) yields

$$\int_\Omega u_h U_j \frac{\partial u_h}{\partial x_j} \, \mathrm{d}\Omega + \nu \int_\Omega \frac{\partial u_h}{\partial x_j} \frac{\partial u_h}{\partial x_j} \, \mathrm{d}\Omega = \int_\Omega f u_h \, \mathrm{d}\Omega. \tag{4.17}$$

Note that

$$\int_\Omega u_h U_j \frac{\partial u_h}{\partial x_j} \, \mathrm{d}\Omega = u^T C u = 0,$$
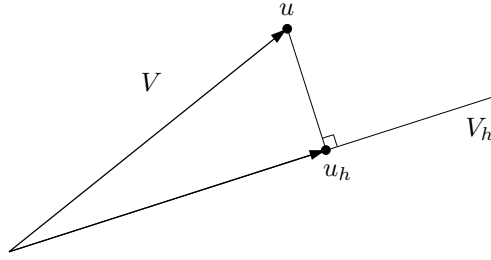
Figure 4.7: An orthogonal projection on a line. Note that the vector $u - u_h$ is the shortest of all vectors $u - v_h$ for $v_h \in V_h$, that is, $\|u - u_h\| \leq \|u - v_h\| \; \forall v_h \in V_h$.

since $C = -C^T$ (using the notation of § 4.1.4).

Expression (4.17) thus becomes

$$\nu \underbrace{\int_\Omega \frac{\partial u_h}{\partial x_i} \frac{\partial u_h}{\partial x_i} \, \mathrm{d}\Omega}_{\|u_h\|_V^2} = \int_\Omega f u_h \, \mathrm{d}\Omega$$

$$(\text{Cauchy–Schwarz}) \leq \left(\int_\Omega f^2 \, \mathrm{d}\Omega\right)^{1/2} \left(\int_\Omega u_h^2 \, \mathrm{d}\Omega\right)^{1/2}$$

$$(\text{Poincaré}) \leq C \left(\int_\Omega f^2 \, \mathrm{d}\Omega\right)^{1/2} \left(\int_\Omega \frac{\partial u_h}{\partial x_i} \frac{\partial u_h}{\partial x_i} \, \mathrm{d}\Omega\right)^{1/2}$$

$$\leq C \|f\|_{L^2(\Omega)} \|u_h\|_V.$$

Dividing with $\|u_h\|_V$ shows that

$$\|u_h\|_V \leq \frac{C}{\nu} \|f\|_{L^2(\Omega)}$$

where the constant C *does not depend on* h. That is, we have shown that u_h cannot blow up as the discretization is refined, which is the same as saying that the numerical solution is stable.

Error bounds

Choosing $v = v_h \in V_h \subset V$ in variational problem (4.6) and subtracting equation (4.7) from (4.6) yields the "Galerkin orthogonality",

$$\nu \int_\Omega \frac{\partial v_h}{\partial x_i} \frac{\partial}{\partial x_i} (u - u_h) \, \mathrm{d}\Omega + \int_\Omega v_h U_j \frac{\partial}{\partial x_j} (u - u_h) \, \mathrm{d}\Omega = 0 \qquad \forall v_h \in V_h.$$

This is a "true" orthogonality condition when $U_j \equiv 0$:

$$\nu \int_\Omega \frac{\partial v_h}{\partial x_i} \frac{\partial}{\partial x_i} (u - u_h) \, \mathrm{d}\Omega = 0 \qquad \forall v_h \in V_h; \tag{4.18}$$

that is, the error $u - u_h$ is orthogonal to each $v_h \in V_h$ with respect to the inner product (4.14). Cf. two orthogonal vectors: $a \cdot b = 0$. Orthogonality relation (4.18) means that u_h is an *orthogonal projection* of u on V_h. As illustrated in figure 4.7, the orthogonal projection on a subspace yields the vector that is closest to the given element in the norm derived from the inner product in which orthogonality is defined. Orthogonality property (4.18) (cf. figure 4.7) implies that the FE approximation is *optimal* in the energy norm when $U_i \equiv 0$:

$$\|u - u_h\|_V \leq \|u - v_h\|_V \qquad \forall v_h \in V_h. \tag{4.19}$$

The matrix A is *symmetric* in this case (since $C = 0$).

When $U_j \neq 0$, the FEM approximation is no longer optimal in this sense. However, there is a $C > 0$ such that *Cea's lemma* holds,

$$\|u - u_h\|_V \leq \frac{C}{\nu} \|u - v_h\|_V \qquad \forall v_h \in V_h. \tag{4.20}$$

Matrix A is *nonsymmetric* in this case. Note that the approximation may be far from optimal when ν is small, as the next example illustrates.
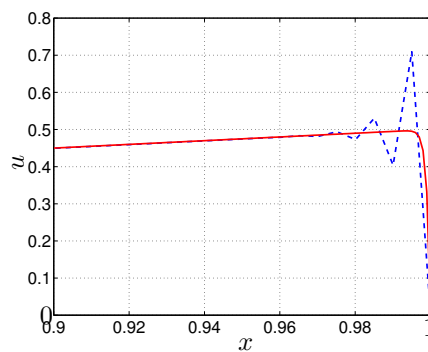
Figure 4.8: Under-resolved standard Galerkin finite-element approximations may yield oscillations for advection-dominated flows.
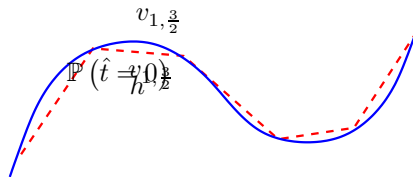

Figure 4.9: An interpolant $\Pi_h v$ interpolates v at the nodal points.

Example 1. Let us consider the above FEM applied to the 1D problem

$$-\frac{1}{500}u'' + u' = 1 \qquad \text{in } (0,1).$$
$$u(0) = u(1) = 0$$

(4.21)

The problem is *advection dominated*, that is, the viscosity coefficient in equation (4.21) is quite small: $\nu = 1/500$. A *boundary layer* with sharp gradients forms close to $x = 1$. The solid line in figure 4.8 shows the FE solution $h = 1/1000$, whereas the dashed line shows the FE approximation for $h = 1/200$. □

The result of example 1 is typical: under-resolved standard Galerkin methods may generate oscillations for advection-dominated flows in areas of sharp gradients.

An explanation of this phenomenon is that the directions of flow is not accounted for in the standard Galerkin approximation. Galerkin approximations correspond to finite-difference *central schemes*. *Upwinding* is used to prevent oscillations for finite-difference methods. Various methods exist to obtain similar effects in FEM, for instance *streamline diffusion* and *discontinuous Galerkin*. Both these methods modify the basic Galerkin idea to account for flow-direction effects.

Approximation and mesh quality

Estimates (4.19) and (4.20) are independent of the choice of V_h! We only used that $V_h \subset V$. Next step in assessing the error is an *approximation problem*: How well can functions in V_h approximate those in V? This step is *independent of the equation in question*. *Interpolants* are used to study approximation properties. The interpolant is the function $\Pi_h v \in V_h$ that interpolates the function $v \in V$ at the node points of the triangulation, as illustrated for a 1D case in figure 4.9.

Approximation theory yields that, for sufficiently smooth v,

$$\|v - \Pi_h v\|_{L^2(\Omega)} \le Ch^2 \qquad \text{"second order error"}$$
$$\|v - \Pi_h v\|_V \le Ch \qquad \text{"first order error"}$$

(4.22)

The constant C contains integrals of the second derivatives of v.

A *mesh quality* assumption is needed for (4.22) to hold. The essential point is: *beware of "thin" triangles*; do not let the shape deteriorate as the mesh is refined. A strategy to preserve mesh quality when refining is to subdivide each triangle into four new triangles by joining the edge midpoints (figure 4.10).

Each of the conditions below is sufficient for approximation property (4.22) to hold:
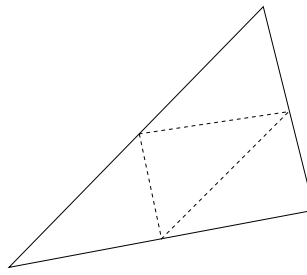
M
m
$m+2$
1
h
V
V_h
u
u_h

1
2
i
j
Inflow
Solid wall
$\mathbf{n}$
Γ_0
Γ_1
Ω
M^2
M
m
$m+2$
1
h
V
V_h
u
u_h

Figure 4.10: A subdivision that preserves mesh quality.

(i) ("Maximum angle condition"). The largest angle of any triangle should not approach $180°$ as $h \to 0$.

(ii) ("Chunkiness parameter condition"). The quotient between the diameter of the largest circle that can be inscribed and the diameter of the triangle should not vanish as $h \to 0$.

Condition (ii) implies (i), but not the reverse.
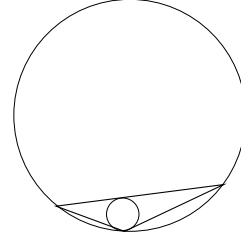
Accuracy of the FE solution

Error bounds (4.19) and (4.20) together with approximation property (4.22) implies

$$\|u - u_h\|_V \le Ch$$

Further analysis (not provided here) yields

$$\|u - u_h\|_{L^2(\Omega)} \le Ch^2.$$

This *second-order convergence rate* requires a mesh quality assumption and a sufficiently smooth solution u (since the constant C contains second derivatives of u). The convergence rate is of *optimal order*: it agrees with approximation property (4.22) and is the best *order* that in general can be obtained for, in this case, piecewise-linear functions. However, the solution may still be bad for a particular, too-crude mesh; recall example 1.

Improving Accuracy

Accuracy is improved by refining the mesh ("*h*-method"). This may be done *adaptively*, where it is needed (say, in areas of large gradients), to prevent the matrices to become too large. Automatic methods for adaptation are common.

An alternative is to keep the mesh fixed and increase the order of the polynomials at each triangle ("*p*-method"). For instance, continuous, piecewise quadratics yields a *third-order-accurate* solution (in the $L^2(\Omega)$ norm).

These strategies may be combined in the "*h–p* method", where the mesh is refined in certain regions and the order of approximations is increased in other.

4.1.6   Alternative Elements, 3D

A *quadrilateral* is a "skewed" rectangle: four points connected by straight lines to form a closed geometric object. Nonoverlapping quadrilaterals can be used to "triangulate" a domain, as in figure 4.11.

An example of a finite-element space V_h on quadrilaterals is globally continuous functions varying *linearly* along the edges. As in the triangular case, the functions are defined uniquely by specifying the function values at each node. The functions are defined by interpolation into the interior of each quadrilateral. This space reduces to piecewise *bilinear* functions for rectangles with edges in the coordinate directions:

$$v_h(x, y) = a + b\,x + c\,y + d\,xy$$

Quadrilaterals on a "logically rectangular domain" (a distorted rectangle, as in figure 4.11) yields a regular structure to the matrix A. This may
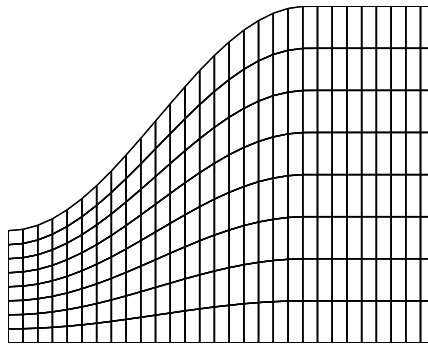
Figure 4.11: A logically rectangular domain triangulated with quadrilaterals.

Figure 4.12: In 3D, triangular and quadrilateral elements generalize to tetrahedral (left) and hexahedral (right) meshes.

- give high accuracy (particularly if the mesh is aligned with the flow), and

- allow efficient solution of the linear system (particularly for uniform, rectangular meshes).

Compared to triangular mesh, it is harder to generate quadrilateral meshes automatically for complicated geometries.

Triangular and quadrilateral meshes generalize in 3D to *tetrahedral* and *hexahedral* meshes (figure 4.12). Advantages and limitations with tetrahedral and hexahedral meshes are similar to corresponding meshes in 2D.
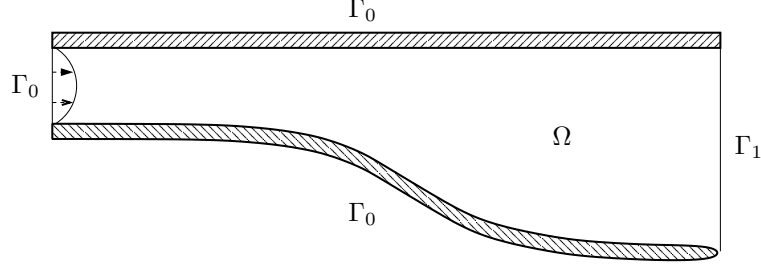

V
V_h Γ_0
u
u_h Inflow
Solid wall
n
Γ_0
Γ_1

Ω

Γ_1

Γ_0

Figure 4.13: A simple example of a domain.

M^2
M
m
m + 2
1
h
V
V_h Γ_0
u
u_h
Γ_0
Γ_1
Ω
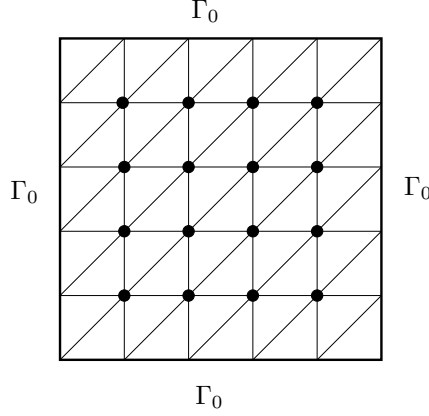
Γ_0

Γ_0

Γ_0

Γ_0

Figure 4.14: The black nodes are the degrees of freedom for the velocity components, which are fewer than the number of triangles.

4.2 FEM for Navier–Stokes

We now turn to the Navier–Stokes equations for the steady flow of an incompressible fluid, and consider a situation with flow in a bounded domain Ω. A simple example domain is depicted in figure 4.13. The mathematical problem in differential form is

$$u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - \frac{1}{\mathrm{Re}} \nabla^2 u_i = f_i \qquad \text{in } \Omega, \qquad (4.23a)$$

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad \text{in } \Omega. \qquad (4.23b)$$

$$u_i = g_i \qquad \text{on } \Gamma_0 \qquad (4.23c)$$

$$-\frac{1}{\mathrm{Re}} \frac{\partial u_i}{\partial x_j} n_j + p n_i = 0 \qquad \text{on } \Gamma_1. \qquad (4.23d)$$

The boundary condition on $\Gamma_1 = \partial\Omega \setminus \Gamma_0$ is of no obvious physical significance, but is useful as an artificial outflow condition.

The Navier–Stokes equations is a system of nonlinear advection–diffusion equations together with the incompressibility condition (4.23b). This condition introduces new complications not present in the advection–diffusion problem of section 4.1. Naive approximations of the Navier–Stokes equations are bound to produce disapping results, as the following example indicates.

Example 2 *("the counting argument").* Consider a case with only Dirichlet boundary conditions—that is, $\Gamma_1 = \emptyset$—and the mesh of figure 4.14. Assume that we use continuous, piecewise-linear approximations u_h, v_h for the x- and y-components of the velocity field. There are I number of panels in each direction and therefore $2(I-1)^2$ degrees of freedom (two times the number of black dots) for the velocity vector field. The left side of equation (4.23b), that is

$$\frac{\partial u_h}{\partial x} + \frac{\partial v_h}{\partial y}$$

is here constant on each triangle. Thus, demanding equation (4.23b) to be satisfied on each triangle yields $2I^2$ equations, *which is more than the degrees of freedom for the velocity*! Thus $u_h = v_h = 0$ is the only FE function satisfying the incompressibility condition (4.23b)! $\qquad\square$

Following strategies can be used to tackle the problem of example 2.

(i) Use more degrees of freedom for the velocity—for instance continuous, piecewise quadratics on each triangle—to balance the large number of equations associated with the incompressibility condition.

(ii) Accept that the discrete velocities only is approximately divergence-free.

(iii) Associate the velocity with the midpoints of *edges* instead of the vertices. This also gives more degrees of freedom for the velocities. (There are $\sim 3I^2$ edges, but only $\sim 2I^2$ vertices).

Strategies (i) and (ii) are the "standard" methods, whereas (iii) leads to "non-conforming" approximations related to finite-difference and finite-volume approximation with "staggered mesh".

4.2.1 A variational form of the Navier–Stokes equations

To derive the variational form, we need new integration-by-part formulas. The product rule of differentiation yields

$$\frac{\partial}{\partial x_j}\left(z_i \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial z_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + z_i \nabla^2 u_i \tag{4.24}$$

and

$$\frac{\partial}{\partial x_j}(z_j p) = z_i \frac{\partial p}{\partial x_i} + p \frac{\partial z_j}{\partial x_j}. \tag{4.25}$$

Integrating expressions (4.24) and (4.25) and applying the the divergence theorem (4.3) on the left sides yields

$$\int_\Gamma n_j z_i \frac{\partial u_i}{\partial x_j}\,\mathrm{d}\Gamma = \int_\Omega \frac{\partial z_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\,\mathrm{d}\Omega + \int_\Omega z_i \nabla^2 u_i\,\mathrm{d}\Omega \tag{4.26}$$

and

$$\int_\Gamma n_j z_j p\,\mathrm{d}\Gamma = \int_\Omega z_i \frac{\partial p}{\partial x_i}\,\mathrm{d}\Omega + \int_\Omega p \frac{\partial z_i}{\partial x_i}\,\mathrm{d}\Omega, \tag{4.27}$$

which are the integration-by-parts formulas needed to derive our variational form.

Now, let z_i be a smooth vector-valued function on $\overline{\Omega}$ that vanishes on Γ_0. Multiply both sides of equation (4.23a) with z_i, integrating over Ω, and utilizing (4.26) and (4.27), we find that

$$\int_\Omega z_i f_i\,\mathrm{d}\Omega = \int_\Omega z_i u_j \frac{\partial u_i}{\partial x_j}\,\mathrm{d}\Omega + \int_\Omega z_i \frac{\partial p}{\partial x_i}\,\mathrm{d}\Omega - \frac{1}{\mathrm{Re}}\int_\Omega z_i \nabla^2 u_i\,\mathrm{d}\Omega$$

$$= \int_\Omega z_i u_j \frac{\partial u_i}{\partial x_j}\,\mathrm{d}\Omega + \frac{1}{\mathrm{Re}}\int_\Omega \frac{\partial z_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\,\mathrm{d}\Omega - \frac{1}{\mathrm{Re}}\int_\Gamma n_j z_i \frac{\partial u_i}{\partial x_j}\,\mathrm{d}\Gamma - \int_\Omega p \frac{\partial z_i}{\partial x_i}\,\mathrm{d}\Omega + \int_\Gamma n_j z_j p\,\mathrm{d}\Gamma \tag{4.28}$$

$$= \int_\Omega z_i u_j \frac{\partial u_i}{\partial x_j}\,\mathrm{d}\Omega + \frac{1}{\mathrm{Re}}\int_\Omega \frac{\partial z_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\,\mathrm{d}\Omega - \int_\Omega p \frac{\partial z_i}{\partial x_i}\,\mathrm{d}\Omega,$$

where, in the last equality, the boundary integrals vanish on Γ_0 since $z_i = 0$ on Γ_0 and on Γ_1 due to boundary condition (4.23d). Moreover, multiplying equation (4.23b) with a smooth function q and integrating yields

$$\int_\Omega q \frac{\partial u_i}{\partial x_i}\,\mathrm{d}\Omega = 0 \tag{4.29}$$

Thus, we conclude that a classical solution (a twice continuously differentiable solution) to the Navier–Stokes equations satisfies the variational forms (4.28) and (4.29) for all smooth functions z_i and q such that z_i vanishes on Γ_0.

Similarly as for the advection–diffusion problem, the variational form will be used to define weak solutions, without explicit reference to the differential equations. A difference with advection–diffusion problem is that we here need *different* function spaces for the velocity components and the pressure. Also note that $u_i = g_i$ on Γ_0 whereas z_i vanishes on Γ_0. To avoid this last complication and simplify the exposition, we will only consider homogeneous boundary conditions, $g_i = 0$. Thus, the equations will be "driven" only by the right-hand side f_i, which, for instance, could be gravity or magnetic forcing for a electrically conducting fluid.

The pressure space is

$$H = L^2(\Omega) = \left\{ q \mid \int_\Omega q^2\,\mathrm{d}\Omega < +\infty \right\},$$

and the space of velocity components is

$$V = \left\{ u \mid \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} \, d\Omega < +\infty \text{ and } u = 0 \text{ on } \Gamma_0 \right\}.$$

For the analysis in section 4.2.4, it will be convenient to work with the space $\boldsymbol{V}$ of the velocity vector $\boldsymbol{u} = (u_1, u_2)$. Saying that $\boldsymbol{u} \in \boldsymbol{V}$ is equivalent to saying that that each velocity component $u_i \in V$. The energy norm for elements of $\boldsymbol{V}$ is denoted

$$\|\boldsymbol{u}\|_{\boldsymbol{V}} = \left(\int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \, d\Omega \right)^{1/2},$$

whereas the $L^2(\Omega)$ norm is

$$\|\boldsymbol{u}\|_{L^2(\Omega)} = \left(\int_\Omega u_i u_i \, d\Omega \right)^{1/2}.$$

We assume that both boundary conditions really are active. In the case of only Dirichlet boundary condition, that is, that Γ_0 constitute the whole boundary, the pressure is only possible to specify uniquely up to an additive constant.

The variational problem defining weak solutions to equation (4.23) is

Find $u_i \in V$ and $p \in H$ such that

$$\int_\Omega z_i u_j \frac{\partial u_i}{\partial x_j} \, d\Omega + \frac{1}{Re} \int_\Omega \frac{\partial z_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \, d\Omega - \int_\Omega p \frac{\partial z_i}{\partial x_i} \, d\Omega = \int_\Omega z_i f_i \, d\Omega \qquad \forall z_i \in V$$

$$\int_\Omega q \frac{\partial u_i}{\partial x_i} \, d\Omega = 0 \qquad \forall q \in H$$

This is a variational problem of *mixed type*: two different spaces are involved. As for the advection–diffusion problem, the boundary conditions are incorporated in the variational formulation. The *essential boundary conditions* are here that $u_i = 0$ on Γ_0 and is explicitly assigned through the definition of V. The *natural boundary conditions* are the boundary conditions on Γ_1, implicitly specified through the variational form. Note that the natural boundary condition is different than for the advection–diffusion problem.

Remark 2. The mathematical properties of the Navier–Stokes equations are complicated, and important issues are still unresolved. In fact, there is a \$ 1 000 000 prize for the person that can develop a theory that can satisfyingly explain their behavior in three space dimensions! A short summary of what is known is the following.

For the *unsteady problem* on a given, finite time interval $(0, T)$, and

- in *two* space dimensions:

    - a unique weak solution exists for any square-integrable f and for any Re,

    - smooth data yield smooth solutions, so that weak solutions are classical solutions if data is smooth enough;

- in *three* space dimensions:

    - a weak solution exists for any square-integrable f and for any Re.

    - It is not known whether the weak solution always is unique (besides during some initial interval $(0, T^*)$, where the size of T^* is unknown).

    - It is not known whether smooth data always yields a smooth solution, or if singularities can evolve from smooth data.

The *steady* problem has at least one weak solution, in 2D as well as 3D. Note that the steady problem may have several solutions, even in 2D. □

Note that the natural boundary condition here not just is a Neumann condition, but the more complicated condition (4.23d).

4.2.2 Finite-element approximations

To obtain a FEM, we choose subspaces $V_h \subset V$, $H_h \subset H$ of piecewise polynomials, and replace V with V_h and H with H_h in the variational form. In 2D and in components, we obtain

Find u_h, $v_h \in V_h$ and $p_h \in H_h$ such that

$$\int_\Omega z_h \left(u_h \frac{\partial u_h}{\partial x} + v_h \frac{\partial u_h}{\partial y} \right) \mathrm{d}\Omega + \frac{1}{\mathrm{Re}} \int_\Omega \frac{\partial z_h}{\partial x_j} \frac{\partial u_h}{\partial x_j} \mathrm{d}\Omega - \int_\Omega p_h \frac{\partial z_h}{\partial x} \mathrm{d}\Omega = \int_\Omega z_h f_1 \mathrm{d}\Omega \qquad \forall z_h \in V_h, \quad (4.30\mathrm{a})$$

$$\int_\Omega z_h \left(u_h \frac{\partial v_h}{\partial x} + v_h \frac{\partial v_h}{\partial y} \right) \mathrm{d}\Omega + \frac{1}{\mathrm{Re}} \int_\Omega \frac{\partial z_h}{\partial x_j} \frac{\partial v_h}{\partial x_j} \mathrm{d}\Omega - \int_\Omega p_h \frac{\partial z_h}{\partial y} \mathrm{d}\Omega = \int_\Omega z_h f_2 \mathrm{d}\Omega \qquad \forall z_h \in V_h, \quad (4.30\mathrm{b})$$

$$\int_\Omega q_h \left(\frac{\partial u_h}{\partial x} + \frac{\partial v_h}{\partial y} \right) \mathrm{d}\Omega = 0. \qquad (4.30\mathrm{c})$$

We postpone for a while the question on how to choose the subspaces V_h and H_h.

4.2.3 The algebraic problem in 2D

Expanding p_h in basis for H_h, we obtain

$$p_h(x, y) = \sum_{n=1}^{N_p} p^{(n)} \psi^{(n)}(x, y). \qquad (4.31)$$

Likewise, expanding u_h, v_h in basis for V_h yields

$$u_h(x, y) = \sum_{n=1}^{N_u} u^{(n)} \varphi^{(n)}(x, y), \qquad v_h(x, y) = \sum_{n=1}^{N_u} v^{(n)} \varphi^{(n)}(x, y). \qquad (4.32)$$

Now insert expansions (4.31) and (4.32) into equations (4.30). In (4.30a) and (4.30b), choose $z_h = \varphi^{(m)}$ for $m = 1, \ldots, N_u$; in (4.30c), choose $q_h = \psi^{(m)}$ for $m = 1, \ldots, N_p$. Then, equations (4.30) become

$$\sum_{n=1}^{N_u} u^{(n)} \int_\Omega \varphi^{(m)} \left(u_h \frac{\partial \varphi^{(n)}}{\partial x} + v_h \frac{\partial \varphi^{(n)}}{\partial y} \right) \mathrm{d}\Omega + \frac{1}{\mathrm{Re}} \sum_{n=1}^{N_u} u^{(n)} \int_\Omega \left(\frac{\partial \varphi^{(m)}}{\partial x} \frac{\partial \varphi^{(n)}}{\partial x} + \frac{\partial \varphi^{(m)}}{\partial y} \frac{\partial \varphi^{(n)}}{\partial y} \right) \mathrm{d}\Omega$$

$$- \sum_{n=1}^{N_p} p^{(n)} \int_\Omega \frac{\partial \varphi^{(m)}}{\partial x} \psi^{(n)} \mathrm{d}\Omega = \int_\Omega \varphi^{(m)} f_1 \mathrm{d}\Omega \qquad m = 1, \ldots, N_u$$

$$\sum_{n=1}^{N_u} v^{(n)} \int_\Omega \varphi^{(m)} \left(u_h \frac{\partial \varphi^{(n)}}{\partial x} + v_h \frac{\partial \varphi^{(n)}}{\partial y} \right) \mathrm{d}\Omega + \frac{1}{\mathrm{Re}} \sum_{n=1}^{N_u} v^{(n)} \int_\Omega \left(\frac{\partial \varphi^{(m)}}{\partial x} \frac{\partial \varphi^{(n)}}{\partial x} + \frac{\partial \varphi^{(m)}}{\partial y} \frac{\partial \varphi^{(n)}}{\partial y} \right) \mathrm{d}\Omega \quad (4.33)$$

$$- \sum_{n=1}^{N_p} p^{(n)} \int_\Omega \frac{\partial \varphi^{(m)}}{\partial y} \psi^{(n)} \mathrm{d}\Omega = \int_\Omega \varphi^{(m)} f_2 \mathrm{d}\Omega \qquad m = 1, \ldots, N_u$$

$$\sum_{n=1}^{N_u} u^{(n)} \int_\Omega \psi^{(m)} \frac{\partial \varphi^{(n)}}{\partial x} \mathrm{d}\Omega + \sum_{n=1}^{N_u} v^{(n)} \int_\Omega \psi^{(m)} \frac{\partial \varphi^{(n)}}{\partial y} \mathrm{d}\Omega = 0 \qquad m = 1, \ldots, N_p$$

Placing the coefficients into column matrices,

$$u = \left(u^{(1)}, \ldots, u^{(Nu)} \right)^T, \qquad v = \left(v^{(1)}, \ldots, v^{(Nu)} \right)^T,$$

$$p = \left(p^{(1)}, \ldots, p^{(N_p)} \right)^T,$$

we can write the system (4.33) in the matrix–vector form

$$\begin{pmatrix} C(u_h, v_h) + \frac{1}{\mathrm{Re}} K & 0 & G^{(x)} \\ 0 & C(u_h, v_h) + \frac{1}{\mathrm{Re}} K & G^{(y)} \\ -G^{(x)T} & -G^{(y)T} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} b^{(x)} \\ b^{(y)} \\ 0 \end{pmatrix}, \qquad (4.34)$$

where the matrix and vector components are

$$K_{mn} = \int_\Omega \left(\frac{\partial \varphi^{(m)}}{\partial x} \frac{\partial \varphi^{(n)}}{\partial x} + \frac{\partial \varphi^{(m)}}{\partial y} \frac{\partial \varphi^{(n)}}{\partial y} \right) \mathrm{d}\Omega, \qquad C_{mn}(u_h, v_h) = \int_\Omega \varphi^{(m)} \left(u_h \frac{\partial \varphi^{(n)}}{\partial x} + v_h \frac{\partial \varphi^{(n)}}{\partial y} \right) \mathrm{d}\Omega,$$

$$G_{mn}^{(x)} = -\int_\Omega \psi^{(n)} \frac{\partial \varphi^{(m)}}{\partial x} \, \mathrm{d}\Omega, \quad G_{mn}^{(y)} = -\int_\Omega \psi^{(n)} \frac{\partial \varphi^{(m)}}{\partial y} \, \mathrm{d}\Omega, \quad b_m^{(x)} = \int_\Omega \varphi^{(m)} f_1 \, \mathrm{d}\Omega, \quad b_m^{(y)} = \int_\Omega \varphi^{(m)} f_2 \, \mathrm{d}\Omega.$$

$$(4.35)$$

Matrix K represents an approximation of the discrete negative Laplacian $-\nabla^2$; $C(u_h, v_h)$ the discrete advection operator $u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$; $G^{(x)}$ and $G^{(y)}$ the discrete gradient ∇; and $G^{(x)T}$ and $G^{(y)T}$ the discrete divergence. That is, the structure of the Navier–Stokes equations are directly reflected in the nonlinear system (4.34). We obtained the same structure of the nonlinear equation for the finite-volume discretization, except that here, it is *guaranteed* the the last block row of the matrix is the negative transpose of the last block column.

4.2.4   Stability

So far, the approximating spaces V_h and H_h have not been specified. The counting argument of example 2 indicates that not any choice is good. We will limit the discussion to the *Stokes equations*, whose finite-element formulation reads

Find $u_i^h \in V_h$ and $p_h \in H_h$ such that

$$\int_\Omega \frac{\partial z_i^h}{\partial x_j} \frac{\partial u_i^h}{\partial x_j} \, \mathrm{d}\Omega - \int_\Omega p_h \frac{\partial z_i^h}{\partial x_i} \, \mathrm{d}\Omega = \int_\Omega z_i^h f_i \, \mathrm{d}\Omega \qquad \forall\, z_i^h \in V_h, \tag{4.36a}$$

$$\int_\Omega q_h \frac{\partial u_i^h}{\partial x_i} \, \mathrm{d}\Omega = 0 \qquad \forall\, q_h \in H_h. \tag{4.36b}$$

The Stokes equations is a limit case of the Navier–Stokes equations for very low Reynolds numbers and assume that inertia can be neglected compared to viscous forces. The Stokes equations may be used to model very viscous flow, creeping flow, lubrication, but also flow of common fluids (like air and water) at the micro scale. Due to the lack of nonlinear advection, these are *linear* equations. Mathematically, the Stokes equations are much easier to analyze than the full Navier–Stokes equations. Indeed, there exists a unique weak solutions for each square-integrable f_i both in two and three space dimension. The reason to start the analysis on the Stokes equation is that a stable Stokes discretization is necessary to obtain a stable Navier–Stokes discretization.

As for the Navier–Stokes equations, we require that $V_h \subset V$ and $H_h \subset H$. Recall that each function in V and H is required to be bounded in the energy and the $L^2(\Omega)$ norm, respectively. Thus, in makes sense to require that the velocity and pressure approximations satisfy the same bounds, that is,

$$\|p_h\|_{L^2(\Omega)} = \left(\int_\Omega p_h^2 \, \mathrm{d}\Omega \right)^{1/2} \le C \left(\int_\Omega f_i f_i \, \mathrm{d}\Omega \right)^{1/2} = C\|\boldsymbol{f}\|_{L^2(\Omega)}, \tag{4.37a}$$

$$\|\boldsymbol{u}_h\|_{\boldsymbol{V}} = \left(\int_\Omega \frac{\partial u_i^h}{\partial x_j} \frac{\partial u_i^h}{\partial x_j} \, \mathrm{d}\Omega \right)^{1/2} \le C\|\boldsymbol{f}\|_{L^2(\Omega)}, \tag{4.37b}$$

where the constant C should not depend on h. This implies that the approximations will not blow up when the mesh is refined.

Choosing $z_i^h = u_i^h$ in equation (4.36a), we obtain

$$\int_\Omega \frac{\partial u_i^h}{\partial x_j} \frac{\partial u_i^h}{\partial x_j} \, \mathrm{d}\Omega - \int_\Omega p_h \frac{\partial u_i^h}{\partial x_i} \, \mathrm{d}\Omega = \int_\Omega \frac{\partial u_i^h}{\partial x_j} \frac{\partial u_i^h}{\partial x_j} \, \mathrm{d}\Omega = \int_\Omega u_i^h f_i \, \mathrm{d}\Omega,$$

where the first equality follows from equation (4.36b). Thus,

$$\|\boldsymbol{u}_h\|_{\boldsymbol{V}}^2 = \int_\Omega u_i^h f_i \, \mathrm{d}\Omega \le \|\boldsymbol{u}\|_{L^2(\Omega)} \|\boldsymbol{f}\|_{L^2(\Omega)} \le C\|\boldsymbol{u}\|_{\boldsymbol{V}} \|\boldsymbol{f}\|_{L^2(\Omega)} \tag{4.38}$$

where the Cauchy–Schwarz inequality yields the first inequality, and the Poincaré inequality the second. We thus conclude from expression (4.38) that

$$\|\boldsymbol{u}_h\|_{\boldsymbol{V}} \le C\|\boldsymbol{f}\|_{L^2(\Omega)}, \tag{4.39}$$

that is, *any* Galerkin approximation of the velocity in the Stokes equations is stable! For bad choices of subspaces, we will of course obtain inaccurate velocity approximations, but they will *not* blow up as the mesh is refined, since their energy norm is bounded by the given data f_i.

Regarding the pressure, the situation is more complicated. Not any combination H_h and $\boldsymbol{V}_h$ yield stable pressure approximations in the sense of (4.37a). We will derive a compatibility condition between the velocity and pressure spaces, *the LBB condition*, necessary to yield stable pressure approximations.

4.2.5 The LBB condition

Rearranging equation (4.36a) implies that for each $z_i^h \in V_h$,

$$
\int_\Omega p_h \frac{\partial z_i^h}{\partial x_i} \, \mathrm{d}\Omega = \int_\Omega \frac{\partial z_i^h}{\partial x_j} \frac{\partial u_i^h}{\partial x_j} \, \mathrm{d}\Omega - \int_\Omega z_i^h f_i \, \mathrm{d}\Omega
$$

$$
\text{(Cauchy–Schwarz)} \le \left(\int_\Omega \frac{\partial z_i^h}{\partial x_j} \frac{\partial z_i^h}{\partial x_j} \, \mathrm{d}\Omega \right)^{1/2} \left(\int_\Omega \frac{\partial u_i^h}{\partial x_j} \frac{\partial u_i^h}{\partial x_j} \, \mathrm{d}\Omega \right)^{1/2} + \left(\int_\Omega z_i^h z_i^h \, \mathrm{d}\Omega \right)^{1/2} \left(\int_\Omega f_i f_i \, \mathrm{d}\Omega \right)^{1/2}
$$

$$
= \|\boldsymbol{z}_h\|_{\boldsymbol{V}} \|\boldsymbol{u}_h\|_{\boldsymbol{V}} + \|\boldsymbol{z}_h\|_{L^2(\Omega)} \|\boldsymbol{f}\|_{L^2(\Omega)}
$$

$$
\text{(Poincaré)} \le \|\boldsymbol{z}_h\|_{\boldsymbol{V}} \|\boldsymbol{u}_h\|_{\boldsymbol{V}} + \|\boldsymbol{z}_h\|_{V} \|\boldsymbol{f}\|_{L^2(\Omega)}
$$

$$
\text{(by (4.39))} \le C \|\boldsymbol{z}_h\|_{\boldsymbol{V}} \|\boldsymbol{f}\|_{L^2(\Omega)} + \|\boldsymbol{z}_h\|_{V} \|\boldsymbol{f}\|_{L^2(\Omega)}.
$$

$$(4.40)$$

Dividing expression (4.40) with $\|\boldsymbol{z}_h\|_V$ yields that

$$
\frac{1}{\|\boldsymbol{z}_h\|_V} \int_\Omega p_h \frac{\partial z_i^h}{\partial x_i} \, \mathrm{d}\Omega \le (C+1) \|\boldsymbol{f}\|_{L^2(\Omega)}
$$

for each $z_i^h \in V_h$, from which it follows that there is a $C > 0$ such that

$$
\max_{0 \ne z_i \in V} \frac{1}{\|\boldsymbol{z}_h\|_V} \int_\Omega p_h \frac{\partial z_i^h}{\partial x_i} \, \mathrm{d}\Omega \le C \|\boldsymbol{f}\|_{L^2(\Omega)}.
$$

$$(4.41)$$

We have thus found a bound on the discrete pressure in the "exotic" norm (4.16). However, we were aiming for the bound (4.37a). Thus, if it happens that

$$
\alpha \|p_h\|_{L^2(\Omega)} \le \max_{0 \ne z_i \in V_h} \frac{1}{\|\boldsymbol{z}_h\|_V} \int_\Omega p_h \frac{\partial z_i^h}{\partial x_i} \, \mathrm{d}\Omega
$$

$$(4.42)$$

for some $\alpha > 0$ independent of h, then it is immediate that $\|p_h\|_{L^2(\Omega)} \le C \|\boldsymbol{f}\|_{L^2(\Omega)}$ for some $C > 0$.

Condition (4.42) says that our exotic norm is *stronger* than the $L^2(\Omega)$-norm for p_h. The LBB-condition is a condition on the pairing of spaces V_h and H_h that requires the exotic norm to be stronger than the $L^2(\Omega)$-norm for *each* function $q_h \in H_h$. The derivation above proves thus that the LBB-condition is sufficient for the pressure to be stably determined by data:

Theorem 3. *Assume that the LBB-condition holds for the spaces H_h, V_h. That is, assume that there is an $\alpha > 0$ independent of h, such that, for each $q_h \in H_h$,*

$$
\alpha \left(\int_\Omega q_h^2 \, \mathrm{d}\Omega \right)^{1/2} \le \max_{0 \ne z_i \in V_h} \frac{1}{\|\boldsymbol{z}_h\|_V} \int_\Omega q_h \frac{\partial z_i^h}{\partial x_i} \, \mathrm{d}\Omega,
$$

$$(4.43)$$

then there is a $C > 0$ such that

$$
\left(\int_\Omega p_h^2 \, \mathrm{d}\Omega \right)^{1/2} \le C \left(\int_\Omega |\boldsymbol{f}|^2 \, \mathrm{d}\Omega \right)^{1/2}
$$

holds, independently of h.

The LBB condition (4.43) is satisfied for $V_h = V$ and $H_h = H$. That is, the pressure (before discretization) is stably determined by data. The expression $\int_\Omega q_h \frac{\partial z_i^h}{\partial x_i} \, \mathrm{d}\Omega$ represents the discrete gradient of q_h (see § 4.2.3). The LBB condition thus means that a zero discrete pressure gradient implies a zero value of the pressure. Conversely, if the LBB condition is *not* satisfied, it means that the discrete pressure gradient may be small even when the pressure is not small: The discrete gradient cannot "feel" that the pressure is large, an effect that usually manifests itself as wild pressure oscillations.
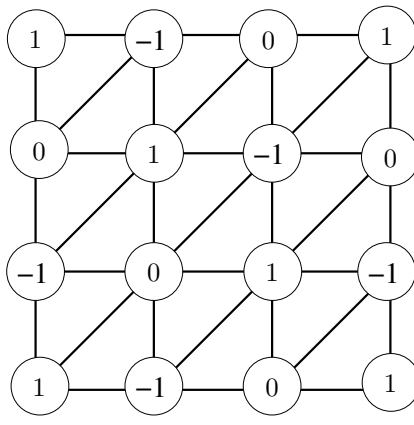
Figure 4.15: A checkerboard mode for equal-order interpolation on a regular triangular mesh.

4.2.6 Mass conservation

The integral $\int_\Gamma \rho\, n_i u_i \,\mathrm{d}\Gamma$ yields the total flux of mass(in kg/s) through the boundary. There is no net mass-flux through the boundary for a constant-density flow, so $\int_\Gamma \rho\, n_i u_i \,\mathrm{d}\Gamma = 0$. A velocity approximation u_i^h is is *globally mass conservative* if

$$\int_\Gamma n_i u_i^h \,\mathrm{d}\Gamma = 0,$$

or, equivalently, if

$$\int_\Omega \frac{\partial u_i^h}{\partial x_i} \,\mathrm{d}\Omega = 0,$$

by the divergence theorem (4.3). Recall from equation (4.30c) that

$$\int_\Omega q_h \frac{\partial u_i^h}{\partial x_i} \,\mathrm{d}\Omega = 0 \qquad \forall q_h \in H_h.$$

A Galerkin approximation is thus globally mass conservative if $1 \in H_h$, that is, if constant functions can be represented exactly in H_h. All reasonable FE approximation satisfy this.

A finite-element approximation is *locally mass conservative* if

$$\int_K \frac{\partial u_i^h}{\partial x_i} \,\mathrm{d}\Omega = 0,$$

for each element K in the mesh. Not all finite-element approximations are locally mass conservative. If the pressure approximations consist of piecewise constants on each element, the FE approximation will be locally mass conservative. However, it will *not* be locally mass conservative if the pressure approximations are continuous and piecewise linear.

4.2.7 Choice of finite elements. Accuracy

Now we consider a *regular triangulation* of a 2D domain Ω, that is, a triangulation satisfying a mesh quality condition of the sort discussed in section 4.1.5.

"Equal-order interpolation"

Let us start by trying the approximation space we used for the advection–diffusion problem (section 4.1.1). That is, we use continuous functions that are linear on each triangle both for the functions in H_h and V_h.

This pair does *not* satisfy the LBB condition and yields an unstable discretization. In particular, the system matrix (4.34) is *singular*: the discrete gradient matrix G has linearly dependent columns. Thus, there is at least one nonzero N_p-vector p^c such that $Gp^c = 0$. Corresponding function $p_h^c \in H_h$, whose nodal values are equal to the elements of p^c is called a *checkerboard mode*.

Figure 4.15 depicts the nodal values of a checkerboard mode for equal-order interpolation. To see this, consider the element contribution, associated with a triangle T_n, to the ith row of vector Gp^c (notice from the expressions in figure 4.5 that $\frac{\partial \phi_i}{\partial x}$ is constant on each triangle):

$$-\int_{T_n} p_h^c \frac{\partial \phi_i}{\partial x} \,\mathrm{d}\Omega = -\frac{\partial \phi_i}{\partial x}\bigg|_{T_n} \int_{T_n} p_h^c \,\mathrm{d}\Omega \equiv 0,$$

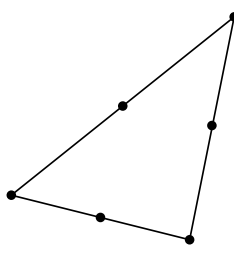h
V
V_h
u
u_h
Γ_0
Γ_1
Ω

Figure 4.16: The evaluation points at each triangle for the continuous, piecewise quadratic velocity approximations of the Taylor–Hood pair.

since the mean value of the function p_h^c vanishes on each triangle, as can be seen from figure 4.15. The right side of the LBB condition (4.43) vanishes for $q_h = p_h^c$, which immediately implies that the LBB cannot hold for a space H_h that includes p_h^c. This (and other) checkerboard mode will pollute the solution of the Stokes as well as Navier–Stokes equations..

Constant pressure, (bi)linear velocities

Now we consider a triangulation either of triangles or rectangles. Let the functions in H_h be piecewise constant on each element. In case of triangular elements, the space V_h will be continuous functions, piecewise linear on each element, whereas for quadrilateral elements, continuous functions that are piecewise bilinear on each rectangle comprise V_h.

Remark 3. The pressure approximations need *not* to be continuous in order for $H_h \subset H$. However, if $V_h \subset V$, the velocity approximations cannot be discontinuous, since ∇v is not a function for discontinuous v.

This choice leads to a locally mass-conserving approximation. However, for triangular elements, the counting argument of example 2 applies here and indicates problems with this choice of elements. Indeed, the LBB conditions is not satisfied for this unstable discretization. Checkerboard modes similar as in figure 4.15 exist. Still, the constant pressure—bilinear velocity space on rectangles is in widespread engineering use! "Fixes" have been developed to take care of pressure oscillations. For instance, the checkerboard mode can be filtered out from the discrete pressures. Moreover, common iterative methods to solve the resulting discrete equations (such as the projection schemes) implies a stabilization of the discretization.

The Taylor–Hood pair

Here, the functions in H_h and V_h are globally continuous and, on each triangle,

- $p_h \in H_h$ is *linear*, whereas

- u_h, $v_h \in V_h$ is *quadratic*.

Thus, on each triangle, we have

$$u_h = a + b\,x + c\,y + d\,xy + e\,x^2 + f y^2$$

The six coefficients a–f are uniquely determined by u_h's values at the triangle vertices and midpoints (figure 4.16).

A basis $\left\{\phi^{(m)}(x_i)\right\}_{m=1}^{N_u}$ of continuous, piecewise quadratic functions interpolates the nodal values of figure 4.16, so that $u_h \in V_h$ can be expressed as $u_h(x_i) = \sum_{m=1}^{N_u} u_i \phi^{(m)}(x_i)$. There are *two* kinds of basis functions here, illustrated in figures 4.17 and 4.18.

The LBB condition is satisfied for the Taylor–Hood pair, and the following error estimates holds for a smooth solution to the Navier–Stokes equation:

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^3$$
$$\|p - p_h\|_{L^2(\Omega)} \leq Ch^2$$

The Taylor–Hood pair is globally, but not locally mass conservative.

The Taylor–Hood pair can be generalized to higher order. Then, the pressure and velocity approximations will be continuous functions, consising of piecewise polynomials of degree k for pressure and $k+1$ for velocity ($k \geq 1$) on each triangle.
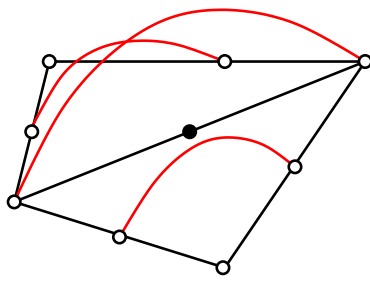
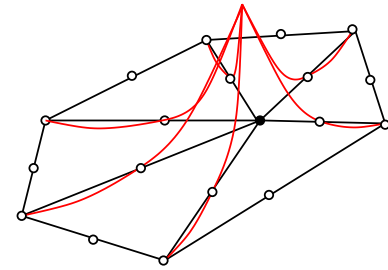Figure 4.17: The basis function $\phi^{(m)}(x_i)$ when m corresponds to an edge-midpoint node

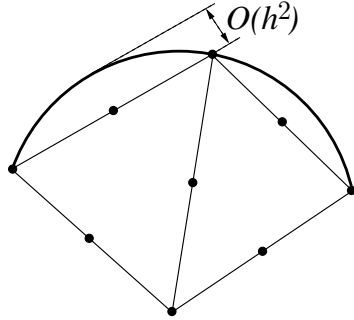Figure 4.18: The basis function $\phi^{(m)}(x_i)$ when m corresponds to a corner node

Figure 4.19: Approximating a curved boundary with a triangle sides leads to an $O(h^2)$ error of the boundary condition.
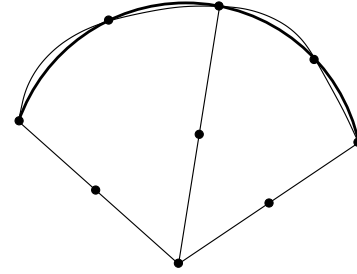

Figure 4.20: Isoparametric elements use polynomials of the same order as used for the FE approximations to approximate the shape of the boundary.

Remark 4.1. Inaccuracies in approximations of the domain boundary may reduce the convergence rate of higher-order approximations. For instance, a straight-forward triangulation of a domain with a curved boundary will lead to a boundary-condition error at the mid nodes of $O(h^2)$ (figure 4.19). This is fine for piecewise-linear approximations since the error anyway is $O(h^2)$. However, this inaccuracy at the boundary results in a *half-order* reduction of the convergence rate of piecewise quadratic approximations. Using *curved elements* restores convergence rate. The most common strategy for curved elements is known as *Isoparametric Elements*: piecewise polynomials of the same degree as used in the FE approximation are used to approximate the shape of the boundary (figure 4.20).

A stable approximation with piecewise-linear velocities and pressures

Instead of using different approximation *orders*, as in the Taylor-Hood pair, we here consider different *meshes* for velocity and pressure. Both the velocity and pressure approximations consist of continuous, piecewise-linear functions on triangles, but each pressure triangle is subdivided into four velocity triangles as in figure 4.21.

This is a stable approximation satisfying the LBB condition, and the following error estimates holds for
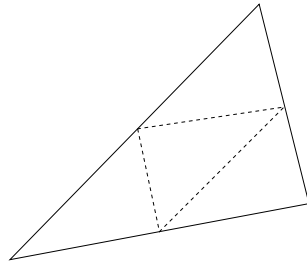

Figure 4.21: Subdividing a pressure triangle into four velocity triangles yields a stable approximation with continuous, piecewise-linear functions for both velocity and pressure.
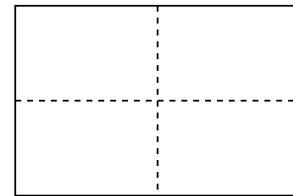

Figure 4.22: Subdividing a pressure rectangle into four velocity rectangles yields a stable approximation with continuous piecewise-bilinear functions for both velocity and pressure.
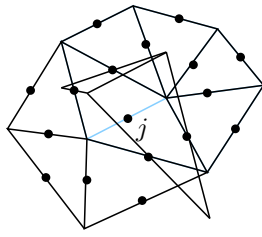
Figure 4.23: The basis functions associated with the velocity approximation for a nonconforming but stable discretization.
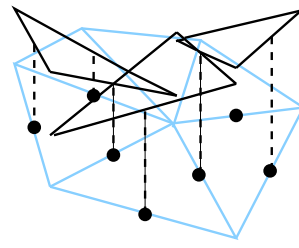

Figure 4.24: Approximations spanned by the basis functions (4.23) yields piecewise-linear functions that are continuous only along at the edge midpoints, in general.

a smooth solution to the Navier–Stokes equation:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} \le Ch^2$$
$$\|p - p_h\|_{L^2(\Omega)} \le Ch$$

These approximations use the same number of unknowns and the same nodal locations as for the Taylor–Hood pair, but are one order less accurate. Calculating the matrix elements (4.35) is however somewhat easier than for the Taylor–Hood pair.

There is also a version of this discretization for rectangular meshes: Each pressure rectangle is then subdivided into four velocity rectangles, as in figure 4.22. Both the velocity and pressure approximations consist of continuous and piecewise-bilinear functions.

A nonconforming method

The discussion after example 2 suggested a strategy to address the problem with the overdetermined incompressibility condition, namely to use nodal points on the *edge mid points* for the velocity. To accomplish this, consider basis functions, associated with the edges, of the type depicted in figure 4.23. Basis function ϕ_j is linear on each triangle and continuous along edge j (but discontinuous along four edges!). If x^i, y^i are the coordinates of the mid point of edge i, we have

$$\phi_j(x^i, y^i) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \ne j. \end{cases}$$

Now define the space of the velocity components V_h from all possible expansions

$$u_h(x,y) = \sum_{j=1}^{N} u_j \phi_j(x,y), \qquad v_h(x,y) = \sum_{j=1}^{N} v_j \phi_j(x,y),$$

Here, N is the total number of edges, excluding those on Γ_1. Note that u_h will be continuous only at edge mid points in general. Thus, $V_h \not\subset V$ since ∇u_h is not a square-integrable (in fact, constant) function. Violating a property like $V_h \subset V$ is called a *variational crime*. However, ∇u_h restricted on each element is a square-integrable function. Letting H_h be piecewise constant on each triangle and V_h defined as above, we define the *nonconforming* FE approximation:

Find u_h, $v_h \in V_h$ and $p_h \in H_h$ such that

$$\sum_{n=1}^{M} \int_{T_n} z_h u_h \frac{\partial u_h}{\partial x}\, d\Omega + \sum_{n=1}^{M} \int_{T_n} z_h v_h \frac{\partial u_h}{\partial y}\, d\Omega + \frac{1}{\mathrm{Re}} \sum_{n=1}^{M} \int_{T_n} \frac{\partial z_h}{\partial x_j}\frac{\partial u_h}{\partial x_j}\, d\Omega - \sum_{n=1}^{M} \int_{T_n} p_h \frac{\partial z_h}{\partial x}\, d\Omega = \int_{\Omega} z_h f_1\, d\Omega \qquad \forall\, z_h \in V_h,$$

$$\sum_{n=1}^{M} \int_{T_n} z_h u_h \frac{\partial v_h}{\partial x}\, d\Omega + \sum_{n=1}^{M} \int_{T_n} z_h v_h \frac{\partial v_h}{\partial y}\, d\Omega + \frac{1}{\mathrm{Re}} \sum_{n=1}^{M} \int_{T_n} \frac{\partial z_h}{\partial x_j}\frac{\partial v_h}{\partial x_j}\, d\Omega - \sum_{n=1}^{M} \int_{T_n} p_h \frac{\partial z_h}{\partial y}\, d\Omega = \int_{\Omega} z_h f_2\, d\Omega \qquad \forall\, z_h \in V_h,$$

$$\sum_{n=1}^{M} \int_{T_n} q_h \left(\frac{\partial u_h}{\partial x} + \frac{\partial v_h}{\partial y} \right) d\Omega = 0 \qquad \forall\, q_h \in H_h,$$

where M is the number of triangles.

The fact that $V_h \not\subset V$ complicates the analysis of this discretization. Nevertheless, the approximation is stable, although not as accurate as the Taylor–Hood pair, for instance. Other nice properties with this method is that it is locally mass conservative, and quite easy to implement.

But maybe the most interesting with this discretization is that it allows for a construction of a *divergence free velocity basis*. In all types of discretizations discussed so far, including the present, we have used the same basis for the u_h and v_h components. By a clever combination of basis functions of the type illustrated in figure 4.23, it is possible to construct a new vector-valued basis functions $\boldsymbol{\eta}_j = (z_h^{(j)}, w_h^{(j)})$ for the velocity having the property

$$\nabla \cdot \boldsymbol{\eta}_j = \frac{\partial z_h^{(j)}}{\partial x} + \frac{\partial w_h^{(j)}}{\partial y} = 0.$$

Expanding the velocity in this basis,

$$\boldsymbol{u}_h = (u_h, v_h) = \sum_{j=1}^{M} u_j \boldsymbol{\eta}_j, \tag{4.44}$$

yields that $\nabla \cdot \boldsymbol{u}_h = 0$. Inserting expansion (4.44) into equations (4.30), and choosing $(z_h, w_h) = \boldsymbol{\eta}_i$, the continuity equation and the pressure variable vanishes, and equations (4.30) reduce to

$$\sum_{j=1}^{M} u_j \int_\Omega \boldsymbol{\eta}_i \cdot (\boldsymbol{u}_h \cdot \nabla) \boldsymbol{\eta}_j \, d\Omega + \nu \sum_{j=1}^{M} u_j \int_\Omega \nabla \boldsymbol{\eta}_i \cdot \nabla \boldsymbol{\eta}_j \, d\Omega + \sum_{j=1}^{dN_u} u_j \int_\Omega \boldsymbol{\eta}_i \cdot (\boldsymbol{u}_h \cdot \nabla) \boldsymbol{\eta}_j \, d\Omega = \int_\Omega \boldsymbol{\eta}_i \cdot \boldsymbol{f} \, d\Omega,$$
$$\tag{4.45}$$

for $i = 1, \ldots, M$. The Navier–Stokes equations then reduces to a vector-valued, non-linear advection–diffusion problem, quite similar to equation (4.10). This leads to a substantial simplification when designing an algorithm to solve the discretized equations; in fact, one of the great challenges for any solver of the Navier–Stokes equations is how to treat the incompressibility condition (4.30c). Unfortunately, the construction of the basis functions $\boldsymbol{\eta}_j$ is not straightforward.

Stabilization of unstable discretizations

Stability of the choices of elements presented in sections 4.2.7–4.2.7 relied on manipulations of the local polynomial order, the meshes, or the conformity of the method. These manipulations complicate the implementation. An alternative is to use the same approximations for velocity and pressure (equal-order interpolation) combined with a *stabilization method*. This circumvents the LBB condition.

We will present a very simple idea of this sort for the Stokes equations. First choose the *same* type of approximations for the pressure and the velocity components, for instance continuous, piecewise-linear functions on a triangular mesh, and replace equations (4.36) with

Find $u_i^h \in V_h$ and $p_h \in H_h$ such that

$$\int_\Omega \frac{\partial z_i^h}{\partial x_j} \frac{\partial u_i^h}{\partial x_j} \, d\Omega - \int_\Omega p_h \frac{\partial z_i^h}{\partial x_i} \, d\Omega = \int_\Omega z_i^h f_i \, d\Omega \qquad \forall z_i^h \in V_h, \tag{4.46a}$$

$$\epsilon \int_\Omega \frac{\partial q_h}{\partial x_i} \frac{\partial p_h}{\partial x_i} \, d\Omega + \int_\Omega q_h \frac{\partial u_i^h}{\partial x_i} \, d\Omega = 0 \qquad \forall q_h \in H_h., \tag{4.46b}$$

for some small $\epsilon > 0$. Equation (4.46b) formally inconsistent by $O(\epsilon)$ with corresponding variational form. More advanced stabilization methods avoids this inconsistency by using a term that vanishes as $h \to 0$.

The main advantage with stabilization methods is the simplified implementation, but the method introduces an extra parameter that may need judicious tuning. Note also that the increased number of pressure unknowns, as compared with the discretization in section 4.2.7, does *not* increase the accuracy. The added resolution introduced is filtered away by the introduction of the stability term.

To see that approximation (4.46) is stable, choose $z_i^h = u_i^h$, $q_h = p_h$ and add equations (4.46a)

and (4.46b),

$$\int_\Omega \frac{\partial u_i^h}{\partial x_i} \frac{\partial u_i^h}{\partial x_i} \, \mathrm{d}\Omega + \epsilon \int_\Omega \frac{\partial p_h}{\partial x_i} \frac{\partial p_h}{\partial x_i} \, \mathrm{d}\Omega = \int_\Omega u_i^h f_i \, \mathrm{d}\Omega$$

$$\text{(Cauchy–Schwarz)} \leq \left(\int_\Omega u_i^h u_i^h \, \mathrm{d}\Omega \right)^{1/2} \left(\int_\Omega f_i f_i \, \mathrm{d}\Omega \right)^{1/2}$$

$$\text{(Poincaré)} \leq C \left(\int_\Omega \frac{\partial u_i^h}{\partial x_i} \frac{\partial u_i^h}{\partial x_i} \, \mathrm{d}\Omega \right)^{1/2} \left(\int_\Omega f_i f_i \, \mathrm{d}\Omega \right)^{1/2}$$

$$\leq C \left(\epsilon \int_\Omega \frac{\partial p_h}{\partial x_i} \frac{\partial p_h}{\partial x_i} \, \mathrm{d}\Omega + \int_\Omega \frac{\partial u_i^h}{\partial x_i} \frac{\partial u_i^h}{\partial x_i} \, \mathrm{d}\Omega \right)^{1/2} \left(\int_\Omega f_i f_i \, \mathrm{d}\Omega \right)^{1/2}.$$

Dividing and squaring yields the stability estimate

$$\int_\Omega \frac{\partial u_i^h}{\partial x_i} \frac{\partial u_i^h}{\partial x_i} \, \mathrm{d}\Omega + \epsilon \int_\Omega \frac{\partial p_h}{\partial x_i} \frac{\partial p_h}{\partial x_i} \, \mathrm{d}\Omega \leq C \int_\Omega f_i f_i \, \mathrm{d}\Omega.$$

Since $C > 0$ does not depend on h, both the velocity and pressure approximations cannot blow up as $h \to 0$, even though the discretization is unstable for $\epsilon = 0$.

$j - \frac{1}{2}$

$j - 1$

A

B

C

$u_{\frac{3}{2},1}$

$p_{1,1}$

0

1

i

j

Inflow

Solid wall

Γ_0

Ω

M^2

M

m

1

h

V

V_h

u

u_h

Γ_0

Γ_1

Ω

Appendix A

Background material

A.1   Iterative solutions to linear systems

Gauss-Seidel method

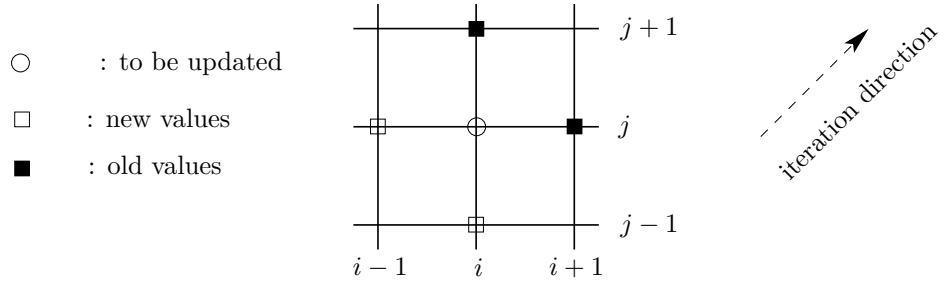The Laplace equation on a Cartesian grids can be discretized as

$$\Phi_{i+1,j} - 2\phi_{i,j} + \Phi_{i-1,j} + \beta^2 \left(\Phi_{i,j+1} - 2\Phi_{i,j} + \Phi_{i,j-1}\right) = 0$$

where $\beta = \Delta x / \Delta y$. We can solve for $\Phi_{i,j}$,

$$\Phi_{i,j} = \frac{\Phi_{i+1,j} + \Phi_{i-1,j} + \beta^2 \left(\Phi_{i,j+1} + \Phi_{i,j-1}\right)}{2\left(1 + \beta^2\right)}$$

Gauss-Seidel uses this expression to update $\Phi_{i,j}$ (using the already updated values calculated in same iteration), i.e.

$$\Phi_{i,j}^{k+1} = \frac{\Phi_{i+1,j}^{k} + \Phi_{i-1,j}^{k+1} + \beta^2 \left(\Phi_{i,j+1}^{k} + \Phi_{i,j-1}^{k+1}\right)}{2\left(1 + \beta^2\right)}$$

$\circ$: to be updated

$\square$: new values

$\blacksquare$: old values

$j + 1$

j

$j - 1$

$i - 1$ i $i + 1$

iteration direction

This iteration method has a slow convergence, it is possible to show

$$\|\mathbf{\Phi} - \mathbf{\Phi}^k\| \leq \rho^m \|\mathbf{\Phi} - \mathbf{\Phi}^0\|$$

where

$$\rho = 1 - \mathcal{O}\left(h^2\right) \qquad h = \Delta x = \Delta y$$

e.i. the error is reduced by $\mathcal{O}\left(h^2\right)$ each iteration. Requiring an error reduction to $\mathcal{O}\left(h^2\right)$ the number of iterations must satisfy

$$\rho^m = \mathcal{O}\left(h^2\right)$$

which implies that

$$m = \mathcal{O}\left(\frac{\ln(h)}{\ln(\rho)}\right) = \mathcal{O}\left(\frac{\ln(h)}{\ln(1 - \mathcal{O}(h^2))}\right) = \mathcal{O}\left(-h^2 \ln(h)\right)$$

To proceed we note that we have $N = n^2$ unknown interior nodes, see Figure A.1. Thus

$$\Omega$$
$$M^2$$
$$M$$
$$h = \Delta x = \Delta y = \frac{1}{n+1} \approx n^{-1} = N^{-1/2}$$

which implies that
$$m$$
$$m+2$$
$$\frac{1}{h}$$
$$m = \mathcal{O}\left(N \ln(N)\right)$$

and that the total work is V
$$V_h$$
$$u$$
$$W = \mathcal{O}\left(N^2 \ln(N)\right)$$

since the work per iteration is $\mathcal{O}(N)$.
$$u_h.$$
$$\Gamma_0$$
$$\Gamma_1$$
$$\Omega$$

: to be updated

: new values

: old values

$i-1$

i

$i+1$

$j-1$

j

$j+1$

iteration direction
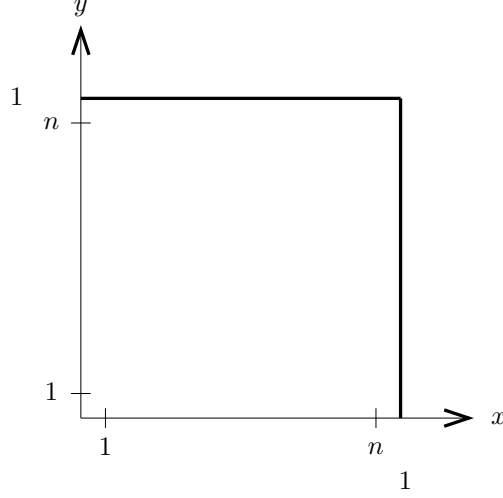
Figure A.1: Domain for the solution of the Laplace equation.

Multigrid method

A general way to accelerate the convergence of e.g. the Gauss-Seidel method is provided by the Multigrid method. We note that the Gauss-Seidel provides good smoothing of the local error, but converges slowly since it takes time for boundary information to propagate into the domain. The idea behind the multigrid method is that a slowly converging low-frequency on a fine grid is a fast converging high-frequency on a coarse grid. Let

$$L\Phi_{i,j} = \frac{\Phi_{i+1,j} - 2\Phi_{i,j} + \Phi_{i-1,j}}{\Delta x^2} + \frac{\Phi_{i,j+1} - 2\Phi_{i,j} + \Phi_{i,j-1}}{\Delta y^2}$$

and define the correction to the solution to the true discrete solution in the following manner

$$
\begin{array}{rcl}
\Phi_{i,j} & = & \Phi_{i,j}^k + \Delta\Phi_{i,j} \\
\Phi_{i,j}^k & - & \text{solution at iteration level } k \\
\Delta\Phi_{i,j} & - & \text{correction to } \Phi_{i,j}^k \text{ to true discrete solution}
\end{array}
$$

This implies that

$$L\Delta\Phi_{i,j} = -L\Phi_{i,j}^k \equiv -R_{i,j}$$

since $L\Phi_{i,j} = 0$. Thus we are left with an equation implying that the Laplace operator on the correction is equal to the negative of the residual. This residual equation is solved on a coarser grid and the correction is interpolated onto finer grid.

In order to define the algorithm we use the following definitions

I transfer operator between grids:

I_1^2 - restriction operator from grid 1 to grid 2

I_2^1 - prolongation operator from grid 2 to grid 1

An example of a restriction operator I_1^2 is to define it as an injection, i.e. we use every other value in each direction. The prolongation operator I_2^1 may be defined by linear interpolation for the intermediate values, i.e. an average of right and left values
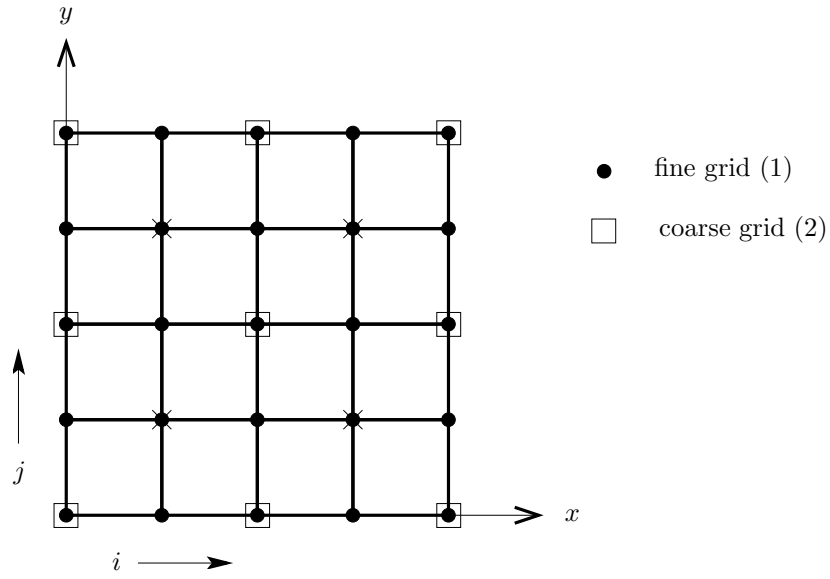
Figure A.2: Two level grid.

and top and bottom values. The values marked with $\times$ in figure A.2 are then taken as an average of the averages of the intermediate values.

Now we can define the following two level scheme:

1. $L\Phi_{i,j}$ iterated k times $\qquad \Rightarrow \qquad L\Phi_{i,j}^k = R_{i,j}$

2. $L\Delta\Phi_{i,j} = -I_1^2 R_{i,j}$ $\qquad$ iterated k times, $\Delta\Phi_{i,j}^0 = 0$ (starting value) (here $\Delta\Phi_{i,j}$ is definied on grid 2)

3. $\Phi_{i,j} = \Phi_{i,j}^k + I_2^1 \Delta\Phi_{i,j}$ $\qquad$ and goto 1.

at the end of 1. convergence is checked.

Wesseling [10] estimated the work of the multigrid method to be $W = \mathcal{O}(N \ln N)$, a substantial improvement over the Gauss-Seidel method.
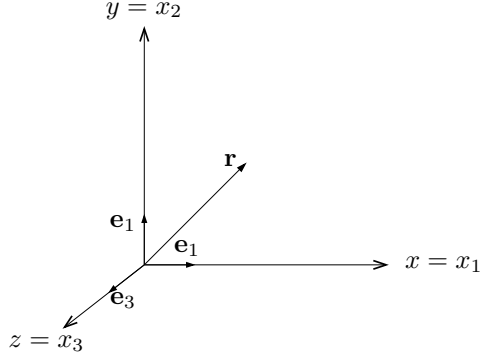
i
$i+1$ i
$j-1$ j
fine grid (1)
y
j coarse grid (2)
$j+1$

iteration direction
x
y
1
n
x
y
i
j
fine grid (1)
coarse grid (2)

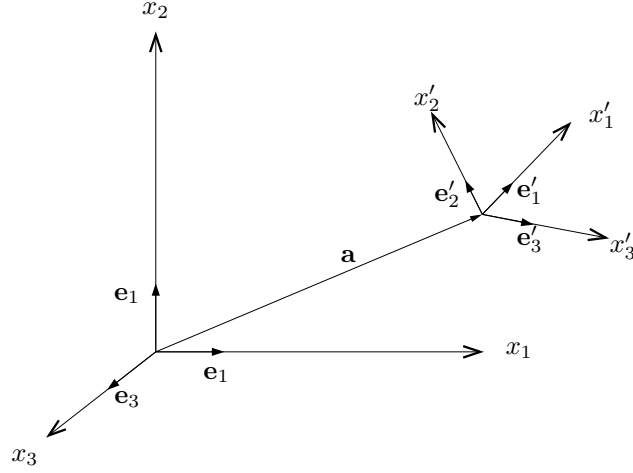Figure A.3: Cartesian coordinate system.

Figure A.4: Two different coordinate systems.

A.2   Cartesian tensor notation

Tensor notation is a compact way to write vector and tensor formulas. Vectors and tensors are objects which are independent of the coordinate system they are represented in.

In the Cartesian coordinate system in Figure A.3 we have the unit vectors

$$|\mathbf{e}_k| = 1 \qquad k = 1, 2, 3$$

The scalar product of two unit vectors is the Kronecker's delta, δ_{kl}

$$\mathbf{e}_k \cdot \mathbf{e}_l = \delta_{kl} = \begin{array}{cc} 1 & k = l \\ 0 & k \neq l \end{array}$$

and the position vector $\mathbf{r}$ is

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \sum_{k=1}^{3} x_k\mathbf{e}_k = x_k\mathbf{e}_k.$$

The last expression make use of *Einstein's summation convention*: when an index occurs twice in the same expression, the expression is implicitly summed over all values for that index.

Example

$$\delta_{kk} = \delta_{ll} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

$$\delta_{ij}a_j = \delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3 = a_i.$$

A.2.1   Orthogonal transformation

Consider two arbitrary oriented coordinate systems as in Figure A.4. Assume identical origin, $\mathbf{a} = 0$. The unit vectors are orthogonal

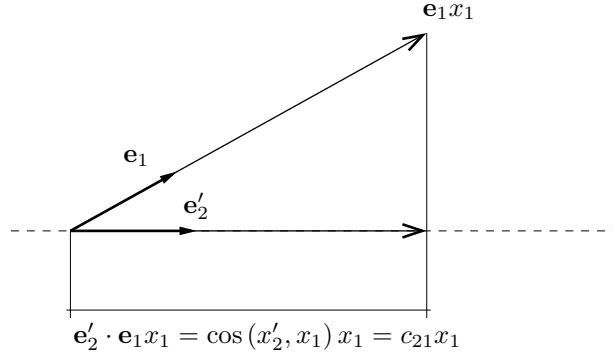$$\mathbf{e}'_k \cdot \mathbf{e}'_l = \delta_{kl}$$

$$\mathbf{e}_2' \cdot \mathbf{e}_1 x_1 = \cos\left(x_2', x_1\right) x_1 = c_{21} x_1$$

Figure A.5: Projection of $\mathbf{e}_1 x_1$ on $\mathbf{e}_2'$.

and a position vector is independent of coordinate system,

$$\mathbf{r} = x_l \mathbf{e}_l = x_l' \mathbf{e}_l'.$$

Component k of $\mathbf{r}$ can be written as

$$\mathbf{e}_k' \cdot \mathbf{r} = x_l \mathbf{e}_k' \cdot \mathbf{e}_l = x_l' \mathbf{e}_l' \cdot \mathbf{e}_k' = x_l' \delta_{kl} = x_k' \qquad \Rightarrow \qquad x_k' = \mathbf{e}_k' \cdot \mathbf{e}_l x_l = c_{k'l} x_l$$

where we have the transformation tensor

$$c_{k'l} = \mathbf{e}_k' \cdot \mathbf{e}_l = \cos(x_k', x_l).$$

Since $\cos(x_2', x_1) \neq \cos(x_1', x_2)$ in general, we have (for $k = 1, l = 2$)

$$c_{l'k} = \mathbf{e}_l' \cdot \mathbf{e}_k = \cos(x_l', x_k) \neq c_{k'l}$$

From

$$\mathbf{e}_k \cdot \mathbf{r} = x_l \mathbf{e}_k \cdot \mathbf{e}_l = x_l' \mathbf{e}_k \cdot \mathbf{e}_l' \qquad \Rightarrow \qquad x_k = \mathbf{e}_l' \cdot \mathbf{e}_k x_l' = c_{l'k} x_l' = c_{lk} x_l'$$

we conclude that **first** index refer to primed system!

Example

$$
\begin{aligned}
x_k' &= c_{kl} x_l = c_{k1} x_1 + c_{k2} x_2 + c_{k3} x_3 = \cos(x_k', x_1) x_1 + \cos(x_k', x_2) x_2 + \cos(x_k', x_3) x_3 \\
&= \mathbf{e}_k' \cdot \mathbf{e}_1 x_1 + \mathbf{e}_k' \cdot \mathbf{e}_2 x_2 + \mathbf{e}_k' \cdot \mathbf{e}_3 x_3
\end{aligned}
$$

which for component 2 reads

$$x_2' = \mathbf{e}_2' \cdot \mathbf{e}_1 x_1 + \mathbf{e}_2' \cdot \mathbf{e}_2 x_2 + \mathbf{e}_2' \cdot \mathbf{e}_3 x_3$$

$\because x_2'$ is projection of $\mathbf{e}_1 x_1, \mathbf{e}_2 x_2, \mathbf{e}_3 x_3$ on $\mathbf{e}_2'$. A visualization of the projection of $\mathbf{e}_1 x_1$ on $\mathbf{e}_2'$ can be seen in Figure A.5.

The relation between c_{kl} and c_{ml} can be studied through

$$\underbrace{x_k'}_{\delta_{km} x_m'} = c_{kl} \underbrace{x_l}_{c_{ml} x_m'} = c_{kl} c_{ml} x_m' \qquad \Rightarrow \qquad (c_{kl} c_{ml} - \delta_{km}) x_m' = 0.$$

In the same manner we have

$$\underbrace{x_k}_{\delta_{km} x_m} = c_{lk} \underbrace{x_l'}_{c_{lm} x_m} = c_{lk} c_{lm} x_m \qquad \Rightarrow \qquad (c_{lk} c_{lm} - \delta_{km}) x_m = 0.$$

A.2.2   Cartesian Tensors

Def. a Cartesian tensor of **rank** R is a set of quantities $T_{klm...}$ which transform as a vector in each of its R indices.

$$T_{klm...}' = c_{kr} c_{ls} c_{mp} \ldots T_{rsp...}$$

Example

For scalars the rank is $R = 0$

$$\mathbf{a} \cdot \mathbf{b} = a'_k \, b'_k = \underbrace{c_{kl}c_{km}}_{\delta_{lm}} a_l b_m = a_l b_l,$$

for vectors $R = 1$

$$a'_k = c_{kl}a_l$$

and for a tensor of second order we have $R = 2$

$$\delta'_{kl} = c_{km}c_{ln}\delta_{mn} = c_{km}c_{lm} = \delta_{kl}.$$

δ_{kl} is **isotropic**, i.e. identical in all coordinate systems.
An outer product generates a tensor

$$T'_{kl}S'_{rs} = c_{km}c_{ln}c_{rp}c_{sq}T_{mn}S_{pq}$$

and an inner product generates a tensor

$$T'_{kl}S'_{ls} = c_{km}\underbrace{c_{ln}c_{lp}}_{\delta_{np}}c_{sq}T_{mn}S_{pq} = c_{km}c_{sq}T_{mn}S_{nq}$$

A.2.3 Permutation tensor

$$\varepsilon_{klm} = \left\{ \begin{array}{c} 1 \\ -1 \\ 0 \end{array} \right\} \text{ if } klm \text{ is } \left\{ \begin{array}{c} \text{even} \\ \text{odd} \\ \text{no} \end{array} \right\} \text{ permutation of } (1,2,3)$$

As examples of permutations of 1,2,3 we have

(1,2,3),	(3,1,2),	(2,3,1)	even permutation	$\varepsilon_{123} = 1$
(2,1,3),	(3,2,1),	(1,3,2)	odd permutation	$\varepsilon_{213} = -1$
at least 2 subscripts equal			no permutation	$\varepsilon_{221} = 0$

A.2.4 Inner products, crossproducts and determinants

The inner product of the vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = \delta_{ij}a_i b_j = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The cross product of $\mathbf{a}$ and $\mathbf{b}$ can be written as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_2 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\mathbf{e}_1 + (a_3 b_1 - a_1 b_3)\mathbf{e}_2 + (a_1 b_2 - a_2 b_1)\mathbf{e}_3 = \varepsilon_{ijk}\mathbf{e}_i a_j b_k$$

and the determinant of the tensor a_{ij} is

$$|\{a_{ij}\}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \varepsilon_{ijk}a_{1i}a_{2j}a_{3k}$$

$$= \frac{1}{6}\varepsilon_{ijk}(a_{i1}a_{j2}a_{k3} + a_{i3}a_{j1}a_{k2} + a_{i2}a_{j3}a_{k1} - a_{i2}a_{j1}a_{k3} - a_{i3}a_{j2}a_{k1} - a_{i1}a_{j3}a_{k2} = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}a_{ip}a_{jq}a_{kr}.$$

We have the permutation relation

$$\varepsilon_{klm}\varepsilon_{ksp} = \delta_{ls}\delta_{mp} - \delta_{lp}\delta_{ms}. \tag{A.1}$$

If we set $s = l$ in (A.1) we get

$$\varepsilon_{klm}\varepsilon_{klp} = (3-1)\delta_{mp} = 2\delta_{mp}$$

and if we also use $p = m$ we end up with

$$\varepsilon_{klm}\varepsilon_{klm} = 6$$

The permutation tensor can also be used to derive the following vector relation

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= \varepsilon_{ijk}a_j b_k \varepsilon_{ilm}c_l d_m = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{lk})a_j b_k c_l d_m \\ &= a_j c_j b_k d_k - a_j d_j b_k c_k = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \end{aligned}$$

A.2.5 Second rank tensors

All second rank tensors can be decomposed into symmetric and antisymmetric parts, $T_{kl} = T_{(kl)} + T_{[kl]}$, where the antisymmetric part is

$$T_{[kl]} = -T_{[lk]} = \frac{1}{2}\left(T_{kl} - T_{lk}\right)$$

and the symmetric part

$$T_{(kl)} = T_{(lk)} = \frac{1}{2}\left(T_{kl} + T_{lk}\right).$$

The symmetric part can be decomposed into isotropic and traceless part, $T_{(kl)} = \overline{\overline{T}}_{kl} + \overline{T}_{kl}$, the isotropic part being

$$\overline{\overline{T}}_{kl} = \frac{1}{3}T_{mm}\delta_{kl}$$

and the traceless part is

$$\overline{T}_{kl} = T_{(kl)} - \frac{1}{3}T_{mm}\delta_{kl}.$$

If we project the isotropic, traceless and entisymmetric parts we get

$$S_{(kl)}T_{[kl]} = \frac{1}{2}\left(S_{(kl)}T_{[kl]} + S_{(lk)}T_{[lk]}\right) = \frac{1}{2}\left(S_{(kl)}T_{[kl]} - S_{(lk)}T_{[kl]}\right) = 0$$

$$\overline{\overline{S}}_{kl}\overline{T}_{kl} = \frac{1}{3}S_{mm}\delta_{kl}\left(T_{(kl)} - \frac{1}{3}T_{mm}\delta_{kl}\right) = \frac{1}{3}S_{mm}\left(T_{kk} - \frac{1}{3}T_{kk}\delta_{ll}\right) = 0$$

$$\Rightarrow$$

$$
\begin{aligned}
S_{kl}T_{[kl]} &= S_{[kl]}T_{[kl]} \\
S_{kl}T_{(kl)} &= S_{(kl)}T_{(kl)} \\
S_{kl}\overline{\overline{T}}_{kl} &= \overline{\overline{S}}_{kl}\overline{\overline{T}}_{kl} \\
S_{kl}\overline{T}_{kl} &= \overline{S}_{kl}\overline{T}_{kl},
\end{aligned}
$$

and each part is invariant under transformation

$$T'_{[kl]} = c_{kr}c_{ls}T_{[rs]} = -c_{kr}c_{ls}T_{[sr]} = -c_{lr}c_{ks}T_{[rs]} = -T'_{[lk]}$$

$$T'_{rr} = c_{rp}c_{rq}T_{pq} = \delta_{pq}T_{pq} = T_{pp}$$

$$\overline{T}'_{kl} = c_{kp}c_{lq}\overline{T}_{pq} = c_{kp}c_{lq}\left(T_{(pq)} - \frac{1}{3}T_{rr}\delta_{pq}\right) = T'_{(pq)} - \frac{1}{3}T_{rr}\delta_{kl}.$$

Since we only need three components to completely describe an antisymmetric tensor of rank $R = 2$ it can be represented by its dual vector

$$d_i = \varepsilon_{ijk}T_{jk} = \underline{\varepsilon_{ijk}T_{(jk)}} + \varepsilon_{ijk}T_{[jk]}$$

The first term on the right hand side is zero since ε_{ijk} is antisymmetric in any two indecies. Multiply both sides by ε_{ilm}

$$\varepsilon_{ilm}d_i = \varepsilon_{ilm}\varepsilon_{ijk}T_{jk} = \left(\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}\right)T_{jk} = T_{lm} - T_{ml} = 2T_{[lm]} \quad \Rightarrow \quad T_{[lm]} = \frac{1}{2}\varepsilon_{ilm}d_i$$

Note: There are 3 independent components in $T_{[lm]}$ and d_i.

In order to sum up, an arbitrary tensor of rank 2 can be divided into the following parts

$$T_{kl} = T_{(kl)} + T_{[kl]} = \overline{\overline{T}}_{kl} + \overline{T}_{kl} + T_{[kl]} = \underbrace{\frac{1}{3}T_{rr}\delta_{kl}}_{\text{isotropic}} + \underbrace{T_{(kl)} - \frac{1}{3}T_{rr}\delta_{kl}}_{\text{traceless}} + \underbrace{\frac{1}{2}\varepsilon_{ikl}d_i}_{\text{antisymmetric}}$$

Properties of the above parts are invariant under transformation. In the Navier-Stokes equations we have the velocity gradient $\frac{\partial u_i}{\partial x_j}$. The antisymmetric part of this tensor describes rotation, the isotropic part the volume change and the traceless part describes the deformation of a fluid element.

Example

We have the second rank tensor

$$\{T_{kl}\} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

which can be divided into a symmetric part

$$\{T_{(kl)}\} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}$$

and an antisymmetric part

$$\{T_{[kl]}\} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}.$$

The symmetric part can then be divided into a isotropic part

$$\{\overline{\overline{T}}_{kl}\} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

and a traceless part

$$\{\overline{T}_{kl}\} = \begin{bmatrix} -4 & 3 & 5 \\ 3 & 0 & 7 \\ 5 & 7 & 4 \end{bmatrix}.$$

The dual vector of the antisymmetric tensor is

$$\{d_i\} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}.$$

A.2.6 Tensor fields

We have the tensor field $T_{ij}(x_1, x_2, x_3) \equiv T_{ij}(x_k)$ with the following properties

- partial derivative transforms like a tensor

$$\frac{\partial}{\partial x'_p} = \frac{\partial x_k}{\partial x'_p} \frac{\partial}{\partial x_k} = c_{pk} \frac{\partial}{\partial x_k} \qquad \text{where} \qquad (x_k = c_{pk} x'_p)$$

- gradient adds one to tensor rank

$$\frac{\partial}{\partial x_k} T_{ij}$$

- divergence decreases tensor rank by one

$$\frac{\partial}{\partial x_k} T_{kl} = \frac{\partial T_{1l}}{\partial x_1} + \frac{\partial T_{2l}}{\partial x_2} + \frac{\partial T_{3l}}{\partial x_3}$$

- curl

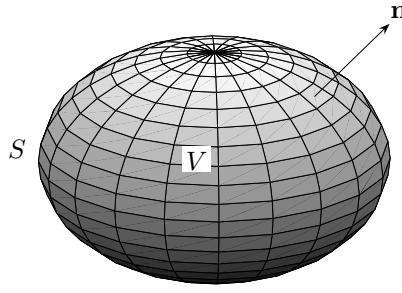$$(\nabla \times \mathbf{u})_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

which can be compared with the expression for the determinant.

Example

Show that $\nabla \cdot (\nabla \times \mathbf{\Psi}) = 0$.

$$\nabla \cdot (\nabla \times \mathbf{\Psi}) = \frac{\partial}{\partial x_k} \varepsilon_{klm} \frac{\partial \Psi_m}{\partial x_l} = \varepsilon_{klm} \frac{\partial^2}{\partial x_k \partial x_l} \Psi_m = 0$$

since ε_{klm} is antisymmetric in kl and $\frac{\partial^2}{\partial x_k \partial x_l}$ is symmetric in kl.

$$\mathrm{d}\mathbf{S} = \mathbf{n}\,\mathrm{d}S$$

Figure A.6: Volume V bounded by the close surface S.

Example

Show that $\nabla f(r) = \frac{1}{r} f'(\mathbf{r})$ where $\{\mathbf{r}\}_k = x_k$ and $r = |\mathbf{r}| = \sqrt{x_k x_k}$.

$$
\begin{aligned}
\{\nabla f(r)\}_k \;=\; & \frac{\partial}{\partial x_k} f(r) = \frac{\mathrm{d}f}{\mathrm{d}r}\frac{\partial r}{\partial x_k} = f' \frac{\partial (x_p x_p)^{1/2}}{\partial x_k} = \\
& f' \frac{1}{2} (x_p x_p)^{-\frac{1}{2}} \frac{\partial}{\partial x_k}(x_p x_p) = f' \frac{1}{2r}(\delta_{pk} x_p + x_p \delta_{pk}) = \frac{1}{r} f' x_k = \left\{ \frac{1}{r} f' \mathbf{r} \right\}_k
\end{aligned}
$$

Example

Show that $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \Delta \mathbf{F}$.

$$
\left(\nabla \times (\nabla \times \mathbf{F}) \right)_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} \frac{\partial}{\partial x_l} F_m = \varepsilon_{ijk}\varepsilon_{klm} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} F_m = \varepsilon_{kij}\varepsilon_{klm} \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l} F_m =
$$

$$
(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l} F_m = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} F_j - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} F_i = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} F_j - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} F_i = \left(\nabla(\nabla \cdot \mathbf{F}) - \Delta \mathbf{F} \right)_i
$$

Example

Show that if $\bar{u} = \bar{\Omega} \times \bar{r}$ then $\mathbf{\Omega} = \dfrac{\nabla \times \mathbf{u}}{2}$. This means, show that $\varepsilon_{ijk} \dfrac{\partial}{\partial x_j} u_k = 2\Omega_i$.

$$
\varepsilon_{ijk} \frac{\partial}{\partial x_j} u_k = \{ u_k = \varepsilon_{klm}\Omega_l r_m \} = \varepsilon_{ijk} \frac{\partial}{\partial x_j}(\varepsilon_{klm}\Omega_l r_m) = \varepsilon_{ijk}\varepsilon_{klm} \frac{\partial}{\partial x_j}\Omega_l r_m =
$$

$$
\varepsilon_{kij}\varepsilon_{klm} \frac{\partial}{\partial x_j}\Omega_l r_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial}{\partial x_j}\Omega_l r_m =
$$

$$
\frac{\partial}{\partial x_m}\Omega_i r_m - \frac{\partial}{\partial x_l}\Omega_l r_i = \{\Omega \text{ independent of } x\} = \Omega_i \frac{\partial r_m}{\partial x_m} - \Omega_l \frac{\partial r_i}{\partial x_l} = \{ \frac{\partial r_m}{\partial x_m} = 1+1+1 = 3 \quad \frac{\partial r_i}{\partial x_l} = \delta_{il} \} =
$$

$$
3\Omega_i - \Omega_l \delta_{il} = 2\Omega_i
$$

A.2.7 Gauss & Stokes integral theorems

Let $\mathbf{n}$ be the outward pointing normal to the closed surface S bounding the simply-connected volume V, see Figure A.6. Gauss integral theorem then states

$$
\oint_S \mathbf{a} \cdot \mathbf{n}\,\mathrm{d}S = \int_V \nabla \cdot \mathbf{a}\,\mathrm{d}V
$$

or

$$
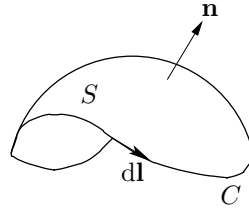\oint_S a_k n_k\,\mathrm{d}S = \int_V \frac{\partial a_k}{\partial x_k}\,\mathrm{d}V
$$

Figure A.7: The open surface S enclosed by the contour C.

for a differentiable vector field $\mathbf{a}$. This can be generalized to

$$\oint_S T_{klm} n_p \, dS = \int_V \frac{\partial}{\partial x_p} T_{klm} \, dV \, .$$

Example

$$\oint_S f \mathbf{n} \, dS = \int_V \nabla f \, dV \, .$$

$$\oint_S \mathbf{a} \times \mathbf{n} \, dS = \int_V \nabla \times \mathbf{a} \, dV \qquad \text{if} \qquad T_{klm} n_m = \varepsilon_{klm} a_l n_m .$$

Stokes theorem states that for a differentiable vector field $\mathbf{a}$ if an open simply-connected surface S is enclosed by a closed curve C, of which $d\mathbf{l}$ is the line element as in Figure A.7, then

$$\oint_C \mathbf{a} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{a}) \cdot \mathbf{n} \, dS$$

or

$$\oint_C a_k \, dx_k = \int_S \varepsilon_{klm} n_k \frac{\partial a_m}{\partial x_k} \, dS$$

this can be generalized to

$$\oint_C T_{klm} \, dx_p = \int_S \varepsilon_{ijp} n_i \frac{\partial}{\partial x_j} T_{klm} \, dS$$

Example
Let $T_{klm} = f$. In the generalized Stokes theorem we then get $\int_S \varepsilon_{ijp} n_i \frac{\partial f}{\partial x_j} \, dS$ in right hand side

$$\oint_C f \, d\mathbf{l} = \int_S \mathbf{n} \times \nabla f \, dS$$

A.2.8 Archimedes principle

Gauss theorem can be used to calculate the buoyancy on an object immersed in a fluid (Archimedes circa 281–212 BC). Consider a water cylinder with volume dV and mass dm, Figure A.8.

The force from the water cylinder is

$$dm \, g = dV \rho g = \underbrace{x_3 \rho g}_{\text{pressure}} \, dA$$

where g is the gravity constant and ρ the water density. The pressure at a distant x_3 from the surface in a fluid is then $p = \rho g x_3 + p_0$ where p_0 is the atmospheric pressure. Let $\mathbf{\Pi}$ be the buoyancy force resulting from the pressure on the object, m the mass of equivalent volume of water and M the mass of the object in Figure A.8. The force on a small element dS of the body resulting from the pressure is
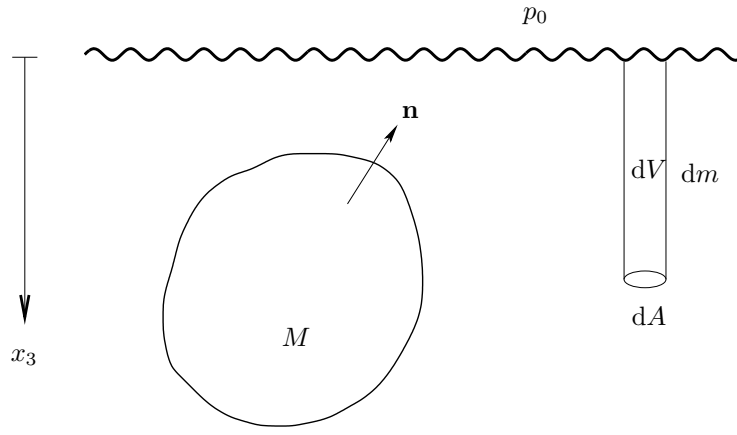
$$d\Pi_k = -g\rho x_3 n_k \, dS$$

Figure A.8: An object with mass M immersed in a fluid illustrating Archimedes principle.
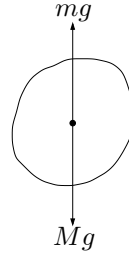
Figure A.9: The forces acting on an object immersed in a fluid. m is the mass of equivalent volume of water and M the mass of the body.

where the minus sign stems from the outward pointing normal $\mathbf{n}$. The buoyancy force can then be calculated through

$$\Pi_k = -\oint_S \rho g x_3 n_k \, dS = -\rho g \int_V \frac{\partial}{\partial x_k} x_3 \, dV = -\rho g \delta_{k3} V.$$

This means that the object loses some weight as the water is displaced. The buoyancy force is

$$\Pi_3 = -\rho g V = -g\, m$$

see Figure A.9.

The moment of a force $\mathbf{F}$ at some arbitrary point with position vector $\mathbf{r}$ from the point of application of the force is $\mathbf{M} = \mathbf{r} \times \mathbf{F}$. The moment around the center of gravity G of the homogenous body produced by the pressure force $d\boldsymbol{\Pi}$ acting on an element dS of the body is $d\mathbf{M}_G = (\mathbf{r} - \mathbf{r}_G) \times d\boldsymbol{\Pi}$ as in Figure A.10. This can then be used to calculate the total moment around G form pressure forces on the body

$$M_{G_k} = -\oint_S \rho g \varepsilon_{klm} (x_l - x_{G_l}) x_3 n_m \, dS = -\rho g \varepsilon_{klm} \int_V \left[\frac{\partial}{\partial x_m}(x_l x_3) - x_{G_l} \frac{\partial x_3}{\partial x_m} \right] dV =$$

$$-\rho g \varepsilon_{klm} \int_V \left[\delta_{lm} x_3 + \delta_{m3} x_l - \delta_{m3} x_{G_l} \right] dV = -\rho g \varepsilon_{kl3} \left[\int_V x_l \, dV - x_{G_l} V \right] = 0$$

If a part of the fluid is frozen (without changing in density) this body is in <u>equilibrium</u> in the fluid: $\mathbf{F} = \mathbf{M} = 0$.

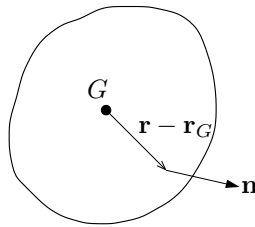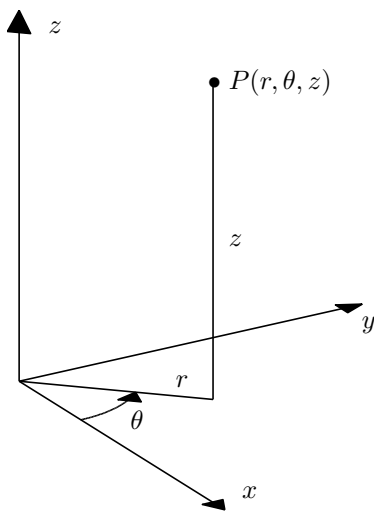Figure A.10: Moment around G from pressure forces acting on the object.

$t = 1$
i z
$i + 1$
$j - 1$ $P(r, \theta, z)$
j
$j + 1$
iteration direction
x
y
1 z
n
x y
y r
i θ
j
fine grid (1)
coarse grid (2) x

Figure A.11: Cylindrical polar coordinates

A.3   Curvilinear coordinates

Cylindrical Polar Coordinates

The cylindrical polar coordinates are (r, θ, z), where θ is the azimuthal angle, see figure A.11. The velocity can be written as

$$\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z, \tag{A.2}$$

where the unit vectors are related to Cartesian coordinates as

$$\begin{aligned}
\mathbf{e}_r &= \mathbf{e}_x \cos\theta + \mathbf{e}_y \sin\theta, \\
\mathbf{e}_\theta &= -\mathbf{e}_x \sin\theta + \mathbf{e}_y \cos\theta, \\
\mathbf{e}_z &= \mathbf{e}_z.
\end{aligned} \tag{A.3}$$

Non-zero derivatives of unit vectors

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r. \tag{A.4}$$

Gradient of a scalar p

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z. \tag{A.5}$$

Laplacian of a scalar p

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2}. \tag{A.6}$$

Divergence of a vector $\mathbf{u}$

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}. \tag{A.7}$$

Advective derivative of a scalar p

$$(\mathbf{u}\cdot\nabla)p = u_r\frac{\partial p}{\partial r} + \frac{u_\theta}{r}\frac{\partial p}{\partial \theta} + u_z\frac{\partial p}{\partial z}. \tag{A.8}$$

Curl of a vector $\mathbf{u}$

$$\nabla \times \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\mathbf{e}_z. \tag{A.9}$$

Incompressible Navier-Stokes equations with no body force

$$\frac{\partial u_r}{\partial t} + (\mathbf{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right), \tag{A.10}$$

$$\frac{\partial u_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right), \tag{A.11}$$

$$\frac{\partial u_z}{\partial t} + (\mathbf{u}\cdot\nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z. \tag{A.12}$$
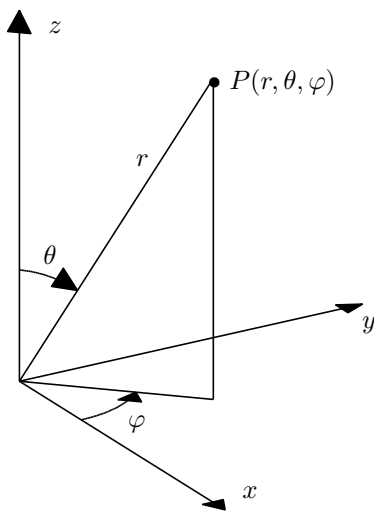
Figure A.12: Spherical polar coordinates

Spherical Polar Coordinates

The spherical polar coordinates are (r, θ, φ), where φ is the azimuthal angle, see figure A.12. The velocity can be written as

$$\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_\varphi \mathbf{e}_\varphi, \tag{A.13}$$

where the unit vectors are related to Cartesian coordinates as

$$\begin{aligned}
\mathbf{e}_r &= \mathbf{e}_x \sin\theta\cos\varphi + \mathbf{e}_y \sin\theta\sin\varphi + \mathbf{e}_z \cos\theta, \\
\mathbf{e}_\theta &= \mathbf{e}_x \cos\theta\cos\varphi + \mathbf{e}_y \cos\theta\sin\varphi - \mathbf{e}_z \sin\theta, \\
\mathbf{e}_\varphi &= -\mathbf{e}_x \sin\varphi + \mathbf{e}_y \cos\varphi.
\end{aligned} \tag{A.14}$$

Non-zero derivatives of unit vectors

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_r}{\partial \varphi} = \mathbf{e}_\varphi \sin\theta, \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r, \quad \frac{\partial \mathbf{e}_\theta}{\partial \varphi} = \mathbf{e}_\varphi \cos\theta, \tag{A.15}$$

$$\frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = -\mathbf{e}_r \sin\theta - \mathbf{e}_\theta \cos\theta. \tag{A.16}$$

Gradient of a scalar p

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \varphi}\mathbf{e}_\varphi. \tag{A.17}$$

Laplacian of a scalar p

$$\nabla^2 p = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial p}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p}{\partial \varphi^2}. \tag{A.18}$$

Divergence of a vector $\mathbf{u}$

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi}. \tag{A.19}$$

Advective derivative of a scalar p

$$(\mathbf{u}\cdot\nabla)p = u_r\frac{\partial p}{\partial r} + \frac{u_\theta}{r}\frac{\partial p}{\partial \theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial p}{\partial \varphi}. \tag{A.20}$$

Curl of a vector $\mathbf{u}$

$$\nabla \times \mathbf{u} = \frac{1}{r\sin\theta}\left(\frac{\partial(u_\varphi \sin\theta)}{\partial \theta} - \frac{\partial u_\theta}{\partial \varphi}\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_r}{\partial \varphi} - \frac{\partial(r u_\varphi)}{\partial r}\right)\mathbf{e}_\theta \tag{A.21}$$

$$+\frac{1}{r}\left(\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\mathbf{e}_\varphi.$$

Incompressible Navier-Stokes equations with no body force

$$\frac{\partial u_r}{\partial t} + (\mathbf{u} \cdot \nabla)u_r - \frac{u_\theta^2 + u_\varphi^2}{r} \tag{A.22}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2 \sin\theta}\frac{\partial(u_\theta \sin\theta)}{\partial\theta} - \frac{2}{r^2 \sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right),$$

$$\frac{\partial u_\theta}{\partial t} + (\mathbf{u} \cdot \nabla)u_\theta + \frac{u_r\, u_\theta}{r} - \frac{u_\varphi^2 \cot\theta}{r} \tag{A.23}$$

$$= -\frac{1}{\rho\, r}\frac{\partial p}{\partial\theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2 \sin^2\theta} - \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right),$$

$$\frac{\partial u_\varphi}{\partial t} + (\mathbf{u} \cdot \nabla)u_\varphi + \frac{u_r\, u_\varphi}{r} + \frac{u_\theta\, u_\varphi \cot\theta}{r} \tag{A.24}$$

$$= -\frac{1}{\rho r \sin\theta}\frac{\partial p}{\partial\varphi} + \nu\left(\nabla^2 u_\varphi + \frac{2}{r^2 \sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2 \sin^2\theta}\right).$$

Appendix B

Recitations 5C1214

B.1 Tensors and invariants

Tensor Notation

Scalar

$$p = p$$

Vector

$$(\bar{u})_i = u_i \quad \text{If } \bar{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \text{ then } u_2 = v$$

Matrix

$$(\mathbf{A})_{ij} = A_{ij} \quad \text{If } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then } A_{23} = a_{23}$$

Kronecker delta

$$\delta_{ij} = (\mathbf{I})_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Einstein summation convention

$$u_i u_i = \sum_{i=1}^{3} u_i u_i$$

Kinetic energy per unit volume

$$\tfrac{1}{2}\rho|\bar{u}|^2 = \tfrac{1}{2}\rho(u^2 + v^2 + w^2) = \tfrac{1}{2}\rho u_i u_i$$

Matrix operations

$$\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i = \delta_{ij} a_i b_j = a_j b_j$$

$$(\mathbf{A}\bar{b})_i = A_{ij} b_j$$

$$(\mathbf{A}\mathbf{B})_{ij} = A_{ik} B_{kj}$$

$$tr(\mathbf{A}) = A_{ii}$$

$$(\mathbf{A})_{ij} = A_{ij} \iff (\mathbf{A}^T)_{ij} = A_{ji}$$

Permutation symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ in cyclic order. } ijk = 123, 231 \text{ or } 312 \\ 0 & \text{if any two indices are equal} \\ -1 & \text{if } ijk \text{ in anticyclic order. } ijk = 321, 213 \text{ or } 132 \end{cases}$$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \bar{e}_1(a_2 b_3 - a_3 b_2) - \bar{e}_2(a_1 b_3 - a_3 b_1) + \bar{e}_3(a_1 b_2 - a_2 b_1) = \varepsilon_{ijk}\bar{e}_i a_j b_k$$

$$(\bar{a} \times \bar{b})_i = \varepsilon_{ijk}a_j b_k$$

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

$$\varepsilon_{ijk}\varepsilon_{ijm} = \{l = j\} = 3\delta_{km} - \delta_{jm}\delta_{kj} = 3\delta_{km} - \delta_{mk} = 2\delta_{km}$$

$$\varepsilon_{ijk}\varepsilon_{ijk} = \{m = k\} = 2 \cdot 3 = 6$$

Rewrite without the cross product:

$$(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = (\bar{a} \times \bar{b})_i (\bar{c} \times \bar{d})_i = \varepsilon_{ijk}a_j \, b_k \varepsilon_{ilm}c_l d_m = \varepsilon_{ijk}\varepsilon_{ilm} \, a_j \, b_k \, c_l \, d_m =$$
$$(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \, a_j \, b_k \, c_l \, d_m = a_j c_j b_k d_k - a_j d_j b_k c_k = (\bar{a} \cdot \bar{c})(\bar{b} \cdot \bar{d}) - (\bar{a} \cdot \bar{d})(\bar{b} \cdot \bar{c})$$

Invariant parts of Tensors

$$T_{ij} = T_{ij}^S + T_{ij}^A \quad \text{Symmetric and Anti-symmetric parts.}$$

$$T_{ij}^S = \tfrac{1}{2}\left(T_{ij} + T_{ji}\right) = T_{ji}^S \quad \text{Symmetric}$$

$$T_{ij}^A = \tfrac{1}{2}\left(T_{ij} - T_{ji}\right) = -T_{ji}^A \quad \text{Anti-symmetric}$$

The symmetric part of the tensor can be divided into two parts:

$$T_{ij}^S = \bar{T}_{ij} + \bar{\bar{T}}_{ij}$$
$$\bar{T}_{ij} = T_{ij}^S - \tfrac{1}{3}T_{kk}\delta_{ij} \quad \text{trace less}$$
$$\bar{\bar{T}}_{ij} = \tfrac{1}{3}T_{kk}\delta_{ij} \quad \text{isotropic}$$

This gives:

$$T_{ij} = T_{ij}^S + T_{ij}^A = \bar{T}_{ij} + \bar{\bar{T}}_{ij} + T_{ij}^A$$

In the NS equations we have the tensor $\dfrac{\partial u_i}{\partial x_j}$. The anti-symmetric part describes rotation, the isotropic part describes the volume change and the trace-less part describes the deformation of a fluid element.

Operators

$$(\nabla p)_i = \frac{\partial}{\partial x_i}p$$

$$\Delta p = \frac{\partial^2}{\partial x_i \partial x_i}p$$

$$\nabla \cdot \bar{u} = \frac{\partial}{\partial x_i}u_i$$

$$(\nabla \times \bar{u})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}u_k$$

Gauss theorem (general)

Gauss theorem:

$$\oint_S \bar{F} \cdot \bar{n} \, dS = \int_V \nabla \cdot \bar{F} \, dV$$

or,

$$\oint_S F_i \, n_i \, dS = \int_V \frac{\partial F_i}{\partial x_i} \, dV$$

In general we can write,

$$\oint_S T_{ijk} \, n_l \, dS = \int_V \frac{\partial}{\partial x_l} T_{ijk} \, dV$$

Example: Put $T_{ijk} \, n_l = T_{ij} \, u_l \, n_l$

$$\oint_S T_{ij} \, u_l \, n_l \, dS = \int_V \frac{\partial}{\partial x_l} (u_l \, T_{ij}) \, dV$$

or,

$$\oint_S \mathbf{T}(\bar{u} \cdot \bar{n}) \, dS = \int_V \left((\nabla \cdot \bar{u}) \mathbf{T} + (\bar{u} \cdot \nabla) \mathbf{T} \right) \, dV$$

Identities

Derive the identity

$$\nabla \times (\bar{F} \times \bar{G}) = (\bar{G} \cdot \nabla)\bar{F} + (\nabla \cdot \bar{G})\bar{F} - (\nabla \cdot \bar{F})\bar{G} - (\bar{F} \cdot \nabla)\bar{G}$$

$$\left(\nabla \times (\bar{F} \times \bar{G}) \right)_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} F_l \, G_m = \varepsilon_{ijk} \varepsilon_{klm} \frac{\partial}{\partial x_j} F_l \, G_m = \varepsilon_{kij} \varepsilon_{klm} \frac{\partial}{\partial x_j} F_l \, G_m =$$

$$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial}{\partial x_j} F_l \, G_m = \frac{\partial}{\partial x_j} F_i \, G_j - \frac{\partial}{\partial x_j} F_j \, G_i = \frac{\partial F_i}{\partial x_j} G_j + \frac{\partial G_j}{\partial x_j} F_i - \frac{\partial F_j}{\partial x_j} G_i + \frac{\partial G_i}{\partial x_j} F_j =$$

$$G_j \frac{\partial F_i}{\partial x_j} + \frac{\partial G_j}{\partial x_j} F_i - \frac{\partial F_j}{\partial x_j} G_i - F_j \frac{\partial G_i}{\partial x_j} = [(\bar{G} \cdot \nabla)\bar{F} + (\nabla \cdot \bar{G})\bar{F} - (\nabla \cdot \bar{F})\bar{G} - (\bar{F} \cdot \nabla)\bar{G}]_i$$

Show that:

$$\nabla \cdot (\nabla \times \bar{F}) = 0$$

$$\nabla \cdot (\nabla \times \bar{F}) = \frac{\partial}{\partial x_i} \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k = \varepsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} F_k$$

Remember that $\varepsilon_{ijk} = -\varepsilon_{jik}$ and that $\dfrac{\partial}{\partial x_i} \dfrac{\partial}{\partial x_j} F_k = \dfrac{\partial}{\partial x_j} \dfrac{\partial}{\partial x_i} F_k$ and thus all terms will cancel.

$$\varepsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} F_k = 0$$

Example

Show that if $\bar{u} = \bar{\Omega} \times \bar{r}$ then $\bar{\Omega} = \dfrac{\nabla \times \bar{u}}{2}$. This means, show that $\varepsilon_{ijk} \dfrac{\partial}{\partial x_j} u_k = 2\Omega_i$.

$$\varepsilon_{ijk} \frac{\partial}{\partial x_j} u_k = \{u_k = \varepsilon_{klm} \Omega_l r_m\} = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\varepsilon_{klm} \Omega_l r_m) = \varepsilon_{ijk} \varepsilon_{klm} \frac{\partial}{\partial x_j} \Omega_l r_m =$$

$$\varepsilon_{kij} \varepsilon_{klm} \frac{\partial}{\partial x_j} \Omega_l r_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial}{\partial x_j} \Omega_l r_m =$$

$$\frac{\partial}{\partial x_m} \Omega_i r_m - \frac{\partial}{\partial x_l} \Omega_l r_i = \{\Omega \text{ independent of } x\} = \Omega_i \frac{\partial r_m}{\partial x_m} - \Omega_l \frac{\partial r_i}{\partial x_l} = \{\frac{\partial r_m}{\partial x_m} = 1 + 1 + 1 = 3 \quad \frac{\partial r_i}{\partial x_l} = \delta_{il}\} =$$

$$3\Omega_i - \Omega_l \delta_{il} = 2\Omega_i$$

Derive the identity

$$\nabla \cdot (\bar{F} \times \bar{G}) = (\nabla \times \bar{F}) \cdot \bar{G} - \bar{F} \cdot (\nabla \times \bar{G})$$

$$\nabla \cdot (\bar{F} \times \bar{G}) = \frac{\partial}{\partial x_i} \varepsilon_{ijk} F_j \, G_k = \varepsilon_{ijk} \frac{\partial}{\partial x_i} F_j \, G_k = \varepsilon_{ijk} \frac{\partial F_j}{\partial x_i} G_k + \varepsilon_{ijk} \frac{\partial G_k}{\partial x_i} F_j = \varepsilon_{kij} \frac{\partial F_j}{\partial x_i} G_k + F_j \, \varepsilon_{ijk} \frac{\partial G_k}{\partial x_i} =$$

$$\varepsilon_{kij} \frac{\partial F_j}{\partial x_i} G_k - F_j \, \varepsilon_{jik} \frac{\partial G_k}{\partial x_i} = (\nabla \times \bar{F}) \cdot \bar{G} - \bar{F} \cdot (\nabla \times \bar{G})$$

Show that:

$$\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \Delta \bar{F}$$

$$\left(\nabla \times (\nabla \times \bar{F})\right)_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} \frac{\partial}{\partial x_l} F_m = \varepsilon_{ijk}\varepsilon_{klm} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} F_m = \varepsilon_{kij}\varepsilon_{klm} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} F_m =$$

$$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} F_m = \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} F_j - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} F_i = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} F_j - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} F_i = \left(\nabla(\nabla \cdot \bar{F}) - \Delta \bar{F}\right)_i$$

B.2 Euler and Lagrange coordinates

Particle path (Lagrange's representation)

Following a fluid particle as time proceeds the path is described by

$$\frac{\partial \bar{r}}{\partial \hat{t}} = \bar{u}, \quad \bar{r}(\hat{t} = 0) = \bar{x}_0$$

Streamline (Euler's representation)

Instantaneously a fluid particles displacement $d\bar{x}$ is tangential to its velocity $\bar{u}$:

$$\frac{d\bar{x}}{|d\bar{x}|} = \frac{\bar{u}}{|\bar{u}|}$$

or if we put $ds = |d\bar{x}|/|\bar{u}|$:

$$\frac{d\bar{x}}{ds} = \bar{u}$$

Exercise 1

Show that the streamlines for the unsteady flow

$$u = u_0, \quad v = kt, \quad w = 0, \quad u_0, k > 0$$

is straight lines, while any fluid particle follows a parabolic path as time proceeds.
Streamline (fix t):

$$\frac{dx}{ds} = u = u_0 \quad \Rightarrow \quad x = u_0 s + a$$

$$\frac{dy}{ds} = u = kt \quad \Rightarrow \quad y = kts + b$$

Then we see that $y(x)$ is a linear function,

$$y = \frac{kt}{u_0} x + d(t)$$

Particle path (fix $\bar{x}_0$):

$$\frac{\partial x}{\partial \hat{t}} = u = u_0 \quad \Rightarrow \quad x = u_0 \hat{t} + x_0$$

$$\frac{\partial y}{\partial \hat{t}} = u = k\hat{t} \quad \Rightarrow \quad x = \frac{1}{2} k \hat{t}^2 + y_0$$

Then we see that $y(x)$ is a parabola,

$$y = \frac{1}{2} \frac{k}{u_0^2} (x^2 - 2x_0 x + x_0^2) + y_0$$

Exercise 2

a) Separate the shear flow $\bar{u} = (\beta y, 0, 0)$ into its (i) local translation, (ii) rotation and (iii) pure straining parts.

Lets denote the different parts as

$$\bar{u} = \bar{u}_T + \bar{u}_R + \bar{u}_D$$

In tensor notation it can be written as

$$u_i^{P_2} = u_i^{P_1} + \xi_{ij} dx_j + e_{ij} dx_j$$

(i) Translation is simply $\bar{u}_T = \{u_i^{P_1}\} = \underline{(\beta y, 0, 0)}$

$j - \frac{1}{2}$ (ii)
$j - 1$

A
B
C
$u_{\frac{3}{2},1}$
$v_{1,\frac{3}{2}}$
$p_{1,1}$

$$\frac{\partial u_i}{\partial x_j} = \begin{pmatrix} 0 & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\begin{pmatrix} 0 & \beta & 0 \\ -\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\bar{u}_R = \xi d\bar{x} = \underline{\frac{1}{2}(\beta dy, -\beta dx, 0)}$$

$\frac{1}{2}$ (iii)

i
j
Inflow
Solid wall
n
Γ_0
Γ_1
Ω
M

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\begin{pmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\bar{u}_D = e d\bar{x} = \underline{\frac{1}{2}(\beta dy, \beta dx, 0)}$$

2b) Find the principal axes of the rate of strain tensor e_{ij} and make a schematic sketch of the decomposition of the flow.

M
m
$m + 2$
1
h
V
V_h
u
u_h

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{symmetric part of} \quad \frac{\partial u_i}{\partial x_j}.$$

$$e_{ij} = \frac{1}{2}\begin{pmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
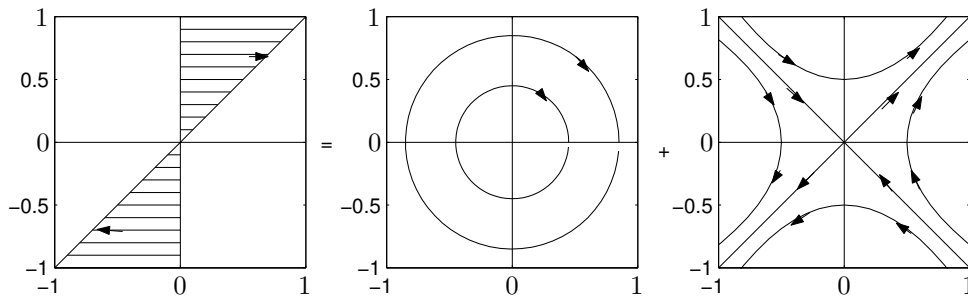
Eigenvalues:

Γ_0
Γ_1
Ω

$$\det(e_{ij} - \lambda\delta_{ij}) = \begin{vmatrix} -\lambda & \beta/2 & 0 \\ \beta/2 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + \beta^2\lambda/4 = -\lambda(\lambda - \beta/2)(\lambda + \beta/2) = 0. \Rightarrow \lambda_1 = \beta/2, \lambda_2 = -\beta/2, \lambda_3 = 0.$$

: to be updated We get the eigenvectors from $e_{ij}k_j = \lambda k_i$:

: new values
: old values
$i - 1$
i
$i + 1$

$$\bar{k}_1 = \frac{1}{\sqrt{2}}(1, 1, 0), \quad \bar{k}_2 = \frac{1}{\sqrt{2}}(-1, 1, 0), \quad \bar{k}_3 = (0, 0, 1)$$

Now sketch the decomposition of the flow

$j - 1$
j
$j + 1$

$$(\beta dy, 0, 0) = \frac{1}{2}(\beta dy, -\beta dx, 0) + \frac{1}{2}(\beta dy, \beta dx, 0)$$

iteration direction

x
y

n

x
y

i
j
fine grid (1)
coarse grid (2)

Exercise 3

ia) Sketch the streamlines for the flow $[u, v, w] = [\alpha x, -\alpha y, 0]$, where $\alpha > 0$.

jStreamlines in parametric form $(x(s), y(s), z(s))$. We have a 2D flow since $w = 0$.

Inflow

Solid wall

n

Γ_0Integrate w.r.t. s,
Γ_1

Ω

M^2Then we see that,

M

mWe have,
$m + 2$

1

hStreamlines:
V

V_h

u

u_h

Γ_0

Γ_1

Ω

to be updated

: new values

: old values

$i - 1$

iRemember that $\frac{dx(s)}{ds} = \alpha x$ and $\frac{dy(s)}{ds} = -\alpha y$. This gives the direction of the flow:

$i + 1$

$j - 1$

j

$j + 1$

ration direction

x

y

n

x

y

i

j

fine grid (1)

coarse grid (2)

$$\frac{dx}{ds} = \alpha x, \quad \frac{dy}{ds} = -\alpha y.$$

$$x(s) = a\, e^{\alpha s}, \quad y(s) = b\, e^{-\alpha s}.$$

$$x(s)\, y(s) = ab = c \quad (\text{ constant }).$$

$$x(s) = \frac{c}{y(s)}, \quad y(s) = \frac{c}{x(s)}$$

$$x > 0,\ y > 0 \quad \Rightarrow \quad \frac{dx}{ds} > 0,\ \frac{dy}{ds} < 0$$

$$x < 0,\ y > 0 \quad \Rightarrow \quad \frac{dx}{ds} < 0,\ \frac{dy}{ds} < 0$$

$$x < 0,\ y < 0 \quad \Rightarrow \quad \frac{dx}{ds} < 0,\ \frac{dy}{ds} > 0$$

$$x > 0,\ y < 0 \quad \Rightarrow \quad \frac{dx}{ds} > 0,\ \frac{dy}{ds} > 0$$
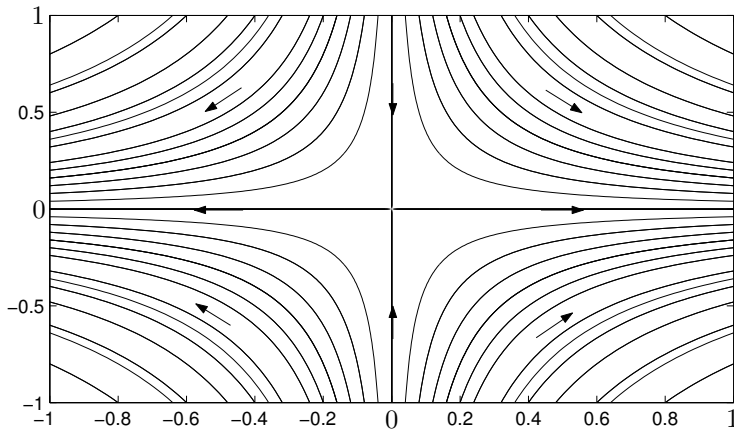
b) The concentration of a scalar is $c(x, y, t) = \beta x^2 y e^{-\alpha t}$. Does this concentration change for a particular fluid particle in time?

Use the material time-derivative

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = -\alpha\beta x^2 y e^{-\alpha t} + \alpha x 2\beta xy e^{-\alpha t} - \alpha y \beta x^2 e^{-\alpha t} = (-1 + 2 - 1)\beta x^2 y e^{-\alpha t} = 0.$$

Thus the concentration $c(x, y, t)$ is constant for a fluid particle.

c) Using the alternative (Lagrangian) description of the flow, show that

$$\frac{\partial \bar{u}}{\partial \hat{t}} = \frac{D\bar{u}}{Dt}$$

and write the concentration as $c = c(x_0, y_0, \hat{t})$.

Particle path:

$$\frac{\partial x}{\partial \hat{t}} = u = \alpha x \quad \Rightarrow \quad x = x_0 e^{\alpha \hat{t}}$$

$$\frac{\partial y}{\partial \hat{t}} = v = -\alpha y \quad \Rightarrow \quad y = y_0 e^{-\alpha \hat{t}}$$

$$\frac{\partial u}{\partial \hat{t}} = \frac{\partial}{\partial \hat{t}} \frac{\partial x}{\partial \hat{t}} = \alpha^2 x_0 e^{\alpha \hat{t}} = \alpha^2 x$$

$$\frac{\partial v}{\partial \hat{t}} = \frac{\partial}{\partial \hat{t}} \frac{\partial y}{\partial \hat{t}} = \alpha^2 y_0 e^{-\alpha \hat{t}} = \alpha^2 y$$

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \alpha^2 x$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \alpha^2 y$$

So we have

$$\frac{\partial \bar{u}}{\partial \hat{t}} = \frac{D\bar{u}}{Dt}$$

Both terms describe the acceleration of a fluid particle.
Now rewrite the concentration

$$c(x, y) = \beta x^2 y e^{-\alpha \hat{t}} = \beta \underbrace{x_0^2 e^{2\alpha \hat{t}}}_{x^2} \underbrace{y_0 e^{-\alpha \hat{t}}}_{y} e^{-\alpha \hat{t}} = \beta x_0^2 y_0 = c(\bar{X}) \quad \text{independent of } \hat{t}!$$

The concentration foes not change as we follow a fluid particle.

B.3   Reynolds transport theorem and stress tensor

Exercise 1

a) Consider a fixed closed surface S in a fluid. Show that conservation of mass implies

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0.$$

The change of mass in the volume is described by the contribution from the mass flux and the change in density and conservation means that this should be zero

$$\int_V \frac{\partial \rho}{\partial t} dV + \oint_S \rho u_i \, n_i \, dS = 0$$

Using Gauss Theorem we can rewrite the surface integral

$$\oint_S \rho u_i \, n_i \, dS = \int_V \frac{\partial}{\partial x_i}(\rho u_i) \, dV.$$

We have

$$\int_V \{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i)\} \, dV = 0.$$

This must be true for an arbitrary volume and this implies

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0.$$

b) Show that this can be written

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0,$$

$$\frac{\partial}{\partial x_i}(\rho u_i) = \frac{\partial \rho}{\partial x_i} u_i + \rho \frac{\partial u_i}{\partial x_i} = u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i}$$

This gives

$$\underbrace{\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i}}_{\frac{D\rho}{Dt}} + \rho \frac{\partial u_i}{\partial x_i} = \frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i}$$

c) Explain what the following means

$$\frac{\partial u_i}{\partial x_i} = 0 \quad \Rightarrow \quad \frac{D\rho}{Dt} = 0$$

If we are following a fluid particle, the density and volume are constant. But in a fix position, the density can change

$$\frac{\partial \rho}{\partial t} = -u_i \frac{\partial \rho}{\partial x_i}$$

Remember the decomposition of $\frac{\partial u_i}{\partial x_j}$ in its invariant parts

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}}_{\text{deformation } \bar{e}_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{rotation } \xi_{ij}} + \underbrace{\frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}}_{\text{volume change } \bar{e}_{ij}}$$

.

Exercise 2

a) Use Reynolds transport theorem to provide an alternative derivation of the conservation of mass equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0.$$

Reynolds Transport Theorem:

$$\frac{D}{Dt} \int_{V(t)} T_{ijk} \, dV = \int_{V(t)} \left(\frac{\partial T_{ijk}}{\partial t} + \frac{\partial}{\partial x_l}(u_l T_{ijk}) \right) dV$$

Put $T_{ijk} = \rho$,

$$\frac{D}{Dt} \int_{V(t)} \rho \, dV = \int_{V(t)} \left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) \right) dV$$

No flow through the surface. Valid for all $V(t)$:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

b) Use this result to show

$$\frac{D}{Dt} \int_{V(t)} \rho F_{ijk} \, dV = \int_{V(t)} \rho \frac{D F_{ijk}}{Dt} \, dV$$

Put $T_{ijk} = \rho F_{ijk}$ in Reynolds transport theorem

$$\frac{D}{Dt} \int_{V(t)} \rho F_{ijk} \, dV = \int_{V(t)} \left(\frac{\partial}{\partial t}(\rho F_{ijk}) + \frac{\partial}{\partial x_l}(u_l \rho F_{ijk}) \right) dV =$$

$$\int_{V(t)} \left(\rho \frac{\partial F_{ijk}}{\partial t} + F_{ijk} \frac{\partial \rho}{\partial t} + F_{ijk} \frac{\partial}{\partial x_l}(\rho u_l) + \rho u_l \frac{\partial F_{ijk}}{\partial x_l} \right) dV =$$

$$\int_{V(t)} \left[F_{ijk} \underbrace{\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_l}(\rho u_l) \right)}_{=0} + \rho \underbrace{\left(\frac{\partial F_{ijk}}{\partial t} + u_l \frac{\partial F_{ijk}}{\partial x_l} \right)}_{= \frac{D F_{ijk}}{Dt}} \right] dV =$$

$$\int_{V(t)} \rho \frac{D F_{ijk}}{Dt} \, dV$$

Exercise 3

Show that the stress tensor $T_{ij} = -p\delta_{ij}$ for the flow

$$\bar{u} = \bar{\omega} \times \bar{x},$$

where $\bar{\omega}$ is constant.

Tensor notation:

$$u_i = \varepsilon_{ikl}\omega_k x_l$$

The stress tensor:

$$T_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) =$$

$$-p\delta_{ij} + \mu \left(\frac{\partial}{\partial x_j}(\varepsilon_{ikl}\omega_k x_l) + \frac{\partial}{\partial x_i}(\varepsilon_{jkl}\omega_k x_l) \right) =$$

$$-p\delta_{ij} + \mu\omega_k \left(\varepsilon_{ikl} \frac{\partial x_l}{\partial x_j} + \varepsilon_{jkl} \frac{\partial x_l}{\partial x_i} \right) =$$

$$-p\delta_{ij} + \mu\omega_k (\varepsilon_{ikl}\delta_{lj} + \varepsilon_{jkl}\delta_{li}) =$$

$$-p\delta_{ij} + \mu\omega_k (\varepsilon_{ikj} + \varepsilon_{jki}) =$$

$$-p\delta_{ij} + \mu\omega_k (-\varepsilon_{ijk} + \varepsilon_{ijk}) = -p\delta_{ij}$$

The stress tensor T_{ij}

The equation for the stress tensor was deduced by Stokes in (1845) from three elementary hypotheses.

$$T_{ij} = -p\delta_{ij} + 2\,\mu e_{ij}$$

Writing this on the form,

$$T_{ij} = -p\delta_{ij} + \tau_{ij}$$

the following statements should be true for the viscous stress tensor τ_{ij} in a Newtonian fluid.

(i) each τ_{ij} should be a linear combination of the velocity gradients $\dfrac{\partial u_i}{\partial x_j}$.

(ii) each τ_{ij} should vanish if the flow involves no deformation of fluid elements.

(iii) the relationship between τ_{ij} and $\dfrac{\partial u_i}{\partial x_j}$ should be isotropic as the physical properties of the fluid are assumed to show no preferred direction.

Exercise 4

If τ_{ij} is a linear function of the components of e_{ij} then it can be written

$$\tau_{ij} = c_{ijkl}\, e_{kl}$$

It can be shown that the most general fourth order isotropic tensor is of the form

$$c_{ijkl} = A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk}$$

where A, B and C are scalars.

a) Use this to show that

$$\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

where λ and μ are scalars.

$$\tau_{ij} = c_{ijkl}\, e_{kl} = (A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk})e_{kl} = A\delta_{ij}e_{kk} + Be_{ij} + Ce_{ji}$$

Say that $A = \lambda$ and that $B = C = \mu$, then since $e_{ij} = e_{ji}$ we have

$$\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

b) Show that if we have a Newtonian fluid where $p = -\dfrac{1}{3}T_{ii}$ then

$$T_{ij} = -\left(p + \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right)\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

We have

$$T_{ij} = -p\delta_{ij} + \tau_{ij} = -p\delta_{ij} + \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

Then

$$T_{ii} = -3p + 3\lambda e_{ii} + 2\mu e_{ii} = -3p + (3\lambda + 2\mu)e_{ii}$$

This gives $\lambda = -\frac{2}{3}\mu$, and thus

$$T_{ij} = -p - \frac{2}{3}\mu e_{kk}\delta_{ij} + 2\mu e_{ij}$$

or

$$T_{ij} = -\left(p + \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right)\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

The Rankine vortex

u_hA simple model for a vortex is that it is a rigid body rotation within a core, and a decay of angular velocity
$\Gamma_0$outside. This can be described by
Γ_1

Ω

: to be updated

: new values

: old values

$i-1$and is called a Rankine vortex.

i

$i+1$
$j-1$

j

$j+1$

iteration direction

x
y

n
x
y

i
j

fine grid (1)

coarse grid (2)

$$u_\theta = \begin{cases} \omega r, & r < a, \\ \dfrac{\omega a^2}{r}, & r > a, \end{cases} \qquad u_r = u_z = 0$$
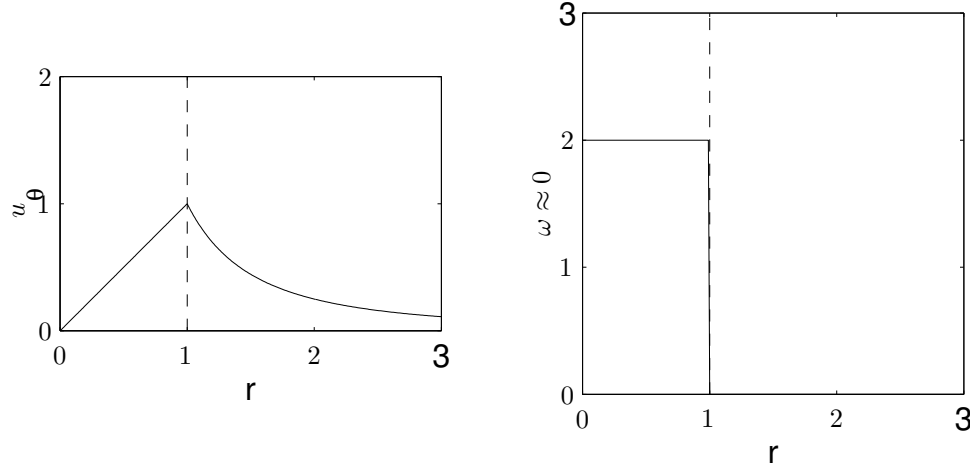

Figure B.1: Velocity and vorticity in a Rankine vortex with $\omega = a = 1$.

Exercise 1

a) Find the pressure inside and outside of a Rankine vortex:

We use Eulers equations for incompressible flow (neglecting viscous effects):

$$\text{Eulers equations} \quad \begin{cases} \dfrac{D\bar{u}}{Dt} = -\dfrac{1}{\rho}\nabla p + \bar{g} \\ \nabla \cdot \bar{u} = 0 \end{cases}$$

$$\frac{D\bar{u}}{Dt} = \underbrace{\frac{\partial \bar{u}}{\partial t}}_{=0} + (\bar{u}\cdot\nabla)\bar{u}$$

We are working with cylindrical coordinates, use the appendix formulas

$$\bar{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}$$

$$(\bar{u}\cdot\nabla)\bar{u} = \frac{u_\theta}{r}\frac{\partial}{\partial\theta}(u_\theta\bar{e}_\theta) = \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}\bar{e}_\theta + \frac{u_\theta}{r}\underbrace{\frac{\partial\bar{e}_\theta}{\partial\theta}}_{=-\bar{e}_r}u_\theta = \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}\bar{e}_\theta - \frac{u_\theta^2}{r}\bar{e}_r$$

$$\nabla p = \frac{\partial p}{\partial r}\bar{e}_r + \frac{1}{r}\frac{\partial p}{\partial\theta}\bar{e}_\theta + \frac{\partial p}{\partial z}\bar{e}_z$$

Insert this into the Euler equations:

$$\frac{u_\theta}{r}\underbrace{\frac{\partial u_\theta}{\partial\theta}}_{=0}\bar{e}_\theta - \frac{u_\theta^2}{r}\bar{e}_r = -\frac{1}{\rho}\left(\frac{\partial p}{\partial r}\bar{e}_r + \frac{1}{r}\frac{\partial p}{\partial\theta}\bar{e}_\theta + \frac{\partial p}{\partial z}\bar{e}_z\right) - g\bar{e}_z$$

Look at the different components:

$$\bar{e}_r: \quad -\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\bar{e}_\theta: \quad 0 = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} \Rightarrow p = p(r,z) \quad \text{only.}$$

$$\bar{e}_z: \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Solve for the pressure when $r < a$:

$$\bar{e}_r: \quad -\omega^2 r = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \Rightarrow \quad p = \rho\omega^2\frac{r^2}{2} + f(z)$$

$$\bar{e}_z: \frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad f(z) = -\rho g z + C_1$$

So we have the pressure:

$$p(r,z) = \rho\omega^2\frac{r^2}{2} - \rho g z + C_1 \quad r < a$$

Solve for the pressure when $r > a$:

$$\bar{e}_r: \quad -\frac{\omega^2 a^4}{r^3} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \Rightarrow \quad p = -\frac{\rho\omega^2 a^4}{2\,r^2} + f(z)$$

$$\bar{e}_z: \frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad f(z) = -\rho g z + C_2$$

So we have the pressure:

$$p(r,z) = -\frac{\rho\omega^2 a^4}{2\,r^2} - \rho g z + C_2 \quad r > a$$

Now determine the difference between the constants by evaluation at $r = a$

$$\frac{\rho\omega^2 a^2}{2} - \rho g z + C_1 = -\frac{\rho\omega^2 a^2}{2} - \rho g z + C_2 \quad \Rightarrow \quad C_2 - C_1 = \rho\omega^2 a^2$$

b) Determine the pressure difference Δp between $r = 0$ and $r \to \infty$

$$\Delta p = p_\infty - p_0 = -\rho g z + C_2 - (-\rho g z + C_1) = C_2 - C_1 = \rho\omega^2 a^2$$

c) Now consider a free surface at atmospheric pressure p_0.
Find the difference in z between $r = 0$ and $r \to \infty$

$$\begin{cases} r = 0: & p_0 = \frac{\rho\omega^2 a^2}{2} - \rho g z_0 + C_1 \\ r \to \infty: & p_0 = -\frac{\rho\omega^2 a^2}{2} - \rho g z_\infty + C_2 \end{cases} \quad \Rightarrow \quad z_\infty - z_0 = \frac{C_2 - C_1}{\rho g} = \frac{\omega^2 a^2}{g}$$

Determine the shape of the free surface:

$$\begin{cases} p_0 = \frac{\rho\omega^2 r^2}{2} - \rho g z + C_1 & r < a \quad z \sim r^2 \Rightarrow z = \frac{\omega^2 r^2}{2g} + \frac{C_1 - p_0}{\rho g} \\ p_0 = -\frac{\rho\omega^2 a^4}{2r^2} - \rho g z + C_2 & r > a \quad z \sim \frac{1}{r} \Rightarrow z = -\frac{\omega^2 a^4}{2gr^2} + \frac{C_2 - p_0}{\rho g} \end{cases}$$

Say that $z = 0$ at $r = 0$ then $C_1 = p_0$ and we get

$$z = \frac{\omega^2 r^2}{2g} \quad r < a$$

and

$$z = -\frac{\omega^2 a^4}{2gr^2} + \frac{C_2 - C_1}{\rho g} = -\frac{\omega^2 a^4}{2gr^2} + \frac{\omega^2 a^2}{g} = \frac{\omega^2 a^2}{g}\left(1 - \frac{a^2}{2r^2}\right) \quad r > a$$

So we have

$$z(r) = \begin{cases} \frac{\omega^2 r^2}{2g} & r < a \\ \frac{\omega^2 a^2}{g}\left(1 - \frac{a^2}{2r^2}\right) & r > a. \end{cases}$$
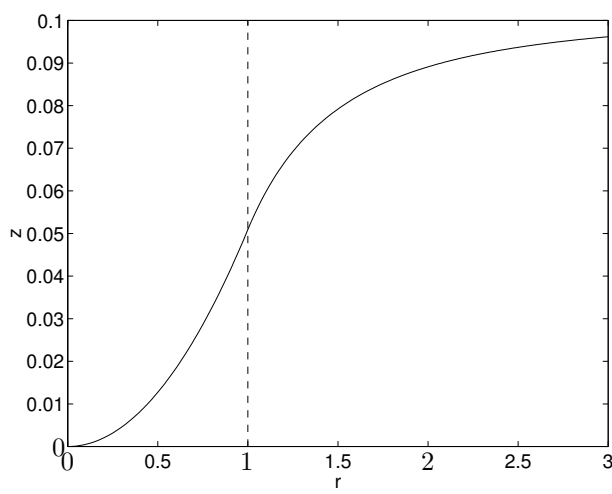
ι
i + 1
j − 1
j
j + 1
iteration direction
x
y
n
x
y
i
j
fine grid (1)
coarse grid (2)

Figure B.2: The free surface of a Rankine vortex with $\omega = a = 1$ and $g = 9.82$.

$$z(r) = \begin{cases} \frac{\omega^2 r^2}{2g} & r < a \\ \frac{\omega^2 a^2}{g}\left(1 - \frac{a^2}{2r^2}\right) & r > a. \end{cases}$$

Exercise 2

Solving the 2D flow around a cylinder involves solving

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla)\bar{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \bar{u}, \quad \nabla \cdot \bar{u} = 0,$$

with the boundary conditions

$$\bar{u} = 0 \quad \text{on } x^2 + y^2 = a^2 \qquad \bar{u} \to (U, 0, 0) \quad \text{as } x^2 + y^2 \to \infty.$$

Rewrite this problem in dimensionless form using the dimensionless variables

$$\bar{x}' = \bar{x}/a \quad \bar{u}' = \bar{u}/U \quad p' = p/\rho U^2 \quad t' = tU/a.$$

Note that the scaling $\bar{x}' = \bar{x}/a$ implies $\nabla' = a\nabla$ and $t' = tU/a$ gives $\frac{\partial}{\partial t'} = \frac{a}{U}\frac{\partial}{\partial t}$.
Change to dimensionless variables

$$\frac{U^2}{a}\frac{\partial \bar{u}'}{\partial t'} + \frac{U^2}{a}(\bar{u}' \cdot \nabla')\bar{u}' = -\frac{\rho U^2}{\rho a}\nabla' p' + \frac{\nu U}{a^2}\nabla'^2 \bar{u}'$$

Divide by U^2/a

$$\frac{\partial \bar{u}'}{\partial t'} + (\bar{u}' \cdot \nabla')\bar{u}' = -\nabla' p' + \underbrace{\frac{\frac{\nu U}{a^2}}{\frac{U^2}{a}}}_{\frac{1}{Re}} \nabla'^2 \bar{u}'$$

The Reynolds number is $Re = \dfrac{\dfrac{U^2}{a}}{\dfrac{\nu U}{a^2}} = \dfrac{\text{Inertial forces}}{\text{Viscous forces}} = \dfrac{U\,a}{\nu}$

$$\frac{\partial \bar{u}'}{\partial t'} + (\bar{u}' \cdot \nabla')\bar{u}' = -\nabla' p' + \frac{1}{Re}\nabla'^2 \bar{u}'$$

The continuity condition

$$\nabla \cdot \bar{u} = 0 \quad \Rightarrow \quad \frac{U}{a}\nabla' \cdot \bar{u}' = 0 \quad \Rightarrow \quad \nabla' \cdot \bar{u}' = 0$$

The boundary conditions

$$\bar{u} = 0 \quad \text{on } x^2 + y^2 = a^2 \Rightarrow \quad U\bar{u}' = 0 \quad \text{on } a^2 x'^2 + a^2 y'^2 = a^2 \Rightarrow \quad \bar{u}' = 0 \quad \text{on } x'^2 + y'^2 = 1.$$

$$\bar{u} \to (U, 0, 0) \quad \text{as } x^2 + y^2 \to \infty \Rightarrow \quad U\bar{u}' \to (U, 0, 0) \quad \text{as } a^2 x'^2 + a^2 y'^2 \to \infty \Rightarrow$$

$$\bar{u}' \to (1, 0, 0) \quad \text{as } x'^2 + y'^2 \to \infty$$

The solutions of this problem will depend on $\bar{x}'$, Re and t' only, and thus the solution of this problem is the same for a specific Re independently of the individual values of U, a and ν.

Exercise 3

Show that the net viscous force per unit volume is proportional to the spatial derivative of vorticity, i.e.

$$\frac{\partial \tau_{ij}}{\partial x_j} = -\mu \varepsilon_{ijk} \frac{\partial \omega_k}{\partial x_j}$$

and discuss its implication for flows with uniform vorticity (as in solid-body rotation).

$$\frac{\partial \tau_{ij}}{\partial x_j} = \mu \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \mu \left(\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_j \partial x_i} \right) = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$-\mu \varepsilon_{ijk} \frac{\partial \omega_k}{\partial x_j} = -\mu \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\varepsilon_{klm} \frac{\partial u_m}{\partial x_l} \right) = -\mu \varepsilon_{kij} \varepsilon_{klm} \frac{\partial^2 u_m}{\partial x_j \partial x_l} = -\mu (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \frac{\partial^2 u_m}{\partial x_j \partial x_l} =$$

$$-\mu \left(\frac{\partial^2 u_j}{\partial x_j \partial x_i} - \frac{\partial^2 u_i}{\partial x_j \partial x_j} \right) = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

Thus

$$\frac{\partial \tau_{ij}}{\partial x_j} = -\mu \varepsilon_{ijk} \frac{\partial \omega_k}{\partial x_j}$$

The net viscous force vanishes when the vorticity is uniform, since no deformation exists.

Exercise 1

Plane Couette flow:

Consider the flow of a viscous fluid between two parallel plates at $y = 0$ and $y = h$. The upper plane is moving with velocity U. Calculate the flow field.

Assume the following:

Steady flow:
$$\frac{\partial}{\partial t} = 0$$

Parallel flow:
$$v = 0, \quad \frac{\partial u_i}{\partial x} = 0$$

Two-dimensional flow:
$$w = 0, \quad \frac{\partial}{\partial z} = 0$$

No pressure gradient:
$$\frac{\partial p}{\partial x_i} = 0$$

Navier–Stokes streamwise momentum equation:
$$\frac{\partial u}{\partial t} + (\bar{u} \cdot \nabla)u = -\frac{1}{\rho}\nabla p + \nu \nabla^2 u$$

We have
$$\frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial y} = A \quad \Rightarrow \quad u = Ay + B$$

Boundary conditions:
$$u(0) = 0 \quad \Rightarrow \quad B = 0, \quad u(h) = U \quad \Rightarrow \quad A = U/h$$

So we get:
$$u(y) = \frac{Uy}{h}$$

Exercise 2 (i)

Plane Poiseuille flow: (Channel flow)

Consider the flow of a viscous fluid between two solid boundaries at $y = \pm h$ driven by a constant pressure gradient $\nabla p = [-P, 0, 0]$. Show that

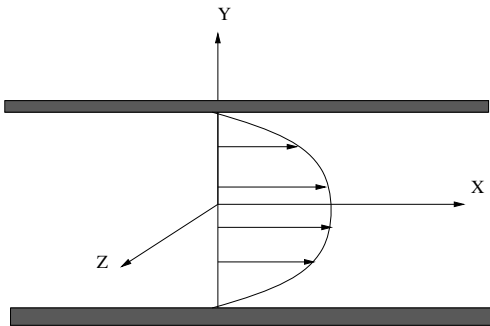$$u = \frac{P}{2\mu}(h^2 - y^2), \quad v = w = 0.$$

Figure B.3: Coordinate system for plane Poiseuille flow

Navier–Stokes equations:
$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla)\bar{u} = -\dfrac{1}{\rho}\nabla p + \nu \nabla^2 \bar{u} \\ \nabla \cdot \bar{u} = 0. \end{cases}$$

Boundary conditions:

$$\bar{u}(y = \pm h) = 0$$

- We are considering a stationary flow and thus $\dfrac{\partial \bar{u}}{\partial t} = 0$.
- The constant pressure gradient implies $\bar{u} = \bar{u}(y)$. Changes of $\bar{u}$ in x, z would require a changing pressure gradient in x, z.
- The continuity equation $\nabla \cdot \bar{u}$ reduces to $\dfrac{\partial v}{\partial y} = 0$. The boundary condition $v(y = \pm h) = 0$ then implies $v = 0$.

Let's study the z component of the Navier–Stokes equations:

$$\underbrace{v}_{=0}\frac{\partial w}{\partial y} = \nu \frac{\partial^2 w}{\partial y^2} \quad \Rightarrow \quad w = c_1 y + c_2$$

The boundary conditions $w(y = -h) = w(y = h) = 0$ imply $c_1 = c_2 = 0$ and thus $w = 0$.

- We can conclude that $\bar{u} = [u(y), 0, 0]$.

Let's study the x component of the Navier–Stokes equations:

$$0 = \frac{P}{\rho} + \nu \frac{\partial^2 u}{\partial y^2} \quad \Rightarrow \quad \frac{\partial u}{\partial y} = -\frac{P}{\nu\rho}y + d_1 \quad \{\mu = \rho\nu\} \Rightarrow u(y) = -\frac{P}{2\,\mu}y^2 + d_1 y + d_2$$

The boundary conditions at $y = \pm h$ give

$$0 = -\frac{P}{2\,\mu}h^2 + d_1 h + d_2 \quad \text{and} \quad 0 = -\frac{P}{2\,\mu}h^2 - d_1 h + d_2$$

We can directly conclude that $d_1 = 0$ and this gives $d_2 = \dfrac{P}{2\,\mu}h^2$ and thus

$$u = \frac{P}{2\,\mu}(h^2 - y^2), \quad v = w = 0.$$

Exercise 2 (ii)

Poiseuille flow: (Pipe flow)

Consider the viscous flow of a fluid down a pipe with a cross-section given by $r = a$ under the constant pressure gradient $P = -\dfrac{\partial p}{\partial z}$. Show that

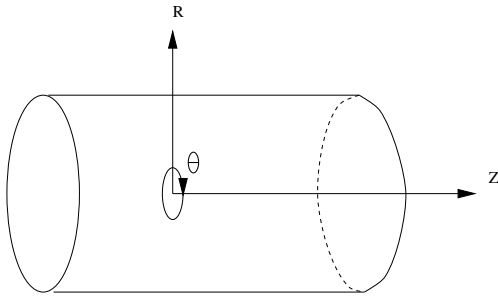$$u_z = \frac{P}{4\mu}(a^2 - r^2) \quad u_r = u_\theta = 0.$$

Figure B.4: Coordinate system for Poiseuille flow

Use the Navier–Stokes equations in cylindrical coordinates

$$\frac{\partial u_r}{\partial t} + (\bar{u} \cdot \nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$\frac{\partial u_\theta}{\partial t} + (\bar{u} \cdot \nabla)u_\theta + \frac{u_r\, u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)$$

$$\frac{\partial u_z}{\partial t} + (\bar{u} \cdot \nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\, u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0.$$

We know that $\dfrac{\partial p}{\partial \theta} = 0$ and $\dfrac{\partial p}{\partial r} = 0$ and can directly see that $u_r = u_\theta = 0$ satisfy the two first equations. From the continuity equation we get

$$\frac{\partial u_z}{\partial z} = 0 \quad \Rightarrow \quad u_z = u_z(r, \theta) \quad \text{only.}$$

Considering a steady flow we get from the Navier–Stokes equations

$$(\bar{u} \cdot \nabla)u_z = u_z\frac{\partial u_z}{\partial z} \quad \Rightarrow \quad u_z\frac{\partial u_z}{\partial z} = \frac{1}{\rho}P + \nu\nabla^2 u_z$$

But we know that $\dfrac{\partial u_z}{\partial z} = 0$ from the continuity equation. We get

$$\nabla^2 u_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = -\frac{P}{\mu} \quad \Rightarrow \quad \frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = -\frac{P}{\mu}r$$

Integrate

$$r\frac{\partial u_z}{\partial r} = -\frac{P}{2\,\mu}r^2 + c_1 \quad \Rightarrow \quad \frac{\partial u_z}{\partial r} = -\frac{P}{2\,\mu}r + \frac{c_1}{r}$$

Integrating again we get

$$u_z = -\frac{P}{4\,\mu}r^2 + c_1\ln(r) + c_2 \quad \text{using the boundary conditions } u_z(r = 0) < \infty \Rightarrow c_1 = 0$$

We also have $u_z = 0$ at $r = a$ and this gives $c_2 = \dfrac{P\,a^2}{4\,\mu}$ and we have

$$u_z = \frac{P}{4\,\mu}(a^2 - r^2)$$

Exercise 3

Calculate the asymptotic suction boundary layer, where the boundary layer over a flat plate is kept parallel of a steady suction V_0 through the plate.

Assumptions.
Two-dimensional flow:
$$\frac{\partial}{\partial z} = 0, \quad w = 0$$

Parallel flow:
$$\frac{\partial}{\partial x} = 0, \quad \text{Continuity satisfied}$$

Steady flow:
$$\frac{\partial}{\partial t} = 0$$

Momentum equations:
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Normal momentum equation gives
$$\frac{\partial p}{\partial y} = 0$$

Boundary conditions:

$$y = 0: \quad u = 0, \quad v = -V_0$$

$$y \to \infty: \quad u \to U_\infty$$

Continuity gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad v = -V_0$$

Streamwise momentum equation at $y \to \infty$

$$-V_0 \frac{\partial U_\infty}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 U_\infty}{\partial y^2}$$

$$\Rightarrow \quad \frac{\partial p}{\partial x} = 0$$

Resulting streamwise momentum equation

$$-V_0 \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -\frac{V_0}{\nu} \frac{\partial u}{\partial y}$$

Characteristic equation

$$\lambda^2 = -\frac{V_0}{\nu} \lambda \quad \Rightarrow \quad \lambda_1 = 0, \lambda_2 = -\frac{V_0}{\nu}$$

$$u(y) = A + B e^{-V_0 y / \nu}$$

Exercise 4

Two incompressible viscous fluids flow one on top of the other down an inclined plane at an angle α. They both have the same density ρ and the viscosities μ_1 and μ_2. The lower fluid has depth h_1 and the upper h_2. Assuming that viscous forces from the surrounding air is negligible and that the pressure on the free surface is constant, show that

$$u_1(y) = [(h_1 + h_2)y - \frac{1}{2}y^2] \frac{g \sin(\alpha)}{\nu_1}.$$

Make the ansatz $\bar{u}_1 = [u_1(y), 0, 0]$ and $\bar{u}_2 = [u_2(y), 0, 0]$. The continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{gives} \quad \frac{\partial v}{\partial y} = 0 \Rightarrow v = c \quad \text{and the boundary condition at } y = 0 \text{ gives } v = 0.$$

- Layer 1:

$$\text{N–S} \cdot \bar{e}_y : 0 = -\frac{1}{\rho} \frac{\partial p_1}{\partial y} - g \cos(\alpha) \quad \Rightarrow \quad p_1 = -\rho g \cos(\alpha) y + f_1(x)$$

$$\text{N–S} \cdot \bar{e}_x : 0 = -\frac{1}{\rho} f_1'(x) + \nu_1 \frac{d^2 u_1}{dy^2} + g \sin(\alpha) \quad \Rightarrow \quad f_1'(x) = c_1$$

- Layer 2:

$$\text{N–S} \cdot \bar{e}_y : 0 = -\frac{1}{\rho} \frac{\partial p_2}{\partial y} - g \cos(\alpha) \quad \Rightarrow \quad p_2 = -\rho g \cos(\alpha) y + f_2(x)$$

$$\text{N–S} \cdot \bar{e}_x : 0 = -\frac{1}{\rho} f_2'(x) + \nu_2 \frac{d^2 u_2}{dy^2} + g \sin(\alpha) \quad \Rightarrow \quad f_2'(x) = c_2$$

The pressure at the free surface $y = h_1 + h_2$ is p_0:

$$p_0 = -\rho g \cos(\alpha)(h_1 + h_2) + f_2(x) \quad \Rightarrow \quad f_2 = p_0 + \rho g(h_1 + h_2)\cos(\alpha) \quad \Rightarrow \quad f_2' = 0$$

The pressure is continuous at $y = h_1$:

$$p_0 + \rho g h_2 \cos(\alpha) = -\rho g h_1 \cos(\alpha) + f_1(x) \quad \Rightarrow \quad f_1 = p_0 + \rho g(h_1 + h_2)\cos(\alpha) \quad \Rightarrow \quad f_1' = 0$$

This gives the pressure:

$$p_1(y) = p_2(y) = p(y) = -\rho g \cos(\alpha) y + p_0 + \rho g \cos(\alpha)(h_1 + h_2)$$

We now have two momentum equations in x:

$$0 = \nu_1 \frac{d^2 u_1}{dy^2} + g \sin(\alpha) \quad (1)$$

$$0 = \nu_2 \frac{d^2 u_2}{dy^2} + g \sin(\alpha) \quad (2)$$

And four boundary conditions:

BC1: No slip on the plane: $\quad u_1(0) = 0$

BC2: No viscous forces on the free surface: $\quad \mu_2 \frac{du_2}{dy}\big|_{y=h_1+h_2} = 0$

BC3: Force balance at the fluid interface: $\quad \mu_1 \frac{du_1}{dy}\big|_{y=h_1} = \mu_2 \frac{du_2}{dy}\big|_{y=h_1}$

BC4: Continous velocity at the interface: $u_1\big|_{y=h_1} = u_2\big|_{y=h_1}$

$$(1) \Rightarrow \quad \frac{du_1}{dy} = -\frac{g}{\nu_1} y \sin(\alpha) + c_{11} \quad \Rightarrow \quad u_1 = -\frac{g}{2\nu_1} y^2 \sin(\alpha) + c_{11} y + c_{12}$$

$$(2) \Rightarrow \quad \frac{du_2}{dy} = -\frac{g}{\nu_2} y \sin(\alpha) + c_{21} \quad \Rightarrow \quad u_2 = -\frac{g}{2\nu_2} y^2 \sin(\alpha) + c_{21} y + c_{22}$$

$$BC1 \Rightarrow \quad c_{12} = 0$$

$$BC2 \Rightarrow \quad \mu_2\left(-\frac{g}{\nu_2}(h_1 + h_2)\sin(\alpha) + c_{21}\right) = 0 \quad \Rightarrow \quad c_{21} = \frac{g}{\nu_2}(h_1 + h_2)\sin(\alpha)$$

$$BC3 \Rightarrow \quad \mu_1\left(-\frac{g}{\nu_1} y \sin(\alpha) + c_{11}\right) = \mu_2\left(-\frac{g}{\nu_2} y \sin(\alpha) + c_{21}\right) \quad \{\mu = \nu\rho\} \quad \Rightarrow c_{11} = \frac{\mu_2}{\mu_1} c_{21} = \frac{g}{\nu_1}(h_1 + h_2)\sin(\alpha)$$

$$BC4 \Rightarrow \quad -\frac{g}{2\nu_1} h_1^2 \sin(\alpha) + \frac{g}{\nu_1}(h_1 + h_2)\sin(\alpha)h_1 = -\frac{g}{2\nu_2} h_1^2 \sin(\alpha) + \frac{g}{\nu_2}(h_1 + h_2)\sin(\alpha)h_1 + c_{22}$$

$$\Rightarrow c_{22} = g \sin(\alpha)\left(\frac{h_1^2}{2} - (h_1 + h_2)h_1\right)\left(\frac{1}{\nu_2} - \frac{1}{\nu_1}\right)$$

This gives us the velocities,

$$u_1(y) = -\frac{g}{2\nu_1} y^2 \sin(\alpha) + \frac{g}{\nu_1}(h_1 + h_2)\sin(\alpha)y = \frac{g \sin(\alpha)}{\nu_1}\left[(h_1 + h_2)y - \frac{1}{2} y^2\right]$$

$$u_2(y) = -\frac{g \sin(\alpha)}{2\nu_2} y^2 + \frac{g \sin(\alpha)}{\nu_2}(h_1 + h_2)y + g \sin(\alpha)\left(\frac{h_1^2}{2} - (h_1 + h_2)h_1\right)\left(\frac{1}{\nu_2} - \frac{1}{\nu_1}\right)$$

$$= \frac{g \sin(\alpha)}{\nu_2}\left((h_1 + h_2)y - \frac{1}{2} y^2\right) + g \sin(\alpha)\left(\frac{h_1^2}{2} - (h_1 + h_2)h_1\right)\left(\frac{1}{\nu_2} - \frac{1}{\nu_1}\right)$$

The velocity in layer 1 does depend on h_2 but not on the viscosity in layer 2. This is due to that the depth is important for the tangential stress boundary condition at the interface but the viscosity is not. There is no acceleration of the upper layer and thus the tangential stress must be equal to the gravitational force on the upper layer which depends on h_2 but not on ν_2.

B.6   More exact solutions to Navier-Stokes

Exercise 1

Oscillating Rayleigh–Stokes flow (or Stokes second problem).
a) Show that the velocity field $\bar{u} = [u(y, t), 0, 0]$ satisfies the equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

Consider the Navier–Stokes equation in the x direction:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

With the current velocity field only terms with y derivatives will remain since there can be no change in the other directions. Further more, the streamwise pressure gradient has to be zero since the streamwise velocity far from the wall is constant, namely zero.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

b) Show that the velocity field is

$$u(y, t) = U e^{-ky} \cos(ky - \omega t), \quad \text{where} \quad k = \sqrt{\frac{\omega}{2\nu}}$$

Make the ansatz: $u = \Re\left[f(y)e^{i\omega t}\right] = f(y)\cos(\omega t)$.

Insert into the equation:

$$i\omega f(y)e^{i\omega t} = \nu f''(y)e^{i\omega t}$$

$$f''(y) - \frac{i\omega}{\nu}f(y) = 0 \quad \text{with } f(y) = e^{\lambda y} \quad \text{gives}$$

$$\lambda^2 - \frac{i\omega}{\nu} = 0 \quad \Rightarrow \quad \lambda = \pm\sqrt{\frac{i\omega}{\nu}} = \pm\sqrt{\frac{\omega}{\nu}}\left(\frac{1+i}{\sqrt{2}}\right)$$

Introduce $k = \sqrt{\dfrac{\omega}{2\nu}}$,

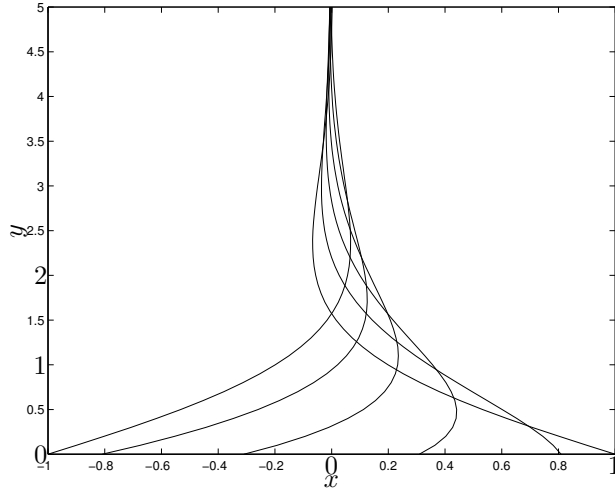$$f(y) = Ae^{yk(1+i)} + Be^{-yk(1+i)}, \quad f(y) \to 0 \quad \text{as} \quad y \to \infty \to A = 0$$

We have

$$u = \Re\left[Be^{-yk(1+i)}e^{iwt}\right] = \Re\left[Be^{-ky}e^{i(\omega t - ky)}\right] = Be^{-ky}\cos(\omega t - ky) = Be^{-ky}\cos(ky - \omega t)$$

Boundary condition at $y = 0$

At $y = 0$ we have $u = U\cos(\omega t) \Rightarrow B = U$

So we have

$$u(y,t) = Ue^{-ky}\cos(ky - \omega t), \quad \text{where} \quad k = \sqrt{\frac{\omega}{2\nu}}$$

c) How thick is the boundary layer thickness?

If we define the thickness δ of the oscillating layer as the position where $u/U = 0.01$ we get that

$$e^{-k\delta} = 0.01 \Rightarrow k\delta \approx 4.6 \Rightarrow \delta \approx 4.6\sqrt{\frac{2\nu}{\omega}} \Rightarrow \delta \approx 6.5\sqrt{\frac{\nu}{\omega}}$$

d) Consider instead the oscillating flow $U_\infty = U\cos(\omega t)$ over a stationary wall. This will simply result in a change of frame of reference to one following the plate instead. If we consider the solution to the previous problem and look at it in this new frame of reference we get

$$u(y,t) = U\cos(\omega t) - Ue^{-ky}\cos(ky - \omega t), \quad \text{where} \quad k = \sqrt{\frac{\omega}{2\nu}}$$

The solution now looks like this
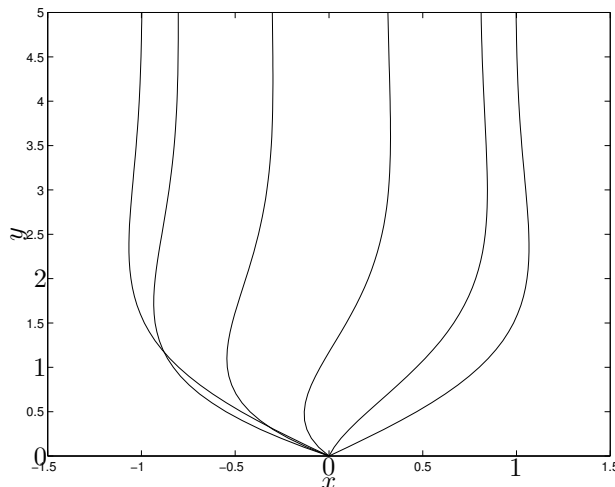
Exercise 2

Consider a long hollow cylinder with inner radius r_1 and a concentric rod with radius r_0 inside it. The rod is moving axially with velocity U_0.

a) Find the velocity field of a viscous fluid occupying the space between the rod and the cylinder.

Assumptions Steady flow:

$$\frac{\partial}{\partial t} = 0$$

Parallel flow and symmetry:

$$\mathbf{u} = u_z(r)\mathbf{e}_z, \quad \frac{\partial}{\partial z} = 0, \quad \frac{\partial}{\partial \theta} = 0$$

No streamwise pressure gradient:

$$\frac{\partial p}{\partial z} = 0$$

We can directly see that $u_r = u_\theta = 0$ satisfy the two first equations. The streamwise momentum equation reduces to

$$(\mathbf{u} \cdot \nabla)u_z = \nu \nabla^2 u_z$$

where

$$(\mathbf{u} \cdot \nabla)u_z = u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = 0$$

$$\nabla^2 u_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)$$

We end up with

$$\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = 0$$

Integrate twice

$$r\frac{\partial u_z}{\partial r} = A \quad \Rightarrow \quad u_z = A\ln r + B$$

Boundary conditions

$$\begin{cases} u_z(r_0) &= U_0 &= A\ln r_0 + B \\ u_z(r_1) &= 0 &= A\ln r_1 + B \quad \Rightarrow \quad B = -A\ln r_1 \end{cases} \Rightarrow$$

$$U_0 = A(\ln r_0 - \ln r_1) \quad \Rightarrow \quad A = \frac{U_0}{\ln \frac{r_0}{r_1}}$$

$$u_z(r) = \frac{U_0}{\ln \frac{r_0}{r_1}}\ln r - \frac{U_0}{\ln \frac{r_0}{r_1}}\ln r_1 = U_0 \frac{\ln \frac{r}{r_1}}{\ln \frac{r_0}{r_1}}$$

b) With what force does one have to pull a rod with length L? Neglect end effects.

Shear stress

$$\tau_{rz} = \mu \frac{\partial u_z}{\partial r} = \frac{\mu U_0}{r \ln \frac{r_0}{r_1}}$$

Force

$$F = 2\pi r_0 L \tau_{rz}(r_0) = \frac{2\pi L \mu U_0}{\ln \frac{r_0}{r_1}}$$

B.7 Axisymmetric flow and irrotational vortices

Axisymmetric flow

Consider incompressible and rotationally symmetric flow with no mass source at the symmetry axis. The velocity component in the direction of the axis of symmetry and the vorticity is given.

$$u_z = \gamma\, z \qquad \text{and} \qquad \bar{\omega} = \omega(r)\, \bar{e}_z.$$

1) Compute $u_r(r, z)$ and $u_\theta(r, z)$ for the two cases when

$$a) \quad \omega(r) = 0 \qquad \text{and} \qquad b) \quad \omega(r) = \omega_0\, e^{-r^2/a^2}.$$

The vorticity is given,

$$\bar{\omega} = \nabla \times \bar{u} = \frac{1}{r}
\begin{vmatrix}
\bar{e}_r & r\bar{e}_\theta & \bar{e}_z \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\
u_r & r\, u_\theta & u_z
\end{vmatrix}$$

Look at the different components:

$$\bar{e}_r: \quad \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} = \bar{\omega} \cdot \bar{e}_r = 0 \quad \Rightarrow \quad u_\theta = u_\theta(r)$$

$$\bar{e}_\theta: \quad \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = \bar{\omega} \cdot \bar{e}_\theta = 0 \quad \Rightarrow \quad u_r = u_r(r)$$

$$\bar{e}_z: \quad \frac{1}{r}\left[\frac{\partial(r\, u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right] = \bar{\omega} \cdot \bar{e}_z = \omega(r)$$

The rotational symmetry implies $\dfrac{\partial}{\partial \theta} = 0$. From the $\bar{e}_z$ component we get,

$$\frac{\partial}{\partial r}(r\, u_\theta) = r\, \omega(r) \tag{B.1}$$

We also have the incompressibility condition:

$$\nabla \cdot \bar{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial u_z}{\partial z}}_{\gamma} = 0$$

From this we get,

$$\frac{\partial}{\partial r}(r u_r) = -r\gamma \tag{B.2}$$

Equation (2) is valid both for case a) and b). Case a) $\omega(r) = 0$

$$(1) \Rightarrow \quad \frac{\partial}{\partial r}(r\, u_\theta) = 0 \quad \Rightarrow \quad u_\theta = \frac{A}{r}$$

$$(2) \Rightarrow \frac{\partial}{\partial r}(r u_r) = -r\gamma \quad \Rightarrow \quad \text{integrate the rhs} \quad r u_r = -\frac{r^2\gamma}{2} + B$$

$$\Rightarrow \quad u_r = -\frac{\gamma\, r}{2} + \frac{B}{r}$$

There should be no mass source at $r = 0$. The mass flow is,

$$Q = \int_0^{2\pi} u_r\, r\, d\theta = 2\pi\, B - 2\pi\frac{\gamma\, r^2}{2}.$$

When $r \to 0$ then $Q \to 2\pi B$ which gives $B = 0$.

$$u_r = -\frac{\gamma\, r}{2} \quad \text{for both a) and b)!}$$

The circulation Γ is,

$$\Gamma = \int_0^{2\pi} u_\theta \, r \, d\theta = 2\pi A \Rightarrow A = \frac{\Gamma}{2\pi}$$

Case b) $\omega(r) = \omega_0 \, e^{-r^2/a^2}$,

$$(1) \Rightarrow \quad \frac{\partial}{\partial r}(r \, u_\theta) = r \, \omega_0 \, e^{-r^2/a^2}$$

Integrate the right hand side:

$$r \, u_\theta = -\frac{\omega_0 \, a^2}{2} e^{-r^2/a^2} + C \quad \Rightarrow \quad u_\theta = -\frac{\omega_0 \, a^2}{2r} e^{-r^2/a^2} + \frac{C}{r}$$

Look at the circulation,

$$\Gamma(r) = \int_0^{2\pi} u_\theta \, r \, d\theta = 2\pi \left(-\frac{\omega_0 \, a^2}{2} e^{-r^2/a^2} + C \right)$$

$$\Rightarrow \quad \text{for } r = 0 \text{ we get} \quad \Gamma_0 = 2\pi \left(-\frac{\omega_0 \, a^2}{2} + C \right) \quad \Rightarrow \quad C = \frac{\Gamma_0}{2\pi} + \frac{\omega_0 \, a^2}{2}$$

This gives

$$u_\theta = \frac{\Gamma_0}{2\pi r} + + \frac{\omega_0 \, a^2}{2r} \left[1 - e^{-r^2/a^2} \right]$$

c) Consider circles $C(t)$ following the flow with a radius $R(t)$ and position $Z(t)$. Compute $R(t)$ and $Z(t)$. Can any of the flow cases be inviscid?

$$R(0) = R_0 \quad Z(0) = Z_0$$

$$\frac{dR}{dt} = u_r(R) = -\frac{1}{2}\gamma R \quad \Rightarrow \quad R(t) = R_0 \, e^{-\frac{1}{2}\gamma t}$$

$$\frac{dZ}{dt} = u_z(Z) = \gamma Z \quad \Rightarrow \quad Z(t) = Z_0 \, e^{\gamma t}$$

The circulation is then,

$$\Gamma(t) = \int_0^{2\pi} u_\theta(R) R \, d\theta = 2\pi R u_\theta(R) = \begin{cases} \Gamma & \text{a)} \\ \Gamma_0 + \pi\omega_0 a^2 \left[1 - e^{-R^2/a^2} \right] & \text{b)} \end{cases}$$

The circulation is constant for a) but not for b). This means that the flow in a) can be inviscid.

Inviscid and irrotational vortices

Consider a circular flow with $\bar{u} = u_\theta(r)\bar{e}_\theta$. Which vortices are inviscid and which vortices are irrotational?

The Navier–Stokes equation for u_θ

$$\frac{\partial u_\theta}{\partial t} = -\frac{1}{\rho r} + \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) - \frac{u_\theta}{r^2} \right)$$

For inviscid flow we require,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) - \frac{u_\theta}{r^2} = 0$$

Make the ansatz $u_\theta = r^n$,

$$\frac{1}{r} \frac{\partial}{\partial r} (r n r^{n-1}) - r^{n-2} = \frac{1}{r} n^2 r^{n-1} - r^{n-2} \quad \rightarrow \quad n^2 - 1 = 0 \quad \Rightarrow n = \pm 1$$

We get the inviscid flow,

$$u_\theta(r) = \underbrace{Ar}_{\text{solid body rotation}} + \underbrace{\frac{B}{r}}_{\text{irrotational}}$$

The vorticity is,

$$\bar{\omega} = \nabla \times \bar{u} = \{A.32\} = \frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) \bar{e}_z$$

$$\bar{\omega} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial r} (r u_\theta) = 0 \quad \Rightarrow \quad u_\theta = \frac{C}{r}$$

Conclusion:

$$\text{Irrotational} \Rightarrow \text{inviscid}$$
$$\text{Inviscid} \nRightarrow \text{irrotational}$$

Exercise 3

Show that the inviscid vorticity equation

$$\frac{D\bar{\omega}}{Dt} = (\bar{\omega} \cdot \nabla)\bar{\mathbf{u}}$$

reduces to the equation

$$\frac{D}{Dt}\left(\frac{\omega}{r}\right) = 0$$

in the case of axisymmetric flow:

$$\bar{\mathbf{u}} = u_r(r, z, t)\bar{\mathbf{e}}_r + u_z(r, z, t)\bar{\mathbf{e}}_z$$

The vorticity in an axisymmetric flow

$$\bar{\omega} = \nabla \times \bar{\mathbf{u}} = \underbrace{\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)}_{\omega}\bar{\mathbf{e}}_\theta = \omega\bar{\mathbf{e}}_\theta$$

Study the right hand side of the inviscid vorticity equation

$$(\bar{\omega} \cdot \nabla) = \frac{\omega}{r}\frac{\partial}{\partial\theta} \quad \Rightarrow \quad (\bar{\omega} \cdot \nabla)\bar{\mathbf{u}} = \frac{\omega}{r}\frac{\partial}{\partial\theta}\left(u_r(r, z, t)\bar{\mathbf{e}}_r + u_z(r, z, t)\bar{\mathbf{e}}_z\right) =$$

$$\frac{\omega}{r}\underbrace{\frac{\partial u_r}{\partial\theta}}_{=0}\bar{\mathbf{e}}_r + \frac{\omega}{r}u_r\underbrace{\frac{\partial\bar{\mathbf{e}}_r}{\partial\theta}}_{=\bar{\mathbf{e}}_\theta} + \frac{\omega}{r}\underbrace{\frac{\partial u_z}{\partial\theta}}_{=0}\bar{\mathbf{e}}_z + \frac{\omega}{r}u_z\underbrace{\frac{\partial\bar{\mathbf{e}}_z}{\partial\theta}}_{=0} = \frac{\omega}{r}u_r\bar{\mathbf{e}}_\theta$$

The left hand side of the inviscid vorticity equation gives

$$\frac{D\bar{\omega}}{Dt} = \frac{\partial\bar{\omega}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla)\bar{\omega} = \{\bar{\omega} = \omega\bar{\mathbf{e}}_\theta\} = \left(\frac{\partial\omega}{\partial t} + \left(u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}\right)\omega\right)\bar{\mathbf{e}}_\theta$$

This gives that the inviscid vorticity equation now is

$$\frac{\partial\omega}{\partial t} + \left(u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}\right)\omega = \frac{\omega}{r}u_r$$

Multiply by $\frac{1}{r}$

$$\frac{\partial}{\partial t}\left(\frac{\omega}{r}\right) + \frac{1}{r}\left(u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}\right)\omega - \frac{\omega}{r^2}u_r = 0$$

Notice that

$$-\frac{\omega}{r^2}u_r = \omega u_r\frac{\partial}{\partial r}\frac{1}{r} \quad \text{and that} \quad \omega u_z\frac{\partial}{\partial z}\frac{1}{r} = 0$$

This means we can write

$$\frac{\partial}{\partial t}\left(\frac{\omega}{r}\right) + \frac{1}{r}\left(u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}\right)\omega + \omega\left(u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}\right)\frac{1}{r} = 0$$

And thus we have

$$\frac{\partial}{\partial t}\left(\frac{\omega}{r}\right) + \frac{1}{r}\left(u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}\right)\frac{\omega}{r} = \frac{D}{Dt}\left(\frac{\omega}{r}\right) = 0$$

B.8 Vorticity equation, Bernoulli equation and streamfunction

Solid body rotation

Consider the flow in a uniformly rotating bucket with velocity

$$\mathbf{u} = (\omega r, 0, 0)$$

a) Use Bernoulli equation to determine the free surface: What is wrong?
Bernoulli:

$$\frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2 + gz = C \quad \text{along streamlines}$$

This gives the surface of constant pressure

$$z = \frac{C - p_0}{\rho g} - \frac{\omega^2 r^2}{2g}$$

The free surface is highest in the center of the bucket, something is wrong. Bernoulli theorem is valid along a streamline in a steady ideal fluid. If the flow had been irrotational, it would have been valid everywhere. But now $\nabla \times \bar{\mathbf{u}} = (0, 0, 2\omega)$.

b) Use Euler equation to determine the free surface:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

this gives

$$\mathbf{e}_r : \quad -\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \Rightarrow \quad \frac{\partial p}{\partial r} = \rho\omega^2 r \quad \Rightarrow \quad p = \frac{1}{2}\rho\omega^2 r^2 + f(z)$$

$$\mathbf{e}_\theta : \quad 0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \quad \Rightarrow \quad \frac{\partial p}{\partial \theta} = 0$$

$$\mathbf{e}_z : 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \quad \Rightarrow \quad \frac{\partial f}{\partial z} = -\rho g \quad \Rightarrow \quad f = \rho g z + p_0$$

Thus

$$p(r, z) = \frac{1}{2}\rho\omega^2 r^2 - \rho g z + p_0$$

This gives the free surface where $p = p_0$

$$z = \frac{\omega^2 r^2}{2g}$$

Flow over a hill (Exam 20020527 question 2)

A hill with the height h has the shape of a half circular cylinder as shown in Figure 1. Far from the hill the wind U_∞ is blowing parallel to the ground in the x-direction and the atmospheric pressure at the ground is p_0.
a) (5) Assume potential flow and show that the stream function in cylindrical coordinates is of the form

$$\psi = f(r)\sin\theta,$$

where $f(r)$ is an arbitrary function. Calculate the velocity field above the hill.
b) (3) Derive an equation for the curve with constant vertical wind velocity V.
c) (2) Assume that the density ρ and the gravitational acceleration g is constant. Calculate the atmospheric pressure at the top of the hill.
a) The stream function satisfies continuity:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

The flow is irrotational:

$$\nabla \times \bar{u} = 0 \quad \Rightarrow \quad \frac{\partial \psi}{\partial r} + r\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \quad (1)$$
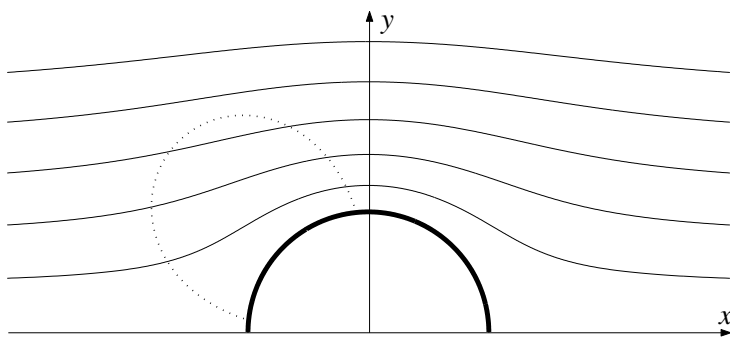
Figure B.5: Streamlines above a hill with $h = 100m$ and $U_\infty = 5m/s$. A paraglider pilot with a sink of $1m/s$ will find lift in the area within the dotted line, while soaring along the hill.

Introduce the ansatz $\psi = f(r)\sin\theta$ into equation (1):

$$f'\sin\theta + rf''\sin\theta - \frac{1}{r}f\sin\theta = 0 \quad \Rightarrow$$

$$f' + rf'' - \frac{1}{r}f = 0$$

Make the ansatz $f = r^n$:

$$nr^{n-1} + rn(n-1)r^{n-2} - \frac{1}{r}r^n = 0 \quad \Rightarrow$$

$$n + n^2 - n - 1 = 0 \quad \Rightarrow \quad n = \pm 1$$

So we have

$$\psi = \left(Ar + \frac{B}{r} \right)\sin\theta$$

We need two boundary conditions.
1. Free stream:

$$Ar\sin\theta = U_\infty r\sin\theta \quad \Rightarrow \quad A = U_\infty$$

2. Streamline on the hill surface:

$$U_\infty h + \frac{B}{h} = 0 \quad \Rightarrow \quad B = -U_\infty h^2$$

So we have:

$$\psi = U_\infty \left(r - \frac{h^2}{r} \right)\sin\theta$$

Now we can calculate the velocity field above the hill:

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U_\infty \left(1 - \frac{h^2}{r^2} \right)\cos\theta$$

$$u_\theta = -\frac{\partial\psi}{\partial r} = -U_\infty \left(1 + \frac{h^2}{r^2} \right)\sin\theta$$

b) Constant vertical wind velocity is described by:

$$V = u_r\sin\theta + u_\theta\cos\theta = U_\infty \left(1 - \frac{h^2}{r^2} \right)\cos\theta\sin\theta - U_\infty \left(1 + \frac{h^2}{r^2} \right)\sin\theta\cos\theta = -2U_\infty\frac{h^2}{r^2}\sin\theta\cos\theta \quad \Rightarrow$$

$$\underline{r = h\sqrt{-2\frac{U_\infty}{V}\sin\theta\cos\theta}}$$

c) Use Bernoulli equation with free stream pressure p_0 at the ground:

$$p_o + \frac{1}{2}\rho U_\infty^2 = p + \frac{1}{2}\rho(2U_\infty)^2 + \rho gh \quad \Rightarrow p = p_o - \frac{3}{2}\rho U_\infty^2 - \rho gh$$

Stokes stream function:

Consider a 2D incompressible flow

$$\nabla \cdot \bar{u} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Define the stream function Ψ such that:

$$u = \frac{\partial \Psi}{\partial y} \quad v = -\frac{\partial \Psi}{\partial x}$$

This means

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y \partial x} = 0,$$

so continuity is fulfilled. Now we can write

$$\bar{u} = \nabla \times \Psi \bar{e}_z = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \Psi \end{vmatrix} = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x}, 0 \right)$$

And

$$\bar{\omega} = \nabla \times \bar{u} = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \Psi}{\partial y} & -\frac{\partial \Psi}{\partial x} & 0 \end{vmatrix} = \left(0, 0, -\frac{\partial \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial y^2} \right) = -\nabla^2 \Psi \bar{e}_z$$

Irrotational flow

$$\nabla^2 \Psi = 0$$

In spherical coordinates for axisymmetrical flow, define Ψ

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \Psi}{\partial \theta} \quad u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial r}$$

Incompressibility is still valid

$$\nabla \cdot \bar{u} = 0$$

Velocity

$$\bar{u} = \nabla \times \frac{\Psi}{r \sin \theta} \bar{e}_\theta$$

Vorticity

$$\bar{\omega} = -\frac{1}{r \sin \theta} \left[\frac{\partial^2 \Psi}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \Psi}{\partial \theta} \right) \right]$$

Irrotational

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \Psi}{\partial \theta} \right) = 0$$

Flow around a sphere

Consider a sphere with $\Psi = 0$ at $r = a$, compute the irrotational velocity distribution when the velocity of the freestream at infinity is U:

$$r \to \infty \quad \Psi \to \frac{1}{2} U r^2 \sin^2 \theta$$

$$\Rightarrow u_r \to U \cos \theta \quad u_\theta \to -U \sin \theta$$

Make the ansatz:

$$\Psi = f(r) \sin^2 \theta$$

For an irrotational flow we get

$$f'' - \frac{2}{r^2} f = 0 \quad \Rightarrow \quad f = Ar^2 + \frac{B}{r}$$

From infinity we get

$$A = \frac{1}{2} U$$

On the sphere

$$\frac{1}{2} U a^2 + \frac{B}{a} = 0 \quad \Rightarrow \quad B = -\frac{1}{2} U a^3$$

This gives

$$\Psi = \frac{1}{2} U \left(r^2 - \frac{a^3}{r} \right) \sin^2 \theta$$

The slip velocity on the sphere is

$$u_\theta = -\frac{3}{2} U \sin \theta$$

The radial velocity is $u_r = 0$.

$i-1$ The flow around a submarine moving at a velocity V can be described by the flow caused by a source and
i a sink with strength Q at a distance $2a$ from each other.

$i+1$
$j-1$ V

j
$j+1$
iteration direction

x
y
1 $p_{1,1}$
n

y
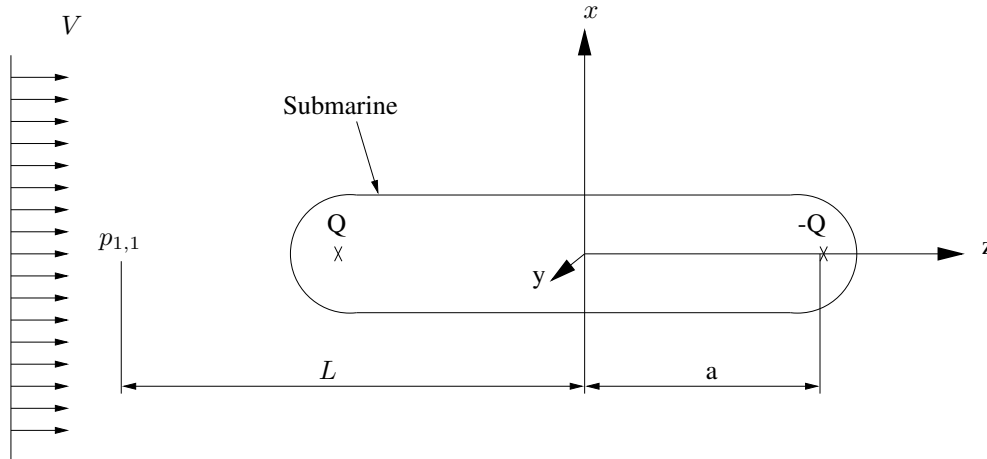i
j
fine grid (1)
coarse grid (2)

Figure B.6: Coordinate system for submarine problem

a) If one wants to construct a pressure sensor that will register an approaching submarine at a distance L, what sensitivity is needed for the sensor? Assume an ideal fluid and that $2a = 80$ m, $Q = 915\,\mathrm{m^3/s}$, $U = 8$ m/s, $L = 200$ m and $\rho = 1000\,\mathrm{kg/m^3}$.

Use a potential flow description

$$\bar{u} = \nabla\phi, \quad u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}$$

The flow is always irrotational due to the definition of the velocity potential

$$\omega = \nabla \times \bar{u} = \nabla \times \nabla\phi = 0, \quad \text{curl(grad)=0}$$

For incompressibility we get

$$\nabla \cdot \bar{u} = \nabla \cdot \nabla\phi = \frac{\partial(\nabla\phi)_i}{\partial x_i} = \Delta\phi = 0$$

The equation is linear and thus superposition can be used. We have freestream plus 3D source plus sink

$$\phi = \underbrace{U\,z}_{\text{freestream}} + \underbrace{\frac{-Q}{4\pi\,r_1}}_{\text{source}} + \underbrace{\frac{Q}{4\pi\,r_2}}_{\text{sink}}$$

The first term is in cylindrical coordinates (R, θ, z) and the two last are in two different spherical coordinate systems with origin in $z-a$ and $z+a$, respectively. Transform the two second terms to cylindrical coordinates

$$\phi = Uz + \frac{-Q}{4\pi\sqrt{(z+a)^2 + R^2}} + \frac{Q}{4\pi\sqrt{(z-a)^2 + R^2}}$$

Velocity

$$\bar{u} = \nabla\phi = \frac{\partial\phi}{\partial R}\bar{e}_R + \frac{1}{R}\frac{\partial\phi}{\partial\theta}\bar{e}_\theta + \frac{\partial\phi}{\partial z}\bar{e}_z =$$

$$\bar{e}_R\left\{\frac{QR}{4\pi((z+a)^2 + R^2)^{3/2}} - \frac{QR}{4\pi((z-a)^2 + R^2)^{3/2}}\right\} + \bar{e}_z\left\{U + \frac{Q(z+a)}{4\pi((z+a)^2 + R^2)^{3/2}} - \frac{Q(z-a)}{4\pi((z-a)^2 + R^2)^{3/2}}\right\}$$

We need to know the distance, b, from the point source to the stagnation point on the submarine nose. Thus we need to know the length of the submarine.

b) How long is the submarine?
Compute where $u_z = 0$ for $R = 0$

$$U + \frac{Q}{4\pi}\left(\frac{1}{(z+a)^2} - \frac{1}{(z-a)^2}\right) = 0$$

$$-\frac{4\pi U}{Q} = \frac{(z-a)^2 - (z+a)^2}{(z-a)^2(z+a)^2} = \frac{-4az}{(z^2 - a^2)^2}$$

Solving this system gives $z = \pm 43.01$ m, $z = \pm 36.99$ m, where the second solution lies inside the submarine. The length is then $2 * 43.01 \approx 86$ m and $b = 3.01$ m.

Now we continue to solve a)
Use Bernoulli equation to determine the pressure fluctuations at $z = -L - b - a, R = 0$

$$p + \frac{1}{2}\rho|\bar{u}|^2 = p_\infty + \frac{1}{2}\rho U^2$$

Evaluate $\bar{u}$ noticing that $u_R = 0$

$$\bar{u}(z = -L - b - a, R = 0) = \bar{e}_z\left\{U + \frac{-Q}{4\pi(L+b)^2} + \frac{Q}{4\pi((L+b+2a)^2}\right\}$$

Inserting the given values gives $|\bar{u}| = 7.99914$ m/s and we get

$$p - p_\infty = \frac{1}{2}\rho(U^2 - |\bar{u}|^2) \approx 6.86 \text{ N/m}^2 \approx 0.07 \text{ mbar}$$

c) How wide is the submarine?
To get this we need to compute the shape of the submarine. The stream function is constant along streamlines and is useful for this. In spherical coordinates the stream function is defined as

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta}$$

$$u_\theta = -\frac{1}{r \sin\theta} \frac{\partial\psi}{\partial r}$$

Transformation between cylindrical and spherical coordinates

$$u_r = u_R \sin\theta + u_z \cos\theta$$

$$R = r\sin\theta, z = r\cos\theta$$

Our velocity field gives

$$\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} = \sin\theta\frac{Q}{4\pi}r\sin\theta\left\{\frac{1}{(r^2+a^2+2ar\cos\theta)^{3/2}} - \frac{1}{(r^2+a^2-2ar\cos\theta)^{3/2}}\right\} +$$

$$\cos\theta\left(U + \frac{Q}{4\pi}\left\{\frac{r\cos\theta+a}{(r^2+a^2+2ar\cos\theta)^{3/2}} - \frac{r\cos\theta-a}{(r^2+a^2-2ar\cos\theta)^{3/2}}\right\}\right)$$

$$= U\cos\theta + \frac{Q}{4\pi}\left(\frac{r+a\cos\theta}{(r^2+a^2+2ar\cos\theta)^{3/2}} - \frac{r-a\cos\theta}{(r^2+a^2-2ar\cos\theta)^{3/2}}\right)$$

This is difficult to integrate.

$^{p_{1,1}}$Simplify to a Rankine body by neglecting the sink and say that $a = 0$

$$u_r = U\cos\theta + \frac{Q}{4\pi}\frac{1}{r^2} = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \quad \Rightarrow$$

$$\Psi = \frac{1}{2}Ur^2\sin^2\theta - \frac{Q}{4\pi}\cos\theta + C$$

Determine C from the stagnation point

$$u_r(\theta = \pi, r_0) = 0 \quad \Rightarrow r_0^2 = \frac{Q}{4\pi U}$$

Since $\Psi = 0$ on the body we get

$$C = -\frac{Q}{4\pi}.$$

The stream function is then

$$\Psi = \frac{1}{2}UR^2\sin^2\theta - \underbrace{\frac{Q}{4\pi}(\cos\theta + 1)}_{\text{source}}$$

The shape is given by $\Psi = 0$. As $r \to \infty$, $\theta \to 0$ then $r\sin\theta \to \dfrac{d}{2}$. This gives
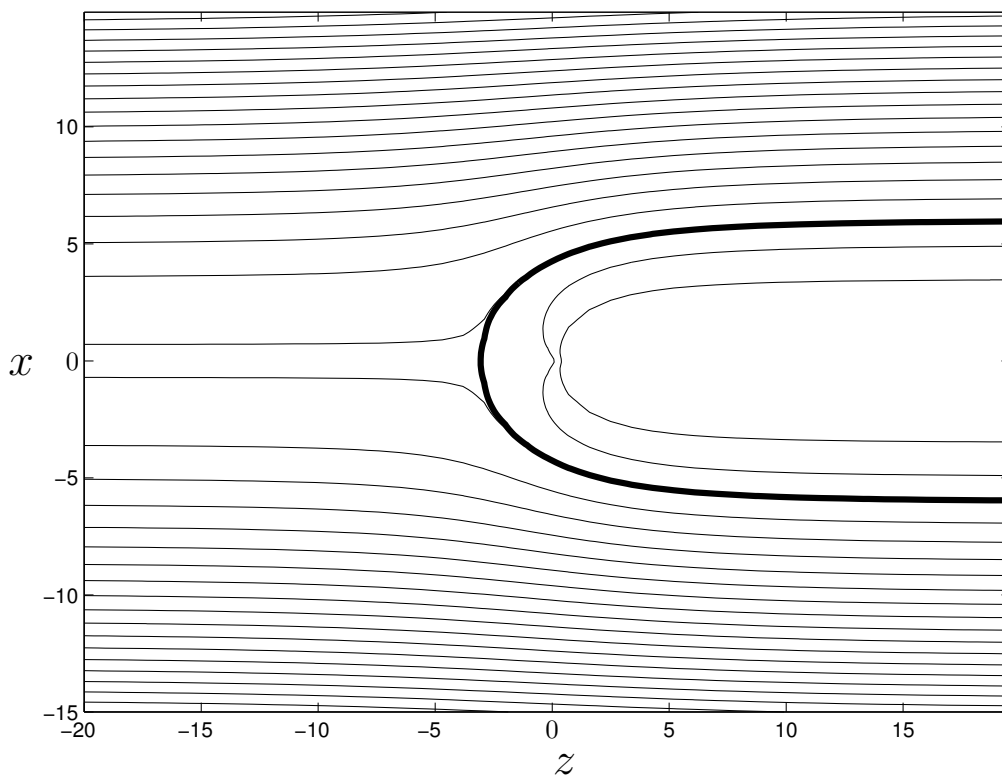
Figure B.7: Rankine body for submarine problem

$$\frac{1}{2}U\frac{d^2}{4} - 2\frac{Q}{4\pi} = 0 \quad \Rightarrow \quad d^2 = \frac{4Q}{U\pi} \quad \Rightarrow \quad d = 2\sqrt{\frac{Q}{U\pi}}$$

There is a simple way of determining the radius as $z \to \infty$ directly. The flow from the source must take up an particular area in the flow at infinity. Since no fluid can cross the streamlines this area must be equal to that of the Rankine body:

$$Q = U\pi\frac{d^2}{2} \quad \Rightarrow \quad d = 2\sqrt{\frac{Q}{U\pi}} = 12.07\,\text{m}$$

We can use the computed stream function for a point source and displace it to $z = -a$. In cylindrical coordinates

$$\Psi = \frac{-Q}{4\pi}\left(\frac{z+a}{\sqrt{(z-a)^2 + R^2}}\right)$$

Transform to spherical coordinates

$$\Psi = \frac{-Q}{4\pi}\left(\frac{r\cos\theta + a}{\sqrt{(r\cos\theta + a)^2 + r^2\sin^2\theta}}\right) = \frac{-Q}{4\pi}\left(\frac{r\cos\theta + a}{\sqrt{r^2 + a^2 + 2ar\cos\theta}}\right)$$

The stream function for the submarine is then

$$\Psi = \frac{1}{2}Ur^2\sin^2\theta + \frac{Q}{4\pi}\left(-\frac{r\cos\theta + a}{\sqrt{r^2 + a^2 + 2ar\cos\theta}} + \frac{r\cos\theta - a}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}\right)$$
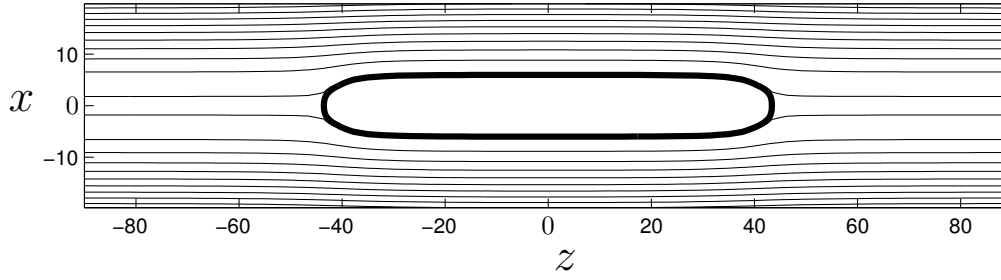
Figure B.8: Submarine body for submarine problem

At $r = R$ and $\theta = \pi/2$ we get

$$\Psi = \frac{1}{2}UR^2 + \frac{Q}{4\pi}\left(-\frac{a}{\sqrt{R^2 + a^2}} + \frac{-a}{\sqrt{R^2 + a^2}}\right)$$

For the body $\Psi = 0$ and we get

$$R^2\sqrt{R^2 + a^2} - \frac{aQ}{\pi U} = 0$$

Multiply by

$$R^2\sqrt{R^2 + a^2} + \frac{aQ}{\pi U} \quad \Rightarrow$$

$$(R^2)^3 + (R^2)^2 a^2 - \frac{a^2 Q^2}{\pi^2 U^2} = 0$$

Computing this

$$d = 2R = 12.0006\,\text{m}$$

$m+2$

$\frac{1}{h}$The lines with constant stream function Ψ are the streamlines. They are orthogonal to the lines of constant
Vvelocity potential ϕ which are equipotential lines. Since both of them satisfy Laplace's equation we can
$V_h$define a complex function

$$F(z) = \phi(x,y) + i\,\Psi(x,y) \quad z = x + iy$$

u

u_h

Γ_0

Γ_1

Ω

: to be updated

: new values

: old values

$i-1$

i

$i+1$

$j-1$

j

$j+1$

iteration direction

x

y

1

n

x

y

i

j

fine grid (1)

coarse grid (2)
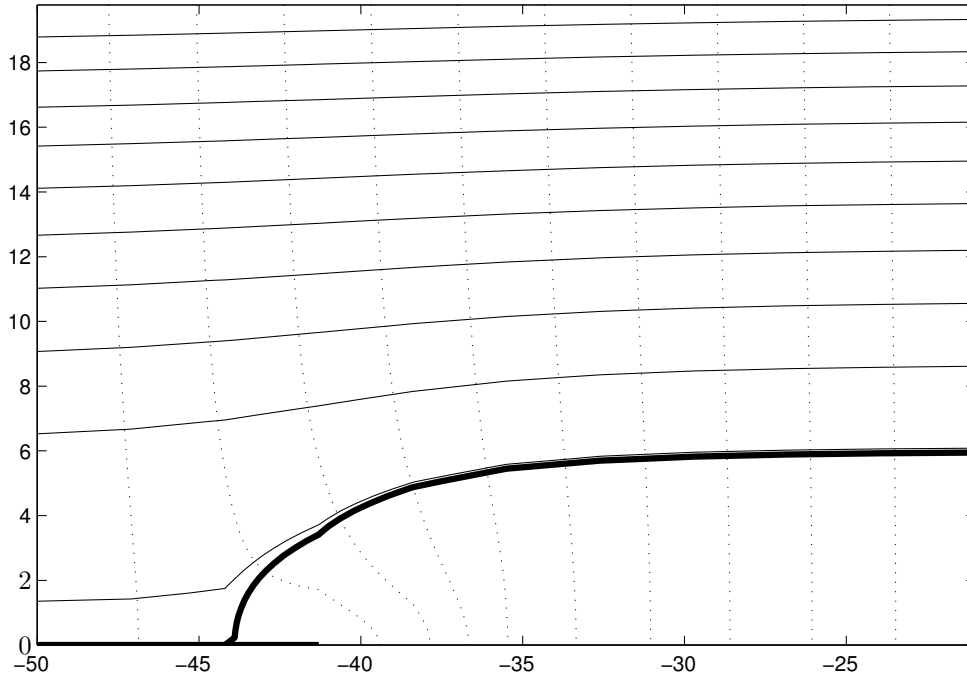
Figure B.9: Complex potential for submarine problem, solid: Ψ , dotted: ϕ

This is an analytical function since the Cauchy–Riemann equation holds

$$\frac{\partial \phi}{\partial x} = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \Psi}{\partial x}$$

The velocity is then

$$w(z) = \frac{dF}{dz} = \frac{\partial \phi}{\partial x} + i\,\frac{\partial \Psi}{\partial x} = u - iv$$

This enables the use of complex analysis, in particular conformal mapping that can be used to compute
the flow over airfoil shapes.

m

$m+2$

1

h

V

V_h

u

u_h

Γ_0

Γ_1

Ω

to be updated

: new values

: old values

$i-1$

i

$i+1$

$j-1$

j

$j+1$

ration direction

x

y

n

x

y

i

j

fine grid (1)

coarse grid (2)

Example:

Flow past a rotating cylinder centered at $z = \lambda$ at an angle of attack α

$$F(z) = U\left[(z+\lambda)e^{-i\alpha} + \frac{(a+\lambda)^2}{(z+\lambda)}e^{i\alpha}\right] - \frac{i\Gamma}{2\pi}\log(z+\lambda)$$

Mapping by $z = \frac{1}{2}Z + \left(\frac{1}{4}Z^2 - a^2\right)^{\frac{1}{2}}$ gives an airfoil shape with the potential F(z). A correct flow is not achieved unless the Kutta–Joukovski condition is satisfied requiring

$$\Gamma = -4\pi U(a+\lambda)\sin\alpha$$
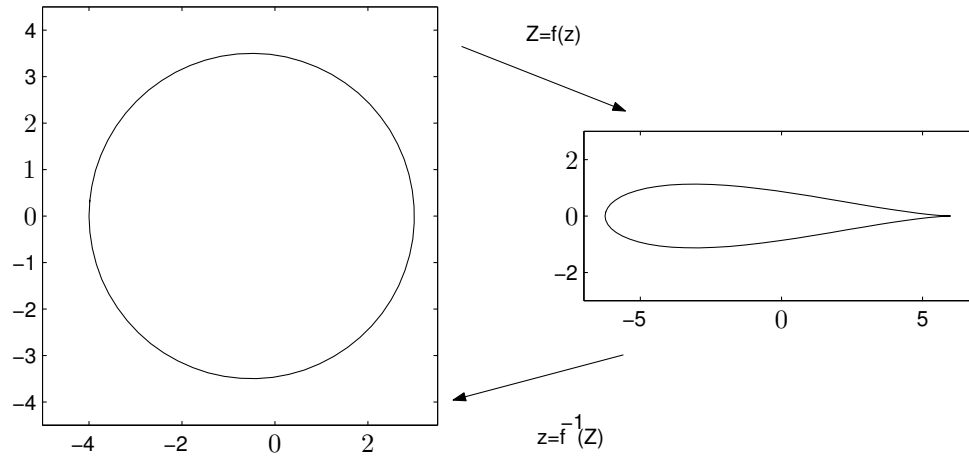
Figure B.10: Conformal mapping from circle to airfoil shape, $(a = 3, \lambda = 0.5)$

B.10   More potential flow

Half body over a wall

A line source of strength Q is located at $(0, a)$ above a flat plate that coincides with the x-axis. A uniform stream with velocity U flows along the x-axis. Calculate the irrotational flow field.

Method of images. Put a line source of equal strength at $(0, -a)$ in order to fulfill the condition of no flow through the plate. Superposition of a uniform flow and the two line sources gives the complex potential

$$F = Uz + \frac{Q}{2\pi} \ln(z - ia) + \frac{Q}{2\pi} \ln(z + ia)$$

Complex velocity

$$W = \frac{dF}{dz} = U + \frac{Q}{2\pi} \left(\frac{1}{z - ia} + \frac{1}{z + ia} \right) \quad \Rightarrow$$

$$W = U + \frac{Q}{2\pi} \left(\frac{1}{x + i(y - a)} + \frac{1}{x + i(y + a)} \right) \quad \Rightarrow$$

$$W = U + \frac{Q}{2\pi} \left(\frac{x - i(y - a)}{x^2 + (y - a)^2} + \frac{x - i(y + a)}{x^2 + (y + a)^2} \right) \quad \Rightarrow$$

$$W = U + \frac{Q}{2\pi} \left[\frac{x}{x^2 + (y - a)^2} + \frac{x}{x^2 + (y + a)^2} - i\left(\frac{y - a}{x^2 + (y - a)^2} + \frac{y + a}{x^2 + (y + a)^2} \right) \right]$$

The velocity field now becomes

$$u = U + \frac{Q}{2\pi} \left(\frac{x}{x^2 + (y - a)^2} + \frac{x}{x^2 + (y + a)^2} \right)$$

$$v = \frac{Q}{2\pi} \left(\frac{y - a}{x^2 + (y - a)^2} + \frac{y + a}{x^2 + (y + a)^2} \right)$$

to be updated

: new values

: old values

$i-1$

i

$i+1$
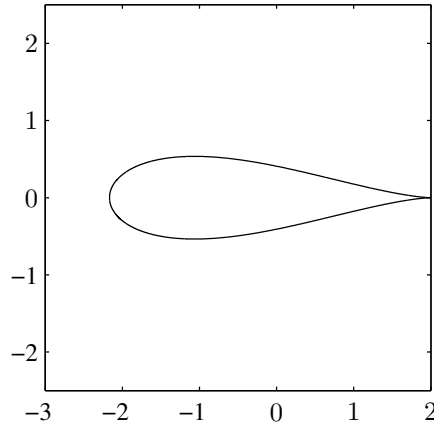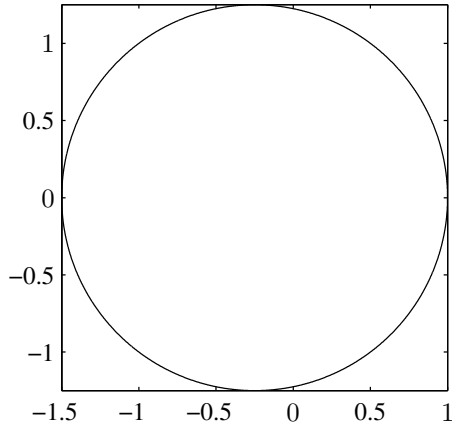
$j-1$

j

$j+1$

ration direction

x

y

n

x

y

i

j

fine grid (1)

coarse grid (2)

a) Use conformal mapping to calculate the irrotational flow field around a symmetric airfoil.

Joukowski transformation

$$\zeta(z) = z + \frac{c^2}{z}$$

Equation for the circle

$$z = -\lambda + (a+\lambda)e^{i\theta}$$

Equation for the airfoil

$$\zeta = -\lambda + (a+\lambda)e^{i\theta} + \frac{a^2}{-\lambda + (a+\lambda)e^{i\theta}}$$

Complex potential in the z-plane

$$F = U(z+\lambda)e^{-i\alpha} + U\frac{(a+\lambda)^2}{(z+\lambda)}e^{i\alpha} + \frac{i\Gamma}{2\pi}\ln(z+\lambda)$$

Complex velocity in the z-plane

$$W = \frac{dF}{dz} = Ue^{-i\alpha} - U\frac{(a+\lambda)^2}{(z+\lambda)^2}e^{i\alpha} + \frac{i\Gamma}{2\pi(z+\lambda)}$$

Complex velocity in the ζ-plane

$$\omega = \frac{dF/dz}{d\zeta/dz} = \left(Ue^{-i\alpha} - U\frac{(a+\lambda)^2}{(z+\lambda)^2}e^{i\alpha} + \frac{i\Gamma}{2\pi(z+\lambda)}\right)\bigg/\left(1 - \frac{a^2}{z^2}\right)$$

The velocity can then be found by introducing the reversed transformation $z = \zeta/2 + \sqrt{\zeta^2/2 - a^2}$ into

$$u_* = Re\{\omega(z)\}, \quad v_* = -Im\{\omega(z)\}$$

b) Calculate the Joukowski condition for the airfoil.

The flow field has a singular point at the trailing edge of the airfoil at $\zeta = 2a$. Resolve the singularity by choosing the circulation Γ so the numerator vanishes at the trailing edge $z = a$

$$Ue^{-i\alpha} - Ue^{i\alpha} + \frac{i\Gamma}{2\pi(a+\lambda)} = 0 \quad \Rightarrow$$

$$\Gamma_k = 4\pi(a+\lambda)U\frac{e^{i\alpha} - e^{-i\alpha}}{2i} = 4\pi(a+\lambda)U\sin\alpha$$

B.11   Boundary layers

Exercise 1

Consider the high Reynolds number flow in a conversing channel. Compute the boundary layer over the surface at $y = 0$.

Assume the free stream:

$$U(x) = -\frac{Q}{x},$$

where Q is the flux. The boundary layer equations read:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{Q}{x^3} + \nu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2)$$

Seek a similarity solution where the dimensionless velocity u/U only depend on the dimensionless wall distance

$$\eta = \frac{y}{\delta} = \frac{y}{\sqrt{\frac{x\nu}{|U|}}} = \frac{y}{x\sqrt{\frac{\nu}{Q}}} = \frac{y}{x}\sqrt{\frac{Q}{\nu}}$$

The stream function satisfies (2):

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Make the stream function dimensionless:

$$f(\eta) = \frac{\psi}{U\delta}$$

Determine the terms in equation (1):

$$u = \frac{\partial \psi}{\partial y} = U\delta\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} = U\delta f'\frac{1}{\delta} = Uf' = \underline{-\frac{Q}{x}f'}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial(U\delta f)}{\partial x} = -U\delta\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} - U\frac{\partial \delta}{\partial x}f - \frac{\partial U}{\partial x}\delta f =$$

$$\frac{Q}{x}x\sqrt{\frac{\nu}{Q}}f'\left(-\frac{\eta}{x}\right) + \frac{Q}{x}\sqrt{\frac{\nu}{Q}}f - \frac{Q}{x^2}x\sqrt{\frac{\nu}{Q}}f = \underline{-\frac{\sqrt{Q\nu}}{x}\eta f'}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{Q}{x}f'\right) = -\frac{Q}{x}\frac{\partial f'}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{Q}{x^2}f' = -\frac{Q}{x}f''\left(-\frac{\eta}{x}\right) + \frac{Q}{x^2}f' = \underline{\frac{Q}{x^2}(\eta f'' + f')}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{Q}{x}f'\right) = -\frac{Q}{x}\frac{\partial f'}{\partial \eta}\frac{\partial \eta}{\partial y} = -\frac{Q}{x}f''\frac{1}{\delta} = \underline{-\frac{Q}{x^2}\sqrt{\frac{Q}{\nu}}f''}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{Q}{x^2}\sqrt{\frac{Q}{\nu}}\frac{\partial f''}{\partial \eta}\frac{\partial \eta}{\partial y} = \underline{-\frac{Q^2}{x^3\nu}f'''}$$

Insert into equation (1):

$$-\frac{Q}{x}f'\frac{Q}{x^2}(\eta f'' + f') + \frac{\sqrt{Q\nu}}{x}\eta f'\frac{Q}{x^2}\sqrt{\frac{Q}{\nu}}f'' = -\frac{Q^2}{x^3} - \nu\frac{Q^2}{x^3\nu}f'''$$

Divide by Q^2/x^3:

$$-\eta f'f'' - f'^2 + \eta f'f'' = -1 - f''' \quad \Rightarrow$$

$$f''' - f'^2 + 1 = 0$$

Boundary conditions:

$$\begin{cases} u(0) = 0 \\ u(\infty) = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} f'(0) = 0 \\ f'(\infty) = 1 \end{cases}$$

M^2
M
m
$m+2$
1
h
V
V_h
u
u_h
Γ_0
Γ_1
Ω
to be updated
: new values
: old values
$i-1$
i
$i+1$
$j-1$
j
$j+1$
ration direction
x
y
n
x
y
i
j
fine grid (1)
coarse grid (2)

Substitute $F(\eta) = f'(\eta)$:

$$\begin{cases} F'' - F^2 + 1 = 0 \\ F(0) = 0 \\ F(\infty) = 1 \end{cases}$$

Solution:

$$F = 3\tanh^2\left(\frac{\eta}{\sqrt{2}} + \sqrt{\frac{2}{3}}\right) - 2$$

Exercise 2

Consider the flow downstream of a 2D streamlined body at high Reynolds number. Study the thin wake downstream of the body where variations in y are much more rapid than variations in the downstream direction. Also assume that the wake is so small that we can write $u = U + u_1$ where u_1 is negative and describes the wake. The length scale in x is L and in y it is δ with $\delta << L$.
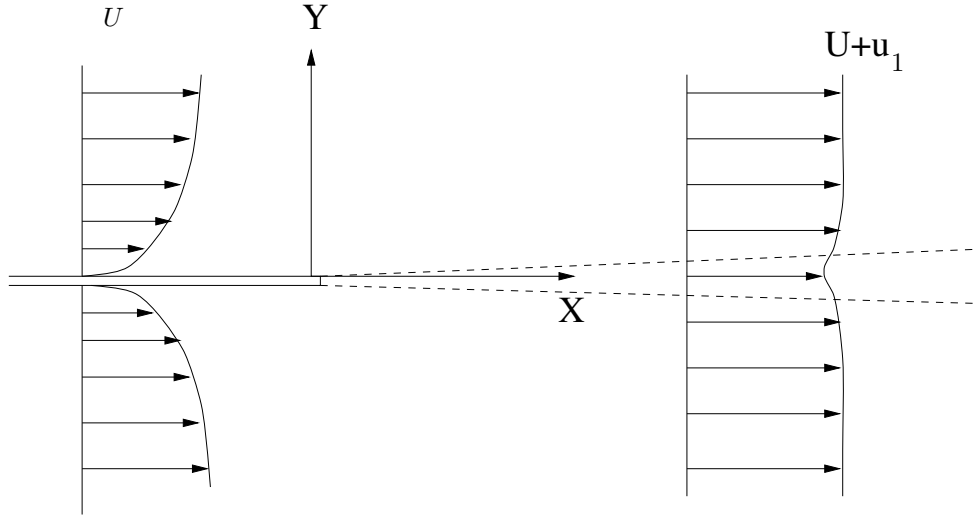
Figure B.11: Coordinate system for wake problem

a) Find the governing (linear) equations:
We start with the boundary layer equations since $\delta << L$,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

In the wake we have no pressure gradient,

$$\frac{\partial p}{\partial x} = 0.$$

From the continuity equation we get,

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}\left(U + u_1\right) = -\frac{\partial u_1}{\partial x} \quad \Rightarrow \quad v \sim \frac{\delta}{L}u_1 << U$$

Inserting $u = U + u_1$ in the boundary layer equations gives,

$$U\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_1}{\partial x} + v\frac{\partial u_1}{\partial y} = 0 + \nu\frac{\partial^2 u_1}{\partial y^2}$$

Neglect the quadratic terms,

$$u_1\frac{\partial u_1}{\partial x} \sim \frac{u_1^2}{L} \quad \text{and} \quad v\frac{\partial u_1}{\partial y} \sim \frac{\delta}{L}u_1\frac{1}{\delta}u_1 \sim \frac{u_1^2}{L}$$

$$\Rightarrow \quad U\frac{\partial u_1}{\partial x} = \nu\frac{\partial^2 u_1}{\partial y^2}$$

b) Show that $\int_{-\infty}^{\infty} u_1 \, dy = $ constant:

Consider the relation,

$$Q_1 = \int_{-\infty}^{\infty} u_1 \, dy$$

which gives the linear contribution from u_1 to the momentum flux.

$$\frac{d}{dx}Q_1 = \frac{d}{dx}\int_{-\infty}^{\infty} u_1 \, dy = \int_{-\infty}^{\infty} \frac{\partial u_1}{\partial x} \, dy = \{\text{From the governing equation}\} = \int_{-\infty}^{\infty} \frac{\nu}{U}\frac{\partial^2 u_1}{\partial y^2} \, dy = \frac{\nu}{U}\left[\frac{\partial u_1}{\partial y}\right]_{-\infty}^{\infty} = 0$$

This means that Q_1 is constant.

c) Find a similarity solution for u_1:

Seek a solution on the form,

$$u_1(x,y) = F(x)f(\eta) \quad \text{where} \quad \eta = \frac{y}{g(x)}$$

This gives,

$$Q_1 = \int_{-\infty}^{\infty} F(x)f(\eta) \, dy = F(x)g(x)\int_{-\infty}^{\infty} f(\eta) \, d\eta$$

$Q_1 = $ constant requires,

$$F(x) = 1/g(x) \quad \text{and} \quad \int_{-\infty}^{\infty} f(\eta) \, d\eta = Q_1 = \text{constant}$$

This means,

$$u_1(x,y) = \frac{1}{g(x)}f(\eta) = \frac{1}{g(x)}f\left(\frac{y}{g(x)}\right)$$

Insert this into the equation of motion,

$$-U\frac{g'(x)}{g(x)^2}\left(\eta f'(\eta) + f(\eta)\right) = \nu\frac{1}{g(x)^3}f''(\eta) \quad \Rightarrow$$

$$\frac{U}{\nu}g'(x)g(x) = -\frac{f''(\eta)}{\eta f'(\eta) + f(\eta)} = C \quad \Rightarrow$$

$$g(x) = \left(2C\frac{\nu x}{U}\right)^{1/2}$$

This gives the equation for f,

$$f''(\eta) + C\eta f'(\eta) + Cf(\eta) = \frac{d}{d\eta}\left(f'(\eta) + C\eta f(\eta) + D\right) = 0$$

From the symmetry condition $\frac{\partial}{\partial y}u(x,0) = 0$ we can determine that $f'(0) = -D = 0$. This gives,

$$f'(\eta) + C\eta f(\eta) = 0$$

Integrating this we get,

$$f(\eta) = ae^{-\frac{1}{2}C\eta^2}$$

We can determine the constant a from the condition that Q_1 is constant,

$$\int_{-\infty}^{\infty} f(\eta) \, d\eta = Q_1 \quad \Rightarrow \quad f(\eta) = \sqrt{\frac{C}{2\pi}}Q_1 e^{-\frac{1}{2}C\eta^2}$$

This now gives u_1:

$$u_1(x,y) = \sqrt{\frac{C}{2\pi}}\frac{Q_1}{g(x)}e^{-\frac{1}{2}C\frac{y^2}{g(x)^2}} =$$

$$\left\{ g(x) = \left(2C\frac{\nu x}{U} \right)^{1/2} \right\} = \underline{\frac{Q_1}{2\sqrt{\pi}} \left(\frac{U}{\nu x} \right)^{1/2} e^{-\frac{U y^2}{4\nu x}}}$$

d) Relate u_1 to the drag F_D:

The drag is

$$F_D = -\rho \int_{-\infty}^{\infty} u_1 (U + u_1) \, dy$$

Remember that $u_1 << U$ and neglect the u_1^2 term,

$$F_D = -\rho \int_{-\infty}^{\infty} U u_1 \, dy = -\rho U Q_1$$

This means that

$$Q_1 = -\frac{F_D}{\rho U} \quad \Rightarrow \quad u_1(x,y) = -\frac{F_D}{2\mu\sqrt{\pi}} \left(\frac{\nu}{U x} \right)^{1/2} e^{-\frac{U y^2}{4\nu x}}$$

Exercise 3

Give an order of magnitude estimate of the Reynolds number for:

   i. Flow past the wing of a jumbo jet at 150 m/s ($\approx$ Mach 0.5)

   ii. A wing profile in salt water with $L = 2$ cm and $U = 5$ cm/s

   iii. A thick layer of golden syrup draining of a spoon.

   iv. A spermatozoan with tail length of 10^{-3} cm swimming at 10^{-1} cm/s in water.

Estimate the boundary layer thickness in case (i).

Fluid	ν cm^2/s
Water	0.01
Air	0.15
Syrup	1200

   i. Flow past the wing of a jumbo jet at 150 m/s ($\approx$ Mach 0.5)

$$U = 150 \, \text{m/s}, \nu = 1.5 \, 10^{-5} \, \text{m}^2/\text{s}, L = 4 \, \text{m} \Rightarrow Re = \frac{UL}{\nu} \approx 4 \times 10^7$$

   ii. A wing profile in salt water with $L = 2$ cm and $U = 5$ cm/s

$$U = 0.05 \, \text{m/s}, \nu = 10^{-6} \, \text{m}^2/\text{s}, L = 0.02 \, \text{m} \Rightarrow Re = \frac{UL}{\nu} \approx 10^3$$

   iii. A thick layer of golden syrup draining of a spoon.

$$U = 0.04 \, \text{m/s}, \nu = 0.12 \, \text{m}^2/\text{s}, L = 0.01 \, \text{m} \Rightarrow Re = \frac{UL}{\nu} \approx 0.003$$

   iv. A spermatozoan with tail length of 10^{-3} cm swimming at 10^{-2} cm/s in water.

$$U = 10^{-4} \, \text{m/s}, \nu = 10^{-6} \, \text{m}^2/\text{s}, L = 10^{-5} \, \text{m} \Rightarrow Re = \frac{UL}{\nu} \approx 10^{-3}$$

The boundary layer thickness in (i) is $\mathcal{O}(1 \, \text{mm})$

$$\frac{\delta}{L} = \mathcal{O}\left(\frac{1}{\sqrt{Re}} \right)$$

B.12   More boundary layers

h
V
Drag on a circular cylinder

V_h Use the integral form of the Navier–Stokes equations, Consider a fix volume in a fluid with velocity $\bar{u}$. The
u continuity equation becomes,
u_h

$$\frac{d}{dt}\int_V \rho\, dV = -\oint_S \rho u_k n_k\, dS.$$

Γ_0
Γ_1
Ω The momentum equation,

$$\frac{d}{dt}\int_V \rho u_i\, dV = -\oint_S \left(\rho u_j n_j u_i + p n_i - \tau_{ij}\, n_j\right) dS + \int_V \rho F_i\, dV$$

: to be updated
  : new values
  : old values
$i-1$ Use this to compute the drag force on a cylinder:

i U_0
$i+1$
$j-1$
j
$j+1$
iteration direction
x
y n
1

x
y
i
j
fine grid (1)
coarse grid (2)

n $U \approx 0$

l d Wake n

L n

The cylinder is stationary and so is the flow $\Rightarrow$ $\dfrac{\partial \bar{u}}{\partial t} = 0$. Incompressible flow with constant density. The continuity equation gives:

$$\oint_S u_k n_k\, dS = 0 \quad\Rightarrow\quad -U_0 l - Lv|_{y=-l/2} + Lv|_{y=l/2} + \int_{l/2}^{l/2} U_w(y)\, dy = 0$$

Due to the symmetry we get,

$$-U_0 l + 2Lv|_{y=l/2} + \int_{l/2}^{l/2} U_w(y)\, dy = 0$$

The momentum equation in x:

$$\oint_S \left[\rho n_j u_j u_x + n_x\, p - n_j \tau_{xj}\right] dS = 0$$

Far away from the cylinder we can assume $p = p_0$ and $\tau_{xx} = \tau_{xy} = 0$. This gives,

$$-\rho U_0^2 l + 2\rho U_0 v|_{y=l/2} L + \int_{-l/2}^{l/2} \rho U_w^2(y)\, dy + \underbrace{\int_{\text{cyl}} [n_x p - n_j \tau_{xj}]\, dS}_{F_D}$$

Multiply the continuity equation by ρU_0,

$$-\rho U_0^2 l + 2\rho U_0 v|_{y=l/2} L = -\rho U_0 \int_{-l/2}^{l/2} U_w(y)\, dy$$

Insert into momentum equation:

$$-\rho U_0 \int_{-l/2}^{l/2} U_w(y)\, dy + \int_{-l/2}^{l/2} \rho U_w^2(y)\, dy + F_D = 0$$

$$F_D = \rho \int_{-l/2}^{l/2} \left(U_0 U_w(y) - U_w^2(y)\right) dy = \rho U_0^2 \int_{-l/2}^{l/2} \left(\frac{U_w(y)}{U_0} - \left(\frac{U_w(y)}{U_0}\right)^2\right) dy$$

Make the variable substitution $r = y/d$,

$$F_D = \rho U_0^2 d \int_{-l/2d}^{l/2d} \left(\frac{U_w(r)}{U_0} - \left(\frac{U_w(r)}{U_0} \right)^2 \right) dr$$

Now if $d << l$ we get,

$$F_D = \rho U_0^2 d \underbrace{\int_{-\infty}^{\infty} \left(\frac{U_w(r)}{U_0} - \left(\frac{U_w(r)}{U_0} \right)^2 \right) dr}_{\theta} = \rho U_0^2 \theta$$

The momentum thickness is called θ and is a measure of the loss of momentum in the flow. Now compute the drag coefficient $C_D = \dfrac{F_D}{\frac{1}{2}\rho U_0^2 d} = 2\dfrac{\theta}{d}$

Glauert wall jet

Compute an ODE for a self-similar wall jet.

The boundary layer equations read:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2)$$

Seek a similarity solution where the dimensionless velocity $u/U_m(x)$, where $U_m(x)$ is the jet core velocity, only depend on the dimensionless wall distance

$$\eta = \frac{y}{g(x)}$$

The stream function satisfies (2):

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Make the stream function dimensionless:

$$f(\eta) = \frac{\psi}{U_m(x)g(x)}$$

Determine the terms in equation (1):

$$u = \frac{\partial \psi}{\partial y} = U_m g \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} = U_m g f' \frac{1}{g} = \underline{U_m f'}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial(U_m g f)}{\partial x} = -U_m' g f - U_m g' f - U_m g f'\left(-\frac{y g'}{g^2}\right) = \underline{-U_m' g f - U_m g' f + U_m g' f' \eta}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(U_m f') = \underline{U_m' f' - U_m f'' \eta \frac{g'}{g}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(U_m f') = \underline{U_m f'' \frac{1}{g}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(U_m f'' \frac{1}{g}\right) = \underline{U_m f''' \frac{1}{g^2}}$$

Insert into equation (1):

$$U_m f'(U_m' f' - U_m \frac{g'}{g}\eta f'') + (U_m g'\eta f' - U_m' g f - U_m g' f)U_m \frac{1}{g}f'' = \nu U_m \frac{1}{g^2}f''' \quad \Rightarrow$$

$$U_m U_m' f'^2 - U_m^2 \frac{g'}{g}\eta f' f'' + U_m^2 \frac{g'}{g}\eta f' f'' - U_m U_m' f f'' - U_m^2 \frac{g'}{g}f f'' = \nu U_m \frac{1}{g^2}f''' \quad \Rightarrow$$

$$f''' + \underbrace{\left(\frac{U_m' g^2}{\nu} + \frac{U_m g' g}{\nu}\right)}_{\alpha} f f'' - \underbrace{\frac{U_m' g^2}{\nu}}_{\beta} f'^2 = 0 \quad \Rightarrow$$

$$f''' + \alpha f f'' + \beta f'^2 = 0$$

Boundary conditions:

$$\begin{cases} u(0) = 0 \\ v(0) = 0 \\ u(\infty) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} f'(0) = 0 \\ f(0) = 0 \\ f'(\infty) = 1 \end{cases}$$

Similarity requires that α and β are constant. Assume the x-dependence:

$$U_m = ax^m, \quad g = bx^n$$

Then we get

$$\beta = -\frac{1}{\nu}amx^{m-1}b^2x^{2n} = -\frac{ab^2}{\nu}m, \quad \underline{2n + m = 1}$$

$$\alpha = \frac{1}{\nu}amx^{m-1}b^2x^{2n} + \frac{1}{\nu}ax^m bn x^{n-1}bx^n = \frac{ab^2}{\nu}(m+n)$$

We can choose the length scale g such that $\alpha = 1$:

$$\frac{ab^2}{\nu} = \frac{1}{m+n} \quad \Rightarrow \quad \beta = -\frac{m}{m+n}$$

So we have:

$$f''' + ff'' + \beta f'^2 = 0 \quad (3)$$

For what β is the boundary conditions fulfilled? Work on (3):

$$f''' + ff'' + \beta f'^2 = 0 \quad \Rightarrow$$

$$f''' + (ff'' + f'^2) + (\beta - 1)f'^2 = 0$$

Integrate:

$$f'' + ff' + (\beta - 1)g = 0, \quad g(\eta) = \int_\eta^\infty f'^2 d\eta > 0$$

Multiply with f':

$$f''f' + ff'^2 + (\beta - 1)gf' = 0 \quad \Rightarrow$$
$$f''f' + (ff'^2 + gf') + (\beta - 2)gf' = 0$$

Integrate:

$$\frac{1}{2}f'^2 + fg + (\beta - 2)\int_\eta^\infty gf'd\eta = 0$$

Put in $\eta = 0$ and use the boundary conditions:

$$(\beta - 2)\int_0^\infty gf'd\eta = 0 \quad \Rightarrow \quad \beta = 2 \quad \Rightarrow$$

$$\beta = -\frac{m}{m+n} = 2 \quad \Rightarrow \quad \underline{3m + 2n = 0}$$

Solve for m and n:

$$m = -\frac{1}{2}, \quad n = \frac{3}{4}$$

B.13 Introduction to turbulence

Exercise 1

Compute the far-field two-dimensional turbulent wake $U(x,y)$ behind a cylinder.

Let
$$U_1(x,y) = U_0 - U(x,y), \quad U_s(x) = U_1(x, y = 0)$$

Assume
$$U_s \ll U_0 \quad \Rightarrow \quad U_1 \ll U_0, \quad u, v \sim U_s, \quad x \sim L, \quad y \sim l$$

Continuity
$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \quad \Rightarrow \quad \underbrace{\frac{\partial V}{\partial y}}_{\sim \frac{V}{l}} = -\frac{\partial}{\partial x}(U_0 - U_1) = \underbrace{\frac{\partial U_1}{\partial x}}_{\sim \frac{U_s}{L}} \quad \Rightarrow \quad V \sim U_s \frac{l}{L}$$

The turbulent boundary layer momentum equation read:

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial P_0}{\partial x} + \nu\frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial y}(\overline{uv}) \quad \Rightarrow$$

Disregard the pressure gradient and insert $U = U_0 - U_1$:

$$(U_0 - U_1)\left(-\frac{\partial U_1}{\partial x}\right) - V\frac{\partial U_1}{\partial y} = -\nu\frac{\partial^2 U_1}{\partial y^2} - \frac{\partial}{\partial y}(\overline{uv}) \quad \Rightarrow$$

Neglect small terms:
$$U_0\frac{\partial U_1}{\partial x} = \frac{\partial}{\partial y}(\overline{uv}) \quad (1)$$

Self-similar hypothesis:
$$\frac{U_1(x,y)}{U_s(x)} = f(\eta)$$

$$\frac{-\overline{uv}}{U_s^2} = g(\eta)$$

where
$$\eta = \frac{y}{l(x)} \quad \Rightarrow \quad \frac{\partial \eta}{\partial x} = -\frac{y}{l^2}l' = -\frac{\eta}{l}l', \quad \frac{\partial \eta}{\partial y} = \frac{1}{l}$$

Determine the terms in equation (1):

$$\frac{\partial U_1}{\partial x} = \frac{\partial}{\partial x}(U_s f) = U_s' f + U_s f'\frac{\partial \eta}{\partial x} = \underline{U_s' f - U_s\frac{l'}{l}\eta f'}$$

$$\frac{\partial}{\partial y}(\overline{uv}) = \frac{\partial}{\partial y}(-U_s^2 g) = -U_s^2 g'\frac{\partial \eta}{\partial y} = \underline{-U_s^2\frac{1}{l}g'}$$

Insert into equation (1):

$$U_0\left(U_s' f - U_s\frac{l'}{l}\eta f'\right) = -U_s^2\frac{1}{l}g' \quad \Rightarrow$$

$$g' = -\frac{U_0 l U_s'}{U_s^2}f + \frac{U_0 l'}{U_s}\eta f'$$

Boundary conditions:
$$\begin{cases} U_1(-\infty) = 0 \\ U_1(\infty) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} f(-\infty) = 0 \\ f(\infty) = 0 \end{cases}$$

Similarity requires:

$$\frac{lU'_s}{U_s^2} = \text{const} \quad \Rightarrow \quad \frac{l'}{U_s} = \text{const}$$

Assume the x-dependence:

$$U_s = Ax^m, \quad l = Bx^n$$

Then we get

$$Bx^n Amx^{m-1} = \text{const} \cdot x^{2m} \quad \Rightarrow \quad n + m - 1 = 2m \quad \Rightarrow \quad m = n - 1 \quad \Rightarrow$$

$$U_s = Ax^{n-1}, \quad l = Bx^n$$

We need one more condition. The momentum loss thickness is constant:

$$\theta = \int_{-\infty}^{\infty} \frac{U}{U_0}\left(1 - \frac{U}{U_0}\right) dy = \int_{-\infty}^{\infty} \frac{U_0 - U_1}{U_0}\left(1 - \frac{U_0 - U_1}{U_0}\right) dy = \int_{-\infty}^{\infty}\left(1 - \frac{U_1}{U_0}\right)\frac{U_1}{U_0} dy \approx$$

$$\frac{1}{U_0}\int_{-\infty}^{\infty} U_1 dy \approx= \frac{1}{2}C_D d = \text{const} \quad \text{(Recitation 12)}$$

$$\frac{1}{U_0}\int_{-\infty}^{\infty} U_1 dy = U_s l \underbrace{\int_{-\infty}^{\infty} f d\eta}_{\text{const}} = \text{const} \quad \Rightarrow$$

$$U_s l = Ax^{n-1}Bx^n = \text{const} \quad \Rightarrow \quad 2n - 1 = 0 \quad \Rightarrow \quad n = \frac{1}{2} \quad \Rightarrow$$

$$U_s = Ax^{-\frac{1}{2}}, \quad l = Bx^{\frac{1}{2}}$$

For a sufficiently large Reynolds number, the wake will be turbulent.

$$Re_l = \frac{lU_s}{\nu} = \frac{AB}{\nu} = \text{const} \quad \Rightarrow$$

A turbulent wake will remain turbulent!

To determine the wake-velocity profile, we need to model the turbulent shear stress. E.g.

$$-\overline{uv} = \nu_T \frac{\partial U}{\partial y} \quad \Rightarrow$$

$$U_s^2 g = \nu_T\left(-\frac{\partial U_1}{\partial y}\right) = -\nu_T U_s f' \frac{1}{l} \quad \Rightarrow$$

$$g = -\frac{\nu_T}{U_s l}f' \quad \Rightarrow \quad g' = -\frac{\nu_T}{U_s l}f'' \quad \Rightarrow$$

$$-\frac{\nu_T}{U_s l}f'' = -\frac{U_0 l U'_s}{U_s^2}f + \frac{U_0 l'}{U_s}\eta f' \quad \Rightarrow$$

$$f'' + \frac{U_0 l' l}{\nu_T}f' - \frac{U_0 l^2 U'_s}{\nu_T U_s}f = 0 \quad \Rightarrow$$

$$f'' + \frac{U_0 B^2}{2\nu_T}(f' + f) = 0 \quad (2)$$

Equation (2) can be solved together with the boundary conditions

$$f(\pm\infty) = 0$$

B.14 Old Exams

1. Relative motion.

   Consider the relative motion of two fluid particles initially separated by the distance dx_i^0. (Here x_i^0 and $\hat{t}$ are the Lagrangian coordinates).

   a) (5) Show that the separation at time $d\hat{t}$ is

   $$dr_i(d\hat{t}) = dx_i^0 + \frac{\partial u_i}{\partial x_j^0} dx_j^0 d\hat{t}.$$

   b) (2) Use this relation to write down the expression for the deformation of the sides of a small cube aligned with the coordinate directions. Thus, show that the components of the side aligned with the x-direction, $R\delta_{i1}$, is deformed into

   $$dr_i^{(1)} = R(\delta_{i1} + \frac{\partial u_i}{\partial x_1^0} d\hat{t}),$$

   with analogous expressions for the other two sides.

   c) (4) Show that the deformation rate of the volume of the cube is given by the divergence of the velocity field. Note that $x_i^0 = x_i$ at $\hat{t} = 0$.

2. Flow between a rod and a cylinder.

   Consider a long hollow cylinder with radius b, a concentric rod with radius a inside the cylinder, and water occupying the space between the rod and the cylinder, see figure B.14. The rod is rotating with the constant angular frequency ω.

   (10) Find the velocity field of the water driven by the rotating rod and the gravitational acceleration g.
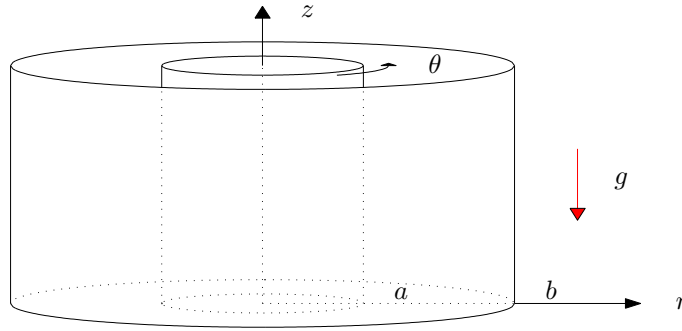

Figure B.12: Water and a concentric rotating rod inside a cylinder.

3. Bernoulli's equation

a) (3) Derive the following formula:

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\frac{\partial}{\partial x_i}(u_j u_j) + \epsilon_{ijk}\omega_j u_k.$$

b) (3) Use this in the momentum equation to derive Bernoulli's equation for unsteady potential flow.

c) (3) Use the result in (a) to derive Bernoulli's equation for steady inviscid flow. Discuss the validity of the derived relation.

4. Laminar wake flow.

Consider the laminar wake flow downstream of a two-dimensional streamlined body at high Reynolds number, see figure B.13. Assume that the wake is so weak that we can write $u = U + u_1$ where u_1 is negative and much smaller than the free-stream velocity U.

a) (3) Motivate the usage of the equation

$$U\frac{\partial u_1}{\partial x} = \nu \frac{\partial^2 u_1}{\partial y^2}.$$

b) (2) Show that the mass flux Q_1, in the direction of the wake is constant in x

$$Q_1 = \int_{-\infty}^{\infty} \rho u_1 \mathrm{d}y. \quad (*)$$

c) (7) A similarity solution can be found by scaling the wake velocity with $U_s(x) = u_1(x,0)$ so that $f(\eta) = u_1(x,y)/U_s(x)$, where $\eta = y/\delta(x)$ is the similarity variable. Show, using $(*)$, that $U_s = k/\delta(x)$, where k is a given constant. Find the similarity solution for u_1.
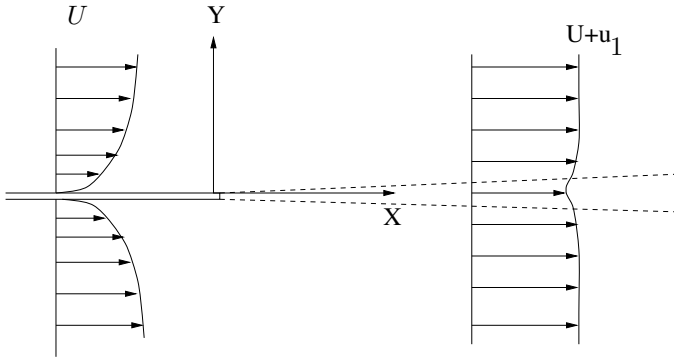

Figure B.13: Wake downstream of a flat plate.

5. Mean heat equation.

The heat equation for the instantaneous turbulent flow is

$$\frac{\partial \tilde{T}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{T}}{\partial x_j} = \kappa \frac{\partial^2 \tilde{T}}{\partial x_j \partial x_j}.$$

(8) Derive the heat equation governing the mean flow.

Answers to exam in Fluid mechanics 5C1214, 2003-10-23

1. See Kundu & Cohen p. 57.

2. Flow between a rod and a cylinder.

$$u_\theta = \frac{\omega a^2}{b^2 - a^2} \left(\frac{b^2 - r^2}{r} \right)$$

$$u_z = \frac{1}{4} \frac{g}{\nu} \left(r^2 - a^2 - (b^2 - a^2) \frac{\ln(r/a)}{\ln(b/a)} \right)$$

3. See Kundu & Cohen pp. 110-114.

4. Laminar wake flow.

$$u_1 = \frac{Q_1}{2\rho} \sqrt{\frac{U}{\pi \nu x}} e^{-\frac{U y^2}{4 \nu x}}$$

5. See Kundu & Cohen pp. 511-512.

1. (5) Consider the *unsteady* flow,

$$u = u_0, \quad v = kt, \quad w = 0,$$

where u_0 and k are positive constants. Show that the streamlines are straight lines, and sketch them at two different times. Also show that any fluid particle follows a parabolic path as time proceeds.

2. (5) For surface waves on water with finite depth h the phase speed is given by the relation $c^2 = \frac{g}{k}\tanh(kh)$. Compare the group velocity to the phase speed in the limit of long and short waves. Make a rough 2D sketch of a wave packet and explain how it is moving.

3. Consider a thin layer of liquid in contact with a solid flat vertical wall. See figure 1. The wall is subject to a constant temperature gradient with increasing temperature in the downward direction. Due to a varying surface tension, as a result of the temperature gradient, a constant shear stress τ at the free surface is acting upward. At the same time, the gravitational acceleration g is acting downward.

a) (7) Assume constant layer thickness h and calculate the velocity field of the liquid layer.

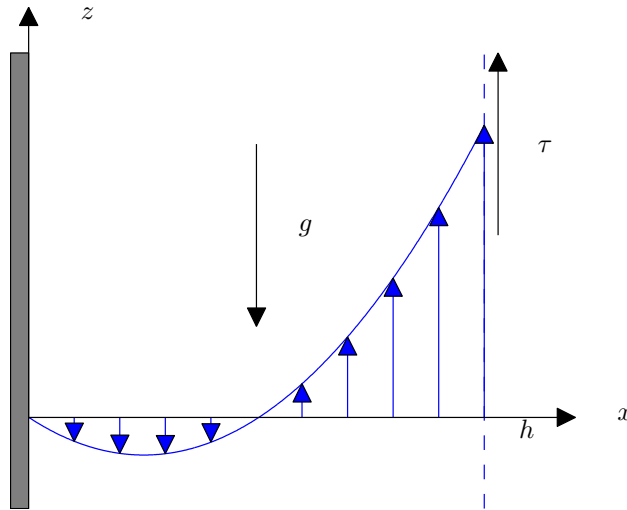b) (3) For what layer thickness h is the flux zero?

Figure B.14: Liquid film on vertical wall driven by gravity and surface tension.

4. a) (3) Show that

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{2} \frac{\partial}{\partial x_i} (u_j u_j) + \epsilon_{ijk} \omega_j u_k$$

b) (4) Derive the vorticity equation starting with the Navier–Stokes equation for incompressible flow.

c) (3) Show that the following relation holds for incompressible flow and discuss its implication for inviscid flow

$$\frac{\partial \tau_{ij}}{\partial x_j} = -\mu \epsilon_{ijk} \frac{\partial \omega_k}{\partial x_j}$$

5. Consider the flow over an oscillating plate.

a) (3) Motivate that the solution can be written as $[u(y,t), 0, 0]$ and that it thus satisfies the equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

b) (7) Show that

$$u(y,t) = U e^{-ky} \cos(ky - \omega t), \quad \text{where} \quad k = \sqrt{\frac{\omega}{2\nu}}.$$

6. a) (5) Derive the Reynolds average equation valid for turbulent flow.

b) (5) Assume that the Reynolds stress can be modeled as a turbulent viscosity and introduce this into the Reynolds equation and simplify.

$j-1$

A
B
C
$u_{\frac{3}{2},1}$
$v_{1,\frac{3}{2}}$
$p_{1,1}$

1

i
j
Inflow
Solid wall
n
Γ_0
Γ_1
Ω
M^2
M
m
$m+2$
1
h
V
V_h
u
u_h
Γ_0
Γ_1
Ω
: to be updated
: new values
: old values
$i-1$
i
$i+1$
$j-1$
j
$j+1$
iteration direction
x
y

n
x
y
i
j
fine grid (1)
coarse grid (2)

Solutions to exam in Fluid mechanics 5C1214, 2004-01-13

1. Streamline (fix t):

$$\frac{dx}{ds} = u = u_0 \quad \Rightarrow \quad x = u_0 s + a$$

$$\frac{dy}{ds} = u = kt \quad \Rightarrow \quad y = kts + b$$

Then we see that $y(x)$ is a linear function,

$$y = \frac{kt}{u_0}x + d(t)$$

Particle path (fix $\bar{x}_0$):

$$\frac{\partial x}{\partial \hat{t}} = u = u_0 \quad \Rightarrow \quad x = u_0\hat{t} + x_0$$

$$\frac{\partial y}{\partial \hat{t}} = u = k\hat{t} \quad \Rightarrow \quad x = \frac{1}{2}k\hat{t}^2 + y_0$$

Then we see that $y(x)$ is a parabola,

$$y = \frac{1}{2}\frac{k}{u_0^2}(x^2 - 2x_0 x + x_0^2) + y_0$$

2. The wave number is defined as $k = 2\pi/\lambda$, where λ denotes the wave length of the wave. For long waves, go to the limit $k \to 0$. For short waves, go to the limit $k \to \infty$.

Phase speed:

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh(kh)} = \sqrt{gh\frac{\tanh(kh)}{kh}}$$

Short waves:

$$\lim_{x\to\infty}\frac{\tanh(x)}{x} = \frac{1}{x} \quad \Rightarrow \quad c = \sqrt{\frac{g}{k}}$$

Long waves:

$$\lim_{x\to 0}\frac{\tanh(x)}{x} = 1 \quad \Rightarrow \quad c = \sqrt{gh}$$

Group velocity:

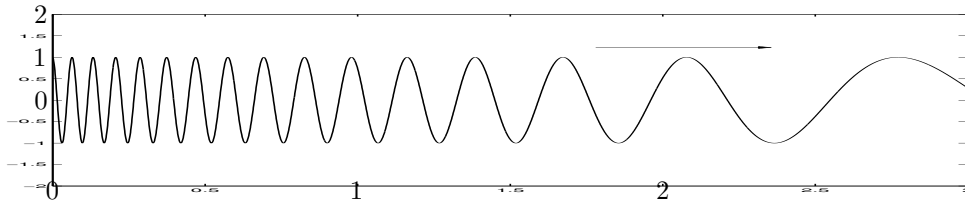$$c_g = \frac{d\omega}{dk} = \frac{c}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right)$$

Short waves:

$$\lim_{x\to\infty}\frac{x}{\sinh(x)} = 0 \quad \Rightarrow \quad c_g = \frac{c}{2}$$

Long waves:

$$\lim_{x\to 0}\frac{x}{\sinh(x)} = 1 \quad \Rightarrow \quad c_g = c$$

Longer waves travel faster.

3. a) Assume steady, parallel flow $\bar{u} = w(x)\bar{e}_z$ and solve the z-momentum equation with boundary conditions:

$$\nu\frac{\partial^2 w}{\partial x^2} - g = 0, \quad w(0) = 0, \quad \frac{\partial w}{\partial x}(h) = \frac{\tau}{\mu}$$

Integrate the equation once and introduce the second boundary condition:

$$\frac{\partial w}{\partial x} = \frac{g}{\nu}x + A, \quad \frac{\partial w}{\partial x}(h) = \frac{gh}{\nu} + A = \frac{\tau}{\mu} \quad \Rightarrow \quad A = \frac{1}{\nu}\left(\frac{\tau}{\rho} - gh\right)$$

Integrate the equation once more and introduce the first boundary condition:

$$w = \frac{1}{2}\frac{g}{\nu}x^2 + Ax + B, \quad w(0) = 0 \quad \Rightarrow \quad B = 0$$

So we get:

$$w = \frac{1}{2}\frac{g}{\nu}x^2 + \frac{1}{\nu}\left(\frac{\tau}{\rho} - gh\right)x$$

b) The flux per unit width of the wall can be written

$$Q = \int_0^h w \, dx = \left[\frac{1}{6}\frac{g}{\nu}x^3 + \frac{1}{2}\frac{1}{\nu}\left(\frac{\tau}{\rho} - gh\right)x^2\right]_0^h = \frac{1}{2}\frac{1}{\nu}\frac{\tau}{\rho}h^2 - \frac{1}{3}\frac{g}{\nu}h^3$$

Zero flux gives:

$$h = \frac{3}{2}\frac{\tau}{\rho g}$$

4. a) Begin with the second term on the right hand side:

$$\epsilon_{ijk}\omega_j u_k = \epsilon_{ijk}\epsilon_{jlm}\frac{\partial}{\partial x_l}(u_m)u_k = (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})u_k\frac{\partial u_m}{\partial x_l} = u_k\frac{\partial u_i}{\partial x_k} - u_k\frac{\partial u_k}{\partial x_i} \quad \Rightarrow$$

$$u_j\frac{\partial u_i}{\partial x_j} = u_j\frac{\partial u_j}{\partial x_i} + \epsilon_{ijk}\omega_j u_k \quad \Rightarrow \quad u_j\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\frac{\partial}{\partial x_i}(u_j u_j) + \epsilon_{ijk}\omega_j u_k$$

b) Navier–Stokes momentum equation for incompressible flow:

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

Use the expression from 4. a):

$$\frac{\partial u_i}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x_i}(u_j u_j) + \epsilon_{ijk}\omega_j u_k = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

Take the curl of the equation. The curl of a gradient vanish:

$$\frac{\partial \omega_i}{\partial t} + \epsilon_{imn}\frac{\partial}{\partial x_m}(\epsilon_{njk}\omega_j u_k) = \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j} \quad \Rightarrow$$

$$\frac{\partial \omega_i}{\partial t} + \epsilon_{nim}\epsilon_{njk}\frac{\partial}{\partial x_m}(\omega_j u_k) = \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j} \quad \Rightarrow$$

$$\frac{\partial \omega_i}{\partial t} + (\delta_{ij}\delta_{mk} - \delta_{ik}\delta_{mj})\frac{\partial}{\partial x_m}(\omega_j u_k) = \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j} \quad \Rightarrow$$

$$\frac{\partial \omega_i}{\partial t} + \frac{\partial}{\partial x_k}(\omega_i u_k) - \frac{\partial}{\partial x_j}(\omega_j u_i) = \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j}$$

Use continuity. We end up with the vorticity equation:

$$\frac{\partial \omega_i}{\partial t} + u_j\frac{\partial \omega_i}{\partial x_j} = \omega_j\frac{\partial u_i}{\partial x_j} + \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j}$$

c) Right hand side:

$$-\mu\epsilon_{ijk}\frac{\partial \omega_k}{\partial x_j} = -\mu\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\epsilon_{klm}\frac{\partial u_m}{\partial x_l}\right) = -\mu\epsilon_{kij}\epsilon_{klm}\frac{\partial^2 u_m}{\partial x_j \partial x_l} =$$

$$-\mu(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial^2 u_m}{\partial x_j \partial x_l} = -\mu\left(\frac{\partial^2 u_j}{\partial x_j \partial x_i} - \frac{\partial^2 u_i}{\partial x_j \partial x_j}\right) = \mu\nabla^2 u_i$$

Left hand side:

$$\frac{\partial \tau_{ij}}{\partial x_j} = \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \mu\nabla^2 u_i$$

Thus:

$$\frac{\partial \tau_{ij}}{\partial x_j} = -\mu\epsilon_{ijk}\frac{\partial \omega_k}{\partial x_j}$$

Irrotational flow is not subject to viscous forces.

5. a) Consider the Navier–Stokes equation in the x direction:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

With the current velocity field only terms with y derivatives will remain since there can be no change in the other directions,

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$$

b) Make the ansatz: $u = \Re\left[f(y)e^{i\omega t}\right] = f(y)\cos(\omega t)$.

Insert into the equation:

$$i\omega f(y)e^{i\omega t} = \nu f''(y)e^{i\omega t}$$

$$f''(y) - \frac{i\omega}{\nu}f(y) = 0 \quad \text{with } f(y) = e^{\lambda y} \quad \text{gives}$$

$$\lambda^2 - \frac{i\omega}{\nu} = 0 \quad \Rightarrow \quad \lambda = \pm\sqrt{\frac{i\omega}{\nu}} = \pm\sqrt{\frac{\omega}{\nu}}\left(\frac{1+i}{\sqrt{2}}\right)$$

Introduce $k = \sqrt{\frac{\omega}{2\nu}}$,

$$f(y) = Ae^{yk(1+i)} + Be^{-yk(1+i)}, \quad f(y) \to 0 \quad \text{as} \quad y \to \infty \to A = 0$$

We have

$$u = \Re\left[Be^{-yk(1+i)}e^{i\omega t}\right] = \Re\left[Be^{-ky}e^{i(\omega t - ky)}\right] = Be^{-ky}\cos(\omega t - ky) = Be^{-ky}\cos(ky - \omega t)$$

Boundary condition at $y = 0$

$$\text{At } y = 0 \text{ we have } u = U\cos(\omega t) \Rightarrow B = U$$

So we have,

$$u(y,t) = Ue^{-ky}\cos(ky - \omega t), \quad \text{where} \quad k = \sqrt{\frac{\omega}{2\nu}}$$

6. a) Turbulent flow is inherently time dependent and chaotic. However, in applications one is not usually interested in knowing full the details of this flow, but rather satisfied with the influence of the turbulence on the averaged flow. For this purpose we define an ensemble average as

$$U_i = \overline{u}_i = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} u_i^{(n)}$$

where each member of the ensemble $u_i^{(n)}$ is regarded as an independent realization of the flow.

We are now going to derive an equation governing the mean flow U_i. Divide the total flow into an average and a fluctuating component u_i' as

$$u_i = \overline{u}_i + u_i' \qquad p = \overline{p} + p'$$

and introduce into the Navier-Stokes equations. We find

$$\frac{\partial u_i'}{\partial t} + \frac{\partial \overline{u}_i}{\partial t} + u_j'\frac{\partial \overline{u}_i}{\partial x_j} + \overline{u}_j\frac{\partial u_i'}{\partial x_j} + \overline{u}_j\frac{\partial \overline{u}_i}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} - \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu\nabla^2 u_i' + \nabla^2 \overline{u}_i$$

We take the average of this equation, using

$$\overline{u_i'} = 0 \qquad \overline{u_j'\overline{u}_i} = \overline{u}_i \cdot \overline{u}_j = 0$$

which gives

$$\begin{cases} \dfrac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \dfrac{\partial \overline{u}_i}{\partial x_j} = -\dfrac{1}{\rho}\dfrac{\partial \overline{p}}{\partial x_i} + \nu \nabla^2 \overline{u}_i - \dfrac{\partial}{\partial x_j}\left(\overline{u'_i u'_j}\right) \\[2mm] \dfrac{\partial \overline{u}_i}{\partial x_i} = 0 \end{cases}$$

where the average of the continuity equation also has been added. Now, let $U_i = \overline{u}_i$, as above, and drop the $'$. We find the Reynolds average equations

$$\begin{cases} \dfrac{\partial U_i}{\partial t} + U_j \dfrac{\partial U_i}{\partial x_j} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial x_i} + \dfrac{\partial}{\partial x_j}\left(\tau_{ij} - \overline{u_i u_j}\right) \\[2mm] \dfrac{\partial U_i}{\partial x_i} = 0 \end{cases}$$

where $\overline{u_i u_j}$ is the Reynolds stress.

b) Assume the Reynolds stress:

$$\overline{u_i u_j} = \frac{2}{3}K\delta_{ij} - 2\nu_T \overline{e}_{ij}$$

The first term on the right hand side is introduced to give the correct value of the trace of $\overline{u_i u_j}$. Here the deformation rate tensor is calculated based on the mean flow, i.e.

$$\overline{e}_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{2}{3}\frac{\partial U_r}{\partial x_r}\delta_{ij}\right)$$

where we have assumed incompressible flow. Introducing this into the Reynolds average equations gives

$$\Rightarrow \quad \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left[-\left(\frac{P}{\rho} + \frac{2}{3}k\right)\delta_{ij} + 2\left(\nu + \nu_T\right)\overline{e}_{ij}\right] = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \left(\nu + \nu_T\right)\nabla^2 U_i$$

where the last equality assumes that ν_T is constant, an approximation only true for very simple turbulent flow. It is introduced here only to point out the analogy between the molecular viscosity and the turbulent viscosity.

Exam 041018

1. The kinetic energy dT of an infinitesimal volume element, with density ρ, which is rotating with angular velocity $\mathbf{\Omega}$ can be expressed

$$dT = \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\rho\, dV = \frac{1}{2}(\mathbf{\Omega}\times\mathbf{x})\cdot(\mathbf{\Omega}\times\mathbf{x})\,\rho\, dV,$$

where $\mathbf{v}$ is the velocity of the volume element and $\mathbf{x}$ is the vector from the center of rotation O to the volume element, see figure B.15.
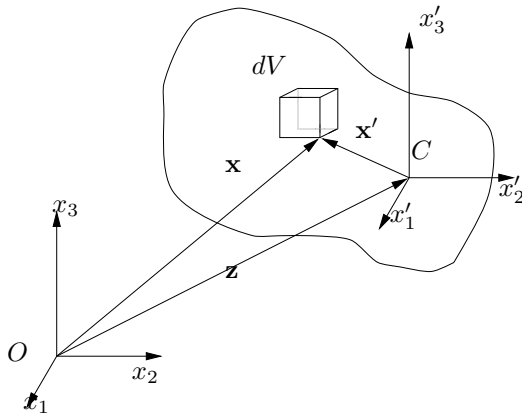

Figure B.15: Coordinate system and definitions

a) (5) Show that the total kinetic energy of the body can be expressed as

$$T = \int_V dT = \frac{1}{2}J_{jl}\Omega_j\Omega_l,$$

where J_{jl} is the moment of inertia tensor with respect to the origin of the unprimed system. It is defined as

$$J_{jl} = \int_V (x_k x_k \delta_{jl} - x_j x_l)\rho\, dV.$$

b) (5) Show that you can relate J_{jl} to the moment of inertia tensor with respect to the center of mass of the body J_{jl}^C in the following way

$$J_{jl} = (z_k z_k \delta_{jl} - z_j z_l)M + J_{jl}^C, \quad J_{jl}^C = \int_V (x_k' x_k' \delta_{jl} - x_j' x_l')\rho\, dV,$$

where M is the total mass of the body and the primed coordinates represent the coordinate system centered at the center of mass C of the body.

Hints: Integrals of the type $\int_V x_k'\rho\, dV$ vanishes as a result of the definition of the center of mass. The two coordinate systems are related by the formula $\mathbf{x} = \mathbf{z} + \mathbf{x}'$.

2. (10) Consider the laminar flow of a viscous fluid down a vertical cylindrical pipe with radius a. The flow is driven by the gravitational acceleration g. Calculate the velocity field.

3. Potential flow and conformal mapping

a) (2) Show that the Kutta-Joukowski transformation $\zeta = z + a^2/z$ maps a circle with radius a to a flat plate with length $4a$.

b) (2) Write down the complex potential of the flow around a circular cylinder with radius a at angle of attack α and with clockwise circulation Γ. The uniform far-field velocity has the magnitude U.

c) (4) Show that a necessary condition for finite velocity at the trailing edge of the flat plate is (Kutta condition)

$$\Gamma = 4\pi U a \sin \alpha.$$

d) (2) How should one modify the circle in order to obtain a finite velocity at the leading edge?

4. Blasius boundary layer

a) (2) Write down the boundary layer equations.

b) (8) Make appropriate assumptions for similarity and reduce the boundary layer equations to an ordinary differential equation for the boundary layer over a flat plate (i.e there is no pressure gradient in the free stream). Also state the boundary conditions for the similarity function.

c) (2) Sketch the streamwise and normal velocity profiles.

5. Turbulent flow

a) (2) Define the velocity and length scales appropriate for turbulent flow near a wall, using the wall shear stress τ_w, the viscosity ν and the density ρ.

b) (2) What is the law of the wall and what is the explicit functional dependence of the turbulent mean velocity in the intermediate region between the wall and the outer flow?

c) (4) In two-equation turbulence models, partial differential equations for the turbulent kinetic energy $\bar{k}$ and the dissipation rate ε are solved. In those equations, the turbulent viscosity $\nu_T = u_T l$, where u_T and l are the turbulent velocity and length scales, respectively. Use dimensional analysis to determine u_T, l and ν_T in terms of $\bar{k}$ and ε.

Solutions to exam in Fluid mechanics 5C1214, 2004-10-18

1. a) The kinetic energy of an infinitesimal volume element

$$dT = \frac{1}{2}\mathbf{v} \cdot \mathbf{v}\rho\,dV = \frac{1}{2}(\mathbf{\Omega} \times \mathbf{x}) \cdot (\mathbf{\Omega} \times \mathbf{x})\,\rho\,dV \Rightarrow$$

$$dT = \frac{1}{2}v_i v_i \rho\,dV = \frac{1}{2}\varepsilon_{ijk}\Omega_j x_k \varepsilon_{ilm}\Omega_l x_m \rho\,dV = \frac{1}{2}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})x_k x_m \Omega_j \Omega_l \rho\,dV =$$

$$\frac{1}{2}(\delta_{jl}x_k x_k - x_j x_l)\Omega_j \Omega_l \rho\,dV \Rightarrow$$

$$T = \int_V dT = \frac{1}{2}\int_V dT(\delta_{jl}x_k x_k - x_j x_l)\rho\,dV\,\Omega_j \Omega_l = \frac{1}{2}J_{jl}\Omega_j \Omega_l,$$

where

$$J_{jl} = \int_V (\delta_{jl}x_k x_k - x_j x_l)\rho\,dV.$$

b) The moment of inertia tensor

$$J_{jl} = \int_V (\delta_{jl}x_k x_k - x_j x_l)\rho\,dV = \int_V [\delta_{jl}(z_k + x'_k)(z_k + x'_k) - (z_j + x'_j)(z_l + x'_l)]\rho\,dV =$$

$$\int_V [\delta_{jl}(z_k z_k + 2z_k x'_k + x'_k x'_k) - (z_j z_l + z_j x'_l + z_l x'_j + x'_j x'_l)]\rho\,dV =$$

$$(z_k z_k \delta_{jl} - z_j z_l)\int_V \rho\,dV + \int_V (x'_k x'_k \delta_{jl} - x'_j x'_l)\rho\,dV+$$

$$2z_k \delta_{jl}\int_V x'_k \rho\,dV + z_j \int_V x'_l \rho\,dV + z_l \int_V x'_j \rho\,dV.$$

The three last terms vanishes due to the definition of the center of mass. So we have

$$J_{jl} = (z_k z_k \delta_{jl} - z_j z_l)M + J_{jl}^C,$$

where

$$M = \int_V \rho\,dV \quad \text{and} \quad J_{jl}^C = \int_V (x'_k x'_k \delta_{jl} - x'_j x'_l)\rho\,dV.$$

2. Poiseuille flow (Pipe flow)

   Use the Navier–Stokes equations (with no volume force) in cylindrical coordinates

   $$\frac{\partial u_r}{\partial t} + (\mathbf{u} \cdot \nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

   $$\frac{\partial u_\theta}{\partial t} + (\mathbf{u} \cdot \nabla)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)$$

   $$\frac{\partial u_z}{\partial t} + (\mathbf{u} \cdot \nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z$$

   $$\frac{1}{r}\frac{\partial}{\partial r}(r\,u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0.$$

   We know that $\frac{\partial p}{\partial \theta} = 0$ and $\frac{\partial p}{\partial r} = 0$ and can directly see that $u_r = u_\theta = 0$ satisfy the two first equations. From the continuity equation we get

   $$\frac{\partial u_z}{\partial z} = 0 \quad \Rightarrow \quad u_z = u_z(r, \theta) \quad \text{only.}$$

   Considering a steady flow we get from the Navier–Stokes equations

   $$(\mathbf{u} \cdot \nabla)u_z = u_z\frac{\partial u_z}{\partial z} \quad \Rightarrow \quad u_z\frac{\partial u_z}{\partial z} = \nu\nabla^2 u_z - g$$

But we know that $\dfrac{\partial u_z}{\partial z} = 0$ from the continuity equation. We get

$$\nabla^2 u_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{g}{\nu} \quad \Rightarrow \quad \frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{g}{\nu}r$$

Integrate

$$r\frac{\partial u_z}{\partial r} = \frac{1}{2}\frac{g}{\nu}r^2 + c_1 \quad \Rightarrow \quad \frac{\partial u_z}{\partial r} = \frac{1}{2}\frac{g}{\nu}r + \frac{c_1}{r}$$

Integrating again we get

$$u_z = \frac{1}{4}\frac{g}{\nu}r^2 + c_1\ln r + c_2 \quad \text{using the boundary conditions } u_z(r=0) < \infty \Rightarrow c_1 = 0$$

We also have $u_z = 0$ at $r = a$ and this gives $c_2 = -\dfrac{ga^2}{4\nu}$ and we have

$$u_z = -\frac{g}{4\nu}(a^2 - r^2)$$

3. Potential flow and conformal mapping

a) Equation for the circle

$$z = ae^{i\theta}$$

Equation for the plate

$$\zeta = z + \frac{a^2}{z} = ae^{i\theta} + ae^{-i\theta} = 2a\frac{e^{i\theta} + e^{-i\theta}}{2} = 2a\cos\theta,$$

which is a real function taking the values from $-2a$ to $2a$, thus a flat plate with length $4a$.

b) Complex potential for a circular cylinder

$$F = Uze^{-i\alpha} + U\frac{a^2}{z}e^{i\alpha} + \frac{i\Gamma}{2\pi}\ln z$$

c) Complex velocity for the circular cylinder

$$W = \frac{dF}{dz} = Ue^{-i\alpha} - U\frac{a^2}{z^2}e^{i\alpha} + \frac{i\Gamma}{2\pi z}$$

Complex velocity in the ζ-plane

$$\omega = \frac{dF/dz}{d\zeta/dz} = \left(Ue^{-i\alpha} - U\frac{a^2}{z^2}e^{i\alpha} + \frac{i\Gamma}{2\pi z}\right)\bigg/\left(1 - \frac{a^2}{z^2}\right)$$

The flow field has singular points at the leading and trailing edges of the flat plate. Resolve the singularity at the trailing edge by choosing the circulation Γ so the numerator vanishes at $z = a$

$$Ue^{-i\alpha} - Ue^{i\alpha} + \frac{i\Gamma}{2\pi a} = 0$$

$$\Gamma_k = 4\pi Ua\frac{e^{i\alpha} - e^{-i\alpha}}{2i} = 4\pi Ua\sin\alpha.$$

d) Move the point $z = -a$ inside of the circle by putting the center at $z = -\lambda$. We still want a sharp trailing edge so the circle crosses the real axis at $z = a$. The radius of such a circle is $a + \lambda$ and the circle is described by

$$z = -\lambda + (a + \lambda)e^{i\theta}$$

4. Blasius boundary layer

a) Boundary layer equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{B.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{B.4}$$

b) Seek a similarity solution where the dimensionless velocity u/U, where U is the constant free-stream velocity, only depends on the dimensionless wall distance

$$\eta = \frac{y}{\delta(x)}$$

The stream function satisfies (2):

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Make the stream function dimensionless:

$$f(\eta) = \frac{\psi(x, y)}{U\delta(x)} \quad \Leftrightarrow \quad \psi = U\delta f$$

Determine the terms in equation (1):

$$u = \frac{\partial \psi}{\partial y} = U\delta \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} = U\delta f' \frac{1}{\delta} = \underline{U f'}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial (U\delta f)}{\partial x} = -U\delta' f - U\delta f' \left(-\frac{y\delta'}{\delta^2} \right) = \underline{U\delta' \eta f' - U\delta' f}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (U f') = \underline{-U\frac{\delta'}{\delta}\eta f''}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} (U f') = \underline{U\frac{1}{\delta}f''}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left(U f'' \frac{1}{\delta} \right) = \underline{U\frac{1}{\delta^2}f'''}$$

$$U\frac{dU}{dx} = 0$$

Insert into equation (1):

$$U f' \left(-U\frac{\delta'}{\delta}\eta f'' \right) + (U\delta'\eta f' - U\delta' f) U\frac{1}{\delta}f'' = \nu U\frac{1}{\delta^2}f''' \quad \Rightarrow$$

$$-U^2\frac{\delta'}{\delta}\eta f' f'' + U^2\frac{\delta'}{\delta}\eta f' f'' - U^2\frac{\delta'}{\delta}f f'' = \nu U\frac{1}{\delta^2}f''' \quad \Rightarrow$$

$$f''' + \underbrace{\left(\frac{U\delta'\delta}{\nu} \right)}_{\alpha} f f'' = 0$$

Similarity requires that α is a constant. We can integrate it with respect to x in order to find δ

$$\frac{U\delta'\delta}{\nu} = \alpha \quad \Rightarrow \quad \frac{1}{2}\delta^2 = \frac{\alpha\nu x}{U} \quad \Rightarrow \quad \delta(x) = \sqrt{\frac{2\alpha\nu x}{U}}$$

The value of the constant α is just a matter of scaling, we can choose it to be $1/2$ so

$$\delta(x) = \sqrt{\frac{\nu x}{U}}.$$

The ordinary differential equation becomes

$$f''' + \frac{1}{2}f f'' = 0.$$

We have the boundary conditions

$$\begin{cases} u(0) = 0 \\ v(0) = 0 \\ u(\infty) = U \end{cases} \quad \Rightarrow \quad \begin{cases} f'(0) = 0 \\ f(0) = 0 \\ f'(\infty) = 1. \end{cases}$$

c) Figure B.16 shows the dimensionless streamwise and normal velocity profiles.

5. Turbulent flow

a) Velocity scale near the wall:

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

Length scale near the wall:

$$l_* = \frac{\nu}{u_\tau}$$

b) Law of the wall

$$U^+ = \frac{U}{u_\tau} = f(y^+), \quad \text{where} \quad y^+ = \frac{y}{l_*}$$

Functional dependence in the intermediate region

$$U^+ = \frac{1}{\kappa} \ln y^+ + B,$$

where κ is the Karmans constant.

c) Dimensions for $\bar{k}$ and ε:

$$\begin{cases} \bar{k} = \frac{1}{2}\overline{u_i u_i} \quad [L^2 T^{-2}] \\ \varepsilon = \nu \overline{\dfrac{\partial u_i}{\partial x_j}\dfrac{\partial u_j}{\partial x_i}} \quad [L^2 T^{-3}] \end{cases}$$

Dimension analysis for u_T:

$$u_T \propto \bar{k}^m \varepsilon^n \quad [LT^{-1}] \quad \Rightarrow$$

$$LT^{-1} = L^{2m}T^{-2m}L^{2n}T^{-3n} \quad \Rightarrow$$

$$\begin{cases} 2m + 2n = 1 \\ -2m - 3n = -1 \end{cases} \quad \Rightarrow \quad \begin{cases} n = 0 \\ m = \frac{1}{2} \end{cases} \quad \Rightarrow$$

$$u_T \propto \sqrt{\bar{k}}$$

Dimension analysis for l:

$$l \propto \bar{k}^m \varepsilon^n \quad [L] \quad \Rightarrow$$

$$L = L^{2m}T^{-2m}L^{2n}T^{-3n} \quad \Rightarrow$$

$$\begin{cases} 2m + 2n = 1 \\ -2m - 3n = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} n = -1 \\ m = \frac{3}{2} \end{cases} \quad \Rightarrow$$
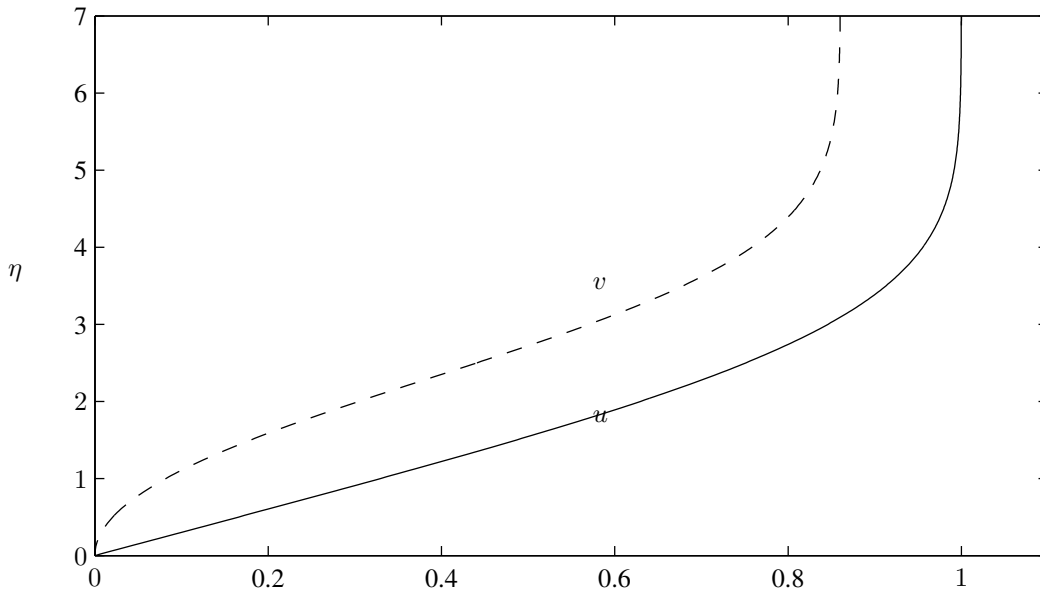

Figure B.16: Velocity profiles of Blasius boundary layer.

$$l \propto \frac{\bar{k}^{\frac{3}{2}}}{\varepsilon}$$

So we have

$$\nu_T = u_T l \propto \sqrt{\bar{k}}\frac{\bar{k}^{\frac{3}{2}}}{\varepsilon} \quad \Rightarrow$$

$$\nu_T \propto \frac{\bar{k}^2}{\varepsilon}$$

or

$$\nu_T = C_\mu \frac{\bar{k}^2}{\varepsilon}.$$

Appendix C

Study questions 5C1212

1. Define and use the Rankine-Hugoniot jump condition to compute the shock speed for the following problem

$$u_t + uu_x = 0 \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \begin{cases} 1 & x \le 0 \\ 0 & \text{otherwise} \end{cases}$$

2. Define the entropy condition for a scalar conservation law.

$$u_t + f(u)_x = 0 \quad -\infty < x < \infty, \quad t > 0$$

with a convex flux function $f(u)$. The shock is moving with speed s and the state to the left is given by u_L and the state to the right by u_R.

Why do we need an entropy condition ?

3. Write a difference approximation (in 1D) on conservative form and define the notation.

4. Solve

$$u_t + (\frac{u^2}{2})_x = 0, \qquad u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x \ge 0 \end{cases} \tag{C.1}$$

by the upwind scheme on the following form,

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} u_j^n (u_j^n - u_{j-1}^n)$$

a) Will the numerical solution converge to the analytical solution?

b) If not, suggest another way of solving C.1.

5. Define a total variation decreasing (TVD) method. Why is this a desirable property ?

6. Investigate the one-sided difference scheme

$$u_j^{n+1} = u_j^n - a\frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)$$

for the advection equation

$$u_t + au_x = 0$$

Consider the cases $a > 0$ and $a < 0$.

a) Prove that the scheme is consistent and find the order of accuracy. Assume k/h constant.

b) Determine the stability requirement for $a > 0$ and show that it is unstable for $a < 0$.

7. Apply Lax-Friedrichs scheme to the linear wave equation

$$u_t + au_x = 0.$$

a) Write down the modified equation.

b) What type of equations is this?

c) What kind of behavior can we expect from the solution?

8. A the three-point centered scheme applied to

$$u_t + au_x = 0, \quad a > 0.$$

yields the approximation

$$u_j^{n+1} = u_j^n + \frac{a\Delta t}{2\Delta x}(u_{j+1} - u_{j-1})$$

Show that this approximation is not stable even though the CFL condition is fulfilled.

9. What does Lax's equivalence theorem state?

10. What is the condition on the $n \times n$ real matrix $A(\mathbf{u})$ for the system

$$\mathbf{u}_t + A\mathbf{u}_x = 0$$

to be hyperbolic ?

11. The barotropic gas dynamic equations

$$\rho_t + \rho u_x = 0 \tag{C.2}$$

$$u_t + uu_x + \frac{1}{\rho}p_x = 0$$

where

$$p = p(\rho) = C\rho^\gamma$$

i and C a constant, can be linearized by considering small perturbations (ρ', u') around a motionless gas.

a) Let $\rho = \rho_0 + \rho'$ and $u = u_0 + u'$ where $u_0 = 0$. Linearize the system (C.2) and show that this yields the following linear system (the primes has been dropped)

$$\rho_t + \rho_0 u_x = 0$$

$$u_t + \frac{a^2}{\rho_0}\rho_x = 0 \tag{C.3}$$

where a is the speed of sound. a and ρ_0 are constants.

b) Is the system given by (C.3) a hyperbolic system? Motivate your answer.

c) Determine the characteristic variables in terms of ρ and u.

d) Determine the partial differential equations the characteristic variables fulfill - characteristic formulation.

e) Given initial conditions at $t = 0$ and let $-\infty < x < \infty$ (no boundaries)

$$\rho(0, x) = sin(x) \quad u(0, x) = 0$$

determine the analytical solution to (C.3) for $t > 0$. Hint: Start from the characteristic formulation.

12. The linearized form of (C.3) is given by

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \underbrace{\begin{pmatrix} 0 & \rho_0 \\ a^2/\rho_0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0, \tag{C.4}$$

where a is the speed of sound. a and ρ_0 are constants.

a) Draw the domain of dependence of the solution to the system (C.4) in a point P in the x-t plane.

b) The system is solved numerically on a grid given by $x_j = j\Delta x, j = 0, 1, 2...$ and $t_n = n\Delta t, n = 0, 1, 2, ...$ using an explicit three-point scheme, see the figure below.

Draw the domain of dependence of the numerical solution at P (in the same figure as a)) of the three-point scheme in the case when

i) the CFL condition is fulfilled

ii) the CFL condition is NOT fulfilled.

Assume that P is a grid point.

$1t_{n+1}$

x
y
i

t_n

fine grid (1)
coarse grid (2)　　$j-1$　　　　j　　　　$j+1$
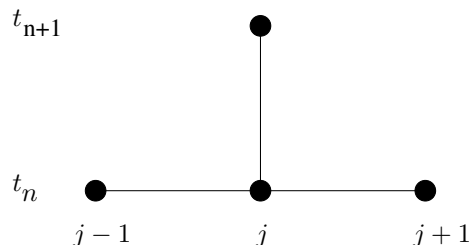
13. To solve Euler equations in 1D

$$\rho_t + \rho u_x + u\rho_x = 0$$
$$u_t + uu_x + \frac{1}{\rho}p_x = 0$$
$$p_t + \rho c^2 u_x + up_x = 0$$

How many boundary conditions must be added at (motivate your answer)

inflow boundary when the flow is

a) Supersonic

b) Subsonic

outflow boundary when the flow is

c) Supersonic

d) Subsonic

14. Describe the ideas behind a flux splitting scheme for solving a non-linear hyperbolic system of equations,
$$\mathbf{U}_t + F(\mathbf{U})_x = 0$$

15. a) Show that a vector field w_i can be decomposed into

$$w_i = u_i + \frac{\partial p}{\partial x_i}$$

where u is is divergence free and parallel to the boundary.

b) Apply this to the Navier-Stokes equations, show that the pressure term disappears and recover an equation for the pressure from the gradient part.

16. From the differential form of the Navier-Stokes equations obtain

a) the Navier-Stokes equations in integral form used in finite-volume discretizations,

b) a variational form of the Navier-Stokes equations used in finite-element discretizations.

17. a) Write down the appropriate function spaces for the pressure and velocity used to define weak solutions to the Navier-Stokes equations.

b) Explain the concept of essential and natural boundary conditions.

18. a) How is a finite-element approximation defined?

b) Explain how to convert a FEM discretization to an algebraic problem, e.g. that the Navier-Stokes equations yield

$$\begin{pmatrix} \nu\mathbf{K} + \mathbf{C}(u) & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

19. a) By choosing $\mathbf{z}_h = \mathbf{u}_h$ in the finite element approximation of the Stokes problem, show that any Galerkin velocity solution is stable.

b) Describe and interpret the LBB condition.

20. Choice of elements. Discrete elements pairs

   a) constant pressure-bilinear velocities,

   b) the Taylor-Hood pair,

   c) a stable choice with piecewise linear velocities.

21. a) Derive the finite volume (FV) discretization on arbitrary grids of the continuity equation ($\partial u_i / \partial x_i = 0$),

   b) derive the FV discretization for Laplace equation on a Cartesian grid,

   c) show that both are equivalent to a central difference approximation for Cartesian grids.

22. Derive the finite element (FEM) discretization for Laplace equation on a Cartesian grid and show that it is equivalent to a central difference approximation.

23. Iterative techniques for linear systems.

   a) Define Gauss-Seidel iterations for the Laplace equations,

   b) Define the 2-level multigrid method for the Laplace equation,

24. State the difficulties associated with the the finite-volume discretizations of the Navier-Stokes equations on a colocated grid? and show the form of the spurious solution which exist.

25. a) Define an appropriate staggered grid that can be used for the discretization of the Navier-Stokes equations,

   b) write down the FV discretization of the Navier-Stokes equations on a staggered cartesian grid,

   c) discuss how to treat noslip and inflow/outflow boundary conditions.

26. Time dependent flows.

   a) Define a simple projection method for the time dependent Navier-Stokes equations

$$\frac{d}{dt} \begin{pmatrix} u \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{N}(u) & \mathbf{G} \\ \mathbf{D} & \mathbf{0} \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

   b) show in detail the equation for the pressure to be solved at each time step and discuss the boundary conditions for the pressure.

27. Time step restriction for Navier-Stokes solutions.

   a) Motivate the use of an appropriate form of the advection-diffusion equation as a model equation for stability analysis,

   b) derive the time step restrictions for the 1D version of that equation,

   c) state the 2D equivalent of that restriction.

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