Homework 2

Finite differences

due February 1, 23:59

Guidelines:

- Use only a **single pdf**. Preferably scan your homework, if you take a photo, make sure it is sharp and bright;
- **Include all plots in the pdf** in the right place;
- Do not present plots without commenting them: always write a (short) description of what the plot tells you and what you can conclude from it;
- You can work in groups up to 3;
- If you work in groups:
  - write the names of all group members at the beginning of the report;
  - you can work and discuss together but each group member is required to submit an individual report written with his/her own words; **copy-paste reports will not be accepted**
- Please use the following naming convention: "**surname_hwX.pdf**" and include all Matlab files as a single **separate** archive: "**surname_hwX.zip**". X is the number of the homework.

Task 1: Integration of an ordinary differential equation

In this problem, the stability and convergence order of three integration methods is examined. The following first order, ordinary, linear differential equation with constant coefficient is considered (also known as the Dahlquist equation):

$$\frac{\mathrm{d}u}{\mathrm{d}t} = A(u) = \lambda u\,, \qquad u(0) = 1\,, \tag{1}$$

where $0 \leq t \leq T$ and $\lambda \in \mathbb{C}$. The time interval $[0, T]$ is split into N parts with the same length Δt. The following integration methods should be considered:

- explicit Euler method:
$$u^{n+1} - u^n = \Delta t A(u^n) \tag{2}$$
- implicit Euler method:
$$u^{n+1} - u^n = \Delta t A(u^{n+1}) \tag{3}$$
- Crank–Nicolson method:
$$u^{n+1} - u^n = \frac{1}{2}\Delta t \left[A(u^{n+1}) + A(u^n)\right] \tag{4}$$

where $n = 0, ..., N$. Compute the solution until $T = 10$ and use $N = 20, 100$ and 500 steps.

1. Derive the analytic solution u_{ex}.
2. For $\lambda = -3/5 + i\pi$, calculate the numerical solution with the given discretizations N and the three integration methods. Plot the real part of the analytic solution and the three numerical solutions for each value of N.

3. Consider $\lambda \in \mathbb{R}$. As $\lambda \to \infty$ the problem becomes more and more stiff. Derive, for the three considered numerical schemes, the expression of the amplification factor, $G(z)$, where $z = \lambda\Delta t \in \mathbb{R}$. Compute the limit $G(z)$ for $z \to \infty$ and plot $G(z)$ as function of z for the three schemes and the analytic solution for $z \in [-10, 0.5]$. Which of the schemes provides a better approximation of the exact (analytic) amplification factor for one time step Δt applied to stiff problems (*i.e.* when $z \to \infty$)? Why are the imaginary part of λ and z irrelevant for the limit?
4. Repeat the analysis of point 2 for $\lambda = -3/5 + i$.
5. Based on the two examples considered above, discuss the usefulness, stability and accuracy of the methods.

Task 2: Finite Difference Schemes

Find the highest order finite difference approximation possible of the first derivative of $u(x)$ at the grid nodes $x = x_i$ based on four grid values u_{i-1}, u_i, u_{i+1} and u_{i+2} where $u_i := u(x_i)$. Assume equidistant grid spacing, *i.e.* $\Delta x := x_{i+1} - x_i = x_i - x_{i-1}$, for all i.

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_i} \approx f(u_{i-1}, u_i, u_{i+1}, u_{i+2})$$

a) Give the approximation of the derivative.

b) What is the leading error term? What is the order of this scheme?

Task 3: Stability Criteria

The range of absolute stability of the Runge–Kutta 4^{th}-order method is studied. This method for an initial value problem of the form $\frac{\mathrm{d}u}{\mathrm{d}t} = f(u,t),\ u(t_0) = u_0$ is

$$u^{n+1} = u^n + \frac{\Delta t}{6}(f^n + 2k_1 + 2k_2 + k_3) \tag{5}$$

where

$$k_0 = f(u^n, t^n) \tag{6}$$

$$k_1 = f(u_1, t^{n+\frac{1}{2}}), \quad u_1 = u^n + \frac{\Delta t}{2} f^n \tag{7}$$

$$k_2 = f(u_2, t^{n+\frac{1}{2}}), \quad u_2 = u^n + \frac{\Delta t}{2} k_1 \tag{8}$$

$$k_3 = f(u_3, t^{n+1}), \quad u_3 = u^n + \Delta t k_2 \tag{9}$$

Consider a simple linear test equation (*Dahlquist equation*):

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u\,. \tag{10}$$

Show that u^{n+1} can be written as a function of u^n and $z = \Delta t\lambda$ as follows

$$u^{n+1} = u^n \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}\right)\,. \tag{11}$$

The absolute stability criterion is

$$|G(z)| = \left|\frac{u^{n+1}}{u^n}\right| \leq 1\,. \tag{12}$$

Draw the region that corresponds to equation (12) on the complex $z-$plane.

(Hint: The curve $|G(z)| = 1$ cuts the imaginary axis at ± 2.83)