Problem Set B.

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1.

1.(a) It only has Neumann boundary condition: $\hat{n} \cdot \nabla u(x) = g$. Only this condition cannot have a unique solution, because $u(x)$ is a solution, $u(x)+c$ is still a solution for any constant $c \in \mathbb{R}$.

1(b). I could not be able to solve this linear system. I have tried for the project 1, when I haven't add Dirichlet condition, the coefficient matrix is singular. In other words, the determinant of this matrix is zero, so it does not have a unique inverse. I got result is Inf or Nan.

2.

2.(a). Derive variational formulation:

$$v \in H^1(\Omega), \quad v(0)=0 \quad v(1)=0.$$
$$= V = H_0^1(\Omega).$$

multiply v to the $-(k(x)u'(x))' + u'(x) = f(x)$.

$$\int_0^1 fv = \int_0^1 -(ku')'v + u' \cdot v$$
$$= \int_0^1 ku'v' + u' \cdot v$$

The variational formulation is: find $u \in V$. such that

$$a(u,v) = L(v) \quad \forall v \in V$$

where $a(u,v) = \int_0^1 ku'v' + u' \cdot v \, dx$

$$L(v) = \int_0^1 f \cdot v$$

(b) show that for $\|w\|_E^2 = a(w,w)$, $\|w\|_E^2 = \int_0^1 k(x)(w'(x))^2 \cdot dx$:
$a(\cdot;\cdot)$ defines the induced energy norm $|||u|||^2 \cdot a(u,u)$.
$|||v|||^2 = \int_\Omega \cdot \nabla v \cdot \nabla v \cdot dx$.

$\|w\|_E^2 = a(w.w) = \int_0^1 w' \cdot w' \cdot dx$.

From (a), we know $a(u.v) = \int_\Omega \cdot ku'v' + u'v \cdot dx$

So $a(w.w) = \int_\Omega kw'w' + w'w \cdot dx$

$= \int_\Omega \cdot k \cdot (w'(x))^2 dx + \int_\Omega w' \cdot w \cdot dx \quad \forall w \in V$

since $\int_\Omega w' \cdot w \cdot dx = \int_{\partial\Omega} w \cdot w \cdot dx - \int_\Omega \cdot w \cdot w' \cdot dx$

$= 0 - \int_\Omega \cdot w' \cdot w \cdot dx$

so $2\int_\Omega w' w \cdot dx = 0$.

$\Rightarrow a(w,w) = \int_\Omega k \cdot w' \cdot w' \cdot dx$

(c): A function must satisfy below properities to be a norm:

1: Non-negativity.

2: Scalar multiplication.

3: Triangle inequality.

1: since $k(x) \geq k_0 \geq 0 \Rightarrow \int k(x) \cdot (w')^2 dx \geq 0$.

fulfill the non-negativity.

2: $\|\lambda \cdot w\|_E^2 = \int_0^1 \lambda^2 \cdot k(x) (w')^2 \cdot dx$.

Scalar multiplication is fulfilled.

3: $\|w+u\|_E^2 = 2a(w,u) + a(w.w) + a(u,u) = \int_0^1 k((w+u)')^2 \cdot dx$

Cauchy inequality $\rightarrow \leq (\|u\|_E + \|w\|_E)^2$

$\Rightarrow \|w+u\|_E \leq \|u\|_E + \|w\|_E$

Triangle inequality is fulfilled.

Thus. $\|w\|_E^2 = a(w.w)$ represents a norm on V.

(c): Prove there exists a unique solution:

a unique solution means solution existence and uniqueness.

Here, Lax-Milgram theorem is recalled.

For a real Hilbert space V and a bilinear form $B: V\times V \to R$,

① that is continuous: i.e. $\beta > 0$. $B(v,w) \le \beta \|v\|_V \|w\|_V$, for $v \in V$

② that is coercive, i.e. $\alpha > 0$. $B(v,v) \ge \alpha \|v\|^2_V$ for all $v \in V$.

Then, there exists exactly one $u \in V$, such that

$B(u,v) = F(v)$ for all $v \in V$.

1: continuous. with L^2 cauchy-schwarz inequality.

$$a(u,v) = \int_\Omega k u' v' + u' v \, dx$$

Cauchy-Schwarz

$$\le \|k\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

Poincaré Friedrichs

$$\le C_1 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + C_2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

$$\le C_3 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

$$\Rightarrow a(u,v) \le C_3 \cdot \|u\|_{H^1(\Omega)} \cdot \|v\|_{H^1(\Omega)}$$

$$L(v) = \int_0^1 f \cdot v \le \|k\|_{L^\infty(\Omega)} \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le C_4 \|\nabla f\|_{L^2(\Omega)} \cdot \|\nabla v\|_{L^2(\Omega)}$$

Cauchy Poincare

$$\le C_5 \|f\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

$$\Rightarrow L(v) \le C_5 \|f\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

2. Coercive: With Poincaré-Friedrichs inequality.

$$a(u,u) = \int_\Omega k \cdot u' u' + u' \cdot u \cdot dx. \qquad k(x) \ge k_0 \ge 0.$$

$$\ge k_0 \cdot \int_\Omega (\nabla u)^2 + \int_\Omega u \cdot \nabla u \cdot dx$$

We need to prove $\exists c > 0$. $a(u,u) \ge \alpha \cdot \|u\|^2_V$.

$$\ge k_0 \cdot \int_\Omega (\nabla u)^2 + \int_\Omega u \cdot \nabla u \cdot dx.$$

proved in 2(b).

$$\ge k_0 \cdot \int_\Omega (\nabla u)^2 + 0.$$

$$\ge \frac{k_0}{2} \int_\Omega |\nabla u|^2 + \frac{k_0}{2} \int_\Omega (\nabla u)^2 = \frac{k_0}{2} \int_\Omega |\nabla u|^2 + \frac{k_0}{\mathrm{diam}(\Omega)^2} \int_\Omega |v|^2$$

$$\ge \alpha \cdot \|u\|^2_{H^1(\Omega)}. \qquad \alpha = k_0 \min\left(\frac{1}{2}, \frac{1}{\mathrm{diam}(\Omega)^2}\right).$$

We summarize the properties:

1. → linearity $\quad a(ku, v) = \int_\Omega \nabla(ku) \cdot \nabla v \, dx$.

2. continuity: $\quad a(u, v) \leq \beta \cdot \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$

3. Coercivity. $\quad a(u, v) \geq \alpha \|v\|^2_{H^1(\Omega)}$

Hence. $a(u,v)$ fulfills the assumptions of the Lax-Milgram theorem.

As for the linear functional: $L(v) = \int_\Omega v$; continuity has been proved.

Above all, the $a(u,v)$ is a continuous and coercive bilinear form.

$L(v)$ is a continuous, linear functional.

Hence, the Lax-Milgram theorem implies that there is a unique solution $u \in H_0^1(\Omega)$ with

$$a(u, v) = L(v) \quad \text{for all } v \in H_0^1(\Omega).$$

(d): Formulate the cG(1) method.

Replace infinite dim space V (e.g. $V = H_0^1(\Omega)$) by finite dim subspace $V_h \subset V$. Find $u_h \in V_h$, such that.

$$a(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in V_h.$$

mesh size $h = h(x)$.

(e): Prove a priori error estimate: $\|u - u_h\|_E \leq c\|hu''\|_{L^2(0,1)}$.

Galerkin orthogonality tell us:

$$a(e_h, v_h) = 0, \quad e_h = u - u_h.$$

Hence, $a(\cdot,\cdot)$ is the best approximation of u in V_h.

$$\|u - u_h\|_E \leq \inf_{v_h \in V_h} \|u - v_h\|_E \qquad \| \ \|_E = \sqrt{a(\cdot,\cdot)} \text{ is the energy norm.}$$

Then, choosing $v_h = \pi_h u$ the interpolant of u.

$$\|u - u_h\|^2_{L^2(0,1)} \leq \|u - \pi_h u\|^2_{L^2(0,1)}$$

$$\leq \sum_{i=1}^{n} \|u - \pi_h u\|^2_{L^2(h_i)} \leq \sum_{i=1}^{n} C \|h u''\|^2_{L^2(0,1)}.$$

$$\Rightarrow \|u - u_h\|_E \leq c \cdot \|h \cdot u''\|_{L^2(0,1)}.$$

(f): steps in an adaptive algorithm:

1: Construct mesh, initial T_h.

2: Solve finite element problem for u_h.

3. Compute local indicators $R_K(u_h)^2$.

4. Compute maximum $m = \max_{K\in T_h} R_K(u_h)^2$.

5. Mark element with error over $r \cdot m$, where $0<r<1$ is a fixed parameter.

6. Refine elements to get new mesh T_h.

7. Return to step 2. (end until N is to large. N is return times or the error is small enough).

Why adaptivity important: Could reduce the error cheaply, if refine the whole mesh, it is expensive. Instead, only refine the mesh locally is fast.

3. Show spatial discretization and identity the entries:

$$\begin{cases} \dot{u} - \Delta u + u = f & , x\in\Omega. \\ u = 0 & , x\in\partial\Omega, t>0 \\ u = u_0 & , x\in\Omega, t=0 \end{cases}$$

variational form:

multiply by $v\in H_0^1(\Omega)$ and integrate over Ω:

$$\left(\frac{\partial u}{\partial t}, v\right) - (\Delta u, v) + (u, v) = (f, v).$$

Green's formula: $\int_\Omega k\cdot\nabla u\cdot\nabla v = -\int_\Omega \nabla\cdot(k\cdot\nabla u)\cdot v + \int_{\partial\Omega} \hat{n}\cdot(k\cdot\nabla u)\cdot v$

Then $\Rightarrow \left(\frac{\partial u}{\partial t}, v\right) + (\nabla u, \nabla v) + (u, v) = (f, v)$.

Find $u(t)\in H_0^1(\Omega)$, such that

$$\left(\frac{\partial u}{\partial t}, v\right) + (\nabla u, \nabla v) + (u, v) = (f, v), \quad \forall v\in H_0^1(\Omega) \quad t\in(0, T].$$

and $u(\cdot, 0) = u_0$.

Semi-discrete:

Replace $H_0^1(\Omega)$ with finite dimensional $V_{h,0}$.

- Mesh T_h: admissible triangulation of Ω of size h.
- Finite element space $V_{h,0}$, Ex: P1 (Lagrange) finite element space.

$V_{h,0} = \{ v \in C^0(\bar{\Omega}) \cap H_0^1(\bar{\Omega}) \ \forall k \in T_h : v|_k \text{ is a polynomial of deg } 1 \}$.

Semi-discrete FEM: Find $u_h(t) \in V_{h,0}$, s.t

$$\left(\frac{\partial u_h}{\partial t}, v\right) + (\nabla u_h, \nabla v) + (u_h, v) = (f, v), \quad \forall v \in V_{h,0},\ t \in (0,T].$$

and $u_h(\cdot, 0) = u_{0,h} \in V_{h,0}$.

Matrix representation of the semi-discrete system:

$u_h(x,t) = \sum_{j \in N_h} \xi_j(t)\, \phi_j(x)$

ϕ_j nodal basis functions of $V_{h,0}$.

N_h number of interior Nodes $N_{h,0}$

Replace u_h and test with ϕ_i, since $\text{span}\{\phi_i\} = V_{h,0}$.

$$\sum_{j=1}^{N_h} \frac{\partial \xi_j}{\partial t} (\phi_j, \phi_i) + \sum_{j=1}^{N_h} \xi_j (\nabla \phi_j, \nabla \phi_i) + \sum_{j=1}^{N_r} \xi_j (\phi_j, \phi_i) = (f, \phi_i) \quad i \in N_h$$

$\xi_j(0) = u_{0,h}(z_j)$, $z_j \in N_{h,0}$.

The identify matrix, mass (M), stiffness (A), Load vector F.

In matrix form:

$$① \ M \frac{\partial \xi}{\partial t} + A\xi + M \cdot \xi(t) = b(t). \qquad ② \ \xi(0) = V_{0,h}$$

$M_{ij} = \int_\Omega (\phi_j, \phi_i)\, dx$ Mass matrix

$A_{ij} = \int_\Omega (\nabla \phi_j, \nabla \phi_i)\, dx$ Stifness matrix

$F_i = \int_\Omega (f(t), \phi_i)\, dx$ Load vector

$(V_{0,h})_j = V_{0,h}(z_j)$ Initial condition.

4. Consider the convection - diffusion - reaction equation, write down a finite element method:

$$\dot{u} - \varepsilon \Delta u + \beta \cdot \nabla u + \alpha \cdot u = f, \quad (x,t) \in \Omega \times (0,T], \quad \varepsilon > 0.$$

Multiply by $v \in H_0^1(\Omega)$ and integrate over Ω.

$$\left(\frac{\partial u}{\partial t}, v\right) - (\varepsilon \Delta u, v) + (\beta \cdot \nabla u, v) + (\alpha u, v) = (f, v)$$

Green's formula: $\int_\Omega k \cdot \nabla u \cdot \nabla v = -\int_\Omega \nabla \cdot (k \cdot \nabla u) \cdot v + \int_{\partial\Omega} \hat{n} \cdot (k \nabla u) \cdot v$.

and assume $u = 0$ on $\partial\Omega$. gives.

$$\left(\frac{\partial u}{\partial t}, v\right) + (\varepsilon \nabla u, \nabla v) + (\beta \nabla u, v) + (\alpha u, v) = (f, v). \quad \forall v \in H_0^1(\Omega), t \in (0,T].$$

$$u(\cdot, 0) = u_0(\cdot)$$

Replace $H_0^1(\Omega)$ with finite dimensional $V_{h,0}$.

$V_{h,0} = \{ v \in C^0(\bar{\Omega}) \cap H_0^1(\bar{\Omega}), \ \forall K \in T_h : v|_K \text{ is a polynomial of deg } 1 \}$

T_h: admissible triangulation of Ω of size h.

Use matrix representation:

$u_h(x,t) = \sum_{j \in N_h} \xi_j(t) \phi_j(x)$.

ϕ_j: nodal basis functions of $V_{h,0}$

N_h: number of interior nodes.

Assume $\begin{cases} u = g_- & \text{on } (T)_- \\ u = g_+ & \text{on } (T)_+ \end{cases}$ T is boundary.

$\beta = (\beta_1, \beta_2)$ representing convection velocity.

$$\begin{cases} T_- = \{ (x,t) \in T : \beta(x,t) \cdot n(x) < 0 \} \\ T_+ = \{ (x,t) \in T : \beta(x,t) \cdot n(x) \geq 0 \} \end{cases}$$

$n(x)$ is the outward normal to T at point x.

$$\sum_{j=1}^{N_h} \frac{\partial \xi_j}{\partial t} (\phi_j, \phi_i) + \sum_{j=1}^{N_h} \varepsilon \xi_j (\nabla \phi_j, \nabla \phi_i) + \sum_{j=1}^{N_h} \beta \xi_j (\nabla \phi_j, \phi_i) + \sum_{j=1}^{1} \alpha \xi_j (\phi_j, \phi_i) = f(\phi_i).$$

$i \in N_h$

$\xi_j(0) = U_{0,h}(z_j). \quad z_j \in N_{h,0}.$

$$\begin{cases} M_{ij} = \int_\Omega (\phi_j \cdot \phi_i) dx \\ A_{ij} = \int_\Omega (\nabla\phi_j \cdot \nabla\phi_i) \cdot dx \\ B_{ij} = \int_\Omega (\nabla\phi_j \cdot \phi_i) \cdot dx \\ F_i = \int_\Omega (f(t) \cdot \phi_i) \cdot dx \\ (u_{0,h})_j = u_{0,h}(z_j) \end{cases}$$

In matrix form:

$$\begin{cases} M \frac{\partial \xi}{\partial t} + \epsilon \cdot A_{ij} \xi + \beta \cdot B_{ij} \xi + \alpha \cdot M \xi = F \\ \xi(0) = u_{0,h} \end{cases}$$

Implicit Euler time stepping: let $0 = t_0 < t_1 < \cdots < t_n = T$, with $k_n = t_{n+1} - t_n$.

$\frac{\partial \xi}{\partial t}$ is approximated by a backward quotient.

$$\begin{cases} M \left(\frac{\xi^{n+1} - \xi^n}{k_n}\right) + \epsilon \cdot A_{ij} \xi^{n+1} + \beta \cdot B_{ij} \xi^{n+1} + \alpha \cdot M \xi^{n+1} = F(t_{n+1}) \quad n \geq 0 \\ \xi^0 = u_{0,h}. \end{cases}$$

Rearranging.

$$\begin{cases} (M + k_n A + k_n \beta \cdot B + k_n \alpha \cdot M) \cdot \xi^{n+1} = M \cdot \xi^n + k_n \cdot F(t_{n+1}). \quad n \geq 0 \\ \xi^0 = u_{0,h} \end{cases}$$

(b): assume $\epsilon = 1$. $\alpha(x) = f(x) = 0$. $\beta = [0,0]^T$, prove that

$$\|u(T)\|^2 + 2\int_0^T \|\nabla u\|^2 \cdot dt = \|u_0\|^2 \quad \forall t > 0.$$

take above information in convection-diffusion-reaction equation.

$$\dot{u} - \epsilon \cdot \Delta u + \beta \cdot \nabla u + \alpha u = f.$$

$$\begin{cases} \dot{u} - \Delta u = 0. \\ u = u_0 \end{cases}$$

$v \in H'(\Omega) \Rightarrow \int_\Omega u' v + \int_\Omega \nabla u \cdot \nabla v = 0 \Rightarrow \int_\Omega \frac{d}{dt}\left(\frac{u^2}{2}\right) + \int_\Omega (\nabla u)^2 = 0$

integrate in T: $\int_0^T \int_\Omega \frac{d}{dt}\left(\frac{u^2}{2}\right) dx \cdot dt + \int_0^T \int_\Omega (\nabla u)^2 dx \, dt = 0.$

$$\Rightarrow \|u(T)\|^2 + 2 \int_0^T \|\nabla u\|^2 dt = \|u_0\|^2 \quad \forall t > 0.$$

$u(T)$ decrease as time increases, since $\|\nabla u\|^2 > 0$, while $\|u(T)\|^2$ need to small to fulfill the above equation.