Problem Set A

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1 (a) i We introduce a test function v that satisfies: $I = (0,1)$
$v \in \mathcal{V} = \{ v : \|v\|_{L^2(I)} < \infty, \|v'\|_{L^2(I)} < \infty, v(0) = 0 \}$

We multiply the test function on both sides of eq(1) and perform an integration by part

$$\int_0^1 -(a(x)u')'v\,dx = 0$$

By integration by part, the left-hand-side becomes

$$-\int_0^1 (a(x)u')'v\,dx = -\left\{ a(x)u'v\Big|_{(0,1)} - \int_0^1 a(x)u'v'dx \right\}$$

$$= \int_0^1 a(x)u'v'dx - \left(a(1)u'(1)v(1) - a(0)u'(0)v(0) \right)$$

$$= \int_0^1 a(x)u'v'dx - 2v(1)$$

Therefore the variational formulation for the exact solution u is:

find $u \in \mathcal{V}$ such that

$$\int_0^1 a(x)u'v'dx = 2v(1) \qquad \forall v \in \mathcal{V}$$

1. (b) We introduce a mesh on the interval I consisting of n subintervals, and the corresponding space V_h of all continuous piecewise linears. We also introduce the subspace $V_{h,0}$ of V_h that satisfies the boundary conditions

$$V_{h,0} = \{ v \in V_h : v(0) = 0 \}$$

We obtain the finite element approximation:

find $u_h \in V_{h,0}$ such that

$$\int_0^1 a(x)\, u_h' v' \, dx = 2v(1) \quad \forall v \in V_{h,0}$$

A basis for $V_{h,0}$ is given by the set of n hat functions $\{\varphi_i\}_{i=1}^{n}$ defined as.

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in (x_{i-1}, x_i) \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in (x_i, x_{i+1}) \\ 0 & \text{else} \end{cases} = \begin{cases} \dfrac{x - x_{i-1}}{h_i} & x \in I_i \\ \dfrac{x_{i+1} - x}{h_{i+1}} & x \in I_{i+1} \\ 0 & \text{else} \end{cases}$$

Thus, the problem is equivalent to find $u_h \in V_{h,0}$ such that

$$\int_0^1 a(x)\, u_h' \varphi_i' \, dx = 2\varphi_i(1) \quad \forall i = 1, 2, \cdots n$$

Since $u_h \in V_{h,0}$, we have

$$u_h = \sum_{j=1}^{n} c_i \varphi_i$$

Thus we get:

$$\sum_{j=1}^{n} c_j \int_I a\, \varphi_j' \varphi_i' dx = 2\varphi_i(1) \quad \forall i = 1, 2, \dots n$$

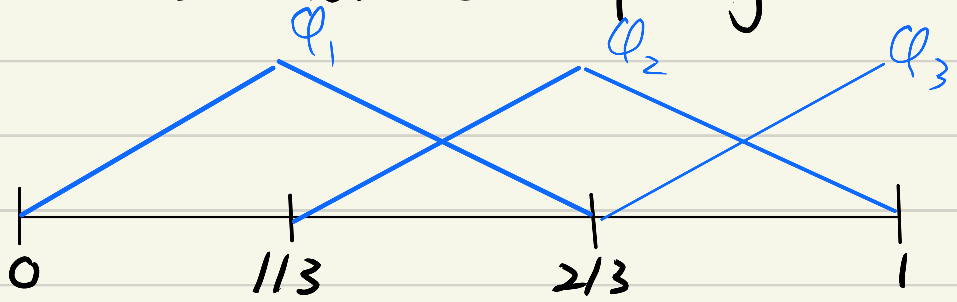
Introduce the notation.

$$A_{ij} = \int_I a \varphi_j' \varphi_i' \, dx$$

$$r_i = 2\varphi_i(1)$$

We have a linear system as $AC = r$

In the case where the domain is equally divided into 3 elements:



By definition of φ_i, for $|i-j|>1$, $A_{ij} = 0$

$$A_{11} = \int_0^{2/3} (1+x)\, \varphi_1'^2 \, dx = 9 \int_0^{2/3} (1+x) dx$$

$$= 9 \times \left[x + \frac{x^2}{2}\right]_0^{2/3} = 9 \times \left(\frac{2}{3} + \frac{4}{18}\right) = 8$$

$$A_{22} = 9 \times \int_{1/3}^{1} (1+x) dx = 9 \times \left[x + \frac{x^2}{2}\right]_{1/3}^{1}$$

$$= 9 \times \left[1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{18}\right] = 10$$

$$A_{33} = 9 \times \left[x + \frac{x^2}{2}\right]_{2/3}^{1} = 9 \times \left[1 + \frac{1}{2} - \frac{2}{3} - \frac{4}{18}\right]$$

$$= \frac{11}{2}$$

$$A_{12} = A_{21} = -9 \times \left[x + \frac{x^2}{2}\right]_{1/3}^{2/3} = -\frac{9}{2}$$

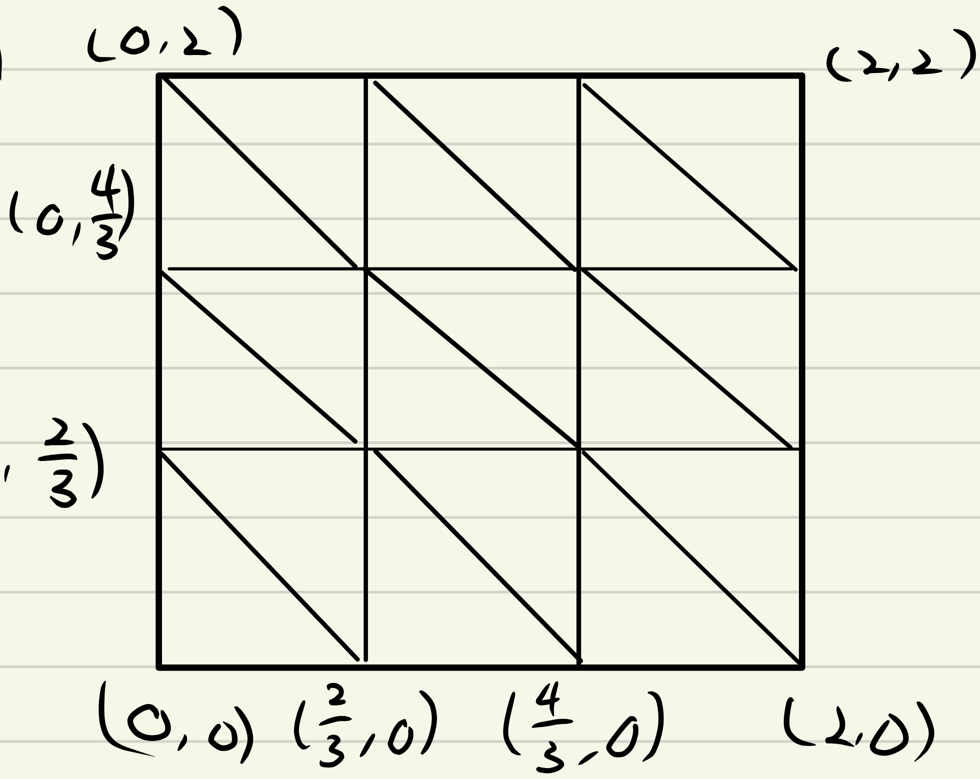
$$A_{23} = A_{32} = -9 \times \left[x + \frac{x^2}{2}\right]_{2/3}^{1} = -\frac{11}{2}$$

$$b_1 = b_2 = 0 \qquad b_3 = 2\varphi_3(1) = 2$$

Therefore, $A = \frac{1}{2}\begin{bmatrix} 16 & -9 & 0 \\ -9 & 20 & -11 \\ 0 & -11 & 11 \end{bmatrix}$ $\quad b = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

2 $\begin{cases} -\Delta u(\underline{x}) = 2 & \underline{x} \in \Omega \\ u(\underline{x}) = 0 & \underline{x} \in \partial\Omega \end{cases}$ $\quad \underline{x} = (x_1, x_2)^T$

(a) We introduce a triangulation $\mathcal{K} = \{K\}$ where K is a set of uniform triangles as shown in the figure and use it as our mesh $\mathcal{T}_h$



(b) $V_h := \{v : v \in C^0(\Omega), \underbrace{v|_K \in P_1(K), \forall K \in \mathcal{T}_h}_{\text{Continuous Piecewise Linear}} ; v|_{\partial\Omega} = 0\}$

(c) $\quad V := \{v : \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)} < \infty, \ v|_{\partial\Omega} = 0\}$.

$$-\int_\Omega \Delta u \, v \, d\underline{x} = \int_\Omega 2v \, d\underline{x} \quad \forall v \in V$$

By Green's formula:

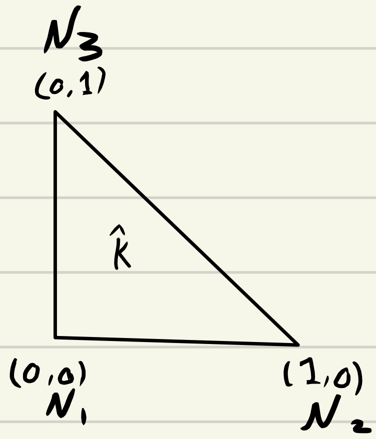
$$-\int_\Omega \Delta u \, v \, d\underline{x} = \int_\Omega \nabla u \cdot \nabla v \, d\underline{x} - \cancelto{0}{\int_{\partial\Omega} n \cdot \nabla u \, v \, ds}$$

We have $\int_\Omega \nabla u \cdot \nabla v \, d\underline{x} = \int_\Omega 2v \, d\underline{x} \quad \forall v \in V$

Thus the finite element approximation is:

$$\boxed{\begin{array}{l} \text{Find } u_h \in V_h \text{ such that} \\ \displaystyle\int_\Omega \nabla u_h \cdot \nabla v \, d\underline{x} = \int_\Omega 2v \, d\underline{x} \quad \forall v \in V_h \end{array}}$$

(d) Compute the element stiffness matrix A and load vector b on the reference triangle $\hat{K}$ with corner $(0,0)$, $(1,0)$ and $(0,1)$



Each basis function is a linear function on $\hat{K}$. Name nodes of $\hat{K}$ as: $N_1 = (x_1, y_1) = (0,0)$, $N_2 = (x_2, y_2) = (1,0)$, $N_3 = (x_3, y_3) = (0,1)$. Then

$$\varphi_i = a_i + b_i x + c_i y \quad \forall i = 1,2,3$$

the coefficients are determined by $\varphi_i(N_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Thus, we have $a_i = \dfrac{x_j y_k - x_k y_j}{2|\hat{K}|}$, $b_i = \dfrac{y_j - y_k}{2|\hat{K}|}$, $c_i = \dfrac{x_k - x_j}{2|\hat{K}|}$

with cyclic permutation of the indices $\{i,j,k\}$ over $\{1,2,3\}$

Here $|\hat{K}|$ is the area of the domain $= \frac{1}{2}$

Hence: $a_1 = x_2 y_3 - x_3 y_2 = 1$; $b_1 = y_2 - y_3 = -1$; $c_1 = x_3 - x_2 = -1$

$a_2 = x_3 y_1 - x_1 y_3 = 0$; $b_2 = y_3 - y_1 = 1$; $c_2 = x_1 - x_3 = 0$

$a_3 = x_1 y_2 - x_2 y_1 = 0$; $b_3 = y_1 - y_2 = 0$; $c_3 = x_2 - x_1 = 1$

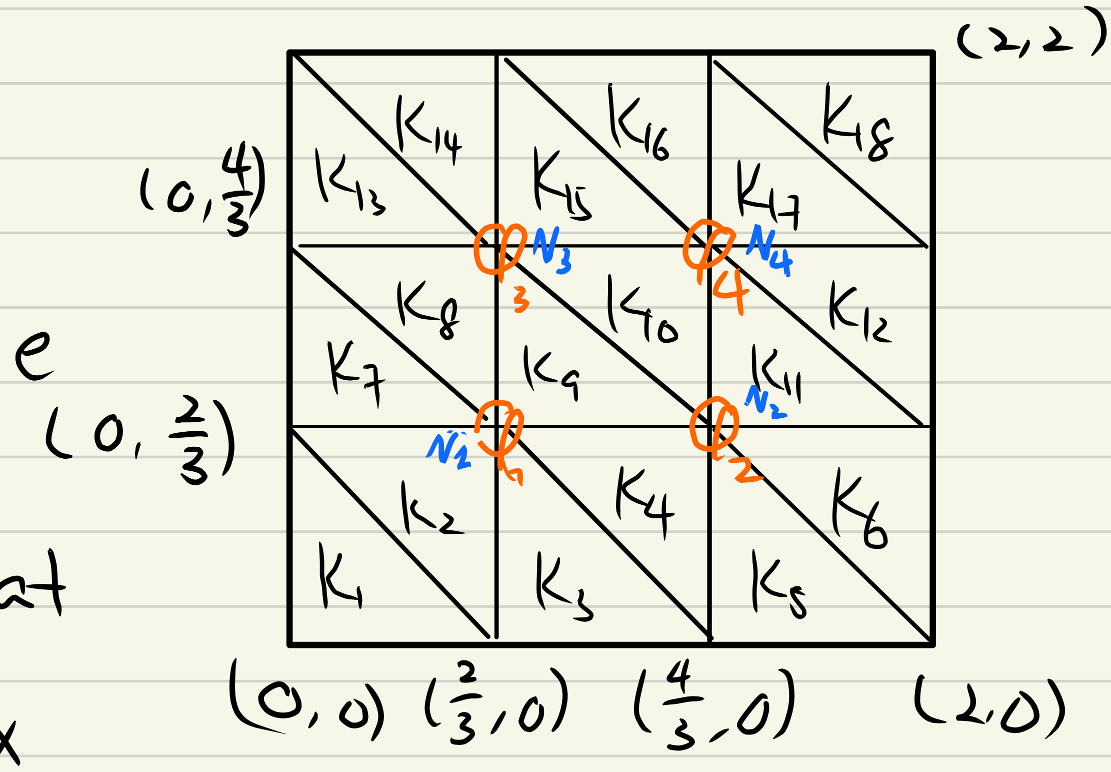
$$\varphi_1 = 1 - x - y \; ; \quad \varphi_2 = x \; ; \quad \varphi_3 = y$$

$$A_{ij}^{\hat{K}} = \int_{\hat{K}} \nabla\varphi_i \nabla\varphi_j \, d\underline{x} = (b_i b_j + c_i c_j) \int_{\hat{K}} d\underline{x} = (b_i b_j + c_i c_j)|\hat{K}| \quad \forall i,j = 1,2,3$$

$$b_i^{\hat{K}} = \int_{\hat{K}} 2\varphi_i \, d\underline{x} = \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} \varphi_i \, dx \, dy$$

$$\boxed{A = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{bmatrix}}$$

(e)



Given the mesh in 2a, space V_h in 2b and weak form in 2c, it is easy to note that A is a 4×4 matrix because of zero boundary condition and b is a 4×1 matrix

$$A_{ij} = \int_{\Omega} \nabla\varphi_i \nabla\varphi_j \, d\underline{x} \qquad \forall\ i,j = 1,2,3,4.$$

From (d) we know that for a given element K

$$A_{ij}^{K} = (b_i b_j + c_i c_j)\,|K|$$

$$A_{11} = A_{11}^{K_2} + A_{11}^{K_3} + A_{11}^{K_4} + A_{11}^{K_7} + A_{11}^{K_8} + A_{11}^{K_9}$$

global mesh

$$= \Big[\underbrace{\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{3}{2}\right)^2}_{K^2} + \underbrace{\left(\tfrac{3}{2}\right)^2}_{K^3} + \underbrace{\left(\tfrac{3}{2}\right)^2}_{K^4} + \underbrace{\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{3}{2}\right)^2}_{K^9} + \underbrace{\left(\tfrac{3}{2}\right)^2}_{K^8} + \underbrace{\left(\tfrac{3}{2}\right)^2}_{K^7}\Big] \times \frac{2}{9}$$

$$= \frac{9}{4} \times 8 \times \frac{2}{9} = 4$$

Similarly, $A_{22} = A_{33} = A_{44} = 4$

$$A_{12} = A_{12}^{K_4} + A_{12}^{K_9}$$

$$= \left(\underbrace{\begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix}^T \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}}_{\nabla\varphi_1 \cdot \nabla\varphi_2} + \begin{bmatrix} -\frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}^T \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix} \right) \times \frac{2}{9} \leftarrow |K|$$

$$= \left(-\frac{9}{4} - \frac{9}{4} \right) \times \frac{2}{9} = -1$$

$$A_{13} = A_{13}^{K_8} + A_{13}^{K_9}$$

$$= \left(\begin{bmatrix} 0 \\ -\frac{3}{2} \end{bmatrix}^T \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}^T \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix} \right) \times \frac{2}{9} = -1$$

$$A_{14} = 0$$

$$A_{23} = A_{23}^{K_9} + A_{23}^{K_{10}}$$

$$= \left(\begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{3}{2} \end{bmatrix}^T \begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix} \right) \times \frac{2}{9}$$

$$= 0$$

$$A_{24} = A_{24}^{K_{10}} \ A_{24}^{K_{11}}$$

$$= \left(\begin{bmatrix} 0 \\ -\frac{3}{2} \end{bmatrix}^T \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}^T \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix} \right) \times \frac{2}{9} = -1$$

$$A_{34} = A_{34}^{K_{10}} + A_{34}^{K_{15}}$$

$$= \left(\begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix}^T \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}^T \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix} \right) \times \frac{2}{9} = -1$$

$$\boxed{\text{Thus} \quad A = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}}$$

$$b_i = 2 \underbrace{\int_\Omega \varphi_i \, dx}_{\text{Volume of the space covered by the base function}}$$

$$b_1 = b_2 = b_3 = b_4 = \frac{1}{3} \times \left(\frac{2}{3}\right)^2 \times \frac{1}{2} \times 1 \times 6$$

$$= \frac{1}{3} \times \frac{4}{9} \times \frac{1}{2} \times 6$$

$$= \frac{4}{9}$$

$$\boxed{\text{Thus } b = \begin{bmatrix} 4/9 \\ 4/9 \\ 4/9 \\ 4/9 \end{bmatrix}}$$