Homework 1 Material derivative and error propagation

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Task 1: Material derivative

according to the description, we could get

$$P_{ct} = P_{co} + \frac{P_f - P_{ct}}{3600s} t$$

$$P(x,t) = kx + b$$

$$P(x,t) = -\frac{P_e - P_{ct}}{R} x + b = -\frac{P_e - (P_{co} + \frac{P_f - P_{ct}}{3600s} t)}{R} x + b = \frac{350 - (40 + \frac{350-40}{3600s} t)}{5000m} x + b$$

$$P(0,0) = 350 \tag{1.1}$$

$$P(5000,0) = 40$$

$$b = 350$$

$$P(x,t) = 350 - \frac{350 - (40 + \frac{350-40}{3600s} t)}{5000m} x$$

Through equation 1.1, we get the function of price, then according to material derivative to calculate when the price is the lowest.

$$P(x,t) = 350 - \frac{350 - (40 + \frac{350-40}{3600s} t)}{5000m} x$$

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} = \frac{31}{1800000} x + u \times (-\frac{31}{500} + \frac{31}{1800000} t)$$

if x=ut, u=2.5 km/h

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} = \frac{31}{1800000} \times ut + (-\frac{31}{500} + \frac{31}{1800000} t) \times u \tag{1.2}$$

$$= \frac{31}{2592000} t^2 - \frac{31}{720} t + 350$$

$$0 = \frac{31}{1296000} t - \frac{31}{720}$$

$$t = 1800\ s$$

Validate: if the x=ut

$$P(x,t) = 350 - \frac{350 - (40 + \frac{350-40}{3600s} t)}{5000m} x \quad \text{then} \quad P(t) = \frac{31}{2592000} t^2 - \frac{31}{720} t + 350, \tag{1.3}$$

$\frac{DP}{Dt} = \frac{31}{1296000} t - \frac{31}{720}$, the result is the same with equation 1.2.

So the man walk **1800 s** and **1250 m** will get the lowest price with **311 kr/kg**. If he has 130 kr, he could buy **0.42 kg**. The life is too complicated for this man.

Task 2: Machine precision

Matlab code:

```
numprec=double(1.0); % Define 1.0 with double precision
numprec=single(1.0); % Define 1.0 with single precision
while(1 < 1 + numprec)
numprec=numprec*0.5;
end
numprec=numprec*2
```

1: Single precision: 1.1921e-07
2: Double precision: 2.2204e-16

Definition: Machine accuracy is the smallest number which machine could store, use and distinguish in one data space. Through the above code, it seems the machine accuracy depends on the storage space of data, double data is smaller than single with double storage space.

Task 3: Round-off Error

2: Analytic derivative of f(x) is

$$f'(x) = -\frac{1}{(2+x)^2} + 2x \tag{1.4}$$

Cenrered difference scheme is

$$f'(n) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} \tag{1.5}$$

So, the relative discretization error is

$$\varepsilon_d = \left| \frac{f'(x) - f'(n)}{f'(x)} \right| = \left| \frac{-\frac{1}{(2+x)^2} + 2x - \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}}{-\frac{1}{(2+x)^2} + 2x} \right| \tag{1.6}$$

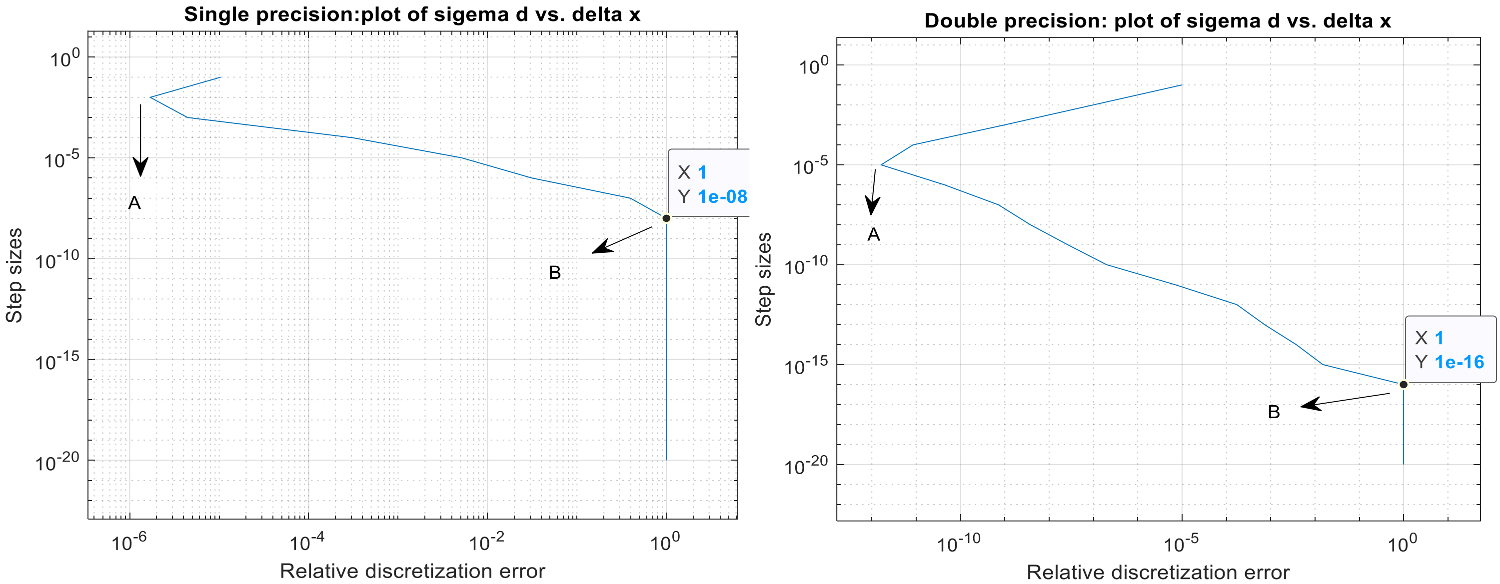

Figure 1 Single and double precision of relative discretization error

As shown in figure 1 left, the single precision of step sizes, when the step sizes are $1*10^{-20}$ to $1*10^{-8}$, the relative discretization error is equal to 1. The reason is computer could not distinguish and these step sizes because they beyond the limit of computer single data. For these cases, step sizes are 0 during calculation and f'(n) is equal to 0, so for relative discretization error, it is always 1. The same results could see in figure 1 right, the double precision of step sizes. Step sizes from $1*10^{-20}$ $1*10^{-20}$ to $1*10^{-16}$, the computer could not distinguish because double precision is near $1*10^{-16}$.

Both for single and double precision of step sizes, after it reaches the computer level, the f'(n) is no longer 0 and starts work. But during calculation, the machinery error still exists, and it decreases with the increase of step sizes since computer could use more effective information of a single data.

For the point A, the relative discretization error increase again, the reason is step sizes are too large and could not show the accurate detail, so the error increasing.

3:

If g(x)=X1+X2, then the propagation error is

$$\varepsilon_p{}^2 = \left(\frac{x_1}{x_1+x_2}\cdot\varepsilon_1\right)^2 + \left(\frac{x_2}{x_1+x_2}\cdot\varepsilon_2\right)^2 \tag{1.7}$$

And

$$f_n(x) = \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x} = \frac{f(x+\Delta x)}{2\Delta x} + \frac{-f(x-\Delta x)}{2\Delta x} \qquad (1.8),$$

Which form is like the g(x)=X1+X2, so the propagation of fn(x) is

$$\varepsilon_p{}^2 = \left[\frac{\frac{f(x+\Delta x)}{2\Delta x}}{\frac{f(x+\Delta x)}{2\Delta x}+\frac{-f(x-\Delta x)}{2\Delta x}}\cdot\varepsilon\right]^2 + \left[\frac{\frac{-f(x-\Delta x)}{2\Delta x}}{\frac{f(x+\Delta x)}{2\Delta x}+\frac{-f(x-\Delta x)}{2\Delta x}}\cdot\varepsilon\right]^2$$

$$\begin{aligned}
&\because \lim_{\Delta x\to 0}\frac{f(x+\Delta x)}{2\Delta x}+\frac{-f(x-\Delta x)}{2\Delta x} \qquad (1.9)\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{2\Delta x}+\frac{f(x)-f(x-\Delta x)}{2\Delta x}\\
&= \frac{1}{2}f^{'}(x)+\frac{1}{2}f^{'}(x)\\
&= f^{'}(x)
\end{aligned}$$

$$\begin{aligned}
\therefore \varepsilon_p{}^2 &= \left[\frac{f(x+\Delta x)}{f^{'}(x)2\Delta x}\cdot\varepsilon\right]^2 + \left[\frac{-f(x-\Delta x)}{f^{'}(x)2\Delta x}\cdot\varepsilon\right]^2\\
&= \frac{f(x+\Delta x)^2+f(x-\Delta x)^2}{[f^{'}(x)2\Delta x]^2}\cdot\varepsilon^2\\
&= \frac{f(x)^2+2f(x)\Delta x+\Delta x^2+f(x)^2-2f(x)\Delta x+\Delta x^2}{[f^{'}(x)2\Delta x]^2}\cdot\varepsilon^2 \qquad (1.10)\\
&= \frac{2f(x)^2+2\Delta x^2}{[f^{'}(x)2\Delta x]^2}\cdot\varepsilon^2
\end{aligned}$$

$$\because \lim_{\Delta x\to 0}$$

$$\therefore \lim_{\Delta x\to 0} 2\Delta x^2 = 0$$

$$\begin{aligned}
\therefore \varepsilon_p{}^2 &= \frac{2f(x)^2+0}{[f^{'}(x)2\Delta x]^2}\cdot\varepsilon^2\\
&= \frac{f(x)^2}{f^{'}(x)^2 2\Delta x^2}\cdot\varepsilon^2 \qquad (1.11)\\
&= \left[\frac{f(x)\varepsilon}{f^{'}(x)\sqrt{2}\Delta x}\right]^2
\end{aligned}$$

For the relative discretization error:

$$\varepsilon_d = \left| \frac{f'(x) - f'(n)}{f'(x)} \right| = \left| \frac{f'(x) - \dfrac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}}{f'(x)} \right| \tag{1.12}$$

Use Taylor expansion to f(x), we get

$$\varepsilon_d = \left| \frac{f'(x) - \dfrac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}}{f'(x)} \right|$$

$$\begin{aligned} f(x+\Delta x) &= \left[f(x) + \frac{f'(x)}{1!}(\Delta x) + \frac{f''(x)}{2!}(\Delta x)^2 + \frac{f'''(x)}{3!}(\Delta x)^3 + \dots \right] \\ f(x-\Delta x) &= \left[f(x) + \frac{f'(x)}{1!}(-\Delta x) + \frac{f''(x)}{2!}(-\Delta x)^2 + \frac{f'''(x)}{3!}(-\Delta x)^3 + \dots \right] \\ f(x+\Delta x) - f(x-\Delta x) &= 2f'(x)\Delta x + \frac{f'''(x)}{3}\Delta x^3 + \dots \\ \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} &= f'(x) + \frac{f'''(x)}{6}\Delta x^2 \end{aligned} \tag{1.13}$$

So

$$\begin{aligned} \varepsilon_d &= \left| \frac{f'(x) - \dfrac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}}{f'(x)} \right| \\ &= \left| \frac{f'(x) - \left[f'(x) + \dfrac{f'''(x)}{6}\Delta x^2 \right]}{f'(x)} \right| \\ &= \left| \frac{f'''(x)\Delta x^2}{6f'(x)} \right| \end{aligned} \tag{1.14}$$

Third question:

$$\begin{aligned} \varepsilon_g^2 &= \varepsilon_p^2 + \varepsilon_d^2 \\ &= \left[\frac{f(x)\varepsilon}{f'(x)\sqrt{2}\Delta x} \right]^2 + \left| \frac{f'''(x)\Delta x^2}{6f'(x)} \right|^2 \\ &= \frac{f(x)^2\varepsilon^2}{f'(x)^2 2\Delta x^2} + \frac{f'''(x)^2\Delta x^4}{6^2 f'(x)^2} \\ &= \frac{18f(x)^2\varepsilon^2 + f'''(x)^2\Delta x^6}{36f'(x)^2\Delta x^2} \end{aligned} \tag{1.15}$$

Then

$$\varepsilon_g^2 = \frac{18 f(x)^2 \varepsilon^2 + f'''(x)^2 \Delta x^6}{36 f'(x)^2 \Delta x^2}$$

$$\varepsilon_g = \left[\frac{18 f(x)^2 \varepsilon^2 + f'''(x)^2 \Delta x^6}{36 f'(x)^2 \Delta x^2}\right]^{\frac{1}{2}}$$

$$f'(\varepsilon_g) = \frac{1}{2\left[\frac{18 f(x)^2 \varepsilon^2 + f'''(x)^2 \Delta x^6}{36 f'(x)^2 \Delta x^2}\right]^{\frac{1}{2}}} \times \left[-2\frac{18 f(x)^2 \varepsilon^2}{36 f'(x)^2 \Delta x^3} + 4\frac{f'''(x)^2 \Delta x^3}{36 f'(x)^2}\right] \tag{1.16}$$

$$= \frac{1}{C1} \times \left[-\frac{f(x)^2 \varepsilon^2}{f'(x)^2 \Delta x^3} + \frac{f'''(x)^2 \Delta x^3}{9 f'(x)^2}\right]$$

The extremum value get when

$$f'(\varepsilon_g) = 0 \tag{1.17}$$

Which means

$$-\frac{f(x)^2 \varepsilon^2}{f'(x)^2 \Delta x^3} + \frac{f'''(x)^2 \Delta x^3}{9 f'(x)^2} = 0$$

then

$$\frac{f'''(x)^2 \Delta x^3}{9 f'(x)^2} = \frac{f(x)^2 \varepsilon^2}{f'(x)^2 \Delta x^3}$$

$$\frac{\Delta x^3}{9} = \frac{f(x)^2 \varepsilon^2}{\Delta x^3} \tag{1.18}$$

$$\Delta x^6 = \frac{9 f(x)^2 \varepsilon^2}{f'''(x)^2}$$

then

$$\Delta x = \sqrt[6]{\frac{9 f(x)^2 \varepsilon^2}{f'''(x)^2}}$$

If x=2, we already now

$$f(x) = \frac{1}{2+x} + x^2$$

$$\varepsilon = 2.2 \times 10^{-16} \tag{1.19}$$

$$f'(2) = 3.9375$$

$$f'''(2) = 0.0234$$

So

$$\Delta x = 4.806 \times 10^{-5} \tag{1.20}$$

Total error:

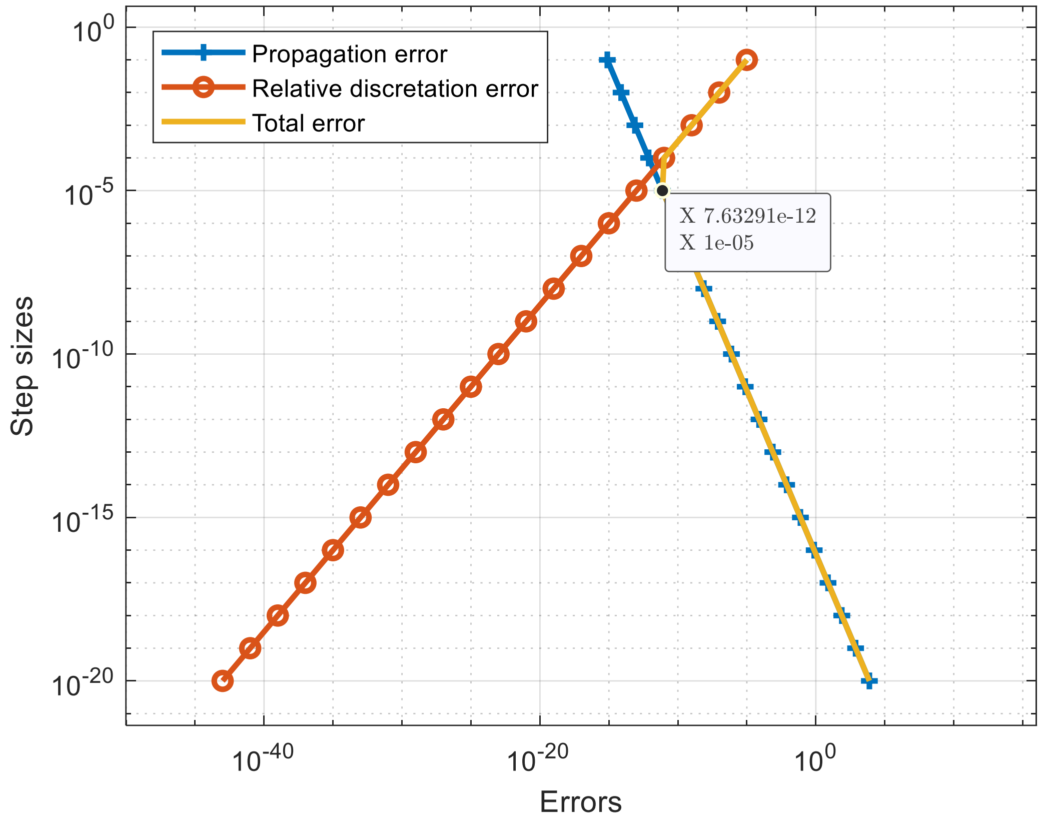

Figure 2 Step sizes and errors

As shown in figure 2, the total error decreases first and then reach the minimum valve, and then increase again. The minimum valve is 1e-05, which is close to our analytical calculation, 4.8e-05. Though they are nearly 5 times, the reason is in Matlab code, the step sizes are 10 time always, etc, 1e-05, 1e-06. While in analytical calculation, the result is exact(It should have small error since in my hand calculation, the significant figures is 4).

Propagation error decreases with increasing of step sizes, the reason is computer could use more effective value in single data, which reduces the effect of machine accuracy.

Relative discretization error increases with increasing of step sizes, the reason is large step sizes could not describe details around x, which increases the discretization error. This feature could see in equation 1.16, the description of discretization error.

Overall, the total error is influenced by propagation error when step sizes are nearly equal to machine accuracy, the reason is the computer could not distinguish theses step sizes and cause many error. Increasing the machine accuracy could reduce the total error which could be get through more byte’s data. Then, with step sizes increasing, the total error is influenced by discretization error since large step sizes taken too much error around the x, which could get through Taylor expansion.