Problem Set B

Aim: The purpose of this Problem Set is to reinforce your understanding of under which circumstances a unique solution exists.

1. On a convex polygonal domain Ω, consider the differential equation

$$-\Delta u(\mathbf{x}) = f, \qquad \mathbf{x} \in \Omega \tag{1}$$
$$\hat{n} \cdot \nabla u(\mathbf{x}) = g, \qquad \mathbf{x} \in \partial\Omega \tag{2}$$

where f and g are sufficiently regular functions that are given.

(a) Why is this problem not well-posed?

(b) If you would implement a standard finite element method for this problem (for example cG(1)[1]), would you be able to solve the linear system? What happens?

2. Consider the following Boundary Value Problem (BVP):

$$-(k(x)u'(x))' + u'(x) = f(x), \qquad x \in (0,1) \tag{3}$$
$$u(x) = u(1) = 0 \tag{4}$$

with the coefficient, $k(x)$, being a continuous function such that $k(x) \geq k_0 \geq 0$ for $x \in [0,1]$. Assume both $k(x)$ and $f(x)$ are given with $f \in L^2(0,1)$.

(a) Derive a variational formulation for the BVP: Find $u \in V$ such that

$$a(u,v) = L(v), \qquad \forall v \in V.$$

i. Define the Hilbert Space $V = \mathcal{H}_0^1(\Omega)$.

ii. Define the bilinear form $a : V \times V \to \mathbb{R}$ and linear form $L : V \to \mathbb{R}$.

(b) Show that for $\|w\|_E^2 = a(w,w)$ we have that

$$\|w\|_E^2 = \int_0^1 k(x)(w'(x))^2\,dx$$

and that this represents a norm on V.

(c) Prove that there exists a unique solution to the variational problem in 2a.

(d) Formulate the cG(1) method for the given BVP on a subdivision of $(0,1)$ with mesh size $h = h(x)$.

(e) Prove the following a priori error estimate:

$$\|u - u_h\|_E \leq C\|hu''\|_{L^2(0,1)}.$$

Here u is the exact solution to the given BVP and u_h is the approximation of u using the cG(1) method.

[1] FEM with a continuous piecewise linear approximation u_h

(f) Describe the steps in an adaptive algorithm for local mesh refinement based on a posteriori error estimation and given a tolerance TOL on $\|u - u_h\|_E$. Why is adaptivity important?

3. Show that a spatial discretization of the equation

$$\begin{aligned} \dot{u} - \Delta u + u =& f, \qquad \mathbf{x} \in \Omega \\ u =& 0, \qquad \mathbf{x} \in \partial\Omega,\ t > 0 \\ u =& u_0, \qquad \mathbf{x} \in \Omega,\ t = 0 \end{aligned}$$

leads to the algebraic system

$$M\dot{\xi}(t) + A\xi(t) + M\xi(t) = b(t).$$

Identify the entries of the involved matrices and vectors.[2]

4. Consider the convection-diffusion-reaction equation

$$\dot{u} - \epsilon\Delta u + \beta \cdot \nabla u + \alpha u = f, \qquad (x,t) \in \Omega \times (0,T], \qquad \epsilon > 0,$$

$\beta = [\beta_1,\ \beta_2]^T$ is a constant vector, and $\alpha(x)$, $f(x)$ are given functions. Here, $x = (x_1, x_2)$, and $\Omega = [0,1] \times [0,1]$. You may assume homogeneous Dirichlet boundary conditions and that the initial solution is $u(x,0) = u_0(x)$.

(a) Write down a finite element method using piecewise linear basis functions in space and implicit Euler time stepping.

(b) Now assume that $\epsilon = 1$, $\alpha(x) = f(x) = 0$, and $\beta = [0,\ 0]^T$. Prove that

$$\|u(T)\|^2 + 2\int_0^T \|\nabla u\|^2\, dt = \|u_0\|^2, \qquad \forall t > 0,$$

meaning that the L^2-norm of the solution $u(t)$ will decrease as time increases [CDE 16.3].

[2]Exercise 5.3 in M. G. Larson and F. Bengzon, *The Finite Element Method: Theory, Implementation, and Applications*