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Examination - Solution to Problem 4
SF2561 and FSF3561 The Finite Element Method
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Problem 3

Consider the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0, & \text{in } \Omega \times (0,T], \\ u = 0, & \text{on } \partial\Omega \times (0,T], \\ u(\cdot,0) = u_0, & \text{in } \Omega, \end{cases} \tag{1}$$

where $u_0 \in L^2(\Omega)$ only. Prove the following stability estimates

$$\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u\|^2_{L^2(\Omega)} \,\mathrm{d}s \le \|u_0\|^2_{L^2(\Omega)},$$

$$\int_0^t s \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} \mathrm{d}s + t\|\nabla u\|^2_{L^2(\Omega)} \le C\|u_0\|^2_{L^2(\Omega)}.$$

Discuss the behavior of the solution to (1) close to zero. *Hint:* For the second estimate, multiply the equation by $t\frac{\partial u}{\partial t}$. (12p)

Solution

For the first inequlaity we multiply the equation by u and integrate over Ω. Using Green's formula we get

$$(\dot{u}, u) + (\nabla u, \nabla u) = 0.$$

Using the identity $(\dot{u}, u) = \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2}$ we arrive at

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2} = 0.$$

Integrate from 0 to T to achieve

$$\|u(T)\|^2_{L^2} + \int_0^T \|\nabla u\|^2_{L^2} \mathrm{d}s \le \|u_0\|^2_{L^2}.$$

For the second inequality we multiply by $t \cdot u$ according to the hint. Using Green's formula we get

$$(\dot{u}, t\dot{u}) + (\nabla u, t\nabla \dot{u}) = 0.$$

Now for the second term

$$(\nabla u, t\nabla \dot{u}) = t(\nabla u, \nabla \dot{u}) = t\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2} = \frac{1}{2}\frac{d}{dt}\big(t\|\nabla u\|^2_{L^2}\big) - \frac{1}{2}\|\nabla u\|^2_{L^2}.$$

Using this we arrive at

$$t\|\dot{u}\|^2_{L^2} + \frac{1}{2}\frac{d}{dt}\big(t\|\nabla u\|^2_{L^2}\big) = \frac{1}{2}\|\nabla u\|^2_{L^2}.$$

Integrate from 0 to t

$$\int_0^t s\|\dot{u}\|^2_{L^2} + t\|\nabla u(t)\|^2_{L^2} - 0 \cdot \|\nabla u_0\|^2_{L^2} \le C\int_0^t \|\nabla u\|^2_{L^2}.$$

Now we can use the first inequality to deduce

$$\int_0^t s\|\dot{u}\|^2_{L^2} + t\|\nabla u(t)\|^2_{L^2} \le C\|u_0\|^2_{L^2}.$$

In particular the second estimate implies

$$\|\nabla u(t)\|^2_{L^2} \le Ct^{-1}\|u_0\|^2_{L^2}$$

so we get blow-up close to zero in the H^1-norm. This is expected since the initial data u_0 is in L^2 only, but for any positive time $t > 0$ the solution is (at least) in H^1.

Problem 4

A Robin boundary problem is given by

$$\begin{cases} -\nabla \cdot (a\nabla u) = f, & \text{in } \Omega, \\ a\nabla u \cdot n + b(u-g) = k, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^d$ is a convex domain, a, f, b, g, k are smooth functions, and $a \geq a_0 > 0$.

a) Formulate the first order Lagrange finite element method (P1-FEM) for (2). (2p)

b) Prove the following a posteriori error bound in L^2-norm

$$\|u - u_h\|_{L^2(\Omega)} \leq C\left(\sum_{K\in\mathcal{T}_h} R_K(u_h)^2\right)^{1/2},$$

where

$$R_K(u_h) = h_K^2\| - \nabla \cdot (a\nabla u_h) - f\|_{L^2(K)} + h_K^{3/2}\|a[n \cdot \nabla u_h]\|_{L^2(\partial K\setminus\partial\Omega)}$$
$$+ h_K^{3/2}\|a\nabla u_h \cdot n + b(u_h - g) - k\|_{L^2(\partial K\cap\partial\Omega)}.$$

Hint: Let $e = u_h - u$ and use the dual problem $-\nabla \cdot (a\nabla\Phi) = e$ in Ω and $a\nabla\Phi \cdot n + b\Phi = 0$ on $\partial\Omega$. You may also want to use the (scaled) trace inequality $\|w\|_{L^2(\partial K)} \leq C(h_K^{-1/2}\|w\|_{L^2(K)} + h_K^{1/2}\|\nabla w\|_{L^2(K)})$. (10p)

c) Give an interpretation of each term in $R_K(u_h)$. (2p)

Solution

The variational formulation of the problem takes the following form; find $u \in H^1$ such that

$$a(u, v) := (a\nabla u, \nabla v) + (bu, v)_{\partial\Omega} = (f, v) + (k + bg, v)_{\partial\Omega}, \quad \forall v \in H^1.$$

By letting V_h denote our classical P1-FEM space on an admissible triangulation $\mathcal{T}_h$ we formulate the P1-FEM; find $u_h \in V_h$ such that

$$a(u_h, v) = (f, v) + (k + bg, v)_{\partial\Omega}, \quad \forall v \in V_h.$$

For the dual problem we have

$$a(\phi, v) = (a\nabla\phi, \nabla v) + (b\phi, v)_{\partial\Omega} = (e, v), \quad \forall v \in H^1.$$

Using this with $v = e = u_h - u \in H^1$ we get

$$\|e\|_{L^2}^2 = a(\phi, e) = (a\nabla\phi, \nabla e) + (b\phi, e)_{\partial\Omega} = (a\nabla e, \nabla\phi) + (be, \phi)_{\partial\Omega}$$
$$\overset{G.O}{=} (a\nabla e, \nabla(\phi - I_H\phi)) + (be, \phi - I_H\phi)_{\partial\Omega}$$

Let us split into a sum over each element $K \in \mathcal{T}_h$

$$(a\nabla e, \nabla(\phi - I_H\phi)) + (be, \phi - I_H\phi)_{\partial\Omega} = \sum_K (a\nabla e, \nabla(\phi - I_H\phi))_K + (be, \phi - I_H\phi)_{\partial\Omega\cap K}$$
$$= \sum_K (a\nabla u_h, \nabla(\phi - I_H\phi))_K + (bu_h, \phi - I_H\phi)_{\partial\Omega\cap K} - (f, \phi - I_H\phi)_K - (k + bg, \phi - I_H\phi)_{\partial\Omega\cap K}$$
$$\overset{Green}{=} \sum_K (-\nabla \cdot a\nabla u_h - f, \phi - I_H\phi)_K + (an \cdot \nabla u_h, \phi - I_H\phi)_{\partial K} + (b(u_h - g) - k, \phi - I_H\phi)_{\partial\Omega\cap K}$$

On the interior edges we can rewrite to get the jump terms. The boundary edges are just added to the "boundary term".

$$\sum_K (-\nabla \cdot a\nabla u_h - f, \phi - I_H\phi)_K + (an \cdot \nabla u_h, \phi - I_H\phi)_{\partial K} + (b(u_h - g) - k, \phi - I_H\phi)_{\partial\Omega\cap K}$$
$$= \sum_K (-\nabla \cdot a\nabla u_h - f, \phi - I_H\phi)_K - \frac{1}{2}(a[n \cdot \nabla u_h], \phi - I_H\phi)_{\partial K\setminus\partial\Omega}$$
$$+ (an \cdot \nabla u_h + b(u_h - g) - k, \phi - I_H\phi)_{\partial\Omega\cap K} =: I + II + III.$$

For I we use the interpolation estimate $\|\phi - I_H\phi\|_{L^2(K)} \le Ch_K^2\|\phi\|_{H^2}$

$$I \le \sum_K \| - \nabla \cdot a\nabla u_h - f\|_K \|\phi - I_H\phi\|_{L^2(K)} \le C\sum_K h_K^2 \| - \nabla \cdot a\nabla u_h - f\|_{L^2(K)}\|\phi\|_{H^2(K)}.$$

for II we use the trace inequality (from the hint), the interpolation estimate above and the estimate for the gradient $\|\nabla(\phi - I_H\phi)\|_{L^2(K)} \le Ch_K\|\phi\|_{H^2}$ to get

$$\begin{aligned}
II &\le \sum_K \frac{1}{2}\|a[n\cdot\nabla u_h]\|_{L^2(\partial K\setminus\partial\Omega)}\|\phi - I_H\phi\|_{L2(\partial K\setminus\partial\Omega)} \\
&\le C\sum_K \frac{1}{2}\|a[n\cdot\nabla u_h]\|_{L^2(\partial K\setminus\partial\Omega)}(h_K^{-1/2}\|\phi - I_H\phi\|_{L^2(K)} + h_K^{1/2}\|\nabla(\phi - I_H\phi)\|_{L^2(K)}) \\
&\le C\sum_K h_K^{3/2}\frac{1}{2}\|a[n\cdot\nabla u_h]\|_{L^2(\partial K\setminus\partial\Omega)}\|\phi\|_{H^2(K)}.
\end{aligned}$$

Similarly, we get for III

$$III \le C\sum_K h_K^{3/2}\|an\cdot\nabla u_h + b(u_h - g) - k\|_{L^2(\partial\Omega\cap K)}\|\phi\|_{H^2(K)}$$

Summing up and using Cauchy-Schwarz gives

$$\begin{aligned}
\|e\|_{L^2}^2 \le C\sum_K \big(h_K^2\| - \nabla\cdot a\nabla u_h - f\|_{L^2(K)} + h_K^{3/2}\|a[n\cdot\nabla u_h]\|_{L^2(\partial K\setminus\partial\Omega)} \\
+ h_K^{3/2}\|an\cdot\nabla u_h + b(u_h - g) - k\|_{L^2(\partial\Omega\cap K)}\big)\|\phi\|_{H^2(K)} \\
\le C\left(\sum_K R_K^2\right)^{1/2}\left(\sum_K \|\phi\|_{H^2(K)}^2\right)^{1/2} \le C\left(\sum_K R_K^2\right)^{1/2}\|\phi\|_{H^2}.
\end{aligned}$$

The estimate now follows by using the (elliptic) regularity $\|\phi\|_{H^2} \le C\|e\|_{L^2}$.

The first term in R_K is the residual on K and describes how well the approximation fits the equation on K, the second term is the jump term, and the third term is the residual on the boundary and describes how well the approximation fits the boundary condition.