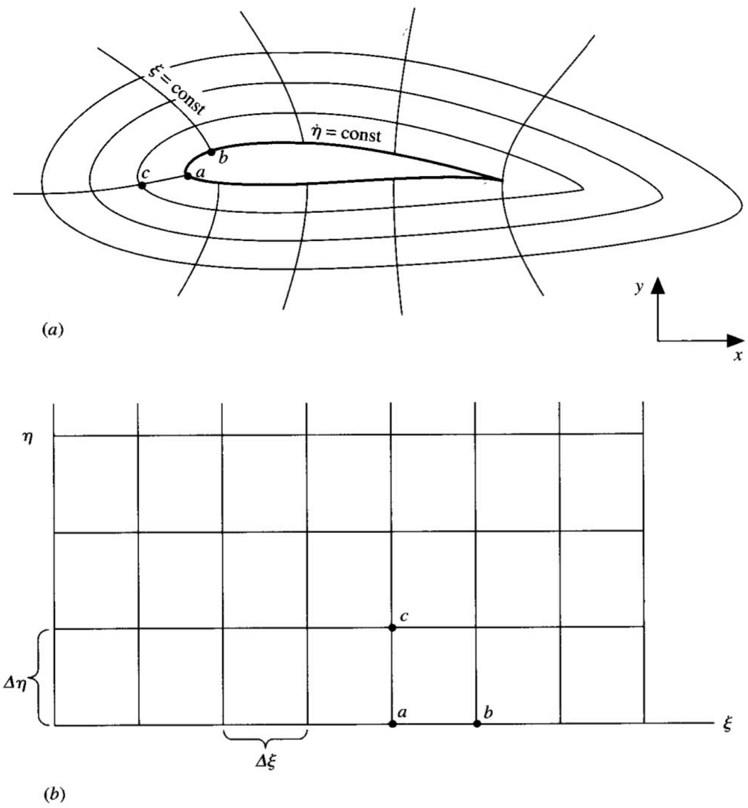
## Coordinate transformation

For finite-differnce discretization on complex geometries requires use of none-cartesian coordinate system. Therefore, a transformation from the none-cartesian to the cartesian system is required.




From Computational Fluid dynamics, Anderson, 1995.

Let the new coordinate system be

$$\xi = \xi(x, y, z)$$
$$\eta = \eta(x, y, z)$$
$$\zeta = \zeta(x, y, z)$$

The spatial derivatives in new coordinate system can be found through chain rule, for example:

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta}$$

In matrix form

$$\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \\ \\ \dfrac{\partial}{\partial z} \end{pmatrix} = \underbrace{\begin{pmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \zeta}{\partial x} \\ \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} & \dfrac{\partial \zeta}{\partial y} \\ \\ \dfrac{\partial \xi}{\partial z} & \dfrac{\partial \eta}{\partial z} & \dfrac{\partial \zeta}{\partial z} \end{pmatrix}}_{J} \begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \\ \\ \dfrac{\partial}{\partial \zeta} \end{pmatrix}$$

where $J$ is called Jacobian matrix. Usually, it is much easier to compute $\left(\dfrac{\partial x}{\partial \xi}\right)_{\eta,\zeta}, \cdots$ instead of $\left(\dfrac{\partial \xi}{\partial x}\right)_{y,z}, \cdots$ . For example, the derivative $\left(\dfrac{\partial \xi}{\partial x}\right)_{y,z}$ should be computed along a line with constant $y$ and $z$, but as it can be seen in figure above, this is not always possible to do numerically, when we move along a constant $\eta$ line, both $x$ and $y$ are changing here. Therefore, we first consider transformation from the cartesian to the non-cartesian coordinates

$$\underbrace{\begin{pmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \\ \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} & \dfrac{\partial z}{\partial \zeta} \end{pmatrix}}_{J^{-1}} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \\ \\ \dfrac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \\ \\ \dfrac{\partial}{\partial \zeta} \end{pmatrix}$$

and then $J$ is computed as the inverse of $J^{-1}$.

$$J = \left(J^{-1}\right)^{-1} = \frac{\text{Transpose of cofactor of } J^{-1}}{\det\left(J^{-1}\right)}$$

$$e.g. \Rightarrow \quad \xi_x = \frac{y_\eta\ z_\zeta - y_\zeta\ z_\eta}{g}, \quad \xi_y = \frac{z_\eta\ x_\zeta - z_\zeta x_\eta}{g}, \quad \xi_z = \frac{x_\eta\ y_\zeta - x_\zeta y_\eta}{g},$$

where $g = \det\left(J^{-1}\right)$.

<u>Ex.: 2D flow</u>

Consider N.-S- equations i conservative form

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0.$$

$$J^{-1} = \begin{pmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{pmatrix} \quad \Rightarrow \quad J = \frac{1}{g} \begin{pmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{pmatrix}$$

where $g = \dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \xi}$.

$$\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{pmatrix} = \frac{1}{g} \begin{pmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \\ \dfrac{\partial}{\partial \eta} \end{pmatrix}$$

$$\Rightarrow \frac{\partial U}{\partial t} + \frac{1}{g}\left(\frac{\partial F}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial F}{\partial \eta}\frac{\partial y}{\partial \xi}\right) + \frac{1}{g}\left(-\frac{\partial G}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial G}{\partial \eta}\frac{\partial x}{\partial \xi}\right) = 0$$

Can be simplified furthere using following relations:

$$\frac{\partial F}{\partial \xi}\frac{\partial y}{\partial \eta} = \frac{\partial}{\partial \xi}\left(F\frac{\partial y}{\partial \eta}\right) - F\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\right),$$

$$\frac{\partial F}{\partial \eta}\frac{\partial y}{\partial \xi} = \frac{\partial}{\partial \eta}\left(F\frac{\partial y}{\partial \xi}\right) - F\frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}\right),$$

$$\frac{\partial G}{\partial \xi}\frac{\partial x}{\partial \eta} = \frac{\partial}{\partial \xi}\left(G\frac{\partial x}{\partial \eta}\right) - G\frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\right),$$

$$\frac{\partial G}{\partial \eta}\frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \eta}\left(G\frac{\partial x}{\partial \xi}\right) - G\frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}\right).$$

$$\Rightarrow \frac{\partial}{\partial t}(gU)+\frac{\partial}{\partial \xi}\left(F\frac{\partial y}{\partial \eta}\right)-\frac{\partial}{\partial \eta}\left(F\frac{\partial y}{\partial \xi}\right)-\frac{\partial}{\partial \xi}\left(G\frac{\partial x}{\partial \eta}\right)+\frac{\partial}{\partial \eta}\left(G\frac{\partial x}{\partial \xi}\right) = 0$$

This can be written as

$$\frac{\partial U'}{\partial t} + \frac{\partial F'}{\partial \xi} + \frac{\partial G'}{\partial \eta} = 0.$$

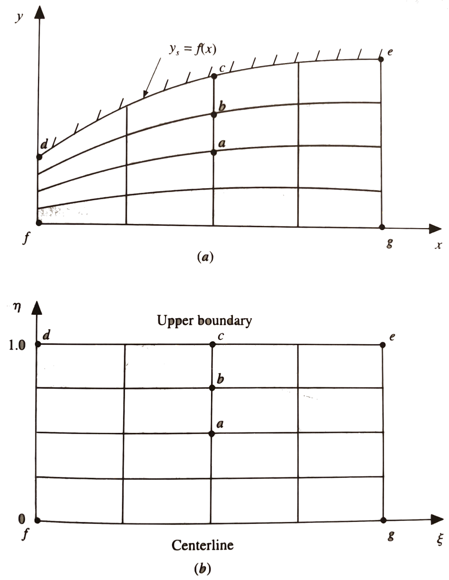
where

$$U' = gU, \quad F' = F\frac{\partial y}{\partial \eta} - G\frac{\partial x}{\partial \eta}, \quad G' = G\frac{\partial x}{\partial \xi} - F\frac{\partial x}{\partial \xi}.$$

## Coordinate transformation: Examples

Algebraic transformation:

Here, the domain extension in the $y$-direction increases with increasing $x$. This kind of mesh is of interest for example for computation of boundary-layer flows. In this case the components of transformation matrix can be computed analytically.




From Computational Fluid dynamics, Anderson, 1995.

$$\xi = x, \quad \eta = \frac{y}{y_s}, \quad y_s = f(x)$$

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} \\ &= \frac{\partial}{\partial \xi} - \frac{1}{y_s^2}\frac{\partial f}{\partial x}\frac{\partial}{\partial \eta} \end{aligned}$$

$$\frac{\partial}{\partial y} = \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \frac{1}{y_s}\frac{\partial}{\partial \eta}$$

Elliptic grid generation: A common type of coordinate transformation used for generation of more complex meshes is the so-called elliptic grid generation. Here the new coordinates are found trough the solution of Laplace equations

$$\xi_{xx} + \xi_{yy} = 0$$

$$\eta xx + \eta_{yy} = 0$$

These can be re-written as a differential equations for $x$ and $y$

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = 0$$

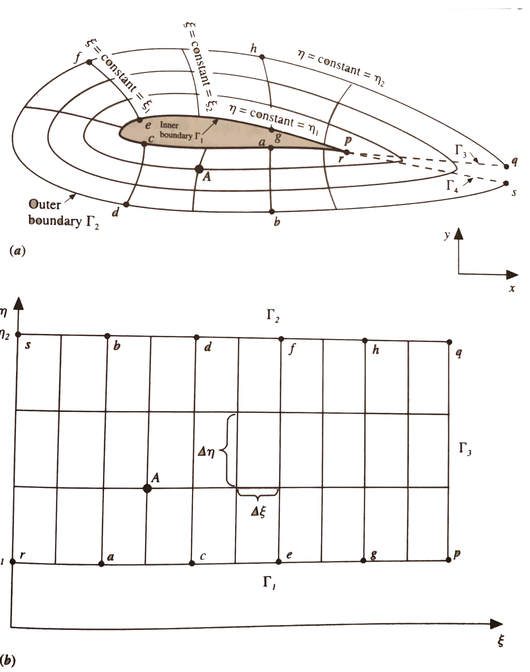
$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = 0$$

where

$$\alpha = x_\eta^2 + y_\eta^2$$

$$\beta = x_\xi x_\eta + y_\xi y_\eta$$

$$\gamma = x_\xi^2 + y_\xi^2$$




From Computational Fluid dynamics, Anderson, 1995.

These equations are then solved along with the boundary conditions given by the known values of $x$ and $y$ along $\Gamma_1$ and $\Gamma_2$ as well as other boundaries of the domain.