Problem Set 1

1. Consider the one-dimensional Poisson equation given by

$$\begin{aligned} -(a(x)u')' =& 0, \quad x \in (0,1) \\ u(0) = 0, \quad & u'(1) = 1, \end{aligned} \tag{1}$$

where $a(x) = 1 + x$.

   (a) In the first part of this exercise, we consider the *exact solution,* u to this problem.
      i. Derive the variational formulation for the *exact solution* u to this problem.
      ii. Define the space in which the exact solution to the variational formulation exists.

   (b) Now consider deriving the finite element formulation.
      i. Define a suitable approximation space for this problem using piecewise linear continuous functions. Both the test and trial spaces for the finite element approximation are chosen to be equal to this space.
      ii. Write the form of the finite element approximation in terms of the approximation space.
      iii. Assume the domain is divided into three elements of equal length. Determine the algebraic system of equations for the finite element method using the approximation space in 1(b)i. Verify the stiffness and load vector are given by

$$A = \frac{1}{2}\begin{bmatrix} 16 & -9 & 0 \\ -9 & 20 & -11 \\ 0 & -11 & 11 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

2. Now consider the two-dimensional Poisson equation given by

$$-\Delta u(\mathbf{x}) = 2, \qquad (\mathbf{x}) \in \Omega \tag{2}$$

$$u(\mathbf{x}) = 0, \qquad x \in \partial\Omega, \tag{3}$$

with $\mathbf{x} = (x_1, x_2)^T$, and $\Omega \in \mathbb{R}^2$. Here, Ω is a rectangular domain with corners $(-1,0)$, $(1,0)$, $(1,1)$, and $(-1,1)$, and $\partial\Omega$ represents the boundary.

   (a) Define a finite element mesh, $\mathcal{T}_h$, over the domain Ω with at least 4 internal (non-boundary) nodes.
   (b) Define a finite element approximation space, V_h, over the mesh $\mathcal{T}_h$.
   (c) Formulate the finite element approximation space using the space V_h.
   (d) Compute the element stiffness matrix and load vector on the reference triangle $\widehat{K}$ with corners $(0,0)$, $(1,0)$, and $(0,1)$.
   (e) Given the mesh in 2a, the space V_h in 2b, and weak form in 2c, compute the stiffness matrix A and load vector b. **Do not solve** the resulting linear system of equations.

3. **Exercise for FSF3561 students (optional otherwise):** Consider the problem:

$$-\nabla \cdot (k(\mathbf{x})\nabla u(\mathbf{x})) = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^3 \tag{4}$$

$$\beta\partial_n u(\mathbf{x}) + \gamma u(\mathbf{x}) = g(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega \tag{5}$$

with $\partial_n u = \nabla u \cdot \hat{n}$, and $\hat{n}$ the unit outward normal of the boundary, $\partial\Omega$. Here, β and γ are given non-negative constants.

(a) State the Lax-Milgram theorem.

(b) Determine if the assumptions of the Lax-Milgram theorem are satisfied in the following cases and state any further assumptions that might be required:

   i. $\beta = 0,\ \gamma = 1,\ g = 0,\ f \in L^2(\Omega)$

   ii. $\beta = 1,\ \gamma = 0,\ g = 0,\ f \in L^2(\Omega)$

   iii. $\beta = 1,\ \gamma = 0,\ g \in L^2(\partial\Omega),\ f = 0$

(c) For the cases where the Lax-Milgram theorem are satisfied, determine the finite element formulation.