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Examination
SF2561 and FSF3561 The Finite Element Method
2021-10-21, 8.00-20.00

Total 50p: 20p for grade E, 25p for grade D, 30p for grade C, 35p for grade B, and 40p for grade A.

Problem 1

Consider the mixed boundary value problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \setminus \{x \in \Omega : x_1 = 1\}, \\ n \cdot \nabla u = 0, & \text{on } \{x \in \Omega : x_1 = 1\}, \end{cases} \tag{1}$$

where $\Omega = \{(x_1, x_2) : -1 < x_1 < 1, 0 < x_2 < 2\}$, and $f = 1$. Note that the Neumann boundary conditions are prescribed on the red part of the mesh in Figure 1. The remaining parts of the boundary have Dirichlet boundary conditions.

a) Approximate (1) using first order Lagrange elements ($P1$-FEM). State the variational form and the finite element approximation. (2p)

b) Given the triangulation in Figure 1, the finite element discretization results in a system system $A\xi = b$, where A is the stiffness matrix and b is the load vector. What is the size of A and b? (2p)

c) Compute A and b in Problem 1 b). (8p)

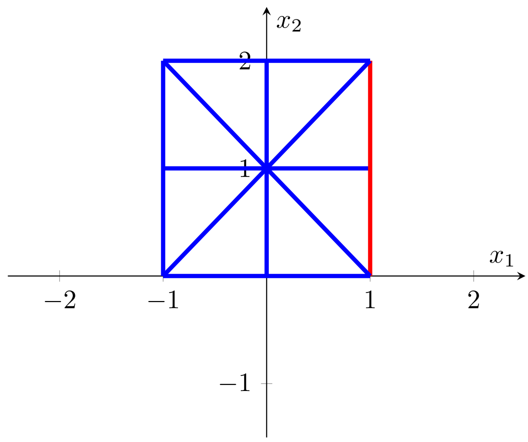

Figure 1: Mesh for Problem 1

Problem 2

Consider the Neumann problem

$$\begin{cases} -\Delta u + cu = f, & \text{in } \Omega, \\ n \cdot \nabla u = g, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $c \in L^\infty(\Omega)$, $c(x) \geq c_0 > 0$, and $f \in L^2(\Omega)$.

a) Derive the variational form and prove that there exists a unique solution (to the variational form). (6p)

b) Give at least three factors that affect the accuracy of finite element methods for the approximation of this problem. (2p)

c) Formulate the first order Lagrange finite element method (P1-FEM) for (2). Derive an a priori bound for the error. (4p)

d) Consider the case when $c = 0$. Prove that the solution to this problem can not be unique. Which part of the Lax-Milgram theorem fails in this case? (2p)

Problem 3

The following initial value problem is given

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + u = f, & \text{in } \Omega \times (0, T], \\ u = 0, & \text{on } \partial\Omega \times (0, T], \\ u(\cdot, 0) = u_0, & \text{in } \Omega. \end{cases} \tag{3}$$

where $\Omega \in \mathbb{R}^d$ is convex and $f, u_0 \in L^2(\Omega)$.

a) Derive the stability estimate

$$\|u(T)\|_{L^2}^2 + \int_0^T \|\nabla u\|_{L^2}^2 \mathrm{d}s + \int_0^T \|u\|_{L^2}^2 \mathrm{d}s \leq C\bigg(\int_0^T \|f\|_{L^2}^2 \mathrm{d}s + \|u_0\|_{L^2}^2\bigg).$$

Assume that there exists a solution to (3). Is it unique? (6p)

b) Formulate the semi-discrete system of ODEs using linear Lagrange finite elements (P1-FEM). (3p)

c) Discretize your system in 3 b) using the backward Euler method. Name two properties of the backward Euler method. (3p)

Problem 4

Let $\Omega \in \mathbb{R}^d$ be a convex domain. Consider the problem

$$\begin{cases} -\nabla \cdot a\nabla u + b \cdot \nabla u + cu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where a, b, and c are smooth functions (in $C^\infty(\bar{\Omega})$) with $a(x) \geq a_0 > 0$, and $c(x) - \frac{1}{2}\nabla \cdot b(x) > 0$.

a) Formulate the first order Lagrange finite element method (P1-FEM) for (4). (2p)

b) Derive an a priori bound in the L^2-norm. (10p)
Hint: Modify the Aubin-Nitsche argument using the dual problem: Find $\psi \in H_0^1(\Omega)$ such that

$$(a\nabla v, \nabla\psi) - (v, b\nabla\psi) - (v, \psi\nabla \cdot b) + (cv, \psi) = (v, e_h), \quad \forall v \in H_0^1(\Omega).$$

where $e_h := u - u_h$ is the error between the exact solution and the finite element approximation in a). You may use the a priori bound in H^1-norm without proof.

Good luck! /Anna