Part 7 - Initial boundary value problems

7.1. Parabolic equations

Initial boundary value problems

A general initial boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} + Au = f, & \text{in } \Omega \times (0, T], \\ Bu(x,t) = g(x,t), & \text{on } \partial\Omega \times (0, T], \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

where

- A - Differential operator (space), B - Boundary operator
- f - Forcing function, g - Boundary function
- u_0 - Initial value

Well posed - If a unique solution exists and satisfies a stability estimate $\|u\| \leq C(\|u_0\| + \|f\| + \|g\|)$.

A parabolic model problem

Heat equation:

$$(HE) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = f, & \text{in } \Omega \times (0, T], \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, T], \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

where $\Omega \in \mathbb{R}^d$, $T > 0$.

Well-posed? - Assume solution exists, we will prove stability and uniqueness.

Heat equation - stability

Multiply by u and integrate over Ω:

$$\left(\frac{\partial u}{\partial t}, u\right) - (\Delta u, u) = (f, u).$$

where $(f, g) = \int_\Omega fg \, \mathrm{d}x$.
Green's formula and $u = 0$ on $\partial\Omega$ gives

$$\left(\frac{\partial u}{\partial t}, u\right) + (\nabla u, \nabla u) = (f, u).$$

Noting that $\left(\frac{\partial}{\partial t} u, u\right) = \frac{1}{2}\frac{\partial}{\partial t}\|u\|_{L_2}^2$ we have

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|_{L_2}^2 + \|\nabla u\|_{L_2}^2 = (f, u) \overset{CS}{\leq} \|f\|_{L_2}\|u\|_{L_2}.$$

Heat equation - stability

From previous slide:

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|_{L_2}^2 + \|\nabla u\|_{L_2}^2 = (f, u) \overset{CS}{\leq} \|f\|_{L_2}\|u\|_{L_2}.$$

Use Poincare-Friedrich $\|\nabla u\|_{L_2}^2 \geq C_{pf}\|u\|_{L_2}^2$ and Young's inequality $ab \leq \frac{1}{2}(\frac{a}{\gamma})^2 + (b\gamma)^2$.

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|_{L_2}^2 + C_{pf}\|u\|_{L_2}^2 \leq \frac{1}{2C_{pf}}\|f\|_{L_2}^2 + \frac{C_{pf}}{2}\|u\|_{L_2}^2.$$

Kick $\frac{C_{pf}}{2}\|u\|_{L_2}^2$ to LHS (and mult. by 2)

$$\frac{\partial}{\partial t}\|u\|_{L_2}^2 + C_{pf}\|u\|_{L_2}^2 \leq \frac{1}{C_{pf}}\|f\|_{L_2}^2.$$

Heat equation - stability

From previous slide:

$$\frac{\partial}{\partial t}\|u\|_{L_2}^2 + C_{pf}\|u\|_{L_2}^2 \leq \frac{1}{C_{pf}}\|f\|_{L_2}^2.$$

Multiply by $e^{C_{pf}t}$

$$\frac{\partial}{\partial t}(e^{C_{pf}t}\|u\|_{L_2}^2) \leq \frac{e^{C_{pf}t}}{C_{pf}}\|f\|_{L_2}^2.$$

Integrate from 0 to t

$$e^{C_{pf}t}\|u(\cdot,t)\|_{L_2}^2 - \|u_0\|_{L_2}^2 \leq \frac{1}{C_{pf}}\int_0^t e^{C_{pf}s}\|f(\cdot,s)\|_{L_2}^2\,\mathrm{d}s.$$

Heat equation - stability

We have the stability estimate:

$$\|u(\cdot,t)\|_{L_2}^2 \le e^{-C_{pf}t}\|u_0\|_{L_2}^2 + \frac{1}{C_{pf}}\int_0^t e^{-C_{pf}(t-s)}\|f(\cdot,s)\|_{L_2}^2\,\mathrm{d}s.$$

In particular if $f = 0$ (no external temperature source)

$$\|u(\cdot,t)\|_{L_2}^2 \le e^{-C_{pf}t}\|u_0\|_{L_2}^2.$$

So energy dissipates exponentially.

Heat equation - well posed

The stability estimate:

$$\|u(\cdot,t)\|_{L_2}^2 \leq e^{-C_{pf}t}\|u_0\|_{L_2}^2 + \frac{1}{C_{pf}}\int_0^t e^{-C_{pf}(t-s)}\|f(\cdot,s)\|_{L_2}^2 \,\mathrm{d}s.$$

1. Note that if a solution exists, then uniqueness follows. If u_1 and u_2 are both solutions to (HE) then $u_1 - u_2$ solves (HE) with $f = u_0 = 0$ and $u_1 = u_2$ follows from the stability.
2. Since we also have stability, the heat equation (HE) is well posed (if solution exists - proof not in this course).

Part 7 - Initial boundary value problems

7.2. Semi-discretization in space

Heat equation - variational form

Multiply (HE) by $v \in H_0^1(\Omega)$ and integrate over Ω.

$$\left(\frac{\partial u}{\partial t}, v\right) - (\Delta u, v) = (f, v),$$

Green's formula and $u = 0$ on $\partial\Omega$ gives

$$\left(\frac{\partial u}{\partial t}, v\right) + (\nabla u, \nabla v) = (f, v).$$

Heat equation - variational form

Find $u(t) \in H_0^1(\Omega)$ such that

$$\left(\frac{\partial u}{\partial t}, v\right) + (\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega),\ t \in (0, T].$$

and $u(\cdot, 0) = u_0$.

Note: The variational form should hold pointwise for all t.
Test functions are not time dependent.

Semi-discrete FEM

Goal: Replace $H_0^1(\Omega)$ with finite dimensional $V_{h,0}$.

- Mesh $\mathcal{T}_h$ - admissible triangulation of Ω of size h.
- Finite element space $V_{h,0}$. Ex: P1 (Lagrange) finite element space

$$V_{h,0} = \{v \in C^0(\bar{\Omega}) \cap H_0^1(\bar{\Omega}) \mid \\ \forall K \in \mathcal{T}_h : v|_K \text{ is a polynomial of deg } 1\}$$

Semi-discrete FEM

Semi-discrete FEM: Find $u_h(t) \in V_{h,0}$ such that

$$\left(\frac{\partial u_h}{\partial t}, v\right) + (\nabla u_h, \nabla v) = (f, v), \quad \forall v \in V_{h,0},\ t \in (0, T].$$

and $u_h(\cdot, 0) = u_{0,h} \in V_{h,0}$.

Note: Initial data u_0 approximated by $u_{0,h} \in V_{h,0}$. For instance:

$$(u_{0,h}, v) = (u_0, v), \quad \forall v \in V_{h,0} \quad L_2\text{-projection},$$
$$(\nabla u_{0,h}, \nabla v) = (\nabla u_0, \nabla v), \quad \forall v \in V_{h,0} \quad \text{Ritz-projection}.$$

Semi-discrete FEM

Goal: Matrix representation of the semi-discrete system.

- $u_h(x,t) = \sum_{j \in N_h} \xi_j(t)\phi_j(x)$.
- ϕ_j nodal basis functions of $V_{h,0}$.
- N_h number of interior nodes $\mathcal{N}_{h,0}$.

Replace u_h and test with ϕ_i (Since $\text{span}\{\phi_i\} = V_{h,0}$)

$$\sum_{j=1}^{N_h} \frac{\partial \xi_j}{\partial t}(\phi_j, \phi_i) + \sum_{j=1}^{N_h} \xi_j(\nabla\phi_j, \nabla\phi_i) = (f, \phi_i), \quad i \in N_h,$$

$$\xi_j(0) = u_{0,h}(z_j), \quad z_j \in \mathcal{N}_{h,0}.$$

We identify a mass (M) and a stiffness (A) matrix.

Semi-discrete FEM

In matrix form:

$$\begin{cases} M\frac{\partial \xi}{\partial t} + A\xi = F, \\ \xi(0) = U_{0,h}. \end{cases}$$

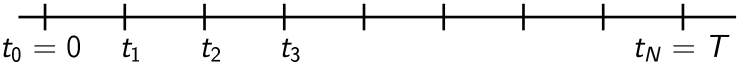
$M_{i,j} = \int_\Omega (\phi_j, \phi_i)\,\mathrm{d}x$, Mass matrix,
$A_{i,j} = \int_\Omega (\nabla\phi_j, \nabla\phi_i)\,\mathrm{d}x$, Stiffness matrix,
$F_i = \int_\Omega (f(t), \phi_i)\,\mathrm{d}x$, Load vector,
$(U_{0,h})_j = u_{0,h}(z_j)$, Initial condition.

Note: This is a system of ordinary differential equations (ODEs).

Part 7 - Initial boundary value problems

7.3. Discretization in time

Discretization in time



Let $0 = t_0 < t_1 < ... < t_N = T$ be partition of $[0, T]$ with time steps $k_n = t_{n+1} - t_n$.

Goal: For each time step find an approximation ξ^n of $\xi(t_n)$. Several schemes will be presented on the next slides.

Backward Euler

$\frac{\partial \xi}{\partial t}$ is approximated by a backward quotient.

$$\begin{cases} M\left(\frac{\xi^{n+1}-\xi^{n}}{k_n}\right) + A\xi^{n+1} = F(t_{n+1}), & n \geq 0 \\ \xi^{0} = U_{0,h}. \end{cases}$$

Rearranging

$$\begin{cases} (M + k_n A)\xi^{n+1} = M\xi^{n} + k_n F(t_{n+1}), & n \geq 0 \\ \xi^{0} = U_{0,h}. \end{cases}$$

Implicit: In every time step we need to solve a system to get ξ^n.First order method - $O(k)$ where $k = \max_n k_n$.

Forward Euler

$\frac{\partial \xi}{\partial t}$ is approximated by a forward quotient.

$$\begin{cases} M\left(\frac{\xi^{n+1}-\xi^n}{k_n}\right) + A\xi^n = F(t_n), \quad n \geq 0 \\ \xi^0 = U_{0,h}. \end{cases}$$

Rearranging

$$\begin{cases} M\xi^{n+1} = (M - k_n A)\xi^n + k_n F(t_n), \quad n \geq 0 \\ \xi^0 = U_{0,h}. \end{cases}$$

Explicit: ξ^n can be obtained at low cost if M is easily inverted (e.g. mass lumping). First order method but only conditionally stable.

Crank-Nicolson

$\frac{\partial}{\partial t}\xi$ is approximated by a centered quotient.

$$\begin{cases} M\left(\frac{\xi^{n+1}-\xi^{n}}{k_n}\right) + A\frac{\xi^{n+1}+\xi^{n}}{2} = \frac{F(t_{n+1})+F(t_n)}{2}, \quad n \geq 0 \\ \xi^0 = U_{0,h}. \end{cases}$$

Rearranging

$$\begin{cases} (M + \frac{k_n}{2}A)\xi^{n+1} = (M - \frac{k_n}{2}A)\xi^{n} + k_n\frac{F(t_{n+1})+F(t_n)}{2}, \quad n \geq 0 \\ \xi^0 = U_{0,h}. \end{cases}$$

Implicit. Second order method $O(k^2)$.

BDF(2) Backward Difference Formula

Alternative to Crank-Nicolson.

$$\begin{cases} M\left(\frac{3\xi^{n+1}-4\xi^{n}+\xi^{n-1}}{2k}\right) + A\xi^{n+1} = F(t_{n+1}), \quad n \geq 1 \\ \xi^0 = U_{0,h}. \end{cases}$$

Rearranging

$$\begin{cases} (3M + 2kA)\xi^{n+1} = 4M\xi^{n} - M\xi^{n-1} + 2kF(t_{n+1}), \quad n \geq 1 \\ \xi^0 = U_{0,h},\ \xi^1 = U_{1,h}. \end{cases}$$

Implicit. Also second order and simple to implement (constant time step k). Requires two starting values.

θ-method

Let $\theta \in [0, 1]$

$$\begin{cases} M\left(\frac{\xi^{n+1}-\xi^n}{k_n}\right) + A(\theta\xi^{n+1} + (1-\theta)\xi^n) = \theta F(t_{n+1}) + (1-\theta)F(t_n), \\ \xi^0 = U_{0,h}. \end{cases}$$

Special cases:

- $\theta = 1 \Rightarrow$ Backward Euler
- $\theta = 0 \Rightarrow$ Forward Euler
- $\theta = 1/2 \Rightarrow$ Crank-Nicolson

Fully discrete FEM

Let us return to the variational form.

Discretize in time with θ-method and let $k_n = k$.

Fully discrete FEM: Find $u_h^n \in V_{h,0}$, such that

$$\left(\frac{u_h^{n+1} - u_h^n}{k}, v\right) + (\nabla u_h^{n+\theta}, \nabla v) = (f^{n+\theta}, v), \quad \forall v \in V_{h,0},$$

for $1 \leq n \leq N$ and $u_h^0 = u_{0,h} \in V_{h,0}$. Where
$u_h^{n+\theta} = \theta u_h^{n+1} + (1-\theta) u_h^n$ and
$f^{n+\theta} = \theta f(\cdot, t_{n+1}) + (1-\theta) f(\cdot, t_n)$

Stability of the θ-method

A stable approximation: Small perturbations in the data causes only small perturbations in the solution.

Goal: Prove that $\|u_h^n\| \leq C(\|u_h^0\| + \|f\|)$

Choose $v = u_h^{n+\theta}$

$$\left(\frac{u_h^{n+1} - u_h^n}{k}, u_h^{n+\theta}\right) + \underbrace{(\nabla u_h^{n+\theta}, \nabla u_h^{n+\theta})}_{=\|\nabla u_h^{n+\theta}\|_{L_2}^2} = (f^{n+\theta}, u_h^{n+\theta}),$$

Note: $u_h^{n+\theta} = \theta u_h^{n+1} + (1-\theta)u_h^n = k(\theta - \frac{1}{2})\frac{u_h^{n+1} - u_h^n}{k} + \frac{u_h^{n+1} + u_h^n}{2}$
Plug into the first term.

Stability of the θ-method

We get

$$k\left(\theta - \frac{1}{2}\right)\left\|\frac{u_h^{n+1} - u_h^n}{k}\right\|_{L_2}^2 + \frac{\|u_h^{n+1}\|_{L_2}^2 - \|u_h^n\|_{L_2}^2}{2k} + \|\nabla u_h^{n+\theta}\|_{L_2}^2 = (f^{n+\theta}, u_h^{n+\theta}),$$

If $\theta \in [1/2, 1]$, then the first term is non-negative. Hence

$$\frac{\|u_h^{n+1}\|_{L_2}^2 - \|u_h^n\|_{L_2}^2}{2k} + \underbrace{\|\nabla u_h^{n+\theta}\|_{L_2}^2}_{\geq C_{pf}\|u_h^{n+\theta}\|_{L_2}^2} \leq \|f^{n+\theta}\|_{L_2}\|u_h^{n+\theta}\|_{L_2},$$

Stability of the θ-method

Young's weighted inequality gives

$$\frac{\|u_h^{n+1}\|_{L_2}^2 - \|u_h^n\|_{L_2}^2}{2k} + C_{pf}\|u_h^{n+\theta}\|_{L_2}^2 \le \frac{1}{2C_{pf}}\|f^{n+\theta}\|_{L_2}^2 + \frac{C_{pf}}{2}\|u_h^{n+\theta}\|_{L_2}^2,$$

and we can kick $\frac{C_{pf}}{2}\|u_h^{n+\theta}\|_{L_2}^2$ to LHS. Multiply by $2k$

$$\|u_h^{n+1}\|_{L_2}^2 - \|u_h^n\|_{L_2}^2 + kC_{pf}\|u_h^{n+\theta}\|_{L_2}^2 \le \frac{k}{C_{pf}}\|f^{n+\theta}\|_{L_2}^2,$$

Sum over n

$$\|u_h^n\|_{L_2}^2 + C_{pf}\sum_{j=0}^{n-1} k\|u_h^{j+\theta}\|_{L_2}^2 \le \|u_h^0\|_{L_2}^2 + \frac{1}{C_{pf}}\sum_{j=0}^{n-1} k\|f^{j+\theta}\|_{L_2}^2,$$

Stability of the θ-method

In particular

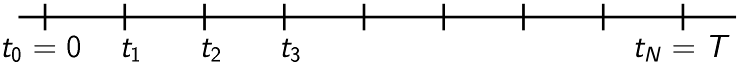
$$\|u_h^n\|_{L_2}^2 \leq \|u_h^0\|_{L_2}^2 + \frac{1}{C_{pf}} \sum_{j=0}^{n-1} k \|f^{j+\theta}\|_{L_2}^2,$$

- For $\theta \in [1/2, 1]$ the method is unconditionally (no limitations on h etc.) stable. Includes Backward Euler and Crank-Nicolson.
- For $\theta \in [0, 1/2)$ the method is conditionally stable ($k \leq ch^2$). Includes Forward Euler.

Part 7 - Initial boundary value problems

7.3. Discretization in time

Discretization in time



Let $0 = t_0 < t_1 < \ldots < t_N = T$ be partition of $[0, T]$ with time steps $k_n = t_{n+1} - t_n$.

Goal: For each time step find an approximation ξ^n of $\xi(t_n)$. Several schemes will be presented on the next slides.

Backward Euler

$\frac{\partial \xi}{\partial t}$ is approximated by a backward quotient.

$$\begin{cases} M\left(\frac{\xi^{n+1}-\xi^{n}}{k_n}\right) + A\xi^{n+1} = F(t_{n+1}), \quad n \geq 0 \\ \xi^0 = U_{0,h}. \end{cases}$$

Rearranging

$$\begin{cases} (M + k_n A)\xi^{n+1} = M\xi^{n} + k_n F(t_{n+1}), \quad n \geq 0 \\ \xi^0 = U_{0,h}. \end{cases}$$

Implicit: In every time step we need to solve a system to get ξ^n.First order method - $O(k)$ where $k = \max_n k_n$.

Forward Euler

$\frac{\partial \xi}{\partial t}$ is approximated by a forward quotient.

$$\begin{cases} M\left(\frac{\xi^{n+1}-\xi^{n}}{k_n}\right) + A\xi^n = F(t_n), & n \geq 0 \\ \xi^0 = U_{0,h}. \end{cases}$$

Rearranging

$$\begin{cases} M\xi^{n+1} = (M - k_n A)\xi^n + k_n F(t_n), & n \geq 0 \\ \xi^0 = U_{0,h}. \end{cases}$$

Explicit: ξ^n can be obtained at low cost if M is easily inverted (e.g. mass lumping). First order method but only conditionally stable.

Crank-Nicolson

$\frac{\partial}{\partial t}\xi$ is approximated by a centered quotient.

$$\begin{cases} M\left(\frac{\xi^{n+1}-\xi^n}{k_n}\right) + A\frac{\xi^{n+1}+\xi^n}{2} = \frac{F(t_{n+1})+F(t_n)}{2}, \quad n \geq 0 \\ \xi^0 = U_{0,h}. \end{cases}$$

Rearranging

$$\begin{cases} (M + \frac{k_n}{2}A)\xi^{n+1} = (M - \frac{k_n}{2}A)\xi^n + k_n\frac{F(t_{n+1})+F(t_n)}{2}, \quad n \geq 0 \\ \xi^0 = U_{0,h}. \end{cases}$$

Implicit. Second order method $O(k^2)$.

BDF(2) Backward Difference Formula

Alternative to Crank-Nicolson.

$$\begin{cases} M\left(\frac{3\xi^{n+1}-4\xi^{n}+\xi^{n-1}}{2k}\right) + A\xi^{n+1} = F(t_{n+1}), \quad n \geq 1 \\ \xi^0 = U_{0,h}. \end{cases}$$

Rearranging

$$\begin{cases} (3M + 2kA)\xi^{n+1} = 4M\xi^{n} - M\xi^{n-1} + 2kF(t_{n+1}), \quad n \geq 1 \\ \xi^0 = U_{0,h},\ \xi^1 = U_{1,h}. \end{cases}$$

Implicit. Also second order and simple to implement (constant time step k). Requires two starting values.

θ-method

Let $\theta \in [0, 1]$

$$\begin{cases} M\left(\frac{\xi^{n+1}-\xi^{n}}{k_n}\right) + A(\theta\xi^{n+1} + (1-\theta)\xi^{n}) = \theta F(t_{n+1}) + (1-\theta)F(t_n), \\ \xi^0 = U_{0,h}. \end{cases}$$

Special cases:

- $\theta = 1 \Rightarrow$ Backward Euler
- $\theta = 0 \Rightarrow$ Forward Euler
- $\theta = 1/2 \Rightarrow$ Crank-Nicolson

Fully discrete FEM

Let us return to the variational form.

Discretize in time with θ-method and let $k_n = k$.

Fully discrete FEM: Find $u_h^n \in V_{h,0}$, such that

$$\left(\frac{u_h^{n+1} - u_h^n}{k}, v\right) + (\nabla u_h^{n+\theta}, \nabla v) = (f^{n+\theta}, v), \quad \forall v \in V_{h,0},$$

for $1 \leq n \leq N$ and $u_h^0 = u_{0,h} \in V_{h,0}$. Where

$u_h^{n+\theta} = \theta u_h^{n+1} + (1-\theta) u_h^n$ and
$f^{n+\theta} = \theta f(\cdot, t_{n+1}) + (1-\theta) f(\cdot, t_n)$

Stability of the θ-method

A stable approximation: Small perturbations in the data causes only small perturbations in the solution.

Goal: Prove that $\|u_h^n\| \leq C(\|u_h^0\| + \|f\|)$

Choose $v = u_h^{n+\theta}$

$$\left(\frac{u_h^{n+1} - u_h^n}{k}, u_h^{n+\theta}\right) + \underbrace{(\nabla u_h^{n+\theta}, \nabla u_h^{n+\theta})}_{=\|\nabla u_h^{n+\theta}\|_{L_2}^2} = (f^{n+\theta}, u_h^{n+\theta}),$$

Note: $u_h^{n+\theta} = \theta u_h^{n+1} + (1-\theta)u_h^n = k(\theta - \frac{1}{2})\frac{u_h^{n+1} - u_h^n}{k} + \frac{u_h^{n+1} + u_h^n}{2}$
Plug into the first term.

Stability of the θ-method

We get

$$k\left(\theta - \frac{1}{2}\right) \left\| \frac{u_h^{n+1} - u_h^n}{k} \right\|_{L_2}^2 + \frac{\|u_h^{n+1}\|_{L_2}^2 - \|u_h^n\|_{L_2}^2}{2k} + \|\nabla u_h^{n+\theta}\|_{L_2}^2 = (f^{n+\theta}, u_h^{n+\theta}),$$

If $\theta \in [1/2, 1]$, then the first term is non-negative. Hence

$$\frac{\|u_h^{n+1}\|_{L_2}^2 - \|u_h^n\|_{L_2}^2}{2k} + \underbrace{\|\nabla u_h^{n+\theta}\|_{L_2}^2}_{\geq C_{pf}\|u_h^{n+\theta}\|_{L_2}^2} \leq \|f^{n+\theta}\|_{L_2}\|u_h^{n+\theta}\|_{L_2},$$

Stability of the θ-method

Young's weighted inequality gives

$$\frac{\|u_h^{n+1}\|_{L_2}^2 - \|u_h^n\|_{L_2}^2}{2k} + C_{pf}\|u_h^{n+\theta}\|_{L_2}^2 \leq \frac{1}{2C_{pf}}\|f^{n+\theta}\|_{L_2}^2 + \frac{C_{pf}}{2}\|u_h^{n+\theta}\|_{L_2}^2,$$

and we can kick $\frac{C_{pf}}{2}\|u_h^{n+\theta}\|_{L_2}^2$ to LHS. Multiply by $2k$

$$\|u_h^{n+1}\|_{L_2}^2 - \|u_h^n\|_{L_2}^2 + kC_{pf}\|u_h^{n+\theta}\|_{L_2}^2 \leq \frac{k}{C_{pf}}\|f^{n+\theta}\|_{L_2}^2,$$

Sum over n

$$\|u_h^n\|_{L_2}^2 + C_{pf}\sum_{j=0}^{n-1} k\|u_h^{j+\theta}\|_{L_2}^2 \leq \|u_h^0\|_{L_2}^2 + \frac{1}{C_{pf}}\sum_{j=0}^{n-1} k\|f^{j+\theta}\|_{L_2}^2,$$

Stability of the θ-method

In particular

$$\|u_h^n\|_{L_2}^2 \leq \|u_h^0\|_{L_2}^2 + \frac{1}{C_{pf}} \sum_{j=0}^{n-1} k \|f^{j+\theta}\|_{L_2}^2,$$

- For $\theta \in [1/2, 1]$ the method is unconditionally (no limitations on h etc.) stable. Includes Backward Euler and Crank-Nicolson.
- For $\theta \in [0, 1/2)$ the method is conditionally stable ($k \leq ch^2$). Includes Forward Euler.

Part 7 - Initial boundary value problems

7.4. Space-time FEM

Space-time FEM - Variational form

Let us now consider the variational form:

Find $u(t) \in C(0, T; H_0^1(\Omega))$ such that

$$\int_0^T \left(\frac{\partial u}{\partial t}, v\right)\mathrm{d}t + \int_0^T (\nabla u, \nabla v)\mathrm{d}t = \int_0^T (f, v)\mathrm{d}t, \quad \forall v \in L^2(0, T; H_0^1(\Omega)).$$

and $u(\cdot, 0) = u_0$.

Now the test functions depends on time!

Space-time FEM - Set up

- Let $0 = t_0 < t_1 < ... < t_N = T$ be partition of $I := [0, T]$ into subintervals $I_n := (t_{n-1}, t_n]$ of length $k_n = t_n - t_{n-1}$.
- Let $P_q(I_n)$ be the space of polynomials of $\deg \leq q$ on I_n.
- Let $\mathcal{T}_n$ be triangulation of Ω with mesh function h_n.
- Let $V_{h_n}^p$ be the space of polynomials of $\deg \leq p$ on $\mathcal{T}_n$.
- On the space-time slab $S_n = \Omega \times I_n$ we define

$$\begin{aligned} W_n^{p,q} &= P_q(I_n) \otimes V_{h_n}^p \\ &= \{v : v(x,t) = \sum_{j=0}^{q} t^j v_j(x),\ v_j(x) \in V_{h_n}^p,\ (x,t) \in S_n\}. \end{aligned}$$

- Let $W^{p,q} = \{v : v|_{S_n} \in W_n^{p,q},\ n = 1, 2, ..., N\}$.

Space-time FEM

In general: Functions in $W^{p,q}$ are discontinuous across the time levels t_n (see picture).

Let $v^n_{+(-)} = \lim_{s\to 0^{+(-)}} v(t_n + s)$ and let $[v^n] = v^n_+ - v^n_-$ denote the jump at t_n.

We define the space-time FEM cG(p)dG(q) as: Find $U \in W^{p,q}$ such that for $n = 1, 2, ..., N$,

$$\int_{I_n} ((\frac{\partial U}{\partial t}, v) + (\nabla U, \nabla v))\,\mathrm{d}t + ([U_{n-1}], v_+^{n-1}) = \int_{I_n} (f, v)\,\mathrm{d}t, \quad \forall v \in W_n^{p,q}.$$

Example - cG(1)dG(0)

Recall cG(p)dG(q):

$$\int_{I_n} \left(\left(\frac{\partial U}{\partial t}, v \right) + (\nabla U, \nabla v) \right) \mathrm{d}t + ([U_{n-1}], v_+^{n-1}) = \int_{I_n} (f, v) \, \mathrm{d}t.$$

Example: cG(1)dG(0), i.e. $p = 1$, $q = 0$.

- For $v \in W_n^{1,0}$ we get $v(x,t) = v(x) \in V_{h_n}^1$ for $(x,t) \in S_n$.
- Thus $\frac{\partial v}{\partial t} = 0$ and $[v^{n-1}] = v_+^{n-1} - v_-^{n-1} = v^n - v^{n-1}$.

cG(1)dG(0): Find $U^n \in V_{h_n}^1$ such that for $n = 1, 2, ..., N$,

$$k_n (\nabla U^n, \nabla v) + (U^n - U^{n-1}, v) = \int_{I_n} (f, v) \, \mathrm{d}t, \quad \forall v \in V_{h_n}^1.$$

Example - cG(1)dG(0)

cG(1)dG(0): Find $U^n \in V^1_{h_n}$ such that for $n = 1, 2, ..., N$,

$$k_n(\nabla U^n, \nabla v) + (U^n - U^{n-1}, v) = \int_{I_n} (f, v)\, \mathrm{d}t, \quad \forall v \in V^1_{h_n}.$$

Compare to Backward Euler: Find $u_h^n \in V_h$ such that

$$(\frac{u_h^n - u_h^{n-1}}{k_n}, v) + (\nabla u_h^n, \nabla v) = (f(t_n), v).$$

Only difference is the right hand side.

Example - cG(1)dG(0)

cG(1)dG(0): Find $U^n \in V^1_{h_n}$ such that for $n = 1, 2, ..., N$,

$$k_n(\nabla U^n, \nabla v) + (U^n - U^{n-1}, v) = \int_{I_n} (f, v)\,\mathrm{d}t, \quad \forall v \in V^1_{h_n}.$$

One can show that:

$$\max_{t \in I} \|u(t) - U(t)\|_{L_2} \leq C \max_n \Big(k_n \max_{t \in I_n} \|\dot{u}(t)\|_{L_2(\Omega)} + h^2 \max_{t \in I_n} \|u(t)\|_{H^2} \Big)$$

Example - cG(1)dG(1)

Example: cG(1)dG(1), i.e. $p = 1$, $q = 1$.

- For $v \in W_n^{1,0}$ we get $v(x,t) = v_0 + \frac{t - t_{n-1}}{k_n} v_1$ for $v_0, v_1 \in V_{h_n}^1$ and $(x,t) \in S_n$.
- Thus $\frac{\partial v}{\partial t} = \frac{v_1}{k_n}$ and $[v^{n-1}] = v_+^{n-1} - v_-^{n-1} = v_0 - v_-^{n-1}$.
- Let $U = \Phi_n + \frac{t - t_{n-1}}{k_n} \Psi_n$ on S_n.

Example - cG(1)dG(1)

Recall cG(q)dG(p):

$$\int_{I_n}\left(\left(\frac{\partial U}{\partial t}, v\right) + (\nabla U, \nabla v)\right) \mathrm{d}t + ([U_{n-1}], v_+^{n-1}) = \int_{I_n}(f, v)\,\mathrm{d}t,$$

With $U = \Phi_n + \frac{t-t_{n-1}}{k_n}\Psi_n$ and $v = v_0 + \frac{t-t_{n-1}}{k_n}v_1$ we get

$$\int_{I_n}\left(\frac{\Psi_n}{k_n}, v_0\right)\mathrm{d}t + \int_{I_n}\left(\nabla\Phi_n + \frac{t-t_{n-1}}{k_n}\Psi_n, \nabla v_0\right)\mathrm{d}t + (\Phi_n - U_-^{n-1}, v_0) = \int_{I_n}(f, v_0)\mathrm{d}t$$

$$\int_{I_n}\left(\frac{\Psi_n}{k_n}, \frac{t-t_{n-1}}{k_n}v_1\right)\mathrm{d}t + \int_{I_n}\left(\nabla\Phi_n + \frac{t-t_{n-1}}{k_n}\Psi_n, \nabla\frac{t-t_{n-1}}{k_n}v_1\right)\mathrm{d}t = \int_{I_n}\left(f, \frac{t-t_{n-1}}{k_n}v_1\right)\mathrm{d}t$$

Example - cG(1)dG(1)

$$\int_{I_n}(\frac{\Psi_n}{k_n}, v_0)\mathrm{d}t + \int_{I_n}(\nabla\Phi_n + \frac{t-t_{n-1}}{k_n}\Psi_n, \nabla v_0)\mathrm{d}t + (\Phi_n - U_-^{n-1}, v_0) = \int_{I_n}(f, v_0)\mathrm{d}t$$

$$\int_{I_n}(\frac{\Psi_n}{k_n}, \frac{t-t_{n-1}}{k_n}v_1)\mathrm{d}t + \int_{I_n}(\nabla\Phi_n + \frac{t-t_{n-1}}{k_n}\Psi_n, \nabla\frac{t-t_{n-1}}{k_n}v_1)\mathrm{d}t = \int_{I_n}(f, \frac{t-t_{n-1}}{k_n}v_1)\mathrm{d}t$$

We compute: $\int_{I_n}\frac{t-t_{n-1}}{k_n}dt = \frac{k_n}{2}$ and $\int_{I_n}\frac{(t-t_{n-1})^2}{{k_n}^2}dt = \frac{k_n}{3}$.

$$(\Psi^n, v_0) + k_n(\nabla\Phi^n, \nabla v_0) + \frac{k_n}{2}(\nabla\Psi^n, \nabla v_0) + (\Phi^n - U_-^{n-1}, v_0) = \int_{I_n}(f, v_0)\,\mathrm{d}t$$

$$\frac{k_n}{2}(\Psi^n, v_1) + \frac{k_n}{2}(\nabla\Phi^n, \nabla v_1) + \frac{k_n}{3}(\nabla\Psi^n, \nabla v_1) = \frac{1}{k_n}\int_{I_n}(t-t_{n-1})(f, v_1)\,\mathrm{d}t$$

Example - cG(1)dG(1)

cG(1)dG(1): Find $U \in W^{p,q}$ such that for $n = 1, 2, ..., N$,

$$(\Psi^n, v_0) + k_n(\nabla\Phi^n, \nabla v_0) + \frac{k_n}{2}(\nabla\Psi^n, \nabla v_0) + (\Phi^n - U_-^{n-1}, v_0) = \int_{I_n}(f, v_0)\,\mathrm{d}t$$

$$\frac{k_n}{2}(\Psi^n, v_1) + \frac{k_n}{2}(\nabla\Phi^n, \nabla v_1) + \frac{k_n}{3}(\nabla\Psi^n, \nabla v_1) = \frac{1}{k_n}\int_{I_n}(t - t_{n-1})(f, v_1)\,\mathrm{d}t$$

where $U_-^0 = U^0$.