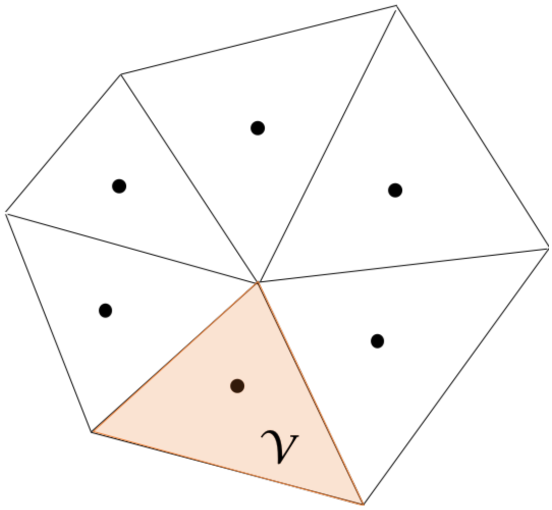
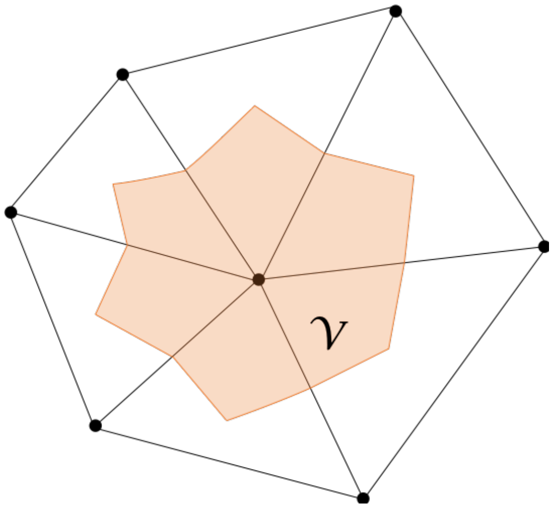
## Unstructured gids

There are two different approaches for finite-volume discretization on unstructured grids: *cell-centered* and *node-centered*. The main difference between these two is the construction of control volumes ($\mathcal{V}$).




In the cell-centered grid metric, the control volumes are identical with primary grid cells.




In the node-centered grid metric, the control volume is given by the so-called *dual grid*.

*Dual grid is usually obtained by connecting midpoint of each edge surrounding a node to center of mass of the cells.*

Some comparison data for cell-centered and node-centered discretization

| | Triangles | Tetrahedra | Prisms | Hexahedra |
|---|---|---|---|---|
| No. of nodes | N | N | N | N |
| No. of cells | 3N | 5.5N | 3N | N |
| No. of edges | ~4.5N | ~6.5N | ~5.5N | 3N |
| No. of faces | | ~11N | ~7.5N | 3N |

Memory requirements (assuming tetrahedra only)

- Node-centered : $C \times N$ arrays
- Cell-centered: $C \times 5.5N$ arrays

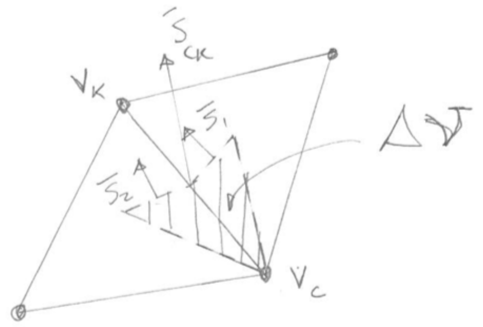
Computational requirements (assuming tetrahedra only)

- Node-centered : $C \times 6.5N$ flops ($\sim$ No. edges)
- Cell-centered: $C \times 5.5N$ flops $\sim$ No. faces)

Here, C is number of variables/equations.

Computation of dual cell volume

$$\Delta\mathcal{V} = \frac{1}{2n_{dim}}\bar{S}_{ck}\cdot(\bar{X}_k - \bar{X}_c)$$




where $n_{dim}$ is the space dimension (2 or 3).

$$\mathcal{V}_c = \frac{1}{2n_{dim}}\sum_{k=1}^{m_c}\bar{S}_{ck}\cdot(\bar{X}_k - \bar{X}_c).$$

$m_c$ is number of neighbouring nodes to $v_c$.

Discretization of first derivative

Consider the scalar equation

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0.$$

Write in integral form

$$\int_\Omega \frac{\partial U}{\partial t}\, d\Omega + a\int_\Omega \frac{\partial U}{\partial x}\, d\Omega = 0$$

$$\Rightarrow \quad \int_\Omega \frac{\partial U}{\partial t}\, d\Omega + a\oint_{\partial\Omega} U\ dy = 0$$

Take the integral over the dual cell

$$\mathcal{V}_c\,\frac{\partial}{\partial t}(U_c) + a\sum_{k=1}^{m_c}\underbrace{\frac{U_c + U_k}{2}}_{\substack{\text{velocity averaged}\\ \text{along } S_{ck}}}\Delta y_k = 0$$

On a rectangular grid with smooth mapping to Cartesian grid, this first-derivative approximation is second-order accurate. It can be shown that on an arbitrary triangular grid it is at least first-order accurate (see Svärd et al., Applied Numerical Mathematics 58 (2008)).

Discretization of second derivative

Consider the equation

$$\frac{\partial U}{\partial t} = \nabla^2 U$$

Written in the integral form

$$\begin{aligned}
\int_\Omega \frac{\partial U}{\partial t}\, d\Omega &= \int_\Omega \frac{\partial}{\partial x_j}\left(\frac{\partial U}{\partial x_j}\right)\, d\Omega \\
&= \oint_{\partial\Omega} \frac{\partial U}{\partial x_j} n_j\, ds \\
&= \oint_{\partial\Omega} \frac{\partial U}{\partial n} ds
\end{aligned}$$

Straight-forward approximation:

$$\mathcal{V}_c \left(\frac{\partial}{\partial t} U\right)_c = \sum_{k=1}^{m_c} \left(\frac{\partial U}{\partial n}\right)_{ck} S_{ck}$$

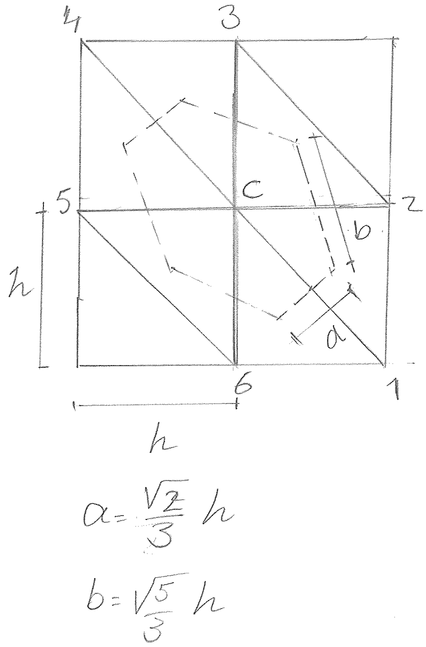
Approximate $\frac{\partial U}{\partial n}$ on boundary of the dual cell by a central difference :

$$\mathcal{V}_c \left(\frac{\partial}{\partial t} U\right)_c = \sum_{k=1}^{m_c} \frac{U_k - U_c}{r_{ck}}\, S_{ck}, \text{ where } r_{ck} = \left|\bar{X}_k - \bar{X}_c\right|$$

Taylor expansion shows this has second-order accuracy on a cartesian grid. But, on a an arbitrary unstructured mesh is *inconsistent* (see Svärd et al., Applied Numerical Mathematics 58 (2008) and HW6).

## An Example of inconsistency

Use Taylor expansion to compute the approximation of second derivative based on scheme above for the grid shown here.




Taylor expansion in 2D reads

$$
\begin{aligned}
& f\left(x_0+\Delta x, y_0+\Delta y\right)= \\
& f\left(x_0, y_0\right)+\Delta x f_x\left(x_0, y_0\right)+\Delta y f_y\left(x_0, y_0\right) \\
& +\frac{\Delta x^2}{2} f_{xx}\left(x_0, y_0\right)+\Delta x \Delta y f_{xy}\left(x_0, y_0\right) \\
& +\frac{\Delta y^2}{2} f_{yy}\left(x_0, y_0\right)+\mathcal{O}\left(\Delta^3\right)
\end{aligned}
$$

Let us compute the contribution along $S_{c1}$

$$
\begin{aligned}
\frac{U_1-U_c}{\sqrt{2} h}= & \left[U_c+h\left(\frac{\partial U}{\partial x}\right)_c-h\left(\frac{\partial U}{\partial y}\right)_c\right. \\
& +\frac{h^2}{2}\left(\frac{\partial^2 U}{\partial x^2}\right)_c-h^2\left(\frac{\partial^2 U}{\partial x \partial y}\right)_c \\
& \left.+\frac{h^2}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)_c+\mathcal{O}\left(h^3\right)-U_c\right] \frac{1}{\sqrt{2} h}
\end{aligned}
$$

Repeating the same along other edges of dual cell and inserting in the approximation for the second derivative yield

$$
\left(\frac{\partial U}{\partial t}\right)_c=\frac{1+\sqrt{5}}{3} \nabla^2 U_c+\frac{2}{3}\left(\frac{\partial^2 U}{\partial x \partial y}\right)_c+\mathcal{O}(h).
$$

**So, the above discretization of second derivative is inconsistent on an arbitrary grid.**