Homework 3 Discretisation errors

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Task 1: Modified wavenumber

a)

$$f' = Df = \frac{1}{\Delta x}\begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & \cdots & & \\ & & & & -1 & 1 \\ 1 & & & & & -1 \end{bmatrix} f \tag{1.1}$$

D is a 20*20 matrix.

b)

$$\begin{aligned}
& f(x) = e^{ikx} \\
& f'_{exact} = ike^{ikx} \\
& f'_{num} = i\tilde{k}e^{ikx} \\
& \Rightarrow \tilde{k}(k) = \frac{f'_{num}}{ie^{ikx}} \\
& f'_{num} = \left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right) = \frac{1}{\Delta x}(e^{ik(x+\Delta x)} - e^{ikx}) \\
& = \frac{1}{\Delta x}e^{ikx}(e^{ik\Delta x} - 1) = i\frac{1}{\Delta x}(i - ie^{ik\Delta x})e^{ikx} = i\tilde{k}e^{ikx} \\
& \Rightarrow \tilde{k} = \frac{1}{\Delta x}\left(i - ie^{ik\Delta x}\right) \\
& \Rightarrow \tilde{k}\Delta x = i - ie^{ik\Delta x} = i(1 - e^{ik\Delta x}) = \sin(k\Delta x) + i2\sin^2(k\frac{\Delta x}{2})
\end{aligned} \tag{1.2}$$

c)

As shown in figure 1 left, the lines are the derivatives of two methods. When N is 50, the first-order finite-difference discretization of the derivative is stable while still exist the error. As shown in figure1 right, the error could not eliminated but could be reduced by larger N, like N=1000, the error will reduce.

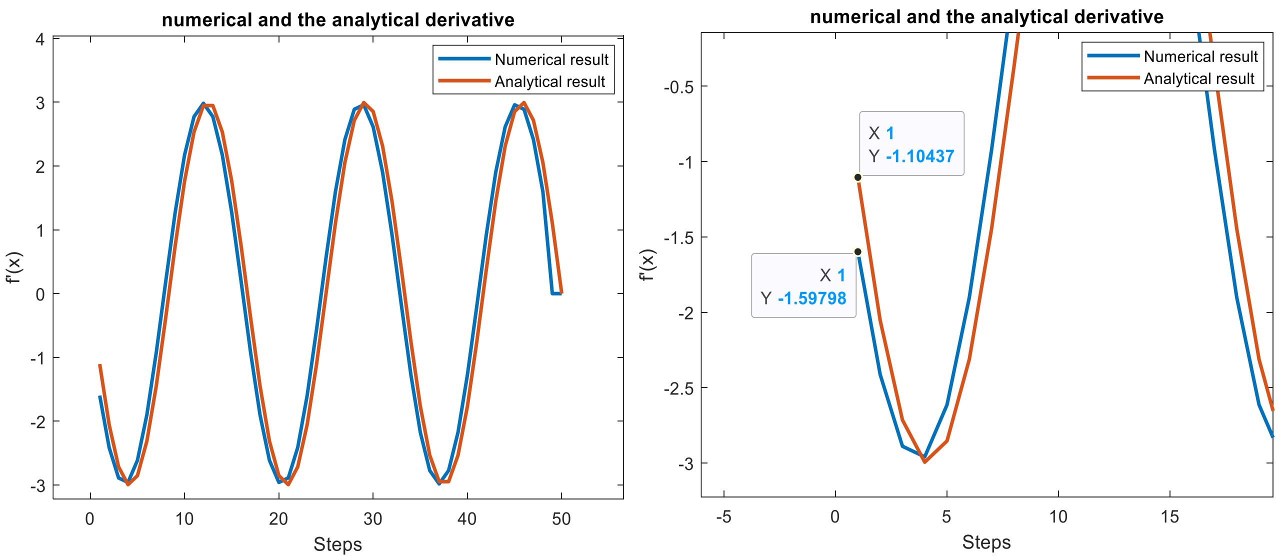

Figure 1 Derivatives of numerical and analtical methods, N=50

d)
Vector u is constant, for N=50, the value is -0.55 + 2.92i.
ik is also -0.55 + 2.92i and k is the modified wavenumber.
These two vectors are totally the same because delta fj is get through the D and f, and wavebumber is get through D. This result confirm that the finite-diference derivative of a Fourier mode e^{ikx} can be found by multiplying the function by the modified wavenumber,

Task 2: Dissipative and dispersion error

Question 1

Discrete the advection equation with FTBS:

$$\frac{u_j^{\,n+1}-u_j^{\,n}}{\Delta t}+a\frac{u_j^{\,n}-u_{j-1}^{\,n}}{\Delta t}=0$$

$$\begin{aligned}&\textit{where}\\ &u_j^{\,n}=\hat{u}_k^n e^{ikx}\\ &u_j^{\,n+1}=\hat{u}_k^{n+1}e^{ikx}\\ &u_{j-1}^{\,n}=\hat{u}_k^n e^{ikx_{j-1}}=\hat{u}_k^n e^{ikx_j}e^{-ik\Delta x}=u_j^{\,n}e^{-ik\Delta x}\end{aligned} \tag{1.1}$$

Thus

$$\begin{aligned}&u_j^{\,n}=\hat{u}_k^n e^{ikx}\\ &u_j^{\,n+1}=\hat{u}_k^{n+1}e^{ikx}\\ &u_{j-1}^{\,n}=\hat{u}_k^n e^{ikx_{j-1}}=\hat{u}_k^n e^{ikx_j}e^{-ik\Delta x}=u_j^{\,n}e^{-ik\Delta x}\\ &\hat{u}_k^{n+1}=\left(1-\frac{\Delta t}{\Delta x}a(1-e^{-ik\Delta x})\right)\hat{u}_k^n=\hat{G}(k)\hat{u}_k^n\\ &\hat{G}(k)=\frac{\hat{u}_k^{n+1}}{\hat{u}_k^n}=1-\frac{\Delta t}{\Delta x}a(1-e^{-ik\Delta x})\end{aligned} \tag{1.2}$$

Second-order central difference

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}+c\frac{u_{j+1}{}^{n}-u_{j-1}{}^{n}}{2\Delta x}=0$$

$$\begin{aligned}
&where\\
&u_j{}^{n}=\hat{u}_k^n e^{ikx}\\
&u_j{}^{n+1}=\hat{u}_k^{n+1} e^{ikx}\\
&u_j{}^{n-1}=\hat{u}_k^{n-1} e^{ikx}\\
&u_{j+1}{}^{n}=\hat{u}_k^n e^{ikx_{j+1}}=\hat{u}_k^n e^{ikx_j}e^{ik\Delta x}=u_j{}^{n}e^{ik\Delta x}\\
&u_{j-1}{}^{n}=\hat{u}_k^n e^{ikx_{j-1}}=\hat{u}_k^n e^{ikx_j}e^{-ik\Delta x}=u_j{}^{n}e^{-ik\Delta x}
\end{aligned}\tag{1.3}$$

Thus

$$\begin{aligned}
&\hat{u}_k^{n-1}-\hat{u}_k^{n+1}=c\frac{\Delta t}{\Delta x}(\hat{u}_k^n e^{ik\Delta x}+\hat{u}_k^n e^{-ik\Delta x})\\
&\Rightarrow\frac{\hat{u}_k^{n-1}}{\hat{u}_k^n}-\frac{\hat{u}_k^{n+1}}{\hat{u}_k^n}=c\frac{\Delta t}{\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})\\
&\Rightarrow\frac{1}{\hat{G}(k)}-\hat{G}(k)=c\frac{\Delta t}{\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})\\
&\hat{G}(k)1=\frac{-c\frac{\Delta t}{\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})+\sqrt{[c\frac{\Delta t}{\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})]^2+4}}{2}\\
&\hat{G}(k)2=\frac{-c\frac{\Delta t}{\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})-\sqrt{[c\frac{\Delta t}{\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})]^2+4}}{2}
\end{aligned}\tag{1.4}$$

Question 2

Show the exact amplification factor for the advection equation:
The exact solution is

$$u=\hat{u}e^{ikx-iwt}\tag{1.5}$$

Thus

$$\begin{aligned}
&u_j{}^{n}=\hat{u}_k^n e^{ikx}\\
&u_j{}^{n+1}=u_j{}^{n}e^{-iw\Delta t}=\hat{u}_k^n e^{ikx}e^{-iw\Delta t}\\
&\hat{G}(k)=\frac{\hat{u}_k^{n+1}}{\hat{u}_k^n}=e^{-iw\Delta t}=e^{-i\sigma\phi}
\end{aligned}\tag{1.6}$$

Where

$$w=kc\quad \sigma\phi=k\Delta x\cdot c\frac{\Delta t}{\Delta x}=kc\Delta t\tag{1.7}$$

Compute amplitude error and phase error:

FTBS:

$$\varepsilon_A = \frac{|\hat{G}_k|}{|\hat{\tilde{G}}_k|} = \frac{\left|1-\frac{\Delta t}{\Delta x}a(1-e^{-ik\Delta x})\right|}{e^{-i\sigma\phi}} = \frac{\sqrt{\left(1-\frac{\Delta t}{\Delta x}a\right)^2+\left(\frac{\Delta t}{\Delta x}ae^{-ik\Delta x}\right)^2}}{e^{-i\sigma\phi}}$$

$$\varepsilon_p = \frac{\Phi}{\tilde{\Phi}} = \frac{\arctan\left(-\frac{\left(1-\frac{\Delta t}{\Delta x}a\right)}{\left(\frac{\Delta t}{\Delta x}ae^{-ik\Delta x}\right)}\right)}{k\pi} = \frac{\arctan\left(-\frac{\Delta x - a\Delta t}{\Delta t a e^{-ik\Delta x}}\right)}{k\pi} \tag{1.8}$$

Second-order central difference

$$\varepsilon_A = \frac{|\hat{G}_k|}{|\hat{\tilde{G}}_k|} = \frac{\left|\frac{-c\frac{\Delta t}{\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})+\sqrt{[c\frac{\Delta t}{\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})]^2+4}}{2}\right|}{e^{-ikc\Delta t}}$$

$$= \frac{\sqrt{\frac{[c\frac{\Delta t}{\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})]^2+4}{4}+\left(-c\frac{\Delta t}{2\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})\right)^2}}{e^{-ikc\Delta t}}$$

$$= \frac{\sqrt{2[c\frac{\Delta t}{2\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})]^2+1}}{e^{-ikc\Delta t}}$$

$$\varepsilon_p = \frac{\Phi}{\tilde{\Phi}} = \frac{\arctan\left(-\frac{\left(\sqrt{[c\frac{\Delta t}{\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})]^2+4}\right)}{\left(-c\frac{\Delta t}{\Delta x}(e^{ik\Delta x}+e^{-ik\Delta x})\right)}\right)}{k\pi} \tag{1.9}$$

Compare the behavior of figure 1, which is given in HW3 description.

First, all the amplitude error and phase error equal to 1 means there is no error exist, the numerical solution is the same with the analytic solution. But the below description, the error increase means the numerical solution is far away with analytic solution.

Second, low σ could reduce amplitude error for the first order scheme, while a low σ could increase the phrase error. There is a balance σ which has acceptable amplitude and phrase error, like σ=0.25 for this case.

Third, for the phase error, different σ nearly has no influence, the performance is the same, when the Φ is small, the error is small while error increases with phase.

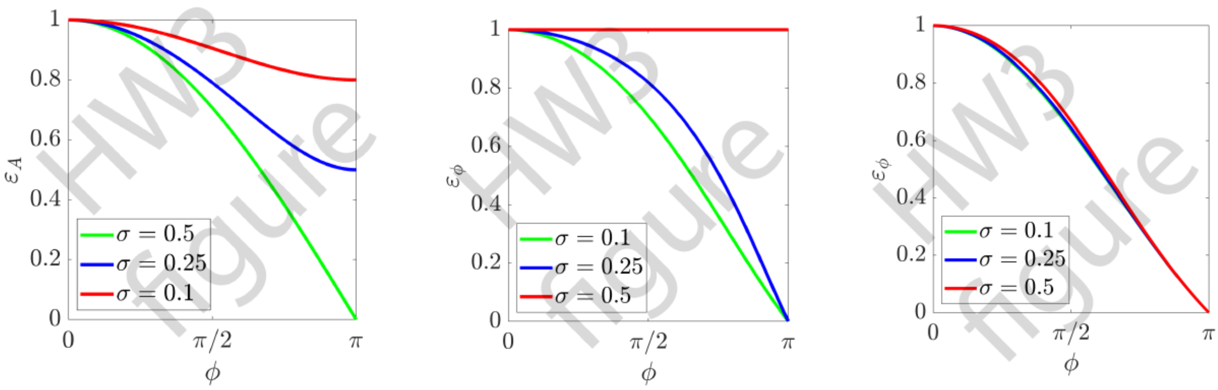

Figure 1: (Left) Dissipative and (centre) dispersive error for the first-order scheme and (right) dispersive error for the second-order scheme, as function of $\phi = k\Delta x$ for different values of $\sigma = c\Delta t/\Delta x$.