Part 4 - Abstract elliptic problems

4.1. Existence and uniqueness of solutions

(Hilbert spaces and Lax-Milgram theorem)

Hilbert spaces

A real Hilbert space V is

- a complete space
  (i.e., every Cauchy sequence in V converges in V)
- with a scalar product $(v, w)_V$ for $v, w \in V$
- and induced norm $\|v\|_V = \sqrt{(v, v)_V}$.

Examples:

- $V = L^2(\Omega)$ with $(\cdot, \cdot)_V = (\cdot, \cdot)_{L^2(\Omega)}$;
- $V = H^1(\Omega)$ with $(\cdot, \cdot)_V = (\cdot, \cdot)_{H^1(\Omega)}$;
- $V = H_0^1(\Omega)$ with $(\cdot, \cdot)_V = (\nabla \cdot, \nabla \cdot)_{L^2(\Omega)}$.

Important inequalities

- Cauchy-Schwarz inequality:

  in any Hilbert space V with scalar product $(\cdot,\cdot)_V$ it holds
  $$|(v,w)_V| \leq \|v\|_V \, \|w\|_V$$
  for all $v, w \in V$.

Important inequalities

Let $\Omega \subset \mathbb{R}^d$ be a domain with diameter $\operatorname{diam}(\Omega) := \sup_{x,y\in\Omega}\{|x-y|\}$.

- Poincaré-Friedrichs inequality:

$$\|v\|_{L^2(\Omega)} \leq \frac{\operatorname{diam}(\Omega)}{\sqrt{2}} \|\nabla v\|_{L^2(\Omega)} \quad \text{for all } v \in \underline{H_0^1(\Omega)}.$$

- Poincaré inequality on convex domains $\Omega \subset \mathbb{R}^d$.
  For all $v \in H^1(\Omega)$ with zero average

$$\int_\Omega v(x)\,dx = 0$$

  it holds

$$\|v\|_{L^2(\Omega)} \leq \frac{\operatorname{diam}(\Omega)}{\pi} \|\nabla v\|_{L^2(\Omega)}.$$

The Lax-Milgram theorem

For a real Hilbert space V and a bilinear form $B : V \times V \to \mathbb{R}$

- that is continuous, i.e., there exists $\beta > 0$ so that

$$B(v, w) \leq \beta \|v\|_V \|w\|_V \qquad \text{for all } v, w \in V,$$

- and coercive, i.e., there exists $\alpha > 0$ so that

$$B(v, v) \geq \alpha \|v\|_V^2 \qquad \text{for all } v \in V.$$

Then for any linear and continuous functional $F : V \to \mathbb{R}$ there exists exactly one $u \in V$ such that

$$B(u, v) = F(v) \qquad \text{for all } v \in V.$$

Recall the general elliptic setting

Let

- $\Omega \subset \mathbb{R}^d$ be a bounded domain
- source term: $f \in L^2(\Omega)$,
- elliptic diffusion coefficient: $\mathbf{k} \in L^\infty(\Omega, \mathbb{R}^{d\times d})$,

$$\mathbf{k}(x)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \mathbf{k}_0 |\boldsymbol{\xi}|^2 \qquad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d$$

Weak formulation of the elliptic problem:

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega k \nabla u \cdot \nabla v = \int_\Omega f\, v \qquad \text{for all } v \in H_0^1(\Omega).$$

Goal: Apply Lax-Milgram to show existence of u.

Application of Lax-Milgram

For the Hilbert space $H_0^1(\Omega)$ with H^1-scalar product

$$(v, w)_{H^1(\Omega)} = (v, w)_{L^2(\Omega)} + (\nabla v, \nabla w)_{L^2(\Omega)},$$

we define the bilinear form

$$B(v, w) := \int_\Omega k \nabla v \cdot \nabla w.$$

- Linearity: obvious.
- Continuity: with L^2-Cauchy-Schwarz inequality

$$\begin{aligned} B(v, w) = \int_\Omega k \nabla v \cdot \nabla w &\le \|k\|_{L^\infty(\Omega)} (\nabla v, \nabla w)_{L^2(\Omega)} \\ &\overset{CS}{\le} \|k\|_{L^\infty(\Omega)} \|\nabla v\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \\ &\le \beta \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} \qquad \text{with } \beta := \|k\|_{L^\infty(\Omega)} \end{aligned}$$

Application of Lax-Milgram

For the Hilbert space $H_0^1(\Omega)$ with H^1-scalar product

$$(v, w)_{H^1(\Omega)} = (v, w)_{L^2(\Omega)} + (\nabla v, \nabla w)_{L^2(\Omega)},$$

we define the bilinear form $B(v, w) := \int_\Omega k \nabla v \cdot \nabla w$.

- Coercivity: with Poincaré-Friedrichs inequality:

$$\begin{aligned} B(v, v) &= \int_\Omega k \nabla v \cdot \nabla v \geq \mathsf{k}_0 \int_\Omega |\nabla v|^2 \\ &= \frac{\mathsf{k}_0}{2} \int_\Omega |\nabla v|^2 + \frac{\mathsf{k}_0}{2} \int_\Omega |\nabla v|^2 \\ &\geq \frac{\mathsf{k}_0}{2} \int_\Omega |\nabla v|^2 + \frac{\mathsf{k}_0}{\operatorname{diam}(\Omega)^2} \int_\Omega |v|^2 \\ &\geq \alpha \|v\|^2_{H^1(\Omega)}, \qquad \text{for } \alpha := \mathsf{k}_0 \min\{2^{-1}, \operatorname{diam}(\Omega)^{-2}\}. \end{aligned}$$

where we used $\mathsf{k}(x)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \mathsf{k}_0 |\boldsymbol{\xi}|^2$ for all $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \{0\}$.

Application of Lax-Milgram

For the Hilbert space $H_0^1(\Omega)$ with H^1-scalar product

$$(v, w)_{H^1(\Omega)} = (v, w)_{L^2(\Omega)} + (\nabla v, \nabla w)_{L^2(\Omega)},$$

we define the bilinear form

$$B(v, w) := \int_\Omega k \nabla v \cdot \nabla w.$$

We summarize the properties:

- Linearity.
- Continuity: $\quad B(v, w) \leq \beta \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}$
- Coercivity: $\quad B(v, v) \geq \alpha \|v\|^2_{H^1(\Omega)}.$

Hence: $B(v, w)$ fulfills the assumptions of the Lax-Milgram theorem.

Application of Lax-Milgram

For the Hilbert space $H_0^1(\Omega)$ with H^1-scalar product

$$(v, w)_{H^1(\Omega)} = (v, w)_{L^2(\Omega)} + (\nabla v, \nabla w)_{L^2(\Omega)},$$

we define linear functional

$$F(v) := \int_\Omega f\, v.$$

- Linearity: obvious.
- Continuity: with L^2-Cauchy-Schwarz inequality

$$F(v) := \int_\Omega f\, v \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}.$$

Reminder: a linear functional $F : V \to \mathbb{R}$ is continuous if $F(v) \leq C\|v\|_V$ for some constant $C > 0$ and all $v \in V$.

Application of Lax-Milgram

We seek $u \in H_0^1(\Omega)$ such that

$$\int_\Omega k \nabla u \cdot \nabla v = \int_\Omega f\, v \qquad \text{for all } v \in H_0^1(\Omega).$$

We have just shown: If we define $V = H_0^1(\Omega)$;

$$B(u, v) := \int_\Omega k \nabla u \cdot \nabla v \qquad \text{and} \qquad F(v) := \int_\Omega f\, v,$$

then $B(\cdot, \cdot)$ is a continuous and coercive bilinear form and $F(\cdot)$ is a continuous, linear functional.

Hence, the Lax-Milgram theorem implies that there is unique solution $u \in H_0^1(\Omega)$ with

$$B(u, v) = F(v) \qquad \text{for all } v \in H_0^1(\Omega).$$

Elliptic problems with Neumann boundary

In the case of Neumann boundary conditions we saw that the weak formulation reads:

Find $u \in H^1(\Omega)$ with

$$\begin{aligned}(k\nabla u, \nabla v)_{L^2(\Omega)} + (u, v)_{L^2(\Omega)} \\ = \ (f, v)_{L^2(\Omega)} + (g_N, v)_{L^2(\partial\Omega)}\end{aligned}$$

for all $v \in H^1(\Omega)$.

Existence and uniqueness follow analogously as before with the Lax-Milgram theorem. The only nontrivial issue is to show that

$$F(v) := (f, v)_{L^2(\Omega)} + (g_N, v)_{L^2(\partial\Omega)}$$

is continuous on $H^1(\Omega)$.

Elliptic problems with Neumann boundary

Find $u \in H^1(\Omega)$ with

$$\begin{aligned}(k\nabla u, \nabla v)_{L^2(\Omega)} + (u, v)_{L^2(\Omega)} \\ = \ (f, v)_{L^2(\Omega)} + (g_N, v)_{L^2(\partial\Omega)}\end{aligned}$$

for all $v \in H^1(\Omega)$.

To apply Lax-Milgram, we need to show the continuity:

$$(g_N, v)_{L^2(\partial\Omega)} \le C\|v\|_{H^1(\Omega)} \qquad \text{for all } v \in H^1(\Omega).$$

This is guaranteed by the trace theorem.

Trace theorem

Let $\Omega \subset \mathbb{R}^d$ be bounded and convex. Then there exists a constant $C_{\text{tr}} > 0$ such that

$$\|v\|_{L^2(\partial\Omega)} \leq C_{\text{tr}} \|v\|_{H^1(\Omega)} \qquad \text{for all } v \in H^1(\Omega).$$

This means that boundary information can be estimated against the H^1-norm of the function.

In the case of Neumann boundary conditions we have with Cauchy-Schwarz:

$$|(g_N, v)_{L^2(\partial\Omega)}| \leq \|g_N\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} \leq \underbrace{\|g_N\|_{L^2(\partial\Omega)} C_{\text{tr}}}_{=:C} \|v\|_{H^1(\Omega)},$$

where

$$C = \|g_N\|_{L^2(\partial\Omega)} C_{\text{tr}} > 0$$

is a constant that does not depend on v.

Part 4 - Abstract elliptic problems

4.2. Galerkin approximations and abstract error estimates

Abstract setting

In the following, we consider the elliptic problem in an abstract setting covered by Lax-Milgram:

For a continuous and coercive bilinear form $B(\cdot, \cdot)$ and a continuous, linear functional F, we seek $u \in V$ with

$$B(u, v) = F(v) \qquad \text{for all } v \in V.$$

Galerkin methods

Find $u \in V$ such that

$$B(u, v) = F(v) \quad \text{for all } v \in V.$$

Numerical approximation?

Idea of Galerkin methods: Replace infinite dim space V (e.g. $V = H_0^1(\Omega)$) by finite dim subspace $V_h \subset V$.

Find $u_h \in V_h$ such that

$$B(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in V_h.$$

Note: exactly what we did for the finite element method!

Galerkin methods

Find $u \in V$ such that

$$B(u, v) = F(v) \quad \text{for all } v \in V.$$

Find $u_h \in V_h$ such that

$$B(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in V_h.$$

How big is the error $e_h = u - u_h$?

Galerkin orthogonality

Since $V_h \subset V$, we have

$$B(u, v_h) = F(v_h) \quad \text{for all } v_h \in V_h,$$
$$B(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in V_h.$$

Subtracting both equations gives
Galerkin orthogonality:

$$B(u - u_h, v_h) = 0 \quad \text{for all } v_h \in V_h.$$

Galerkin orthogonality

Why do we call the following statement Galerkin orthogonality?

$$B(u - u_h, v_h) = 0 \quad \text{for all } v_h \in V_h.$$

Assume that $B(\cdot,\cdot)$ is symmetric (i.e., $B(v,w) = B(w,v)$), then

it is a symmetric and positive definite (=coercive) bilinear form,

or in other words, $B(\cdot,\cdot)$ is scalar product in V.

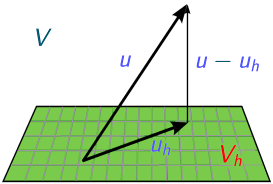
Galerkin orthogonality

Why do we call the following statement Galerkin orthogonality?

$$B(u - u_h, v_h) = 0 \quad \text{for all } v_h \in V_h.$$

If $B(\cdot,\cdot)$ is a <u>scalar product</u> in V, then we can interpret the statement geometrically as:

the error $u - u_h$ is $B(\cdot,\cdot)$-orthogonal on V_h.



Reminder: orthogonal projection

For a Hilbert space $(V, (\cdot,\cdot)_V)$ with subspace $V_h \subset V$,

the orthogonal projection onto V_h is $\quad P_h : V \to V_h.$

This means: for $u \in V$, the projection $P_h(u) \in V_h$ fulfills

$$(P_h(u), v_h)_V = (u, v_h)_V \qquad \text{for all } v_h \in V_h.$$

In other words, the $P_h(u) - u$ is orthogonal on V_h:

$$(P_h(u) - u, v_h)_V = 0 \qquad \text{for all } v_h \in V_h. \tag{1}$$

We can see, that $P_h(u)$ is the function in V_h that has the smallest distance to u, because:

$$\|u - P_h(u)\|_V^2 = (u - P_h(u), u - P_h(u))_V \overset{(1)}{=} (u - P_h(u), u)_V$$

$$\overset{(1)}{=} (u - P_h(u), u - v_h)_V \overset{\text{CS}}{\leq} \|u - P_h(u)\|_V \|u - v_h\|_V.$$

Hence:

$$\|u - P_h(u)\|_V \leq \|u - v_h\|_V \qquad \text{for all } v_h \in V_h.$$

Galerkin orthogonality

Galerkin orthogonality (for error $e_h = u - u_h$):

$$B(u - u_h, V_h) = B(e_h, V_h) = 0.$$

Geometric interpretation: if $B(\cdot,\cdot)$ is symmetric, then

- error e_h is $B(\cdot,\cdot)$-orthogonal to the subspace V_h;
- u_h is $B(\cdot,\cdot)$-orthogonal projection of u onto V_h;
- Hence:
  u_h is the $B(\cdot,\cdot)$-best approximation of u in V_h:

$$|||u - u_h||| = \inf_{v_h \in V_h} |||u - v_h|||,$$

where $||| \cdot ||| := \sqrt{B(\cdot,\cdot)}$ is the energy norm.

Céa's lemma

Galerkin orthogonality (for error $e_h = u - u_h$):

$$B(e_h, V_h) = 0.$$

Estimate for the non-symmetric case?

$$\alpha \|e_h\|^2_{H^1(\Omega)} \leq B(e_h, e_h) = B(e_h, u - u_h)$$

$$\overset{\text{G.O.}}{=} \quad B(e_h, u - v_h) \ \leq \ \beta \, \|e_h\|_{H^1(\Omega)} \, \|u - v_h\|_{H^1(\Omega)}.$$

Consequently (dividing by $\|e_h\|_{H^1(\Omega)}$):

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{\beta}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)},$$

i.e. u_h is a H^1-quasi best approximation of u in V_h.

Summary: Galerkin methods and Céa's lemma

Find $u \in V$ such that

$$B(u, v) = F(v) \quad \text{for all } v \in V.$$

Find $u_h \in V_h$ such that

$$B(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in V_h.$$

The error is quasi-optimal in the H^1-norm, i.e.,

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{\beta}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)}.$$

Can we use this to derive convergence rates?

Part 4 - Abstract elliptic problems

4.3. H^1-a priori error estimates for the Finite Element Method

Reminder: $P1$-finite element space

Let $\mathcal{T}_h$ be a non-overlapping, simplicial partition of the convex polygonal domain $\Omega \subset \mathbb{R}^d$ that is also shape regular and quasi-uniform (and without hanging nodes).

On $\mathcal{T}_h$ the $P1$ finite element space is

$$V_{h,0} := \{v \in C^0(\Omega) \cap H_0^1(\Omega) | \\ \forall K \in \mathcal{T}_h : \ v_{|K} \text{ is polynomial of degree } 1\}.$$

The FE space $V_{h,0}$ is spanned by the nodal basis, i.e.,

$$V_{h,0} = \operatorname{span}\{\phi_z | \ z \in \mathcal{N}_{h,0}\}.$$

Galerkin method (summary)

Find $u \in H_0^1(\Omega)$ such that

$$B(u, v) = F(v) \quad \text{for all } v \in H_0^1(\Omega).$$

Galerkin approximation in $V_{h,0} \subset H_0^1(\Omega)$:
Find $u_h \in V_{h,0}$ such that

$$B(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in V_{h,0}.$$

Abstract error estimate:

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{\beta}{\alpha} \inf_{v_h \in V_{h,0}} \|u - v_h\|_{H^1(\Omega)} = ?.$$

(H^1-quasi-best approximation)

Quasi-interpolation estimates - Part 1

The Clément quasi-interpolation operator

$$I_h : H^1_0(\Omega) \to V_{h,0}$$

is given by

$$I_h(v) := \sum_{z \in \mathcal{N}_{h,0}} \frac{(v, \phi_z)_{L^2(\Omega)}}{(1, \phi_z)_{L^2(\Omega)}} \phi_z$$

and fulfills the estimates for all $v \in H^1_0(\Omega)$

$$\|I_h(v) - v\|_{L^2(\Omega)} \leq Ch\|v\|_{H^1(\Omega)}$$

and

$$\|I_h(v)\|_{H^1(\Omega)} \leq C\|v\|_{H^1(\Omega)}.$$

Quasi-interpolation estimates - Part 2

The Clément quasi-interpolation $I_h : H_0^1(\Omega) \rightarrow V_{h,0}$ with

$$I_h(v) := \sum_{z \in \mathcal{N}_{h,0}} \frac{(v, \phi_z)_{L^2(\Omega)}}{(1, \phi_z)_{L^2(\Omega)}} \phi_z$$

fulfills the estimates for all $v \in H_0^1(\Omega) \cap H^2(\Omega)$:

$$\|I_h(v) - v\|_{L^2(\Omega)} \leq C h^2 \|v\|_{H^2(\Omega)}$$

and

$$\|I_h(v) - v\|_{H^1(\Omega)} \leq C h \|v\|_{H^2(\Omega)}.$$

Here, $H^2(\Omega)$ is the space of two-times weakly differentiable functions with

$$H^2(\Omega) := \{v | \sum_{i+j \leq 2} \|\partial_{x_i x_j} v\|_{L^2(\Omega)} < \infty\}.$$

A priori error estimate - $H^2(\Omega)$ case

Conclusion:

Let $V_{h,0}$ be the $P1$-FEM space, then we have the error estimate

$$\|u - u_h\|_{H^1(\Omega)} \;\leq\; \frac{\beta}{\alpha} \inf_{v_h \in V_{h,0}} \|u - v_h\|_{H^1(\Omega)} \;\leq\; \frac{\beta}{\alpha} \|u - I_h(u)\|_{H^1(\Omega)}.$$

If $u \in H^1_0(\Omega) \cap H^2(\Omega)$ we have

$$\|u - I_h(u)\|_{H^1(\Omega)} \leq C\, h \|u\|_{H^2(\Omega)}$$

and hence (by combining the estimates)

$$\|u - u_h\|_{H^1(\Omega)} \;\leq\; C\, h \|u\|_{H^2(\Omega)},$$

i.e., P1-FEM converges with <u>linear order</u> in the $H^1(\Omega)$-norm.

A priori error estimate - $H^1(\Omega)$ case

Conclusion:

Let $V_{h,0}$ be the P1-FEM space, then we have the error estimate

$$\|u - u_h\|_{H^1(\Omega)} \;\leq\; \frac{\beta}{\alpha} \inf_{v_h \in V_{h,0}} \|u - v_h\|_{H^1(\Omega)}.$$

If only $u \in H_0^1(\Omega)$ we have by density arguments

$$\lim_{h \to 0} \|u - u_h\|_{H^1(\Omega)} \;\leq\; \frac{\beta}{\alpha} \lim_{h \to 0} \inf_{v_h \in V_{h,0}} \|u - v_h\|_{H^1(\Omega)} \;=\; 0.$$

This means, even if the exact solution u is only in $H^1(\Omega)$, the FEM is still guaranteed to converge.

A priori error estimate

For $u \in H_0^1(\Omega) \cap H^2(\Omega)$ we have

$$\|u - u_h\|_{H^1(\Omega)} \; \leq \; C\, h \|u\|_{H^2(\Omega)},$$

and if only $u \in H_0^1(\Omega)$ we have

$$\lim_{h \to 0} \|u - u_h\|_{H^1(\Omega)} \; = \; 0.$$

In fact, depending on the regularity of u the convergence rate can be anything between arbitrarily slow and full linear order.

Typically the geometry of Ω (e.g., not convex and small corners) and a jumping coefficient k have the biggest impact on a bad regularity. This is not uncommon.

When can we guarantee that $H_0^1(\Omega) \cap H^2(\Omega)$?

Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain; $f \in L^2(\Omega)$ a source term and $\mathbf{k} \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ an elliptic diffusion coefficient that is also Lipschitz continuous.

Then there is exists unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$ to

$$\int_\Omega k \nabla u \cdot \nabla v = \int_\Omega f\, v \qquad \text{for all } v \in H_0^1(\Omega).$$

Furthermore, there is constant $C > 0$ that depends on Ω and $\mathbf{k}$, such that

$$\|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

Summary: a priori error estimate

Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain; $f \in L^2(\Omega)$ a source term and $\mathbf{k} \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ an elliptic diffusion coefficient that is also Lipschitz continuous.

Then we have optimal linear convergence for the H^1-error:

$$\|u - u_h\|_{H^1(\Omega)} \;\leq\; C\, h \|u\|_{H^2(\Omega)} \;\leq\; C\, h \|f\|_{L^2(\Omega)}.$$

Part 4 - Abstract elliptic problems

4.4. L^2-a priori error estimates for the Finite Element Method

Aubin-Nitsche lemma

L^2-error estimates - why is there an issue?

Céa lemma (H^1-quasi optimality):

$$\|u - u_h\|_{H^1(\Omega)} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)}.$$

But, it does **not** hold

$$\|u - u_h\|_{L^2(\Omega)} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{L^2(\Omega)}.$$

So we cannot derive L^2-error estimates the same way as in the H^1-case.

Aubin-Nitsche duality argument - Part 1

We consider the elliptic problem with solution $u \in H_0^1(\Omega)$ and Galerkin approximation $u_h \in V_h$:

$$B(u, v) = F(v) \quad \text{for all } v \in H_0^1(\Omega),$$
$$B(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in V_h.$$

With the error $e_h := u - u_h$, we consider a dual auxiliary problem: find $\psi \in H_0^1(\Omega)$ with

$$B(v, \psi) = (v, e_h)_{L^2(\Omega)} \quad \text{for all } v \in H_0^1(\Omega).$$

Selecting $v = e_h$ gives us:

$$\|e_h\|_{L^2(\Omega)}^2 = B(e_h, \psi).$$

Aubin-Nitsche duality argument - Part 2

We just saw:

$$\|e_h\|^2_{L^2(\Omega)} = B(e_h, \psi). \tag{1}$$

Recall Galerkin orthogonality:

$$B(e_h, v_h) = 0 \quad \text{for all } v_h \in V_h.$$

Selecting $v_h = I_h(\psi)$ for the Clément interpolation $I_h : H^1_0(\Omega) \to V_{h,0}$ yields

$$B(e_h, I_h(\psi)) = 0.$$

Subtracting this from (1) gives:

$$\|e_h\|^2_{L^2(\Omega)} = B(e_h, \psi - I_h(\psi)) \leq \beta \|e_h\|_{H^1(\Omega)} \|\psi - I_h(\psi)\|_{H^1(\Omega)}.$$

Aubin-Nitsche duality argument - Part 3

We just saw:

$$\|e_h\|^2_{L^2(\Omega)} \leq C\|e_h\|_{H^1(\Omega)}\|\psi - I_h(\psi)\|_{H^1(\Omega)}.$$

We already know

$$\|e_h\|_{H^1(\Omega)} \leq C h \|f\|_{L^2(\Omega)}$$

and we know the Clément interpolation estimate

$$\|\psi - I_h(\psi)\|_{H^1(\Omega)} \leq C h \|\psi\|_{H^2(\Omega)}.$$

We conclude

$$\|e_h\|^2_{L^2(\Omega)} \leq C h^2 \|f\|_{L^2(\Omega)}\|\psi\|_{H^2(\Omega)}.$$

Aubin-Nitsche duality argument - Part 4

We just saw:

$$\|e_h\|^2_{L^2(\Omega)} \leq C h^2 \|f\|_{L^2(\Omega)} \|\psi\|_{H^2(\Omega)}.$$

Recalling that $\psi \in H^1_0(\Omega)$ solves

$$B(v, \psi) = (v, e_h)_{L^2(\Omega)} \quad \text{for all } v \in H^1_0(\Omega).$$

and the regularity result that $\psi \in H^1_0(\Omega) \cap H^2(\Omega)$ with

$$\|\psi\|_{H^2(\Omega)} \leq C \|e_h\|_{L^2(\Omega)},$$

we conclude

$$\|e_h\|^2_{L^2(\Omega)} \leq C h^2 \|f\|_{L^2(\Omega)} \|e_h\|_{L^2(\Omega)}.$$

L^2-a priori error estimate

We summarize: the L^2-error can be estimated as

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^2 \|f\|_{L^2(\Omega)}.$$

This means, the P1 finite element method shows a second order convergence in the L^2-norm.

Note: this result requires again sufficient regularity, which is for example fulfilled if Ω is convex and **k** Lipschitz-continuous.