Navier-Stokes Equations for Incompressible Flows

For an incompressible flow rate of change of density of material element is zero:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} = 0.$$

Continuity eq. reads

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \frac{\partial u_i}{\partial x_i} = 0 \quad \Rightarrow \quad \frac{\partial u_i}{\partial x_i} = 0 \ (u_i \text{ is divergence free}).$$

One can show that divergence of the velocity field gives the rate of change of volume per unit volume

$$\nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i} = \frac{1}{V} \frac{\mathrm{D}V}{\mathrm{D}t}.$$

For a *homogeneous*, *incompressible fluid*, we also have

$$\frac{\partial \rho}{\partial x_i} = 0 \quad \Rightarrow \frac{\partial \rho}{\partial t} = 0, \quad \rho = constant$$

Further, usually it is assumed that material properties of an incompressible fluid are constant. This means that if we choose fluid properties at a point in the domain as the reference values, the **non-dimensional** values of density, dynamic viscosity, thermal conductivity and coefficient of specific heat will be equal to 1:

$$\rho^* = 1, \ \mu^* = 1, \ \kappa^* = 1, \ c_p^* = 1.$$

Starting from the non-dimensional momentum equations for compressible flows, above-mentioned simplifications result to

$$\frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*} = -\frac{\partial p^*}{\partial x_i^*} + \frac{1}{Re} \frac{\partial}{\partial x_j^*} \left[\left(\frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} \right) + \lambda^* \frac{\partial u_k^*}{\partial x_k^*} \delta_{ij} \right]$$

$$\left\{ \frac{\partial u_i^*}{\partial x_i^*} = 0 \right\} \Rightarrow \frac{\partial u_i^*}{\partial t^*} + \frac{\partial}{\partial x_j^*}(u_j^* u_i^*) = -\frac{\partial p^*}{\partial x_i^*} + \frac{1}{Re} \frac{\partial}{\partial x_j^*} \left(\frac{\partial u_i^*}{\partial x_j^*} \right)$$

Energy equation in non-dimensional form reads

$$\frac{\partial T^*}{\partial t^*} + u_i^* \frac{\partial T^*}{\partial x_i^*} = \frac{1}{Re\ Pr} \frac{\partial}{\partial x_i^*} \left(\frac{\partial T^*}{\partial x_i^*} \right).$$

This equation is decoupled from the momentum and continuity equation and can be solved when the flow filed is computed.

In following, we skip * for non-dimensional quantities.

Observe that the continuity equation does not include a time derivative anymore. This causes a need for different solution algorithms compared to those used for compressible flows.

Integral form of the incompressible Navier-Stokes eqs.

For the finite-volume discretization the integral form of the conservation equations are used.

Continuity:

$$\int_V \frac{\partial u_i}{\partial x_i}\, dV = \int_S u_i n_i \, dS = 0.$$

Momentum equation:

$$\begin{aligned}\int_V \frac{\partial u_i}{\partial t}\, dV &= -\int_V \left[\frac{\partial}{\partial x_j}(u_j u_i) + \frac{\partial p}{\partial x_i} - \frac{1}{Re}\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right)\right]\, dV \\ &= -\int_V \frac{\partial}{\partial x_j}\left[u_j u_i + p\delta_{ij} - \frac{1}{Re}\frac{\partial u_i}{\partial x_j}\right]\, dV \\ &= -\int_S \left[u_i u_j n_j + p n_i - \frac{1}{Re}\frac{\partial u_i}{\partial x_j} n_j\right]\, dS\end{aligned}$$

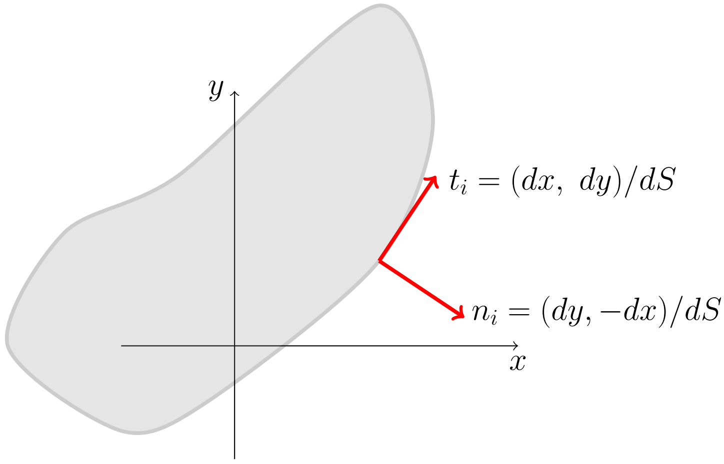
Finite-volume method on arbitrary grids

The differential equations are approximated in integral form.

Equation with first derivatives:

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \int_V \frac{\partial u_i}{\partial x_i}\, dv = \int_S u_i n_i \, dS = 0.$$

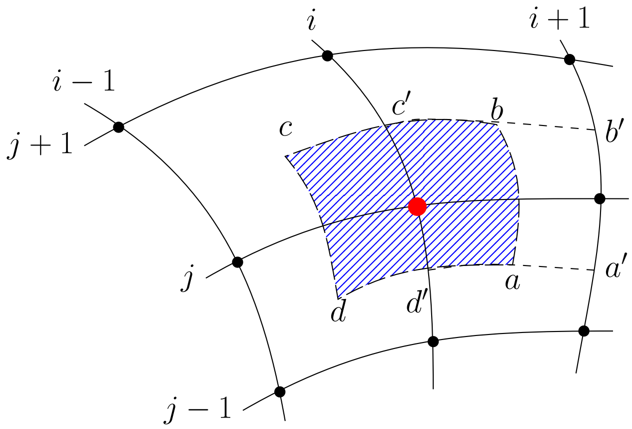
2D case: $n_i \, dS = (dy, -dx)$
(normal with length dS).



$$\Rightarrow \int_S u_i n_i \, dS = \int_S (u \; dy - v \; dx) = 0.$$

Apply to arbitrary grid: Consider volume surrounding node i, j ($abcd$)

$$\int_S (u \; dy - v \; dx) \approx$$
$$(u_{ab} \; \Delta y_{ab} - v_{ab} \; \Delta x_{ab}) +$$
$$(u_{bc} \; \Delta y_{bc} - v_{bc} \; \Delta x_{bc}) +$$
$$(u_{cd} \; \Delta y_{ab} - v_{cd} \; \Delta x_{cd}) +$$
$$(u_{da} \; \Delta y_{da} - v_{da} \; \Delta x_{da}) +$$



where

$$u_{ab} \approx \frac{1}{2}(u_{i+1,j} + u_{i,j}), \;\; \Delta x_{ab} = x_b - x_a,$$
$$v_{ab} \approx \frac{1}{2}(v_{i+1,j} + v_{i,j}), \;\; \Delta y_{ab} = y_b - y_a,$$

etc.

Assume cartesian grid: $\begin{cases} \Delta x_{ab} = \Delta x_{cd} = \Delta y_{bc} = \Delta y_{da} = 0; \\ \Delta y_{cd} = -\Delta y_{ab}, \;\; \Delta x_{bc} = -\Delta x_{da}. \end{cases}$

$$\begin{aligned} \Rightarrow \int_S (u\ dy - v\ dx) \;&\approx\; (u_{ab} - u_{cd})\Delta y_{ab} + (v_{bc} - v_{da})\Delta x_{da} \\ &\approx\; \frac{1}{2}(u_{i+1,j} - u_{i-1,j})\Delta y_{ab} + \frac{1}{2}(v_{i,j+1} - v_{i,j-1})\Delta x_{da} \end{aligned}$$

Observe that division by $\Delta y_{ab} \cdot \Delta x_{da}$ gives a central-difference discretization of continuity equation, i.e.

$$\frac{\partial u}{\partial x}\Big|_{i,j} + \frac{\partial v}{\partial y}\Big|_{i,j} \approx \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}$$

(Second-order accurate discretization.)

Equation with second derivatives:

Ex.: Laplace operator ($\nabla^2 \phi = 0$)

$$0 = \int_V \frac{\partial^2 \phi}{\partial x_k \partial x_k}\, dv = \int_S \frac{\partial \phi}{\partial x_k} n_k\, dS$$

for 2D case

$$\begin{aligned}\int_S \frac{\partial \phi}{\partial x_k} n_k \, dS \;\approx\; & \left[\frac{\partial \phi}{\partial x}\right]_{ab} \Delta y_{ab} - \left[\frac{\partial \phi}{\partial y}\right]_{ab} \Delta x_{ab} \\ & + \left[\frac{\partial \phi}{\partial x}\right]_{bc} \Delta y_{bc} - \left[\frac{\partial \phi}{\partial y}\right]_{bc} \Delta x_{bc} \\ & + \left[\frac{\partial \phi}{\partial x}\right]_{cd} \Delta y_{cd} - \left[\frac{\partial \phi}{\partial y}\right]_{cd} \Delta x_{cd} \\ & + \left[\frac{\partial \phi}{\partial x}\right]_{da} \Delta y_{da} - \left[\frac{\partial \phi}{\partial y}\right]_{da} \Delta x_{da}.\end{aligned}$$

First derivatives are evaluated as a mean value over adjacent control volumes/areas. Use Gauss theorem

$$\left[\frac{\partial \phi}{\partial x_k}\right] \approx \frac{1}{V} \int_V \frac{\partial \phi}{\partial x_k} \, dV = \frac{1}{V} \int_S \phi \, n_k \, dS.$$

Consider area $a'b'c'd'$

$$\left(\left[\frac{\partial \phi}{\partial x}\right]_{ab}, \left[\frac{\partial \phi}{\partial y}\right]_{ab} \right) \approx \frac{1}{A_{a'b'c'd'}} \int_{a'b'c'd'} (\phi \, dy, \; -\phi \, dx)$$

where

$$\int_{a'b'c'd'} \phi \, dy \approx \phi_{i+1,j} \, \Delta y_{a'b'} + \phi_b \, \Delta y_{b'c'} + \phi_{i,j} \, \Delta y_{c'd'} + \phi_a \, \Delta y_{d'a'}.$$

The value of ϕ_a and ϕ_b are taken as average over adjacent nodes:

$$\phi_a = \frac{1}{4}(\phi_{i,j} + \phi_{i+1,j} + \phi_{i+1,j-1} + \phi_{i,j-1})$$

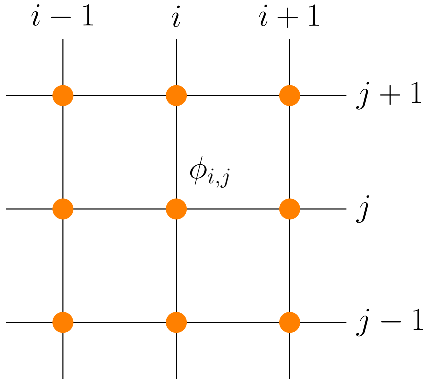
$$\phi_b = \frac{1}{4}(\phi_{i,j} + \phi_{i+1,j} + \phi_{i+1,j+1} + \phi_{i,j+1})$$

The area A is evaluated as half magnitude of cross product of diagonals, i.e.

$$A_{a'b'c'd'} = \frac{1}{2} \left| \Delta x_{d'b'} \, \Delta y_{a'c'} - \Delta y_{d'b'} \, \Delta x_{a'c'} \right|.$$

Evaluating other combinations of $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$, we obtain a 9-point formula of form:

$$
\begin{aligned}
& A_{i,j}\phi_{i-1,j+1} + B_{i,j}\phi_{i,j+1} + C_{i,j}\phi_{i+1,j+1} \\
+ & D_{i,j}\phi_{i-1,j} + E_{i,j}\phi_{i,j} + F_{i,j}\phi_{i+1,j} \\
+ & G_{i,j}\phi_{i-1,j-1} + H_{i,j}\phi_{i,j-1} + I_{i,j}\phi_{i+1,j-1}
\end{aligned}
$$



For a 2D cartesian grid we have

$$
\begin{aligned}
\int_V \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \, dV &= \int_S \left(\frac{\partial \phi}{\partial x} \, dy - \frac{\partial \phi}{\partial y} \, dx \right) \\
&= \left(\left[\frac{\partial \phi}{\partial x} \right]_{ab} - \left[\frac{\partial \phi}{\partial x} \right]_{cdj} \right) \Delta y + \left(\left[\frac{\partial \phi}{\partial y} \right]_{bc} - \left[\frac{\partial \phi}{\partial y} \right]_{da} \right) \Delta x \\
&= \left\{ \left[\frac{\partial \phi}{\partial x} \right]_{ab} = (\phi_{i+1,j} - \phi_{i,j}) \Delta y \, \frac{1}{\Delta x \, \Delta y} \right\} = \\
&= \cdots \\
&= (\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}) \frac{\Delta y}{\Delta x} + (\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}) \frac{\Delta x}{\Delta y}.
\end{aligned}
$$

Dividing by the area $\Delta x \cdot \Delta y$ results to the usual 5-point Laplace formula, i.e.

$$
\frac{\partial^2 \phi}{\partial x^2}\Big|_{i,j} + \frac{\partial^2 \phi}{\partial y^2}\Big|_{i,j} \approx \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2}
$$

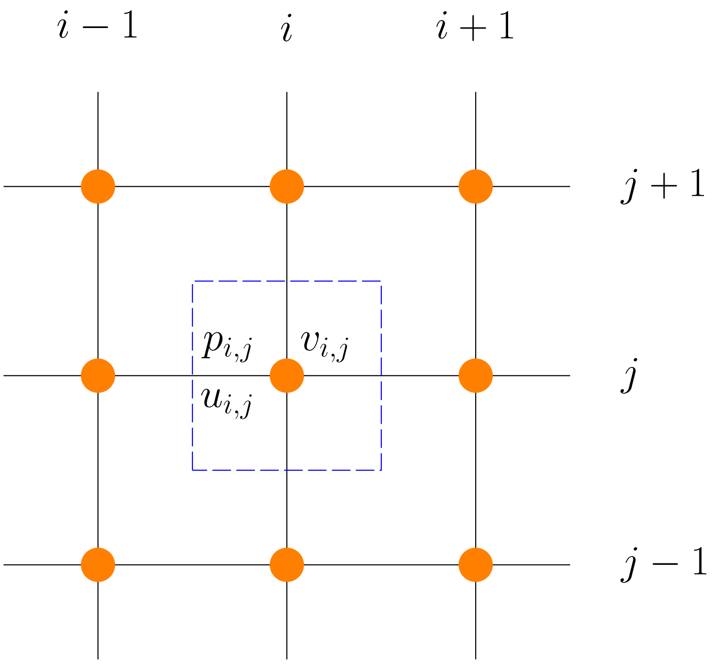
(Second-order accurate discretization.)

Finite-volume or finite-difference discretization of 2D Navier-Stokes equations on a cartesian grid

2D N.-S.- eqs. read:

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0 \\ \dfrac{\partial u}{\partial t}+\dfrac{\partial u^2}{\partial x}+\dfrac{\partial uv}{\partial y}+\dfrac{\partial p}{\partial x}-\dfrac{1}{Re}\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)=0 \\ \dfrac{\partial v}{\partial t}+\dfrac{\partial uv}{\partial x}+\dfrac{\partial v^2}{\partial y}+\dfrac{\partial p}{\partial y}-\dfrac{1}{Re}\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)=0 \end{cases}$$

Consider a co-located cartesian grid:



The discretized equations read as below.

Continuity:

$$\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}+\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}=0,$$

x-momentum:

$$\frac{\partial u_{i,j}}{\partial t}+\frac{u^2_{i+1,j}-u^2_{i-1,j}}{2\Delta x}+\frac{(uv)_{i,j+1}-(uv)_{i,j-1}}{2\Delta y}+\frac{p_{i+1,j}-p_{i-1,j}}{2\Delta x}$$

$$-\frac{1}{Re}\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2}+\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}\right)=0$$

y-momentum:

$$\frac{\partial v_{i,j}}{\partial t}+\frac{(uv)_{i+1,j}-(uv)_{i-1,j}}{2\Delta x}+\frac{v^2_{i,j+1}-v^2_{i,j-1}}{2\Delta y}+\frac{p_{i,j+1}-p_{i,j-1}}{2\Delta y}$$

$$-\frac{1}{Re}\left(\frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{\Delta x^2}+\frac{v_{i,j+1}-2v_{i,j}+v_{i,j-1}}{\Delta y^2}\right)=0$$

Spurious checkerboard modes

If $\Delta x = \Delta y$, a solution of type

$$u_{i,j}=v_{i,j}=(-1)^{i+j}g(t), \quad p_{i,j}=(-1)^{i+j},$$

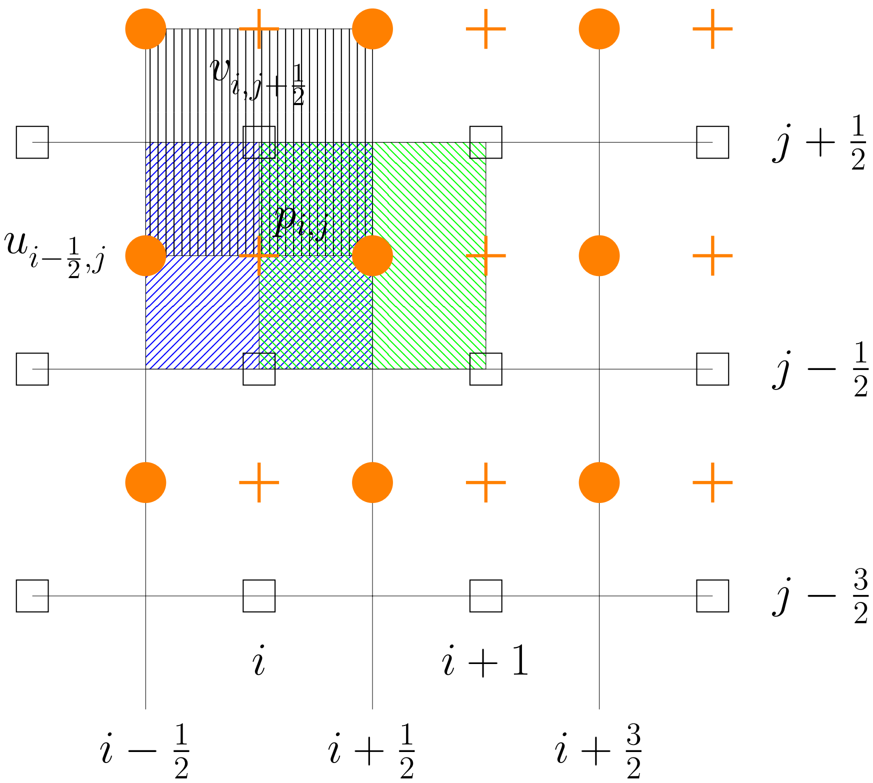
satisfies the divergence-free condition. The momentum equations give

$$\frac{dg}{dt}+\frac{-8t}{Re\Delta x^2}\,g=0 \quad \Rightarrow g=\exp(\frac{-8t}{Re\Delta x^2}).$$

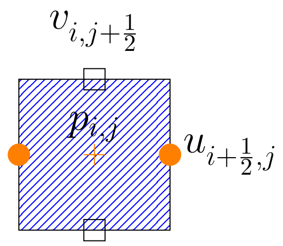
Checkerboard modes decay slowly for velocities and not at all for pressure.

As it can be seen above, every other point appears in all derivatives except for viscous terms. This is called "*even-odd decoupling*". This may be avoided if one-side differences are used for $\nabla\cdot\mathbf{u}$ and ∇p.

FD discretization on a staggered cartesian grid

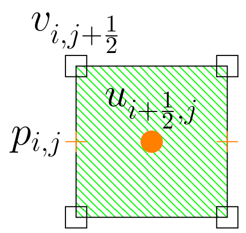


In a staggered grid u, v and p are represented at different locations on the mesh.



The control volume for the continuity equation is centered around the pressure point

$$\frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{\Delta x} + \frac{v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}}}{\Delta y} = 0$$

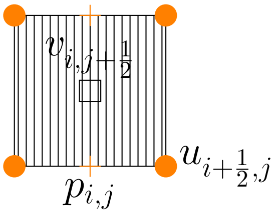


The control volume for the streamwise momentum is centered around the streamwise velocity point

$$\frac{\partial u_{i+\frac{1}{2},j}}{\partial t} + \frac{u^2_{i+1,j} - u^2_{i,j}}{\Delta x} + \frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} + \frac{p_{i+1,j} - p_{i,j}}{\Delta x} -$$

$$\frac{1}{Re}\frac{u_{i+\frac{3}{2},j} - 2u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}}{\Delta x^2} - \frac{1}{Re}\frac{u_{i+\frac{1}{2},j+1} - 2u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j-1}}{\Delta y^2} = 0$$

where the u^2 and (uv) terms need to be interpolated from the points where the corresponding velocities are defined.



The control volume for the normal momentum is centered around the normal velocity point

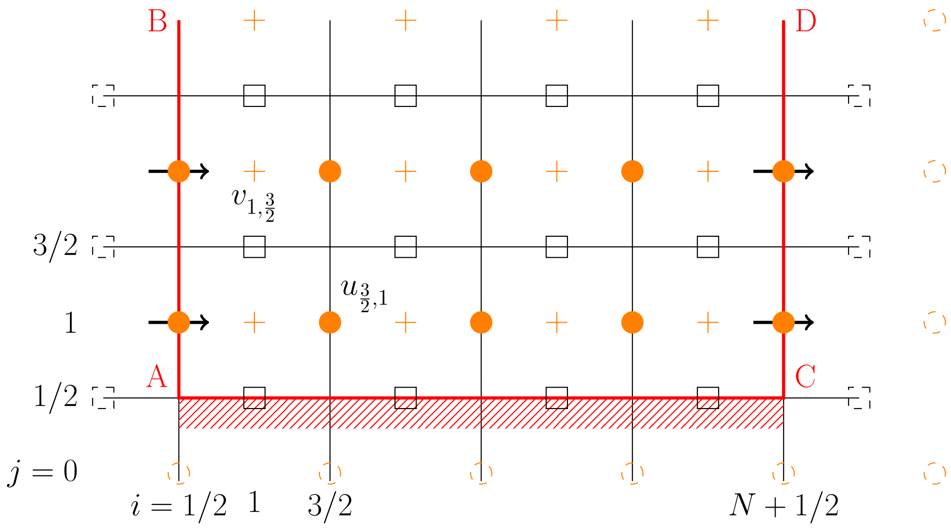
$$\frac{\partial v_{i,j+\frac{1}{2}}}{\partial t} + \frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} + \frac{v^2_{i,j+1} - v^2_{i,j}}{\Delta y} + \frac{p_{i,j+1} - p_{i,j}}{\Delta y} -$$

$$\frac{1}{Re}\frac{v_{i+1,j+\frac{1}{2}} - 2v_{i,j+\frac{1}{2}} + v_{i-1,j+\frac{1}{2}}}{\Delta x^2} - \frac{1}{Re}\frac{v_{i,j+\frac{3}{2}} - 2v_{i,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}}}{\Delta y^2} = 0$$

where the (uv) and v^2 terms need to be interpolated from the points where the corresponding velocities are defined.

For this discretization no checkerboard modes possible, because there is no even-odd decoupling in the divergence constraint, the pressure or the convective terms.

Boundary conditions for a staggered cartesian grid



- Inflow boundary (AB): u and v given

$$u_{\frac{1}{2},1},\ u_{\frac{1}{2},2}, \cdots = \text{known}, \quad v_{\frac{1}{2},\frac{1}{2}}, v_{\frac{1}{2},\frac{3}{2}}, \cdots = \text{known}.$$

Evaluation of y-momentum eq. at $(1, j + \frac{1}{2})$ requites values of $v_{0,j+\frac{1}{2}}$

$$v_{\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}(v_{1,j+\frac{1}{2}} + v_{0,j+\frac{1}{2}}) \Rightarrow \underbrace{v_{0,j+\frac{1}{2}}}_{\text{ghost point}} = 2v_{\frac{1}{2},j+\frac{1}{2}} - v_{1,j+\frac{1}{2}}.$$

- Solid wall (AC): $u = v = 0$

$$u_{\frac{1}{2},\frac{1}{2}},\ u_{\frac{3}{2},\frac{1}{2}}, \cdots = 0, \quad v_{1,\frac{1}{2}}, v_{2,\frac{1}{2}}, \cdots = 0.$$

Evaluation of x-momentum eq. at $(i + \frac{1}{2}, 1)$ requites values of $u_{i+\frac{1}{2},0}$

$$u_{i+\frac{1}{2},\frac{1}{2}} = \frac{1}{2}(u_{i+\frac{1}{2},1} + u_{i+\frac{1}{2},0}) = 0 \Rightarrow u_{i+\frac{1}{2},0} = -u_{i+\frac{1}{2},1}.$$

- Outflow boundary (CD): $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$ evaluated at $i = N + \frac{1}{2}$. Evaluation of x-momentum eq. at $(N + \frac{1}{2}, j)$ requires values of $u_{N+\frac{3}{2},j}$

$$u_{N+\frac{3}{2},j} = u_{N-\frac{1}{2},j}, \quad v_{N+1,j+\frac{1}{2}} = v_{N,j+\frac{1}{2}}.$$

- No reference is made to pressure points outside of domain, no BC for pressure is needed.

Summary of equations:

The discretized equations derived for the staggered grid can be written in the following form

$$\begin{cases} \dfrac{\partial u_{i+\frac{1}{2},j}}{\partial t} + A_{i+\frac{1}{2},j} + \dfrac{p_{i+1,j} - p_{i,j}}{\Delta x} = 0 \\ \dfrac{\partial v_{i,j+\frac{1}{2}}}{\partial t} + B_{i,j+\frac{1}{2}} + \dfrac{p_{i,j+1} - p_{i,j}}{\Delta y} = 0 \\ D_{i,j} = 0 \end{cases}$$

where the expression for the $A_{i+\frac{1}{2},j}$ can be written

$$\begin{aligned}
&A_{i+\frac{1}{2},j} = \\
&= \frac{u_{i+1,j}^2 - u_{i,j}^2}{\Delta x} + \frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} \\
&-\frac{1}{Re}\frac{u_{i+\frac{3}{2},j} - 2u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}}{\Delta x^2} - \frac{1}{Re}\frac{u_{i+\frac{1}{2},j+1} - 2u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j-1}}{\Delta y^2} \\
&= \ \{\text{expand and interpolate using grid values}\} \\
&= a(u,v)_{i+\frac{1}{2},j} u_{i+\frac{1}{2},j} + \\
&\quad +(\cdots)u_{i+\frac{3}{2},j} + (\cdots)u_{i-\frac{1}{2},j} + (\cdots)u_{i+\frac{1}{2},j+1} + (\cdots)u_{i+\frac{1}{2},j-1} \\
&= a(u,v)_{i+\frac{1}{2},j} u_{i+\frac{1}{2},j} + \sum_{nb} a(u,v)_{nb} u_{nb}
\end{aligned}$$

The last line summarizes the expressions, where the sum over nb

indicates a sum over the nearby nodes. The expression for $B_{i,j+\frac{1}{2}}$ can in a similar way be written

$$
\begin{aligned}
&B_{i,j+\frac{1}{2}} = \\
&= \frac{(uv)_{i+\frac{1}{2},j+\frac{1}{2}} - (uv)_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} + \frac{v^2_{i,j+1} - v^2_{i,j}}{\Delta y} \\
&-\frac{1}{R}\frac{v_{i+1,j+\frac{1}{2}} - 2v_{i,j+\frac{1}{2}} + v_{i-1,j+\frac{1}{2}}}{\Delta x^2} - \frac{1}{R}\frac{v_{i,j+\frac{3}{2}} - 2v_{i,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}}}{\Delta y^2} \\
&= \ \{\text{expand and interpolate using grid values}\} \\
&= b(u,v)_{i,j+\frac{1}{2}} v_{i,j+\frac{1}{2}} + \sum_{nb} b(u,v)_{nb} v_{nb}
\end{aligned}
$$

These equations, including the boundary conditions, can be assembled into matrix form. We have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \\ 0 \end{pmatrix} + \begin{pmatrix} A(u,v) & 0 & G_x \\ 0 & B(u,v) & G_y \\ D_x & D_y & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} f_u \\ f_v \\ 0 \end{pmatrix}
$$

Here we we have used the definitions

u	-	vector of unknown streamwise velocities
v	-	vector of unknown normal velocities
p	-	vector of pressure unknowns
$A(u,v)$	-	non-linear operator from advective and viscous terms of u-eq.
$B(u,v)$	-	non-linear operator from advective and viscous terms of v-eq.
G_x	-	linear operator from stremwise pressure gradient
G_y	-	linear operator from normal pressure gradient
D_x	-	linear operator from x-part of divergence constraint
D_y	-	linear operator from y-part of divergence constraint
f	-	vector of source terms from BC

Simpler form of above:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ 0 \end{pmatrix} + \begin{pmatrix} N(u) & G \\ D & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$