

# Finite Difference Schemes

Computational Fluid Dynamics SG2212

## 1 Finite differences for the integration of ODEs

Ordinary differential equation:

$$\frac{du}{dt} = f(u, t), \quad u^n = u(t^n), \quad f^n = f(u^n, t^n), \quad t^n = n\Delta t \quad (1)$$

Explicit Euler scheme, order  $\mathcal{O}(\Delta t)$ :

$$u^{n+1} = u^n + \Delta t \cdot f^n \quad (2)$$

Implicit Euler scheme, order  $\mathcal{O}(\Delta t)$ :

$$u^{n+1} = u^n + \Delta t \cdot f^{n+1} \quad (3)$$

(Generalised) Crank-Nicolson scheme:

$$u^{n+1} = u^n + \Delta t([1 - \theta] \cdot f^n + [\theta] \cdot f^{n+1}), \quad 0 \leq \theta \leq 1 \quad (4)$$

The standard Crank-Nicolson scheme is given by  $\theta = 0.5$  with order  $\mathcal{O}(\Delta t^2)$ ; the explicit and implicit Euler schemes are obtained with  $\theta = 0$  and  $\theta = 1$ , respectively.  
Standard Runge-Kutta scheme (RK4), order  $\mathcal{O}(\Delta t^4)$ :

$$u^{n+1} = u^n + \frac{\Delta t}{6}(f^n + 2k_1 + 2k_2 + k_3) \quad (5)$$

$$\text{with: } u_1 = u^n + \frac{\Delta t}{2}f^n, \quad k_1 = f(u_1, t^{n+\frac{1}{2}}), \quad t^{n+\frac{1}{2}} = t^n + \frac{\Delta t}{2} \quad (6)$$

$$u_2 = u^n + \frac{\Delta t}{2}k_1, \quad k_2 = f(u_2, t^{n+\frac{1}{2}}) \quad (7)$$

$$u_3 = u^n + \Delta t k_2, \quad k_3 = f(u_3, t^{n+1}) \quad (8)$$

## 2 Finite difference formulas for first derivatives

Left-sided finite difference scheme first order:

$$\left. \frac{\partial u}{\partial x} \right|_{x_i} = \frac{u_i - u_{i-1}}{\Delta x} + \frac{\Delta x}{2} \left. \frac{\partial^2 u}{\partial x^2} \right|_{x_i} + \dots \quad (9)$$

Left-sided finite difference scheme second order:

$$\left. \frac{\partial u}{\partial x} \right|_{x_i} = \frac{3u_i - 4u_{i-1} + u_{i-2}}{2\Delta x} + \frac{\Delta x^2}{3} \left. \frac{\partial^3 u}{\partial x^3} \right|_{x_i} + \dots \quad (10)$$

Right-sided finite difference scheme first order:

$$\left. \frac{\partial u}{\partial x} \right|_{x_i} = \frac{u_{i+1} - u_i}{\Delta x} - \frac{\Delta x}{2} \left. \frac{\partial^2 u}{\partial x^2} \right|_{x_i} + \dots \quad (11)$$

Right-sided finite difference scheme second order:

$$\left. \frac{\partial u}{\partial x} \right|_{x_i} = \frac{-3u_i + 4u_{i+1} - u_{i+2}}{2\Delta x} + \frac{\Delta x^2}{3} \left. \frac{\partial^3 u}{\partial x^3} \right|_{x_i} + \dots \quad (12)$$

Central finite difference scheme second order:

$$\left. \frac{\partial u}{\partial x} \right|_{x_i} = \frac{u_{i+1} - u_{i-1}}{2\Delta x} - \frac{\Delta x^2}{6} \left. \frac{\partial^3 u}{\partial x^3} \right|_{x_i} + \dots \quad (13)$$

Central finite difference scheme fourth order:

$$\left. \frac{\partial u}{\partial x} \right|_{x_i} = \frac{-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}}{12\Delta x} + \frac{\Delta x^4}{30} \left. \frac{\partial^5 u}{\partial x^5} \right|_{x_i} + \dots \quad (14)$$

### 3 Finite difference formulas for second derivatives

Left-sided finite difference scheme first order:

$$\left. \frac{\partial^2 u}{\partial x^2} \right|_{x_i} = \frac{u_i - 2u_{i-1} + u_{i-2}}{\Delta x^2} + \Delta x \left. \frac{\partial^3 u}{\partial x^3} \right|_{x_i} + \dots \quad (15)$$

Left-sided finite difference scheme second order:

$$\left. \frac{\partial^2 u}{\partial x^2} \right|_{x_i} = \frac{2u_i - 5u_{i-1} + 4u_{i-2} - u_{i-3}}{\Delta x^2} - \frac{11\Delta x^2}{12} \left. \frac{\partial^4 u}{\partial x^4} \right|_{x_i} + \dots \quad (16)$$

Right-sided finite difference scheme first order:

$$\left. \frac{\partial^2 u}{\partial x^2} \right|_{x_i} = \frac{u_{i+2} - 2u_{i+1} + u_i}{\Delta x^2} - \Delta x \left. \frac{\partial^3 u}{\partial x^3} \right|_{x_i} + \dots \quad (17)$$

Right-sided finite difference scheme second order:

$$\left. \frac{\partial^2 u}{\partial x^2} \right|_{x_i} = \frac{2u_i - 5u_{i+1} + 4u_{i+2} - u_{i+3}}{\Delta x^2} + \frac{11\Delta x^2}{12} \left. \frac{\partial^4 u}{\partial x^4} \right|_{x_i} + \dots \quad (18)$$

Central finite difference scheme second order:

$$\left. \frac{\partial^2 u}{\partial x^2} \right|_{x_i} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} - \frac{\Delta x^2}{12} \left. \frac{\partial^4 u}{\partial x^4} \right|_{x_i} + \dots \quad (19)$$

Central finite difference scheme fourth order:

$$\left. \frac{\partial^2 u}{\partial x^2} \right|_{x_i} = \frac{-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}}{12\Delta x^2} + \frac{\Delta x^4}{90} \left. \frac{\partial^6 u}{\partial x^6} \right|_{x_i} + \dots \quad (20)$$

### 4 Finite difference formulas for third derivatives

Central finite difference scheme second order:

$$\left. \frac{\partial^3 u}{\partial x^3} \right|_{x_i} = \frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2\Delta x^3} - \frac{\Delta x^2}{4} \left. \frac{\partial^5 u}{\partial x^5} \right|_{x_i} + \dots \quad (21)$$

Central finite difference scheme fourth order:

$$\left. \frac{\partial^3 u}{\partial x^3} \right|_{x_i} = \frac{-u_{i+3} + 8u_{i+2} - 13u_{i+1} + 13u_{i-1} - 8u_{i-2} + u_{i-3}}{8\Delta x^3} + \frac{7\Delta x^4}{120} \left. \frac{\partial^7 u}{\partial x^7} \right|_{x_i} + \dots \quad (22)$$

### 5 Finite difference formulas for fourth derivatives

Central finite difference scheme second order:

$$\left. \frac{\partial^4 u}{\partial x^4} \right|_{x_i} = \frac{u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}}{\Delta x^4} - \frac{\Delta x^2}{6} \left. \frac{\partial^6 u}{\partial x^6} \right|_{x_i} + \dots \quad (23)$$