Homework 3

Discretisation errors

due February 8, 23:59

Guidelines:

- Use only a **single pdf**. Preferably scan your homework, if you take a photo, make sure it is sharp and bright;
- **Include all plots in the pdf** in the right place;
- Do not present plots without commenting them: always write a (short) description of what the plot tells you and what you can conclude from it;
- You can work in groups up to 3;
- If you work in groups:
  - write the names of all group members at the beginning of the report;
  - you can work and discuss together but each group member is required to submit an individual report written with his/her own words; **copy-paste reports will not be accepted**
- Please use the following naming convention: "**surname_hwX.pdf**" and include all Matlab files as a single **separate** archive: "**surname_hwX.zip**". X is the number of the homework.

Task 1: Modified wavenumber

On an equidistant grid, the finite-difference derivative of a Fourier mode e^{ikx} can be found by multiplying the function value on each node with the so-called modified wavenumber $\tilde{k}(k)$.

To better understand this concept, we compare the matrix representation and the modified wavenumber representation of a finite-difference scheme. Let's consider a periodic function

$$f(x + pm) = f(x), \quad m \in \mathbb{Z}.$$

where p is the period length. Let the vector $\underline{f}$ be the discrete representation of $f(x)$ on the equidistant grid where $x_j := j\Delta x$, $\Delta x := p/N$, $j = 0, 1, \dots, N-1$,

$$\underline{f} := [f_0, f_1, \dots, f_{N-1}]^\top, \quad f_0 = f_N, \quad \text{where } f_j := f(x_j).$$

For this task consider $p = 2\pi$ and $N = 20$.

a) A first-order finite-difference discretisation of the derivative $f'(x)$ can be written as

$$\underline{f}'_{num} := [\delta f_0, \delta f_1, \dots, \delta f_{N-1}]^\top = \underline{\underline{D}}\ \underline{f},$$

where

$$\delta f_j := \frac{f_{j+1} - f_j}{\Delta x}.$$

Use MATLAB to assemble the system matrix $\underline{\underline{D}}$ (remember that $f_0 = f_N$). Include $\underline{\underline{D}}$ in the written report.

b) Consider $f(x) = e^{ikx}$ and derive the expression for the modified wavenumber $\tilde{k}$ for the finite-difference scheme. Non-dimensionalise the wavenumber with the grid spacing, *i.e.* derive the expression for $\tilde{k}\Delta x$.

c) From now on assume that $k = 3$ (*i.e.* consider a specific Fourier mode). Compute the derivative in a discrete (δf_j) and analytical ($f'(x)|_{x=x_j}$) manner at every grid point. Use the previously defined $\underline{\underline{D}}$ for the discrete derivative. Plot the real part for both the numerical and the analytical derivative as a function of x, then comment on their differences. (*Hint:* use a higher value of N to obtain smoother curves)

d) Compute the vector $\underline{\mu}$ with the elements

$$\mu_j := \frac{\delta f_j}{f_j}$$

and compare it with the complex number $i\tilde{k}$, where $\tilde{k}$ is the modified wavenumber for the right-sided finite differences as derived in b). Does this result confirm that the finite-difference derivative of a Fourier mode e^{ikx} can be found by multiplying the function by the modified wavenumber, *i.e.* does $i\tilde{k}\underline{f} = \underline{\underline{D}}\ \underline{f}$ hold?

Task 2: Dissipative and dispersion error

In this task we will examine the dissipative and dispersion error introduced by a first-order and a second-order numerical scheme. Let us consider the advection equation,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0\ . \tag{1}$$

Question 1: Discretise the advection equation employing the following schemes:

- first-order forward in time and backward in space (*upwind* scheme, FTBS);
- second-order central difference for both time and space (*leapfrog* scheme):

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0 \tag{2}$$

Perform for both the von Neumann analysis and write the real and imaginary part of the amplification factor per wavenumber, $\hat{G}_k$.
(*Hint:* to compute $\hat{G}_k$ for the leapfrog scheme, note that: $\hat{u}^{n+1}/\hat{u}^n = \hat{u}^n/\hat{u}^{n-1}$)

Question 2: We will employ the von Neumann analysis to investigate the relation between amplitude (or **dissipative**) and phase (or **dispersion**) error. For both the numerical and the exact solution, denoted $\hat{\tilde{u}}$, it is possible to write

$$\hat{u}^n = \hat{G}\hat{u}^{n-1} = \hat{G}^2\hat{u}^{n-2} = \cdots = \hat{G}^n\hat{u}^0 \tag{3}$$

$$\hat{\tilde{u}}^n = \hat{\tilde{G}}\hat{\tilde{u}}^{n-1} = \hat{\tilde{G}}^2\hat{\tilde{u}}^{n-2} = \cdots = \hat{\tilde{G}}^n\hat{\tilde{u}}^0\ . \tag{4}$$

Thus, it is possible to define an amplitude and a phase error as ratio between, respectively, the amplitude difference and the phase difference for each wave number k for the discretised and the exact differential equation

$$\text{amplitude error: } \varepsilon_A = \frac{|\hat{G}_k|}{|\hat{\tilde{G}}_k|} \qquad \text{phase error: } \varepsilon_P = \frac{\Phi}{\tilde{\Phi}}\ , \tag{5}$$

where $|\hat{G}_k|$ is the module of the amplification factor ($|\hat{G}_k| = \sqrt{\Im(\hat{G}_k)^2 + \Re(\hat{G}_k)^2}$) and $\tan\Phi = -\Im(\hat{G}_k)/\Re(\hat{G}_k)$.
Show that the exact amplification factor for the advection equation is

$$\hat{\tilde{G}}_k = e^{-i\sigma\phi} \qquad \text{where} \quad \phi = k\Delta x \quad \text{and} \quad \sigma = c\Delta t/\Delta x\ . \tag{6}$$

(*Hint:* What is the analytic solution for a generic initial condition? Use it for each Fourier mode).
Then, compute ε_A and ε_P for the upwind and the leapfrog scheme and use this result to discuss qualitatively how a single sinusoidal wave will be affected by the two discretisation schemes.

You can compare the behaviour of the amplitude and phase error for the first-order scheme and the phase error for the second-order scheme with the solution given in Figure 1.

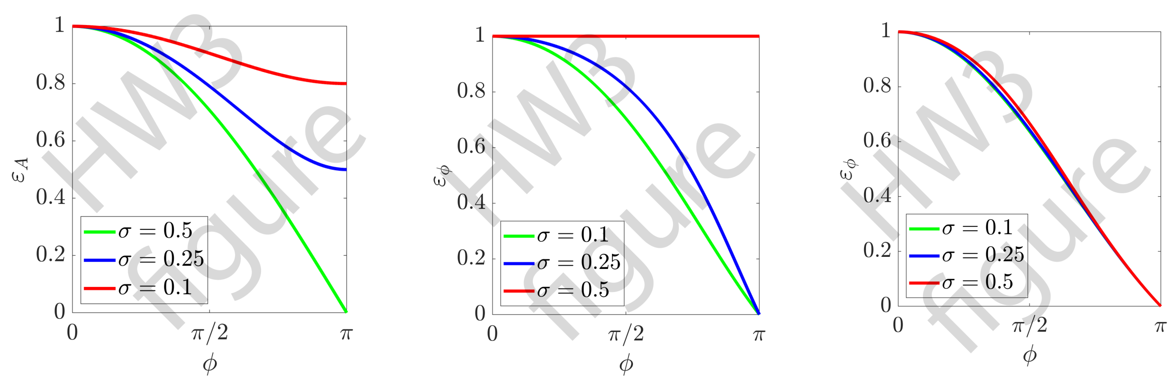

Figure 1: (Left) Dissipative and (centre) dispersive error for the first-order scheme and (right) dispersive error for the second-order scheme, as function of $\phi = k\Delta x$ for different values of $\sigma = c\Delta t/\Delta x$.