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Examination
SF2561 and FSF3561 The Finite Element Method
2020-10-15, 8.00-20.00

Total 50p: 20p for grade E, 25p for grade D, 30p for grade C, 35p for grade B, and 40p for grade A.

Problem 1

Consider the problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega = \{(x_1, x_2) : -1 < x_1 < 2, 0 < x_2 < 2\}$, and

$$f = \begin{cases} 1 & x_1 \leq 0, \\ 2 & x_1 > 0. \end{cases}$$

a) Approximate (1) using first order Lagrange elements ($P1$-FEM). State the variational form and the finite element approximation. Given the triangulation in Figure 1, derive the discrete system $A\xi = b$, where A is the stiffness matrix and b is the load vector. (8p)

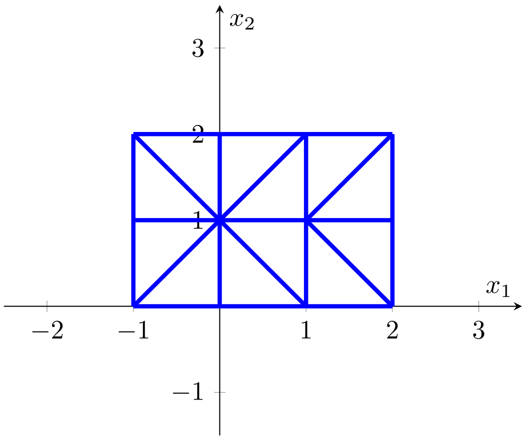

Figure 1: Mesh for Problem 1

b) Consider the same problems as in a), but with the Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on the part of the boundary where $x_1 = 2$. (Use the Dirichlet condition $u = 0$ at the remaining parts of the boundary $\partial\Omega \setminus \{x_1 = 2\}$). (4p)

Problem 2

Let $\Omega \subset \mathbb{R}^d$ be a convex domain. Consider the convection-diffusion problem

$$\begin{cases} -\Delta u + b \cdot \nabla u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $b \in \mathbb{R}^d$ is a constant and $f \in L^2(\Omega)$.

a) Derive the variational form and prove that there exists a unique solution (to the variational form). (6p)

b) Formulate the first order Lagrange finite element method (P1-FEM) for (2). Derive an a priori bound for the error. (6p)

Problem 3

Consider the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0, & \text{in } \Omega \times (0,T], \\ u = 0, & \text{on } \partial\Omega \times (0,T], \\ u(\cdot,0) = u_0, & \text{in } \Omega, \end{cases} \tag{3}$$

where $u_0 \in L^2(\Omega)$ only. Prove the following stability estimates

$$\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u\|^2_{L^2(\Omega)} \, \mathrm{d}s \le \|u_0\|^2_{L^2(\Omega)},$$

$$\int_0^t s \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} \mathrm{d}s + t\|\nabla u\|^2_{L^2(\Omega)} \le C\|u_0\|^2_{L^2(\Omega)}.$$

Discuss the behavior of the solution to (3) close to zero. *Hint:* For the second estimate, multiply the equation by $t\frac{\partial u}{\partial t}$. (12p)

Problem 4

A Robin boundary problem is given by

$$\begin{cases} -\nabla \cdot (a\nabla u) = f, & \text{in } \Omega, \\ a\nabla u \cdot n + b(u-g) = k, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^d$ is a convex domain, a, f, b, g, k are smooth functions, and $a \ge a_0 > 0$.

a) Formulate the first order Lagrange finite element method (P1-FEM) for (4). (2p)

b) Prove the following a posteriori error bound in L^2-norm

$$\|u - u_h\|_{L^2(\Omega)} \le C \left(\sum_{K \in \mathcal{T}_h} R_K(u_h)^2 \right)^{1/2},$$

where

$$R_K(u_h) = h_K^2 \| -\nabla \cdot (a\nabla u_h) - f\|_{L_2(K)} + h_K^{3/2} \|a[n \cdot \nabla u_h]\|_{L^2(\partial K \setminus \partial\Omega)}$$
$$+ h_K^{3/2} \|a\nabla u_h \cdot n + b(u_h - g) - k\|_{L^2(\partial K \cap \partial\Omega)}.$$

Hint: Let $e = u_h - u$ and use the dual problem $-\nabla \cdot (a\nabla\Phi) = e$ in Ω and $a\nabla\Phi \cdot n + b\Phi = 0$ on $\partial\Omega$. You may also want to use the (scaled) trace inequality $\|w\|_{L^2(\partial K)} \le C(h_K^{-1/2}\|w\|_{L^2(K)} + h_K^{1/2}\|\nabla w\|_{L^2(K)})$. (10p)

c) Give an interpretation of each term in $R_K(u_h)$. (2p)