Homework 6 Classification of differential equations and shock tube due march

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Task 1: Classification

a)

$$\underline{\underline{A}}\frac{\partial u}{\partial x}+\underline{\underline{B}}\frac{\partial u}{\partial y}=0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ u & 0 & 1 \\ 0 & u & 0 \end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial p}{\partial x} \end{bmatrix}+\begin{bmatrix} 0 & 1 & 0 \\ v & 0 & 0 \\ 0 & v & 1 \end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \\ \frac{\partial p}{\partial y} \end{bmatrix}=0 \tag{0.1}$$

$$\underline{\underline{A}}=\begin{bmatrix} 1 & 0 & 0 \\ u & 0 & 1 \\ 0 & u & 0 \end{bmatrix}$$

$$\underline{\underline{B}}=\begin{bmatrix} 0 & 1 & 0 \\ v & 0 & 0 \\ 0 & v & 1 \end{bmatrix}$$

b)

$$\begin{aligned} \underline{\underline{B}}-\lambda\underline{\underline{A}} &= \begin{bmatrix} -\lambda & 1 & 0 \\ v-\lambda u & 0 & -\lambda \\ 0 & v-\lambda u & 1 \end{bmatrix}=0 \\ &= -\lambda\times\lambda\times(v-\lambda u)-(v-\lambda u) \\ &= (v-\lambda u)(1+\lambda^2) \\ \lambda 1 &= i \\ \lambda 2 &= -i \\ \lambda 3 &= \frac{v}{u} \end{aligned} \tag{0.2}$$

Since eigenvalues are both real and complex, it is the mixed PDEs type. The results could be expected as marching properties. And all boundary should be know to solve this kind of equations.

Task 2: Shock tube

a)

$$u_t+f_x=0 \qquad f=au \tag{0.3}$$

$$
\begin{aligned}
&u_j^* = u_j^n - \lambda(f(u_{j+1}^n) - f(u_j^n)) \\
&u_j^{n+1} = \frac{1}{2}(u_j^n + u_j^*) - \frac{1}{2}\lambda(f(u_j^*) - f(u_{j-1}^*)) \\
&\sigma = \left| a\frac{\Delta t}{\Delta x} \right| \le 1
\end{aligned}
\tag{0.4}
$$

For the linear advection equation, or this advection equation, the MacCormach scheme is identical to the Lax-Wendroff scheme, second order scheme in time and space.

b)

$$
\begin{aligned}
&\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ K\gamma\rho^{\gamma-2} & u \end{pmatrix}\begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0 \\
&\Rightarrow \begin{pmatrix} u & \rho \\ K\gamma\rho^{\gamma-2} & u \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0 \Rightarrow \begin{vmatrix} u-\lambda & \rho \\ \dfrac{c^2}{\rho} & u-\lambda \end{vmatrix} = 0 \\
&(u-\lambda)^2 - c^2 = 0 \Rightarrow \lambda = u \pm c
\end{aligned}
\tag{0.5}
$$

c)

analytical shock speed s is

$$
s = \frac{\rho_L u_L - \rho_R u_R}{\rho_L - \rho_R} = \frac{0.5336*249 - 0.2233*0.4162}{0.5336 - 0.2233} = 427.8 \text{ m/s}
\tag{0.6}
$$

All the values are taken from the figure 1. While for the measured shock speed:
t=50 time steps * dt=50*6.1307e-5=3.0653e-3 s
x=2.775-1.5=1.275 m
v=415 m/s
These two shock speeds are nearly the same.

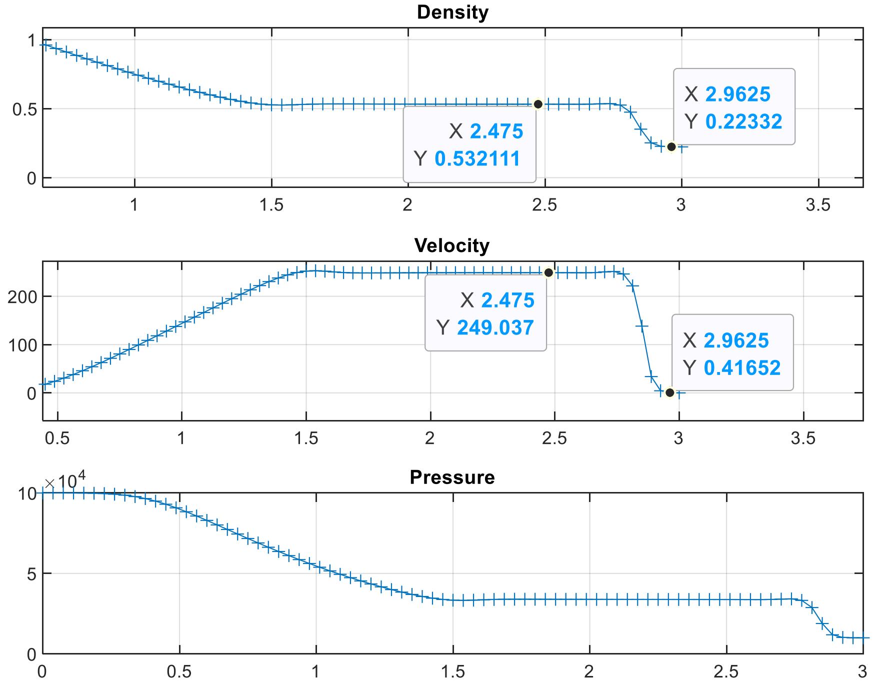

Figure 1 Computed states of shock, where are sensity and speed comes from to calculate the analytical shock speed.

d)
different Cn

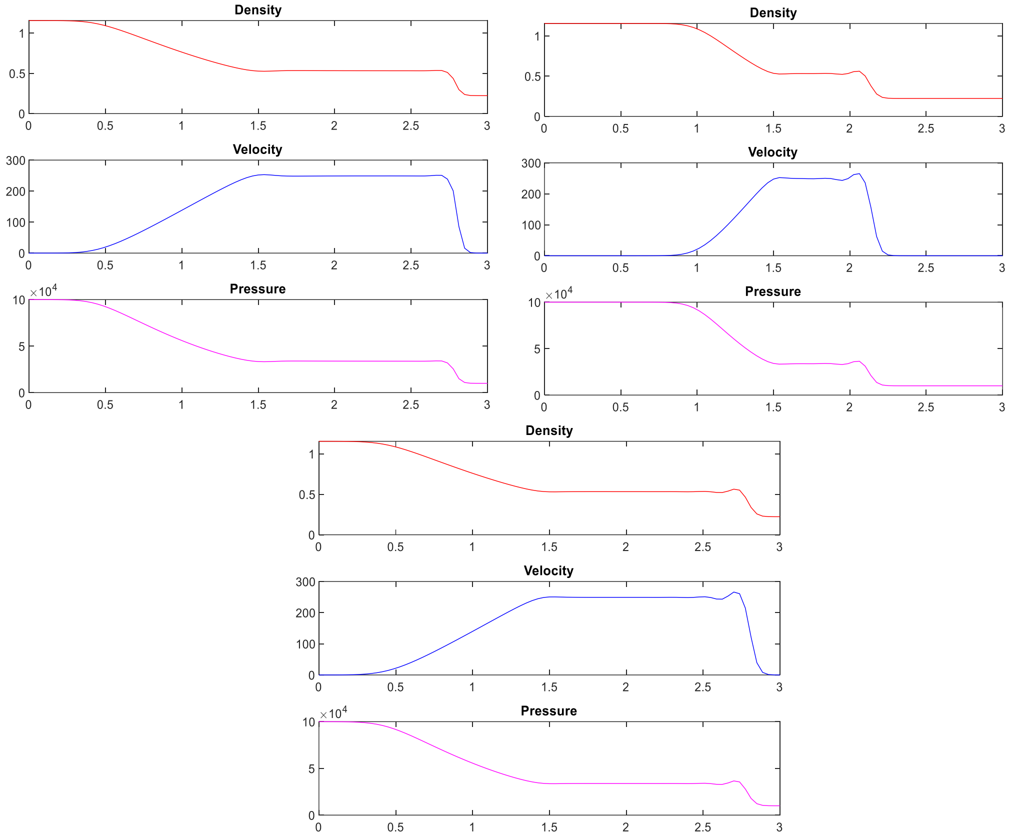

Figure 2   a: Cn=0.9 Nstp=50   b) Cn=0.45 Nstp=50   c) Cn=0.45 Nstp=100

As shown in figure 2 is different Cn. The final time tf is nearly 0.003 s. For case b, Cn=0.45 and Nstp=50, the final time tf is 0.0015 s. The reason is Cn decreased to half so the dt decreased to half, thus the total final time is half of 0.003s. As shown in case 3, when the Nstp increase 2 times to 100, the final time tf is 0.003s and the shock is the nearly the same. But for case c, in the edge of the shock, there is small bump, since the dt is smaller than case a, the result in case c is more accurate. We could say small Cn means high accurate.

Different C0 and C2   Cn=0.9 Nstp=50

As show in figure 3, different C0 and C2. Through the comparison of C0=0.05 C2=0.05 and C0=0.45, C2=0.05, it could be concluded that when C0 is small, the shock shape is sharp in the edge, while C0 is large could induce a smooth edge of shock. The reason is Co is a background diffusion parameter, large C0 means large artificial viscosity and the shock pressure diffusion faster, so no large difference of velocity and pressure in the shock edges.

As for the C2, it seems there are not very significant influence on the shock shape. While for the large density filed, it could damp the oscillations. As shown in C0=0.05 C2=0.4 and C0=0.05 C2=0.05, for the velocity, in the large density derivative area, large C2 has smooth velocity. **Above all, the C0=0.05, C2=0.25 is the best case.**

As for the tf=0.01 for the last case, after the shock is reflected, it could see that at the large density derivative area, the oscillations increase which could be seen as a large bump compares with previous tf=0.003.

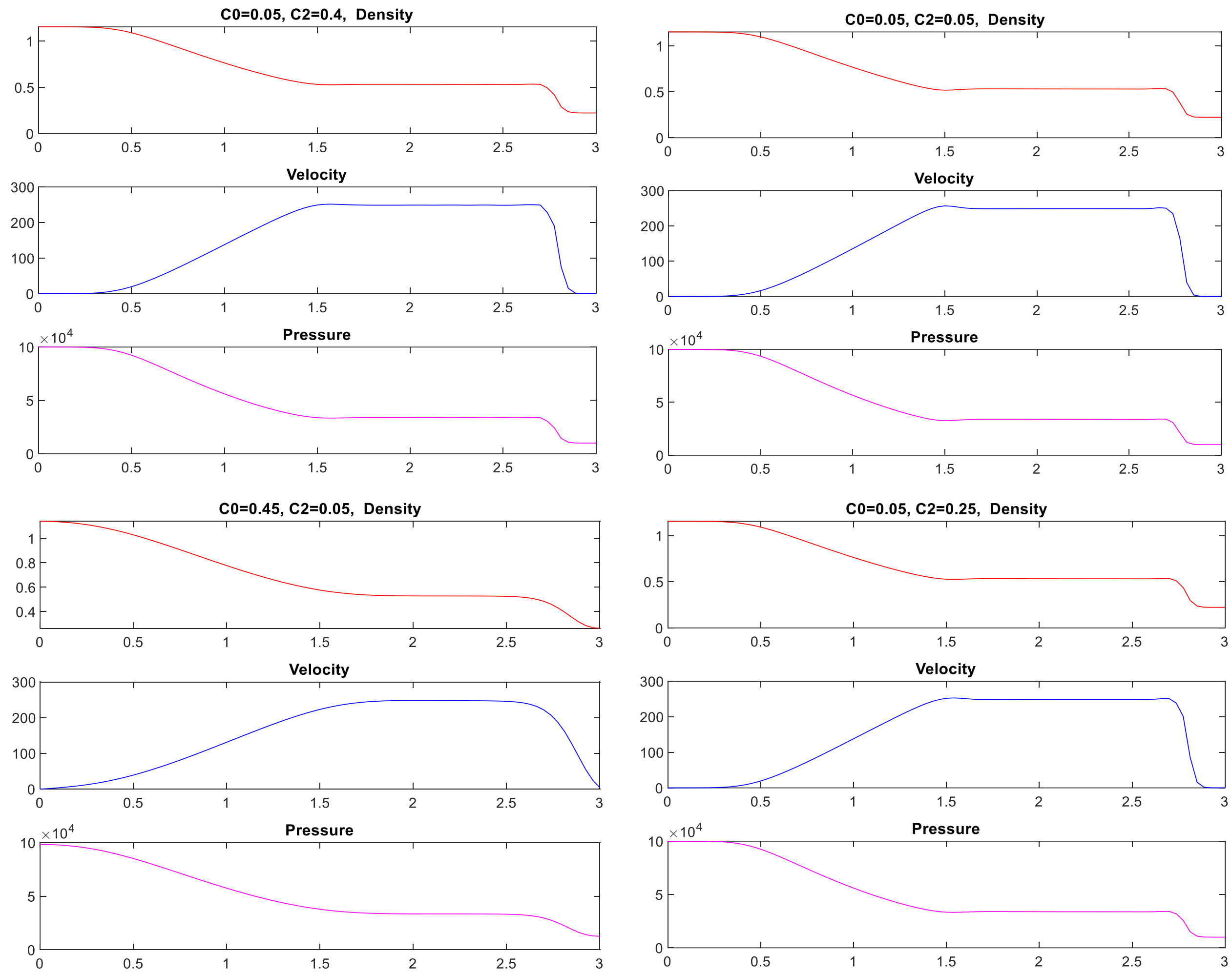

Figure 3 Different C0 and C2

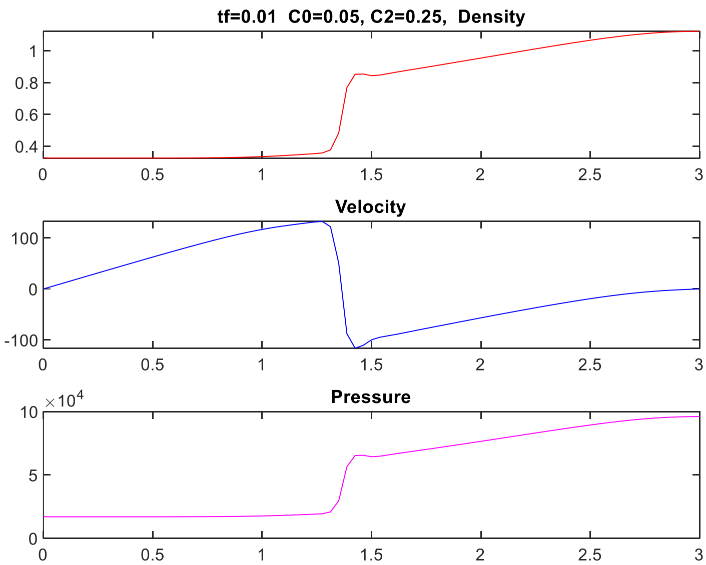

Figure 4 Tf=0.01 for C0=0.05 C2=0.25