## Solution algorithms for incompressible Navier-Stokes equations

There are no time derivatives or pressure terms presented in continuity equation $\Rightarrow$ Numerical methods for evolution equations cannot directly be applied as in compressible flows.

An example of difficulties: Using MacCormack's scheme for discretization yields:

$$\Delta t \leq \frac{1}{|u|/\Delta x + |v|/\Delta y + a\sqrt{1/\Delta x^2 + 1/\Delta y^2}}$$

where '$a$' is speed of sound. For incompressible flow '$a$' is theoretically infinite $\Rightarrow \Delta t \rightarrow 0$.

In general, 'compressible' codes have difficulty to converge for low Mach numbers and there is no guarantee that the solution converges to the correct result for incompressible flow. (A cure is to use pre-conditioning.)

Some methods to solve Navier-Stokes equations for incompressible flows

- Stream function-vorticity formulation (originally for 2D flows)
- Artificial compressibility (originally derived for stationary flow)
- Pressure-correction (projection) methods.

## Stream function-vorticity formulation

The vorticity-stream function approach has been a popular methods for solving the 2-D incompressible Navier-Stokes equations, which also has been extended to three-dimensional by using an *extended* definition of stream function.

Introduce the stream function $\psi$ (2D case):

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

This implies that continuity equation is automatically satisfied:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = 0$$

by taking the curl ($\nabla\times$) of the momentum equations we find a transport equation for the vorticity:

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

where

$$\omega = \nabla \times \mathbf{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

In this way the pressure terms are eliminated ($\nabla \times \nabla p = 0$).

Using definition of the stream function in the expression for the vorticity we get a Poisson equation for $\psi$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega.$$

Replacing velocity with derivatives of stream function results in

$$\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \qquad (*)$$

Equation above together with vorticity transport equation and definition of stream function close the system. The above set of equations are then solved as following:

1. initilaize the vorticity field $\omega$ at $t = 0$,
2. find $\psi$ by solving $\nabla^2\psi = -\omega$,
3. solve equation $(*)$ to find $\omega$ at the next time step,
4. go to step 2.

Note that the above equations should be computed with associated boundary conditions, which we are not going to discuss here.

## Artificial compressibility (steady flows)

This method has originally been developed for steady flows $(\partial/\partial t = 0)$. Here, the idea is to, in an artificial way, introduce an evolution equation for pressure.

Let us introduce an artificial density $\rho'$ and artificial time $t'$ such that $\partial\rho'/\partial t' \to 0$, and write the continuity equation as

$$\frac{\partial \rho'}{\partial t'} + \frac{\partial u_i}{\partial x_i} = 0.$$

Let $p = \beta\rho'$ be artificial eq. of state.

$$\text{N.-S. eqs.} \Rightarrow \begin{cases} \dfrac{\partial p}{\partial t'} + \beta \dfrac{\partial u_i}{\partial x_i} = 0 \\[2ex] \dfrac{\partial u_i}{\partial t'} + \dfrac{\partial}{\partial x_j}(u_j u_i) = -\dfrac{\partial p}{\partial x_i} + \dfrac{1}{Re}\nabla^2 u_i. \end{cases}$$

When these equations are integrated in time, the solution should converge to the steady state where $\partial/\partial t' = 0$, which corresponds to the steady incompressible flow motion.

Now, to investigate characteristics of these equations, we rewrite the above equations as

$$\frac{\partial \underset{\sim}{\boldsymbol{u}}}{\partial t} + \frac{\partial \underset{\sim}{\boldsymbol{e}}}{\partial x} + \frac{\partial \underset{\sim}{\boldsymbol{f}}}{\partial y} + \frac{\partial \underset{\sim}{\boldsymbol{g}}}{\partial z} = \frac{1}{Re}\nabla^2 \underset{\approx}{\mathbf{D}} \underset{\sim}{\boldsymbol{u}}$$

where

$$\underset{\sim}{\boldsymbol{u}} = \begin{bmatrix} p \\ u \\ v \\ w \end{bmatrix} \qquad \underset{\sim}{\boldsymbol{e}} = \begin{bmatrix} \beta u \\ p+u^2 \\ uv \\ uw \end{bmatrix} \qquad \underset{\sim}{\boldsymbol{f}} = \begin{bmatrix} \beta v \\ uv \\ p+v^2 \\ vw \end{bmatrix} \qquad \underset{\sim}{\boldsymbol{g}} = \begin{bmatrix} \beta w \\ uw \\ vw \\ p+w^2 \end{bmatrix}$$

$$\underset{\approx}{\boldsymbol{D}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let us introduce matrices $\underset{\approx}{\boldsymbol{A}}$, $\underset{\approx}{\boldsymbol{B}}$ and $\underset{\approx}{\boldsymbol{C}}$ as

$$\underset{\approx}{\boldsymbol{A}} = \frac{\partial \underset{\sim}{\boldsymbol{e}}}{\partial \underset{\sim}{\boldsymbol{u}}} = \begin{bmatrix} 0 & \beta & 0 & 0 \\ 1 & 2u & 0 & 0 \\ 0 & v & u & 0 \\ 0 & w & 0 & u \end{bmatrix}, \qquad \left(\frac{\partial \underset{\sim}{\boldsymbol{e}}}{\partial \underset{\sim}{\boldsymbol{u}}}\right)_{i,j} = \frac{\partial \underset{\sim}{\boldsymbol{e}}_i}{\partial \underset{\sim}{\boldsymbol{u}}_j}$$

$$\underset{\approx}{\boldsymbol{B}} = \frac{\partial \underset{\sim}{\boldsymbol{f}}}{\partial \underset{\sim}{\boldsymbol{u}}} = \begin{bmatrix} 0 & 0 & \beta & 0 \\ 0 & v & u & 0 \\ 1 & 0 & 2v & 0 \\ 0 & 0 & w & v \end{bmatrix}, \qquad \underset{\approx}{\boldsymbol{C}} = \frac{\partial \underset{\sim}{\boldsymbol{g}}}{\partial \underset{\sim}{\boldsymbol{u}}} = \begin{bmatrix} 0 & 0 & 0 & \beta \\ 0 & w & 0 & u \\ 0 & 0 & w & v \\ 1 & 0 & 0 & 2w \end{bmatrix}.$$

Then we can write

$$\Rightarrow \qquad \frac{\partial \underset{\sim}{\boldsymbol{u}}}{\partial t} + \underset{\approx}{\boldsymbol{A}} \frac{\partial \underset{\sim}{\boldsymbol{u}}}{\partial x} + \underset{\approx}{\boldsymbol{B}} \frac{\partial \underset{\sim}{\boldsymbol{u}}}{\partial y} + \underset{\approx}{\boldsymbol{C}} \frac{\partial \underset{\sim}{\boldsymbol{u}}}{\partial z} = \frac{1}{Re} \nabla^2 \underset{\approx}{\boldsymbol{D}} \underset{\sim}{\boldsymbol{u}}$$

The eigenvalues of $\underset{\approx}{\boldsymbol{A}}, \underset{\approx}{\boldsymbol{B}}, \underset{\approx}{\boldsymbol{C}}$ are the wave speeds of plane waves in the respective coordinate directions, they are

$$\left(u, u, u \pm \sqrt{u^2+\beta}\right), \ \left(v, v, v \pm \sqrt{v^2+\beta}\right) \text{ and } \left(w, w, w \pm \sqrt{w^2+\beta}\right)$$

$\sqrt{\beta}$ is the artificial speed of sound (compare to $a = \sqrt{(\partial p/\partial \rho)_s}$). Effective acoustic wave speed is $\sqrt{u_i^2 + \beta}$.

Typical values of $\beta$ are $0.1 - 10$ (if too small solution will not converge, if too big causes stiffness problem).

**Projection on a divergence-free space. ("Helmholtz-Hodge" decomposition)**

Let $\Omega$ be a region in space with smooth boundary $\delta\Omega$.

**Theorem 1** *Any $w_i$ in $\Omega$ can uniquely be decomposed into*

$$w_i = u_i + \frac{\partial p}{\partial x_i}$$

$$where \quad \frac{\partial u_i}{\partial x_i} = 0, \quad u_i \cdot n_i = 0 \quad on \quad \partial\Omega.$$

*i.e. into a function $u_i$ that is divergence free and parallel to the boundary and the gradient of a function, here called $p$.*

*Proof:*

1- Stablish the orthogonality relation:

$$\left\langle u_i, \frac{\partial p}{\partial x_i} \right\rangle = \int\limits_{\Omega} u_i \frac{\partial p}{\partial x_i} \mathrm{dV} = \left\{ \frac{\partial u_i}{\partial x_i} = 0 \right\} = \int\limits_{\Omega} \frac{\partial}{\partial x_i} (u_i p) \, \mathrm{dV}$$
$$= \oint\limits_{\partial\Omega} p \underbrace{u_i n_i}_{=0} \mathrm{d}S = 0$$

2- Uniqueness:

Let

$$w_i = u_i^{(1)} + \frac{\partial p^{(1)}}{\partial x_i} = u_i^{(2)} + \frac{\partial p^{(2)}}{\partial x_i}$$
$$\Rightarrow u_i^{(1)} - u_i^{(2)} + \frac{\partial}{\partial x_i} \left( p^{(1)} - p^{(2)} \right) = 0 \quad (\star)$$

The inner product between $(\star)$ and $u_i^{(1)} - u_i^{(2)}$ gives

$$0 = \int\limits_{\Omega} \left[ \left( u_i^{(1)} - u_i^{(2)} \right)^2 + \left( u_i^{(1)} - u_i^{(2)} \right) \frac{\partial}{\partial x_i} \left( p^{(1)} - p^{(2)} \right) \right] \mathrm{dV} = 0$$

$$\left\{ \int\limits_{\Omega} u_i^{(k)} \frac{\partial p^{(m)}}{\partial x_i} \mathrm{dV} = 0 \right\} \Rightarrow \int\limits_{\Omega} \left( u_i^{(1)} - u_i^{(2)} \right)^2 \mathrm{dV} = 0$$

$$\Rightarrow u_i^{(1)} = u_i^{(2)}. \quad (\star\star)$$

$$(\star) \text{ and } (\star\star) \Rightarrow \frac{\partial p}{\partial x_i}^{(1)} = \frac{\partial p}{\partial x_i}^{(2)} \Rightarrow p^{(1)} = p^{(2)} + C$$

3- Existence:

To find $p$ use properties of $u_i$

$$\begin{cases} \dfrac{\partial u_i}{\partial x_i} = \dfrac{\partial w_i}{\partial x_i} - \dfrac{\partial^2 p}{\partial x_i \partial x_i} = 0 \Rightarrow \nabla^2 p = \dfrac{\partial w_i}{\partial x_i} \\ \\ u_i n_i = w_i n_i - \dfrac{\partial p}{\partial n} = 0 \Rightarrow \dfrac{\partial p}{\partial n} = w_i n_i \text{ on } \partial\Omega \end{cases}$$

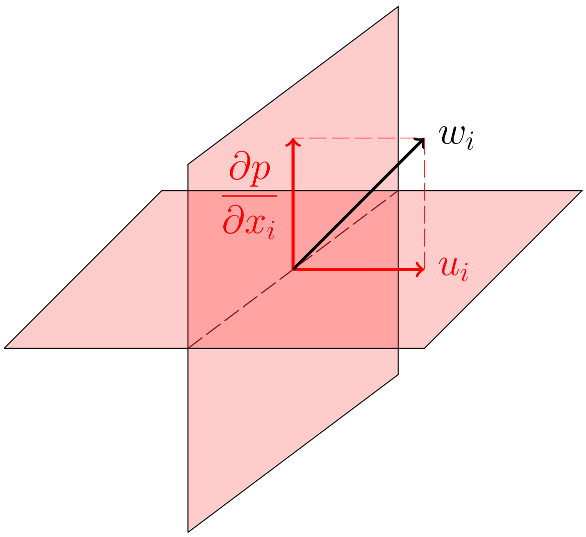
i.e. a problem for $p$ with a unique solution (can be proved).

In summary:

to project $w_i$ on divergence free space let $w_i = u_i + \dfrac{\partial p}{\partial x_i}$

1) solve for $p$ : $\nabla^2 p = \dfrac{\partial w_i}{\partial x_i}, \quad \dfrac{\partial p}{\partial n} = w_i n_i$ on $\delta\Omega$

2) let $u_i = w_i - \dfrac{\partial p}{\partial x_i}$




Note also that $(w_i - u_i) \perp u_i$.

## Apply to Navier-Stokes equations

Let $\mathbb{P}$ be the orthogonal projector which maps $w_i$ on the divergence-free part $u_i$, i.e. $\mathbb{P}w_i = u_i$. We then have the following relations

$$\begin{cases} \mathbb{P}u_i = u_i, \\ \\ \mathbb{P}\dfrac{\partial p}{\partial x_i} = 0 \quad (\dfrac{\partial p}{\partial x_i} \text{ normal to } u_i) \end{cases}$$

Let $\mathbb{P}$ act on the Navier-Stokes equations

$$\mathbb{P}\left(\frac{\partial u_i}{\partial t} + \frac{\partial p}{\partial x_i}\right) = \mathbb{P}\left(-u_j\frac{\partial u_i}{\partial x_j} + \frac{1}{Re}\nabla^2 u_i\right)$$

We have

$$\mathbb{P}\frac{\partial u_i}{\partial t} = \frac{\partial u_i}{\partial t} \text{ } (u_i \text{ divergence free and parallel to boundary})$$

$$\mathbb{P}\nabla^2 u_i \neq \nabla^2 u_i \text{ } (\nabla^2 u_i \text{ not necessarily parallel to the boundary})$$

Thus we have

$$\frac{\partial u_i}{\partial t} = \mathbb{P}\underbrace{\left(-u_j\frac{\partial u_i}{\partial x_j} + \frac{1}{Re}\nabla^2 u_i\right)}_{w_i},$$

an evolution equation without the pressure.

Pressure can be recovered from Poisson eq.

$$\nabla^2 p = \frac{\partial}{\partial x_i}\underbrace{\left(-u_j\frac{\partial u_i}{\partial x_j} + \frac{1}{Re}\nabla^2 u_i\right)}_{w_i} = \left\{\frac{\partial u_i}{\partial x_i} = 0\right\} = -\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}$$

(alternatively found by taking divergence of momentum eqs.)

$\Rightarrow$ Pressure satisfies elliptic Poisson equation, which links the velocity field in the whole domain instantaneously. This can be interpreted such that the information in incompressible flow spreads infinitely fast, i.e. we have an infinite wave speed for pressure waves.

| | |
|---|---|
| Interpretation for N.-S. eqs.: | We make $w_i = -u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{Re}\nabla^2 u_i$ divergence free by subtracting $\frac{\partial p}{\partial x_i}$ where $p$ can be found from a Poisson eq. ($u_i = w_i - \frac{\partial p}{\partial x_i}$). |

## Projection/correction methods (time-dependent flows)

Discretize the time derivative with a first order explicit method

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} + N\left(u^n\right) u^n + G\, p^{n+1} = f\left(t^n\right) \\ \\ D u^{n+1} = 0 \end{cases}$$

Make a prediction of the velocity field $u^*$ not necessary satisfying continuity, and then a correction involving $p$ such that $Du^{n+1} = 0$.

$$\Rightarrow \begin{cases} \dfrac{u^* - u^n}{\Delta t} + N\left(u^n\right) u^n = f\left(t^n\right) & \text{(prediction step)} \\ \dfrac{u^{n+1} - u^*}{\Delta t} + G\, p^{n+1} = 0 & \text{(correction step)} \\ D u^{n+1} = 0 \end{cases}$$

Apply the divergence operator $D$ to the correction step

$$\Rightarrow DG\, p^{n+1} = \frac{1}{\Delta t} D\, u^*$$

Here $DG$ represents $div \cdot grad =$ Laplacian, i.e. discrete version of pressure Poisson equation. The velocity at the next time level becomes

$$u^{n+1} = u^* - \Delta t G\, p^{n+1}$$

$u^{n+1}$ is projection of $u^*$ on divergence-free space (see notes on projection on a divergence-free space).

To discuss BC for the pressure we write the prediction and correction step explicitly

$$\begin{cases} \dfrac{u^*_{i+\frac{1}{2},j} - u^n_{i+\frac{1}{2},j}}{\Delta t} + A^n_{i+\frac{1}{2},j} = 0 \\[2ex] \dfrac{v^*_{i,j+\frac{1}{2}} - v^n_{i,j+\frac{1}{2}}}{\Delta t} + B^n_{i,j+\frac{1}{2}} = 0 \end{cases} \quad \text{(prediction step)}$$

$$\begin{cases} \dfrac{u^{n+1}_{i+\frac{1}{2},j} - u^*_{i+\frac{1}{2},j}}{\Delta t} + \dfrac{p^{n+1}_{i+1,j} - p^{n+1}_{i,j}}{\Delta x} = 0 \\[2ex] \dfrac{v^{n+1}_{i,j+\frac{1}{2}} - v^*_{i,j+\frac{1}{2}}}{\Delta t} + \dfrac{p^{n+1}_{i,j+1} - p^{n+1}_{i,j}}{\Delta y} = 0 \end{cases} \quad \text{(correction/projection step)}$$

Continuity equation:

$$Du^{n+1}_{i,j} = \frac{u^{n+1}_{i+\frac{1}{2},j} - u^{n+1}_{i-\frac{1}{2},j}}{\Delta x} + \frac{v^{n+1}_{i,j+\frac{1}{2}} - v^{n+1}_{i,j-\frac{1}{2}}}{\Delta y} = 0$$

Substitute projection step in the continuity equation

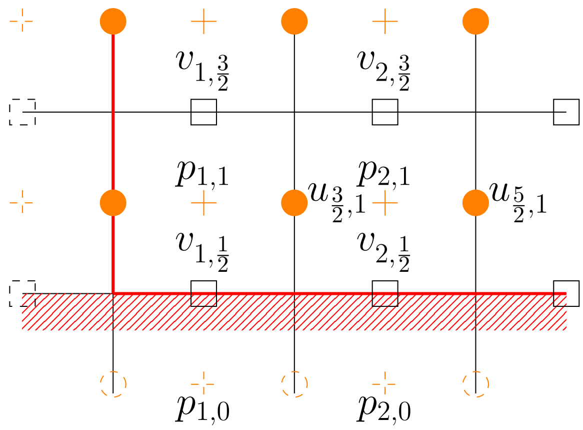
$$\Rightarrow \frac{p^{n+1}_{i+1,j} - 2p^{n+1}_{i,j} + p^{n+1}_{i-1,j}}{\Delta x^2} + \frac{p^{n+1}_{i,j+1} - 2p^{n+1}_{i,j} + p^{n+1}_{i,j-1}}{\Delta y^2} = \frac{1}{\Delta t} Du^*_{i,j}$$

is the discrete pressure Poisson equation.

If $i = 1$ or $j = 1$, we need pressure points outside the domain. It is important to chose pressure B.C. such that divergence constraint is not violated.

B.C. for pressure

Look at projection step (y-momentum)




$$\frac{p_{2,1}^{n+1} - p_{2,0}^{n+1}}{\Delta y} = -\frac{1}{\Delta t}\left(v_{2,\frac{1}{2}}^{n+1} - v_{2,\frac{1}{2}}^{*}\right) \qquad (\star)$$

$v_{2,\frac{1}{2}}^{*}$ is unknown in the prediction step at the boundary.

In general, it is important to choose this boundary condition such that discrete velocity after the correction step is divergence free at the boundary.

Poisson eq.

$$\frac{p_{3,1}^{n+1} - 2p_{2,1}^{n+1} + p_{1,1}^{n+1}}{\Delta x^2} + \frac{p_{2,2}^{n+1} - 2p_{2,1}^{n+1} + p_{2,0}^{n+1}}{\Delta y^2} = \frac{1}{\Delta t}\left(\frac{u_{\frac{5}{2},1}^{*} - u_{\frac{3}{2},1}^{*}}{\Delta x} + \frac{v_{2,\frac{3}{2}}^{*} - v_{2,\frac{1}{2}}^{*}}{\Delta y}\right)$$

Insert $(\star)$ in to Poisson eq.

$$\frac{p_{3,1}^{n+1} - 2p_{2,1}^{n+1} + p_{1,1}^{n+1}}{\Delta x^2} + \frac{p_{2,2}^{n+1} - p_{2,1}^{n+1} + \Delta y/\Delta t\left(v_{2,\frac{1}{2}}^{n+1} - \cancel{v_{2,\frac{1}{2}}^{*}}\right)}{\Delta y^2} = \frac{1}{\Delta t}\left(\frac{u_{\frac{5}{2},1}^{*} - u_{\frac{3}{2},1}^{*}}{\Delta x} + \frac{v_{2,\frac{3}{2}}^{*} - \cancel{v_{2,\frac{1}{2}}^{*}}}{\Delta y}\right)$$

The term $v^*_{2,\frac{1}{2}}$ on the both side of eq. above cancel each other and can be chosen arbirtary.

$$\text{Set } v^*_{2,\frac{1}{2}} = v^{n+1}_{2,\frac{1}{2}} \quad \Rightarrow \quad \frac{p^{n+1}_{2,1} - p^{n+1}_{2,0}}{\Delta y} = 0$$

Neumann B.C. for pressure (numerical artefact).