Homework 2 Finite differences

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Task 1: Integration of an ordinary differential equation

1. Derive the analytic solution u_{ex}.

$$\begin{aligned}
&\frac{du}{dt} = \lambda u \\
&\Rightarrow \frac{du}{u} = \lambda dt \\
&\Rightarrow \ln u = \lambda t + c \\
&\Rightarrow u = e^{\lambda t + c} \\
&u(0) = 1 \Rightarrow C = 0 \\
&\Rightarrow u = e^{\lambda t}
\end{aligned} \tag{1.1}$$

2. Calculate the numerical solution

Explicit

$$\begin{aligned}
&u^{n+1} - u^{n} = \Delta t \lambda u^{n} \\
&\Rightarrow u^{n+1} = (1 + \Delta t \lambda) u^{n}
\end{aligned} \tag{1.2}$$

Implicit

$$\begin{aligned}
&u^{n+1} - u^{n} = \Delta t \lambda u^{n+1} \\
&\Rightarrow u^{n+1} = \frac{u^{n}}{(1 - \Delta t \lambda)}
\end{aligned} \tag{1.3}$$

Crank-Nicolson

$$\begin{aligned}
&u^{n+1} - u^{n} = \frac{1}{2} \Delta t \left[\lambda u^{n+1} + \lambda u^{n} \right] \\
&\Rightarrow u^{n+1} = \left(1 + \frac{2\lambda \Delta t}{2 - \lambda \Delta t} \right) u^{n}
\end{aligned} \tag{1.4}$$

As shown in figure 1, the comparison of above three methods with analytic solution. The conclusion is with discretization into more parts, saying, larger N, the numerical results are closer to analytical. Also, the Crank-Nicolson method has higher accuracy than explicit and implicit Euler methods. Explicit Euler method blow up when the N is 20, it is unstable.

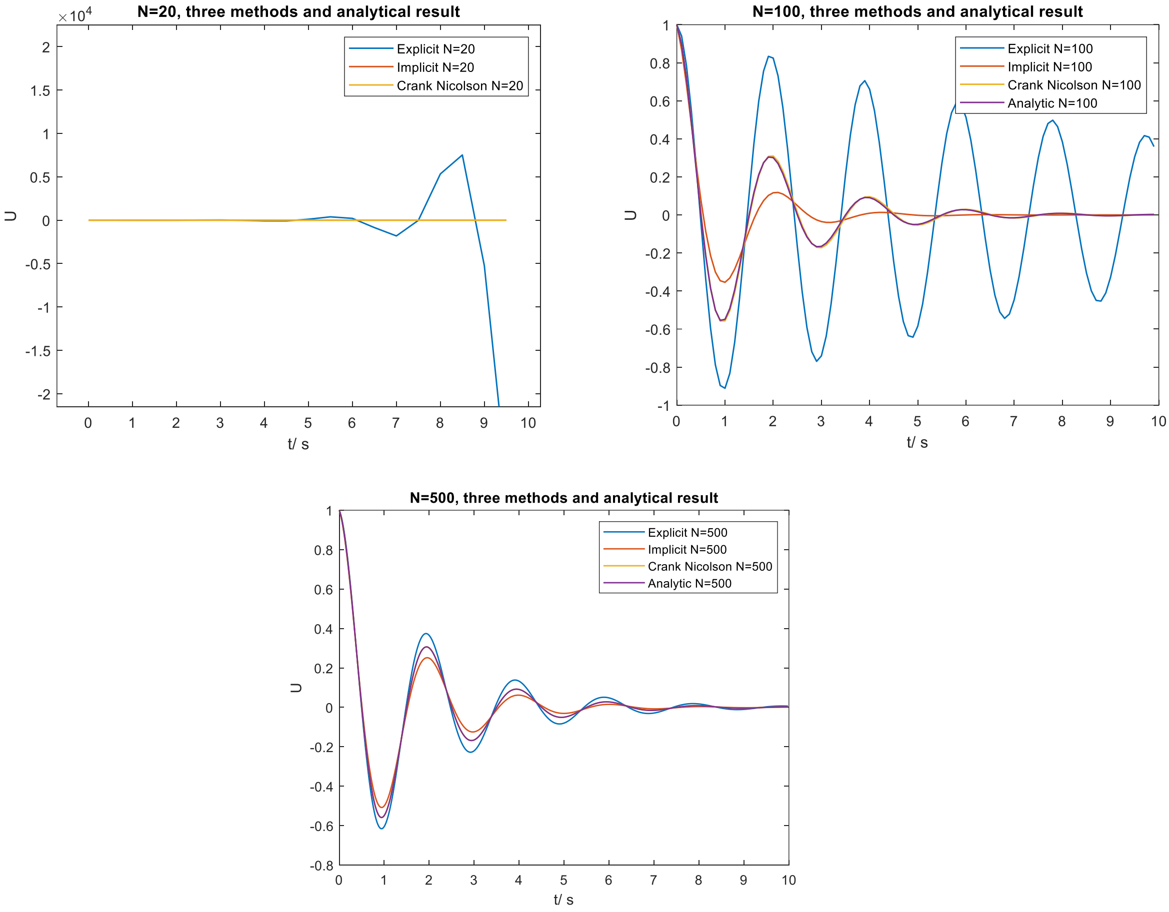

Figure 1 Comparison of three numerical methods with analytic solution

3. Derive the expression of the G(z).

Explicit

$$u^{n+1} = (1+\Delta t\lambda)u^n \Rightarrow G(z) = 1+z \tag{1.5}$$

Implicit

$$u^{n+1} = \frac{u^n}{(1-\Delta t\lambda)} \Rightarrow G(z) = \frac{1}{1-z} \tag{1.6}$$

Crank-Nicolson

$$u^{n+1} = \left(1+\frac{2\lambda\Delta t}{2-\lambda\Delta t}\right)u^n \Rightarrow G(z) = 1+\frac{2z}{2-z} \tag{1.7}$$

When z is infinity,

Explicit

$$G(z) = 1+z \to \infty \tag{1.8}$$

Implicit

$$G(z) = \frac{1}{1-z} \to 0 \tag{1.9}$$

Crank-Nicolson

$$G(z) = 1 + \frac{2z}{2-z} \to -1 \tag{1.10}$$

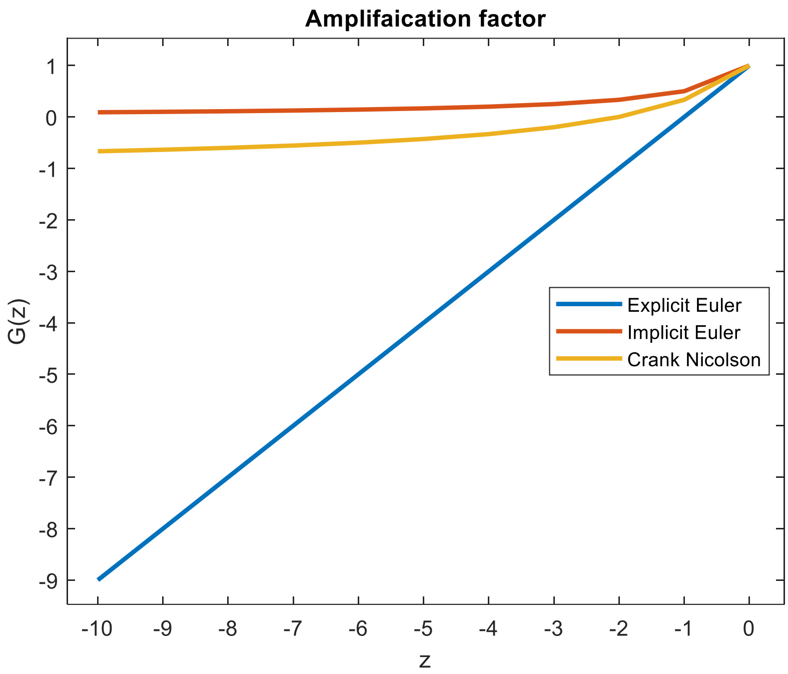

Figure 2 Amplification factor for three different numerical methods

As shown in figure 2, when Z towards to negative infinitive, the G(z) of explicit decreases linearly while implicit and Crank-Nicolson method tend to 0. It means explicit Euler method is unstable while other two is stable. Especially for implicit Euler method, the amplification factor is close to 0, which means the result is very stable. Crank Nicolson method towards to -1, which is not very stable.

The imaginary part of lambda related to frequency of vibration, so it is irrelevant for the limitation. Amplitude is the required solution.

4 and 5: As shown in figure 3, the lambda has change. The difference is imaginary has changed from πi to i. Comparing the figure 2 and figure 3, it could see explicit Eluer method is stable now, the reason is that G(z) is small than 1 now.

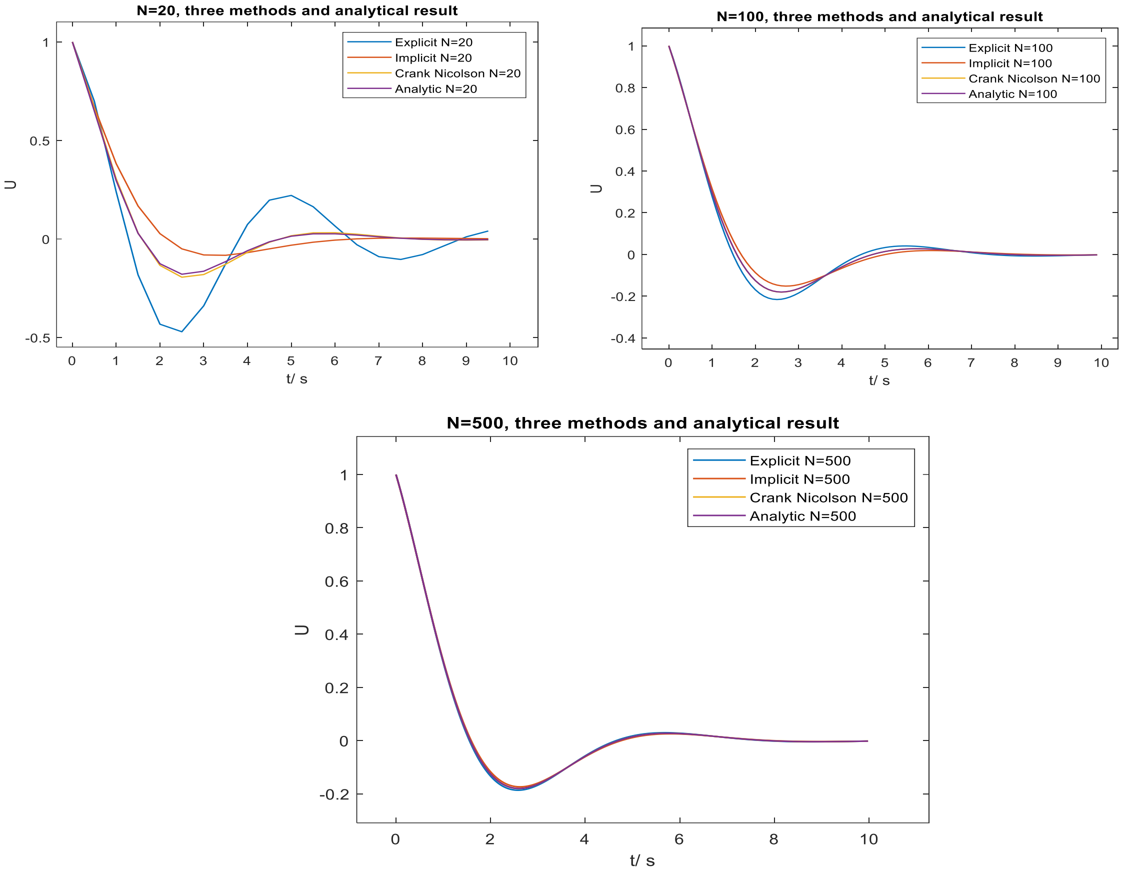

Figure 3 Repeat analysis of point 2 while imagnary part is i

Task 2: Finite Difference Schemes

1: Approximation of the derivative
First, according to Taylor expansion:

$$u_{i-1} = u_i - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{4!}\frac{\partial^4 u}{\partial x^4} + \ldots$$

$$u_{i+1} = u_i + \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{4!}\frac{\partial^4 u}{\partial x^4} + \ldots \quad (1.11)$$

$$u_{i+2} = u_i + 2\Delta x \frac{\partial u}{\partial x} + \frac{(2\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{(2\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{(2\Delta x)^4}{4!}\frac{\partial^4 u}{\partial x^4} + \ldots$$

Then, we try to eliminate.

$$\frac{\partial^2 u}{\partial x^2} \text{ and } \frac{\partial^3 u}{\partial x^3} \quad (1.12)$$

This could be done by following process

$$2u_{i-1}+2u_{i+1}-u_{i+2}=3u_i+2\Delta x\frac{\partial u}{\partial x}-\frac{(2\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}-\frac{\Delta x^4}{2}\frac{\partial^4 u}{\partial x^4}+\dots$$

$$u_{i+1}-u_{i-1}=2\Delta x\frac{\partial u}{\partial x}+\frac{2(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}+\dots$$

$$\Rightarrow (2u_{i-1}+2u_{i+1}-u_{i+2})+4(u_{i+1}-u_{i-1})=3u_i+6\Delta x\frac{\partial u}{\partial x}-\frac{\Delta x^4}{2}\frac{\partial^4 u}{\partial x^4}+\dots \qquad (1.13)$$

$$\Rightarrow -2u_{i-1}+6u_{i+1}-u_{i+2}-3u_i=6\Delta x\frac{\partial u}{\partial x}-\frac{\Delta x^4}{2}\frac{\partial^4 u}{\partial x^4}+\dots$$

$$\Rightarrow \left.\frac{\partial u}{\partial x}\right|_{x=xi}=\frac{-2u_{i-1}-3u_i+6u_{i+1}-u_{i+2}}{6\Delta x}+\frac{\Delta x^3}{12}\frac{\partial^4 u}{\partial x^4}+\dots$$

2): The leading error term is

$$\frac{\Delta x^3}{12}\frac{\partial^4 u}{\partial x^4} \qquad (1.14).$$

The order of this scheme is the third order, which is better than second order. This four grid values are better than right or left sided and central finite differences. If we have u(i-2), we could reach fourth order by central finite difference scheme.

Task 3: Stability Criteria

$$u^{n+1}=u^n+\frac{\Delta t}{6}(f^n+2k_1+2k_2+k_3)$$

$$f^n=\lambda u^n$$

$$=u^n+\frac{\Delta t}{6}\{\lambda u^n+2\lambda(u^n+\frac{\Delta t}{2}\lambda u^n)+2\lambda[u^n+\Delta t(\lambda u^n+\frac{\Delta t}{2}\lambda^2 u^n)]+\lambda u^n+\Delta t\lambda[\lambda u^n+\lambda\Delta t(u^n+\frac{\Delta t}{2}\lambda^2 u^n)]\}$$

$$u^{n+1}=u^n(1+\lambda\Delta t+\frac{(\lambda\Delta t)^2}{2}+\frac{(\lambda\Delta t)^3}{6}+\frac{(\lambda\Delta t)^4}{24})$$

$$z=\lambda\Delta t$$

$$\Rightarrow u^{n+1}=u^n(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24})$$

(1.15)

As shown in figure 4, is the z value when G(z) smaller than 1. It could be seen that the boundary cuts the imaginary axis at 2.83 and -2.83, it also cuts the real axis at -2.83. So, through the proper z, as shown in blue area of figure 4, the Runger-Kutta 4th order method will stable. It stability boundary is the boundary of figure 4, for bule area.

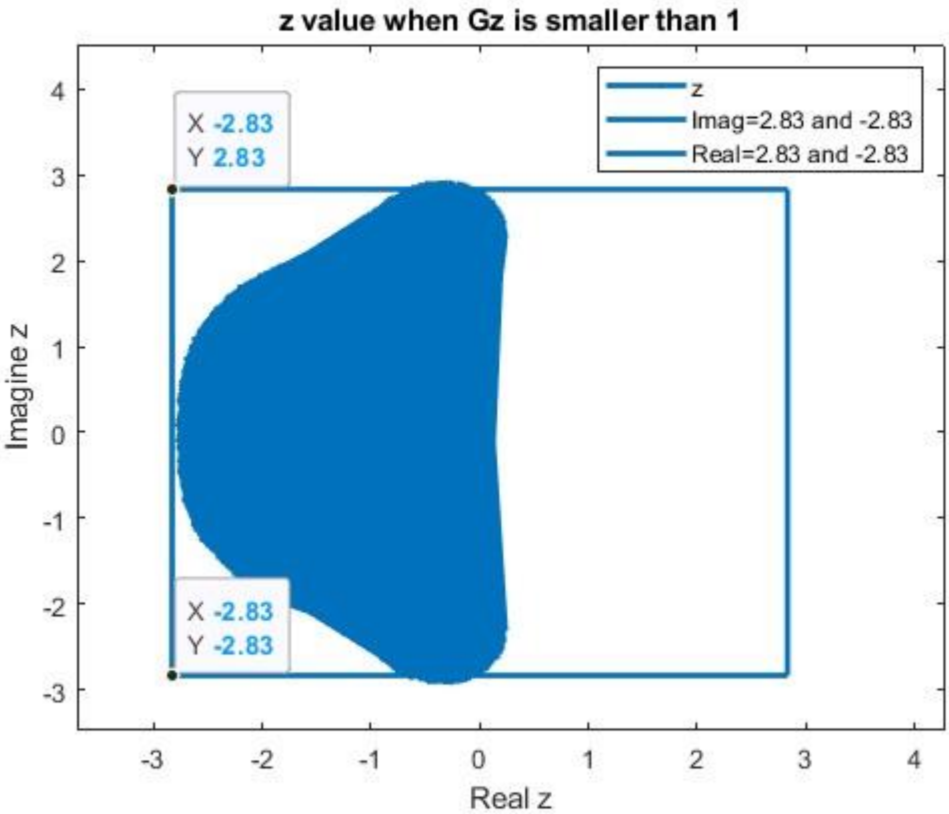

Figure 4 z value when curve G(z) small than 1