What is a finite element?

In general, a finite element is a triple consisting of

1. a geometric object K (e.g. a simplex),
2. a finite local basis on this geometric object (basis functions) and
3. a finite number of local degrees of freedom (the nodes).

In the following we give a general definition of a simplex (as a particular geometric object).

Part 6 - General (Higher order) Lagrange Finite Elements

6.1. Simplicial triangulations revisited

Goal

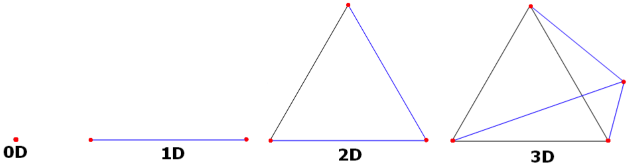
- In the following: state general definitions for finite elements of so-called Lagrange type.
- Considerations are restricted to simplicial elements (as already earlier in the course).
- A corresponding Lagrange finite element space consists of globally continuous functions which are polynomials when restricted to one of the simplices.

Simplicial Lagrange elements

Goal (as before):
Construction of finite element spaces based on simplicial partitions of Ω.

Simplex:

- A d-simplex is a d-dimensional polytope formed by the convex hull of $d+1$ points.
- These points are called vortices of the simplex.



Simplex - A mathematical definition

Let $k \in \{0, \ldots, d\}$.
We call $K \subset \mathbb{R}^d$ a non-degenerate k-simplex if

- it is constructed from $k+1$ local vortices $z_0, z_1, \ldots, z_k \in \mathbb{R}^d$,
- the vectors $(z_0 - z_i)$ are linearly independent for $i \in \{1, \ldots, k\}$,
- and K is (closed) convex hull of $\{z_0, z_1, \ldots, z_k\}$, i.e.

$$K = \left\{ \left(\sum_{i=0}^{k} \lambda_i z_i \right) \in \mathbb{R}^d \middle| \, \lambda_i \in [0,1], \sum_{i=0}^{k} \lambda_i = 1 \right\}.$$

Note: a 0-simplex is called vertex, a 1-simplex an edge and a 2-simplex a face.

Simplex - A mathematical definition

Let $k \in \{0, \ldots, d\}$.
We call $K \subset \mathbb{R}^d$ a non-degenerate k-simplex if

- it is constructed from $k+1$ local vortices $z_0, z_1, \ldots, z_k \in \mathbb{R}^d$,
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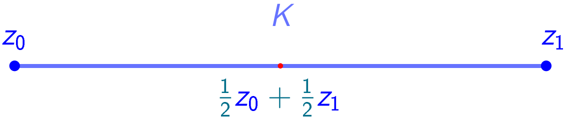
Note 2: The quantities $\lambda_0, \ldots, \lambda_k \in [0,1]$ are the intrinsic coordinates (called barycentric coordinates) of the sub-simplex.

Simplex - Example $k = 1$

We call $K \subset \mathbb{R}^d$ a non-degenerate 1-simplex (an edge) if

- it is constructed from 2 local vortices z_0 and z_1,
- the vector $(z_0 - z_1)$ is not zero,
- and K is (closed) convex hull of z_0 and z_1, i.e.

$$K = \{\ (1-\lambda)z_0 + \lambda z_1 \mid \lambda \in [0,1]\}\,.$$



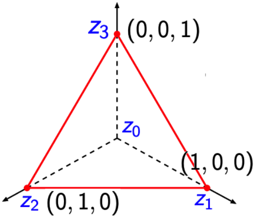
Unit simplex

If $K_0 \subset R^d$ has the units corners

$$z_0 = 0, \; z_i = \mathbf{e}_i \quad \text{for } i = 1, \ldots, d,$$

then we call K_0 the d-dimensional unit simplex.

Example for $d = 3$:



Remark

- In $\mathbb{R}^2$ a 2-dimensional simplex is a triangle with
  3 corners z_0, z_1, z_2 and
  3 1-dimensional edges
- in $\mathbb{R}^3$ a 3-dimensional simplex is a tetrahedra with
  4 corners z_0, z_1, z_2, z_3,
  6 1-dimensional edges and
  4 2-dimensional faces.
- General: An s-dimensional simplex has
  $\binom{s+1}{r+1}$ r-dimensional sub-simplexes.

Triangulation - A mathematical definition

We call $\mathcal{T}_h$ an admissible triangulation of a polygonal domain $\Omega \subset \mathbb{R}^d$ if

1. $\mathcal{T}_h$ consists of non-degenerate d-simplices in $\mathbb{R}^d$;
2. the union of the elements of $\mathcal{T}_h$ is the closure of Ω, i.e. $\overline{\Omega} = \bigcup \mathcal{T}_h$;
3. intersection of two simplices from $\mathcal{T}_h$ is either empty or a simplex itself, i.e. for all $K_1, K_2 \in \mathcal{T}_h$: either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is k-simplex in $\mathbb{R}^d$; $k \in \{0, \ldots, d\}$.

Note that 3. excludes hanging nodes!

Reminder: Mesh size

We denote:

- diameter of a simplex K by $h_K := \mathrm{diam}(K)$,
- maximum diameter of an element of a triangulation $\mathcal{T}_h$ is

$$h := \max_{K \in \mathcal{T}_h} h_K.$$

- We call h the mesh size or the degree of resolution of $\mathcal{T}_h$.

Reminder: Shape regularity

To quantify the regularity of a simplex $K \in \mathcal{T}_h$ define

- the diameter of the incircle by $\rho_K := \mathrm{diam}(B_K)$
- where B_K is the largest ball contained in K.

With this, the local shape regularity parameter σ_K is

$$\sigma_K := \frac{h_K}{\rho_K} \geq 1.$$

The global shape regularity parameter $\sigma > 0$ is then

$$\sigma = \max_{K \in \mathcal{T}} \sigma_K.$$

Reminder: Shape regularity

The local shape regularity parameter σ_K is

$$\sigma_K := \frac{h_K}{\rho_K} \geq 1.$$

- If σ_K becomes large, then the longest edge of K is much larger than then diameter of incircle.
- This element must have a very small interior angle and the simplex degenerates for $h \to 0$.
- Such triangulations should be avoided in practice since arising approximations are typically of very poor quality.
- Later: reflected in error estimates, where σ appears as a scaling factor for guaranteed error bounds.

Reference maps

In order to highly improve the structure and efficiency of finite element codes, it is possible to construct the finite element basis only once on the reference element (unit simplex) K_0 and then transform it to each element K on the finite element mesh.

This is particularly useful when using higher order elements.

Reference maps

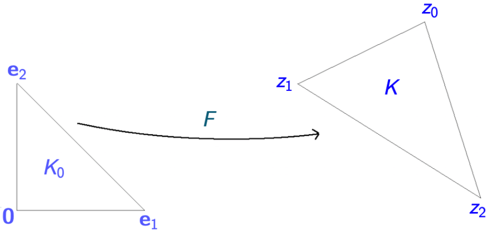
Let K_0 denote the unit simplex in $\mathbb{R}^s$.

Every s-dimensional simplex K is affine-equivalent to K_0 trough a unique affine reference transformation

$$F : K_0 \to K$$

with

$$F(\mathbf{0}) = z_0 \qquad \text{and} \qquad F(\mathbf{e}_j) = z_j, \quad \text{for } j = 1, \ldots, s.$$



Reference maps

Let K_0 denote the unit simplex in $\mathbb{R}^s$.

Every s-dimensional simplex K is affine-equivalent to K_0 trough a unique affine transformation

$$F : K_0 \to K \qquad \text{with} \qquad F(x) = Ax + b,$$

where

- $A \in \mathbb{R}^{s \times s}$ describes the geometric deformation;
- $b \in \mathbb{R}^s$ describes the translation.
- Note: F is invertible if K is not degenerate.

Reference maps

Affine transformation:

$$F : K_0 \to K \qquad \text{with} \qquad F(x) = Ax + b,$$

where

- $A \in \mathbb{R}^{s \times s}$ describes the geometric deformation;
- $b \in \mathbb{R}^s$ describes the translation.
- $|\det A|$ measures the volume deformation.

  If $|\det A| = 0$ the element is degenerate and if $|\det A|$ is very small relative to the volume h_K^s, then the element has a poor shape regularity.

  It can be shown: $\sigma_K^{-s} h_K^s \lesssim |\det A| \lesssim h_K^s.$

Part 6 - General (Higher order) Lagrange Finite Elements

6.2. Linear Lagrange elements revisited

Linear Lagrange elements

- Before giving a general definition of Lagrange finite elements, we recall linear Lagrange elements.
- Linear elements are most common one and support the understanding of the general case.

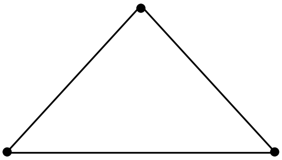


Figure: Linear Lagrange element

Linear simplicial Lagrange element

- $K \subset \mathbb{R}^d$ is a d-simplex

formed as convex hull of $d+1$ local vortices

$$\mathcal{N}_h(K) := \{z_{K,0}, z_{K,1}, \ldots, z_{K,d}\}.$$

Any $p \in \mathbb{P}^1(K)$ is uniquely defined by the values in the vortices $z_{K,i}$, i.e.

- $\mathcal{N}_h(K)$ forms local degrees of freedom.

A basis of $\mathbb{P}^1(K)$ are the (local) nodal basis functions

- $\Phi_K := \{\phi_{K,i} \in \mathbb{P}^1(K) | \, \phi_{K,i}(z_{K,j}) = \delta_{ij}; \; 0 \leq i, j \leq d\}$.

Obviously, Φ_K contains $d+1$ linear independent functions.

The triple $(K, \Phi_K, \mathcal{N}_h(K))$ defines the <u>first order simplicial Lagrange finite element</u> on K.

Example - Linear Lagrange element for $d = 2$

Consider the unit simplex K_0 with corners

$$\tilde{N}_0 := (0,0), \quad \tilde{N}_1 := (1,0) \quad \text{and} \quad \tilde{N}_2 := (0,1).$$

The corresponding unit nodal basis functions are

$$\begin{aligned}
\tilde{\phi}_0(x_1,x_2) &= 1 - x_1 - x_2, \\
\tilde{\phi}_1(x_1,x_2) &= x_1, \\
\tilde{\phi}_2(x_1,x_2) &= x_2.
\end{aligned}$$

Any linear function $\tilde{p} \in \mathbb{P}^1(K_0)$ can be written as

$$\tilde{p}(x_1,x_2) = \sum_{i=0}^{2} \tilde{p}(N_i)\, \tilde{\phi}_i(x_1,x_2).$$

Example - Linear Lagrange element for $d = 2$

For arbitrary triangle $K \subset \mathbb{R}^2$ the linear Lagrange element is obtained using the reference map

$$F_K : K_0 \to K.$$

The shape functions $\phi_{K,i}$ for $i = 0, 1, 2$ are given by

$$\phi_{K,i}(x_1, x_2) := \tilde{\phi}_i(F_K^{-1}(x_1, x_2)).$$

Note: Since F_K is affine, we have $\phi_{K,i} \in \mathbb{P}^1(K)$. In the corners $z_{K,0}$, $z_{K,1}$, $z_{K,2}$ of K we have for i,j =0,1,2

$$\phi_{K,i}(z_{K,j}) = \tilde{\phi}_i(F_K^{-1}(z_{K,j})) = \tilde{\phi}_i(\tilde{N}_j) = \delta_{ij}.$$

Example - Linear Lagrange element for $d = 2$

For arbitrary triangle $K \subset \mathbb{R}^2$ the linear Lagrange element is obtained using the reference map

$$F_K : K_0 \to K.$$

The shape functions $\phi_{K,i}$ for $i = 0, 1, 2$ are given by

$$\phi_{K,i}(x_1, x_2) := \tilde{\phi}_i(F_K^{-1}(x_1, x_2)).$$

An arbitrary linear function $p \in \mathbb{P}^1(K)$ can be hence expressed by its corner values:

$$p(x_1, x_2) = \sum_{i=0}^{2} p(z_{K,i})\, \phi_{K,i}(x_1, x_2).$$

Lagrange elements - Remark 1

- Example shows: it is sufficient to define a finite element on a reference geometry.
- Using the reference map we can obtain a corresponding class of elements of arbitrary geometry.

Lagrange elements - Remark 2

- A finite element just defines a local function space on a simplex K.
- Coupling the local degrees of freedom to a set of global degrees of freedom we obtain the global linear Lagrange finite element space as a subspace of $H_0^1(\Omega)$.

Linear Lagrange finite element space V_h

Let $\Omega \subset \mathbb{R}^d$ denote a polygonally bounded domain and $\mathcal{T}_h$ a corresponding admissible triangulation.

The set of global vortices is given by the union of the local vortex sets, i.e.

$$\mathcal{N}_h := \bigcup_{K \in \mathcal{T}_h} \mathcal{N}_h(K).$$

$\mathcal{N}_h$ determines a global number of degrees of freedom, in the sense that any function in

$$V_h := \{v_h \in C^0(\overline{\Omega}) | \forall K \in \mathcal{T}_h : \ v_h|_K \in \mathbb{P}^1(K)\}$$

is uniquely defined by its values in $\mathcal{N}_h$.

Remark

Note: the enforced continuity of functions in the global space

$$V_h := \{v_h \in C^0(\overline{\Omega}) | \forall K \in \mathcal{T}_h : \; v_h|_K \in \mathbb{P}^1(K)\}$$

couples the local degrees of freedom (represented by $\mathcal{N}_h(K)$) to the global degrees of freedom (represented by $\mathcal{N}_h$).

Representation of functions in V_h

Global representation of $v_h \in V_h$ by global nodal basis:

$$v_h = \sum_{z \in \mathcal{N}_h} v_h(z)\,\phi_z.$$

Local representation of $v_h \in V_h$:

For $K \in \mathcal{T}_h$ with corners $z_{K,i}$ (for $0 \leq i \leq d$) it holds

$$v_h|_K(x) = \sum_{i=0}^{d} v_h(z_{K,i})\,\tilde{\phi}_i(F_K^{-1}(x)),$$

- $(K_0, \tilde{\Phi}, \tilde{N})$: linear Lagrange element on unit simplex,
- $F : K_0 \to K$: reference map,
- $\tilde{\phi}_i \in \tilde{\Phi}$: basis functions on reference element K_0.

Remark

The Local representation

$$v_h|_K(x) = \sum_{i=0}^{d} v_h(z_{K,i})\, \tilde{\phi}_i(F_K^{-1}(x)),$$

is very convenient from an implementation point of view, since

- it only requires the generation of local shape functions $\tilde{\phi}_i$ on the unit simplex K_0.
- Elements K are represented by the reference maps

$$F : K_0 \to K.$$

Part 6 - General (Higher order) Lagrange Finite Elements

6.3. Higher Order Lagrange elements

General Lagrange Finite Elements

- Next: generalize linear case to case of general Lagrange elements of arbitrary polynomial degree $k \in \mathbb{N}$.
- For mathematical proofs on well-posedness see e.g. the book by *Brenner and Scott*.

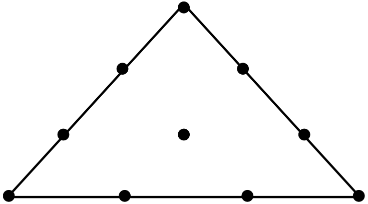


Figure: Cubic ($\mathbb{P}^3$) Lagrange element.

k'th order Lagrange Grid

Let $K \subset \mathbb{R}^d$ denote a d-dimensional simplex.

The local k'th order Lagrange grid $\mathcal{N}_h^k(K)$ on K is given by

$$\mathcal{N}_h^k(K) := \left\{ x = \sum_{j=0}^{d} \lambda_j z_{K,j} \middle| \ \lambda_j \in \{\frac{m}{k} | \ m = 0, \dots, k\}, \ \sum_{j=0}^{d} \lambda_j = 1 \right\}.$$

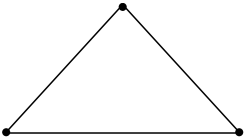


Figure: Linear ($\mathbb{P}^1$) Lagrange element.

$$\mathcal{N}_h^1(K) := \left\{ x = \sum_{j=0}^{d} \lambda_j z_{K,j} \middle| \ \lambda_j \in \{0, 1\}, \ \sum_{j=0}^{d} \lambda_j = 1 \right\} = \{z_{K,0}, \dots, z_{K,d}\}.$$

k'th order Lagrange Grid

Let $K \subset \mathbb{R}^d$ denote a d-dimensional simplex.

The local k'th order Lagrange grid $\mathcal{N}_h^k(K)$ on K is given by

$$\mathcal{N}_h^k(K) := \left\{ x = \sum_{j=0}^{d} \lambda_j z_{K,j} \,|\, \lambda_j \in \{\frac{m}{k} \,|\, m = 0, \ldots, k\}, \; \sum_{j=0}^{d} \lambda_j = 1 \right\}.$$

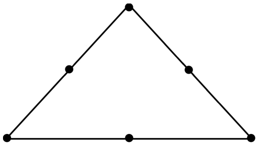


Figure: Quadratic ($\mathbb{P}^2$) Lagrange element.

$$\mathcal{N}_h^2(K) := \left\{ x = \sum_{j=0}^{d} \lambda_j z_{K,j} \,|\, \lambda_j \in \{0, \frac{1}{2}, 1\}, \; \sum_{j=0}^{d} \lambda_j = 1 \right\}.$$

k'th order Lagrange Grid

Let $K \subset \mathbb{R}^d$ denote a d-dimensional simplex.

The local k'th order Lagrange grid $\mathcal{N}_h^k(K)$ on K is given by

$$\mathcal{N}_h^k(K) := \left\{ x = \sum_{j=0}^{d} \lambda_j z_{K,j} \,\middle|\, \lambda_j \in \{\frac{m}{k} | \; m = 0, \ldots, k\}, \; \sum_{j=0}^{d} \lambda_j = 1 \right\}.$$

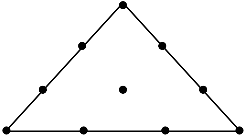


Figure: Cubic ($\mathbb{P}^3$) Lagrange element.

$$\mathcal{N}_h^3(K) := \left\{ x = \sum_{j=0}^{d} \lambda_j z_{K,j} \,\middle|\, \lambda_j \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}, \; \sum_{j=0}^{d} \lambda_j = 1 \right\}.$$

k'th order Lagrange Grid

Let $K \subset \mathbb{R}^d$ denote a d-dimensional simplex.

The local k'th order Lagrange grid $\mathcal{N}_h^k(K)$ on K is

$$\mathcal{N}_h^k(K) := \left\{ x = \sum_{j=0}^{d} \lambda_j z_{K,j} | \ \lambda_j \in \{ \frac{m}{k} | \ m = 0, \ldots, k \}, \ \sum_{j=0}^{d} \lambda_j = 1 \right\}.$$

Note: The k'th order Lagrange grid $\mathcal{N}_h^k(K)$ contains

$$g_k = \begin{pmatrix} d+k \\ k \end{pmatrix} \qquad \text{points.}$$

General Lagrange Element

Let $K \subset \mathbb{R}^d$ denote a d-dimensional simplex.

The local k'th order Lagrange grid $\mathcal{N}_h^k(K)$ on K is

$$\mathcal{N}_h^k(K) := \left\{ x = \sum_{j=0}^{d} \lambda_j z_{K,j} |\ \lambda_j \in \{\frac{m}{k} |\ m = 0, \ldots, k\},\ \sum_{j=0}^{d} \lambda_j = 1 \right\}.$$

By prescribing values $p(N_\ell)$ in the Lagrange points $N_\ell \in \mathcal{N}_h^k(K)$ (with $\ell = 0, \ldots, g_k - 1$) a function $p \in \mathbb{P}^k(K)$ is uniquely defined. A nodal basis of $\mathbb{P}^k(K)$ is given by

$$\Phi_K := \{\phi_i \in \mathbb{P}^k(K) |\ \phi_i(N_\ell) = \delta_{ik},\ \ i, \ell = 1, \ldots, g_k\}$$

We call the triple $(K, \Phi_K, \mathcal{N}_h^k(K))$ a simplicial Lagrange element of degree k.

General Lagrange finite element space

Let: $\Omega \subset \mathbb{R}^d$ bounded polygonal domain and $\mathcal{T}_h$ admissible triangulation of Ω.

(i) The space of Lagrange finite elements of order $k \in \mathbb{N}$ is

$$V_h^k := \{v_h \in C^0(\overline{\Omega}) |\ v_h|_K \in \mathbb{P}^k(K), \forall K \in \mathcal{T}_h\}.$$

(ii) The global Lagrange grid $\mathcal{N}_h^k$ of order $k \in \mathbb{N}$ is the union of the local Langrange grids, i.e.

$$\mathcal{N}_h^k := \{N_\ell |\ \ell = 1, \ldots, N\} := \bigcup_{K \in \mathcal{T}_h} \mathcal{N}_h^k(K).$$

A function $v_h \in V_h^k$ is uniquely defined by its values $v_h(N_\ell)$ in the Lagrange points $N_\ell \in \mathcal{N}_h^k$.

A basis of V_h^k is given by $\phi_i \in V_h^k$, with $\phi_i(N_\ell) = \delta_{i\ell}$ for $1 \leq i, \ell \leq N$.

General Lagrange finite element space

(iii) If $(K_0, \tilde{\Phi}, \tilde{\mathcal{N}})$ denotes the k'th order Lagrange element on the unit simplex K_0 and if $v_h \in V_h^k$ is

$$v_h(x) = \sum_{\ell=1}^{N} v_h(N_\ell)\phi_\ell(x),$$

then for any simplex $K \in \mathcal{T}_h$ with local Lagrange points $N_{K,i} \in \mathcal{N}_h^k(K)$ we have the representation:

$$v_h|_K(x) = \sum_{i=1}^{g_k} v_h(N_{K,i})\, \tilde{\phi}_i(F_K^{-1}(x)).$$

Here, $F_K : K_0 \to K$ is the reference map and $\tilde{\phi}_i \in \tilde{\Phi}$ are the local basis functions of the k'th order reference Lagrange element K_0.

General Lagrange finite element space

(iv) As for the linear case, we define the space with zero boundary values by

$$V_{h,0}^k := V_h^k \cap H_0^1(\Omega).$$

Remark

Note:

- The definition of a finite element can be generalized;
- there are much more finite elements than only the simplicial Lagrange element.

Bi-/tri-linear Lagrange element

Example:

- Instead of simplices we can also use other geometries.
- On rectangles ($d = 2$) or hexahedrons ($d = 3$) we cannot use linear shape functions, but we need to use bi-linear and tri-linear functions instead.
- On the unit cube in $\mathbb{R}^d$ we can use the function space

$$Q^1([0,1]^d) := \bigotimes_{i=1}^{d} \mathbb{P}^1([0,1]).$$

For $d = 2$, the polynomials $p \in Q^1([0,1]^2)$ have the form

$$p(x_1, x_2) = a + bx_1 + cx_2 + dx_1 x_2.$$

Resulting Lagrange FE space is often denoted by Q_h^1.

Part 6 - General (Higher order) Lagrange Finite Elements

6.4. A priori error estimates

A priori error estimates

We can use again the abstract setting (see Part 4) to derive an a priori error estimate, based on Cea's lemma and quasi-interpolation estimates.

Without proof, we just state the result on the next slide.

A priori error estimates

Let $B : H_0^1(\Omega) \times H_0^1(\Omega) \to$ be a continuous and coercive bilinear form and $f \in L^2(\Omega)$.

We want to find $u \in H_0^1(\Omega)$, such that

$$B(u, v) = (f, v)_{L^2(\Omega)} \qquad \text{for all } v \in H_0^1(\Omega).$$

The finite element approximation in the k'th order Lagrange space is given by $u \in V_{h,0}^k$ such that

$$B(u_h, v_h) = (f, v_h)_{L^2(\Omega)} \qquad \text{for all } v_h \in V_{h,0}^k.$$

If $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$ then we have k'th order convergence

$$\|u - u_h\|_{H^1(\Omega)} \leq C \, \sigma h^k \|u\|_{H^{k+1}(\Omega)}.$$

Note the shape regularity parameter σ in the estimate.

Refined a priori error estimates

It is possible to even derive the following localized error estimate:

$$\|u - u_h\|_{H^1(\Omega)} \leq C \left(\sum_{K \in \mathcal{T}_h} \sigma_K^2 h_K^{2k} \|u\|_{H^{k+1}(K)}^2 \right)^{1/2}$$

Here the local influence of the mesh size h_K and the shape regularity σ_K becomes apparent.

Reminder: if a simplex K is degenerate ("flat"), then σ_K is a very large number and we can see how this destroys the convergence order of the method.