Problem Set B

Aim: The purpose of this Problem Set is to reinforce your understanding of under which circumstances a unique solution exists.

1. On a convex polygonal domain Ω, consider the differential equation

$$-\Delta u(\mathbf{x}) = f, \qquad \mathbf{x} \in \Omega \tag{1}$$

$$\hat{n} \cdot \nabla u(\mathbf{x}) = g, \qquad \mathbf{x} \in \partial\Omega \tag{2}$$

   where f and g are sufficiently regular functions that are given.

   (a) Why is this problem **not** well-posed?

   (b) If you would implement a standard finite element method for this problem (for example cG(1)[1]), would you be able to solve the linear system? What happens?

2. Consider the following Boundary Value Problem (BVP):

$$-(k(x)u'(x))' + u'(x) = f(x), \qquad x \in \Omega \tag{3}$$

$$u = 0, \qquad x \in \partial\Omega \tag{4}$$

   with the coefficient, $k(x)$, being a continuous function such that $k(x) \geq k_0 \geq 0$ for $x \in \Omega = (0, 1)$. Assume both $k(x)$ and $f(x)$ are given with $f \in L^1(0, 1)$.

   (a) Derive a variational formulation for the BVP: Find $u \in V$ such that

$$a(u, v) = L(v), \qquad \forall v \in V.$$

      i. Define the Hilbert Space $V = \mathcal{H}_0^1(\Omega)$.

      ii. Define the bilinear form $a : V \times V \to \mathbb{R}$ and linear form $L : V \to \mathbb{R}$.

   (b) Show that for $\|w\|_E^2 = a(w, w)$ we have that

$$\|w\|_E^2 = \int_\Omega k(x)(w'(x))^2 \, dx$$

   and that this represents a norm on E.

   (c) Prove that there exists a unique solution to the variational problem in 2a.

   (d) Formulate the cG(1) method for the given BVP on a subdivision of $(0, 1)$ with mesh size $h = h(x)$.

   (e) Prove the following a priori error estimate:

$$\|u - u_h\|_E \leq C\|hu''\|_{L^2(0,1)}.$$

   Here u is the exact solution to the given BVP and u_h is the approximation of u using the cG(1) method.

[1] FEM with a continuous piecewise linear approximation u_h

(f) Describe the steps in an adaptive algorithm for local mesh refinement based on a posteriori error estimation and given a tolerance TOL on $\|u - u_h\|_E$. Assume TOL is real. Why is adaptivity important?

3. Show that a spatial discretization of the equation

$$\begin{aligned} \dot{u} + \Delta u + u =& f, \qquad \mathbf{x} \in \Omega \\ u =& 0, \qquad \mathbf{x} \in \partial\Omega,\ t > 0 \\ u =& u_0, \qquad \mathbf{x} \in \Omega,\ t = 0 \end{aligned}$$

leads to the algebraic system

$$M\ddot{\xi}(t) + A\xi(t) + M\xi(t) = b(t).$$

Identify the entries of the involved matrices and vectors.[2]

[2]Exercise 5.3 in M. G. Larson and F. Bengzon, *The Finite Element Method: Theory, Implementation, and Applications*