Upwind discretization

Consider the model problem:

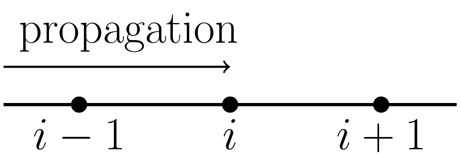
$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad (a \text{ is the convective velocity})$$

Use an explicit first-order Euler for discretization of the time derivative

$$u_i^{n+1} = u_i^n - \Delta t \; a(u_x)_i^n$$

It is known that a central-difference discretization of $(u_x)_i^n$ gives an unstable scheme.

if $a > 0$

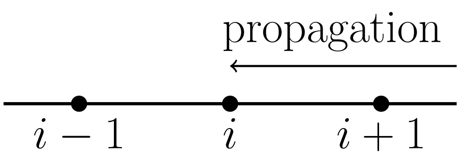


then use a backward discretization when $a > 0$

$$u_i^{n+1} = u_i^n - a\frac{\Delta t}{\Delta x}(u_i^n - u_{i-1}^n).$$

This equation is stable for $0 \leq a\dfrac{\Delta t}{\Delta x} \leq 1$.

if $a < 0$



then use a forward discretization

$$u_i^{n+1} = u_i^n - a\frac{\Delta t}{\Delta x}(u_{i+1}^n - u_i^n).$$

This equation is stable for $-1 \leq a\dfrac{\Delta t}{\Delta x} \leq 0$.

None of these discretizations are stable for both negative and positive wave propagation or convection velocity.

Let us define

$$\begin{aligned} a^+ &= \max(a,0) = \frac{1}{2}(a+|a|), \\ a^- &= \min(a,0) = \frac{1}{2}(a-|a|). \end{aligned}$$

These yield

$$\begin{aligned} &\text{if} \;\; a>0 \;\; \text{then } a^- = 0, \\ &\text{if} \;\; a<0 \;\; \text{then } a^+ = 0. \end{aligned}$$

Then write

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[a^+(u_i^n - u_{i-1}^n) + a^-(u_{i+1}^n - u_i^n)\right].$$

This is first-order accurate upwind scheme for scaler form of Euler equations.

Flux splitting

Scalar problem

Split the convetive flux $u\phi$ as

$$u\phi = f^+ + f^-,$$

where

$$f^+ = \frac{1}{2}(u + |u|)\phi,$$
$$f^- = \frac{1}{2}(u - |u|)\phi.$$

If $u > 0$ then $f^+ = u\phi,\ f^- = 0$

If $u < 0$ then $f^+ = 0,\ f^- = u\phi$

Then, one can write

$$[(u\phi)_x]_i = (f_x^+ + f_x^-)_i \approx \frac{f_i^+ - f_{i-1}^+}{\Delta x} + \frac{f_{i+1}^+ - f_i^+}{\Delta x} \quad \text{(first-order accurate).}$$

Euler equation

1D case: $\dfrac{\partial \underset{\sim}{U}}{\partial t} + \dfrac{\partial \underset{\sim}{F}}{\partial x} = 0.$

$$\underset{\sim}{U} = \begin{pmatrix} \rho \\ \\ \rho u \\ \\ E_t \end{pmatrix}, \qquad \underset{\sim}{F} = \begin{pmatrix} \rho u \\ \\ \rho u^2 + p \\ \\ (E_t + p)u \end{pmatrix}.$$

Can be written as $\dfrac{\partial \underset{\sim}{U}}{\partial t} + \underset{\approx}{A} \dfrac{\partial \underset{\sim}{U}}{\partial x} = 0$, where $\underset{\approx}{A} = \dfrac{\partial \underset{\sim}{F}}{\partial \underset{\sim}{U}}$.

One can write $\underset{\sim}{F} = \underset{\approx}{A}\underset{\sim}{U}$.

Apply eigendecomposition on $\underset{\approx}{A} \Rightarrow \underset{\approx}{A} = \underset{\approx}{T}\underset{\approx}{\Lambda}\underset{\approx}{T}^{-1}$, where

$\underset{\approx}{T}$ is a matrix with eigenvectors of $\underset{\approx}{A}$ as columns.

$\underset{\approx}{\Lambda}$ is a matrix with eigenvalues of $\underset{\approx}{A}$ on diagonal.

Decompose $\underset{\approx}{\Lambda}$ into $\underset{\approx}{\Lambda}^{+} + \underset{\approx}{\Lambda}^{-}$, where

$\underset{\approx}{\Lambda}^{+}$ contains only positive elements (associated with right-running characteristics)

$\underset{\approx}{\Lambda}^{-}$ contains only negative elements (associated with left-running characteristics)

Here, the eigenvalues of matrix $\underset{\approx}{A}$ are $(u+c,\ u,\ u-c)$ which yield

$$\Lambda^{+} = \begin{bmatrix} u+c & & \\ & u & \\ & & 0 \end{bmatrix}, \quad \Lambda^{-} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & u-c \end{bmatrix}$$

Then, we have $\underset{\approx}{A} = \underset{\approx}{T}\underset{\approx}{\Lambda}^{+}\underset{\approx}{T}^{-1} + \underset{\approx}{T}\underset{\approx}{\Lambda}^{-}\underset{\approx}{T}^{-1} = \underset{\approx}{A}^{+} + \underset{\approx}{A}^{-}$

$$\Rightarrow \underset{\sim}{F} = \underset{\approx}{A}^{+}\underset{\sim}{U} + \underset{\approx}{A}^{-}\underset{\sim}{U} = \underset{\sim}{F}^{+} + \underset{\sim}{F}^{-}.$$

Now we can write

$$\frac{\partial \underset{\sim}{U}}{\partial t} + \frac{\partial \underset{\sim}{F}^{+}}{\partial x} + \frac{\partial \underset{\sim}{F}^{-}}{\partial x} = 0.$$

$\underset{\sim}{F}^{+}$ and $\underset{\sim}{F}^{-}$ are associated with wave propagation in positive and negative directions, respectively. The discretized equations are then

$$\underset{\sim}{U}_{i}^{n+1} = \underset{\sim}{U}_{i}^{n} - \frac{\Delta t}{\Delta x}\left[(\underset{\sim}{F}_{i}^{+} - \underset{\sim}{F}_{i-1}^{+}) + (\underset{\sim}{F}_{i+1}^{-} - \underset{\sim}{F}_{i}^{-})\right].$$

This figure illustrates the idea behind the flux-splitting scheme discussed above.