Problem Set A

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1. (a) i. Introduce a test function v, that satisfies:

$I = (0,1)$. $\quad v \in V = \{ v : \|v\|_{L^2(I)} < \infty, \|v'\|_{L^2(I)} < \infty, v(0) = 0 \}$

multiply the test function and perform an integration by part.

$$\int_0^1 -(a(x) \cdot u')' \cdot v \cdot dx = 0.$$

$$LHS = -a(x) \cdot u' \cdot v \big|_0^1 + \int_0^1 a(x) \cdot u' \cdot v' \cdot dx$$

$$= \int_0^1 a(x) \cdot u' \cdot v' \cdot dx - (a(1) \cdot u'(1) \cdot v(1) - a(0) \cdot u'(0) \cdot v(0)).$$

$$= \int_0^1 a(x) \cdot u' \cdot v' dx - 2v(1).$$

Therefore, the exact solution u is:

fin $u \in V$. such that:

$$\int_0^1 a(x) \cdot u' \cdot v' dx = 2v(1). \quad \forall v \in V$$

1. (b). Introduce a mesh on the interval I, consisting n subintervals, and the corresponding space V_h of all continuous piecewise linears. Also introduce the subspace $V_{h,0}$ of V_h that satisfies the boundary conditions:

$$V_{h,0} = \{ v \in V_h, v(0) = 0 \}$$

Then the finite element approximation is:

find $u_h \in V_{h,0}$, such that:

$$\cancel{\int_0^1 a(x) \cdot u_h' \cdot v' dx - 2v(1)}$$

$$\int_0^1 a(x) \cdot u_h' \cdot v' dx = 2v(1). \quad \forall v \in V_{h,0}$$

Basis of $V_{h,0}$ is given by the set of n hat functions. $\{\varphi_i\}_{i=1}^n$ defined as.

$$\varphi_i(x). \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in (x_{i-1}, x_i) \\[2ex] \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in (x_i, x_{i+1}). \end{cases}$$

Thus, the problem could be equivalent to find.

$u_h \in V_{h,0}$,

$$\int_0^1 a(x) \cdot u_h' \cdot \varphi_i' \cdot dx = 2\varphi_i(1). \quad \forall i=1,2,\dots n.$$

Since $u_h \in V_{h,0}$, so.

$$u_h = \sum_{j=1}^{n} \xi_i \cdot \varphi_i$$

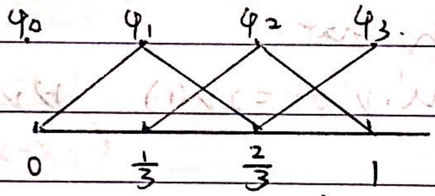
Thus:

$$\int_0^1 a(x) \cdot \sum_{j=1}^{n} \xi_j \cdot \varphi_j' \cdot \varphi_i' \cdot dX = 2\varphi_i(1).$$

$$\sum_{j=1}^{n} \xi_j \cdot \int_0^1 a \cdot \varphi_j' \cdot \varphi_i' \cdot dx = 2\varphi_i(1). \quad \forall i = 1,2,\dots,n.$$

Introduce the notation: $\begin{cases} A_{ij} = \int_I a \cdot \varphi_j' \cdot \varphi_i' \cdot dX \\ r_i = 2\varphi_i(1). \end{cases}$

Then $A \cdot \xi = b$.



for $|i-j|>1$, $A_{ij}=0$. $\quad h=\frac{1}{3}$

Then, as for, A_{ij}. $i=j$.

$$\begin{cases} A_{11} = \frac{1}{h^2}\int_0^{\frac{2}{3}} (1+x)\cdot dx = 9\times(x+\frac{x^2}{2})\Big|_0^{\frac{2}{3}} = 9\times(\frac{2}{3}+\frac{4}{18}) = 8 \\ A_{22} = \frac{1}{h^2}\int_{\frac{1}{3}}^{1} (1+x)\, dx = 9\times(x+\frac{x^2}{2})\Big|_{\frac{1}{3}}^{1} = 9\times(\frac{3}{2}-\frac{1}{3}-\frac{1}{18}) = 10. \\ A_{33} = \frac{1}{h^2}\int_{\frac{2}{3}}^{1} (1+x)\cdot dx = 9\times(x+\frac{x^2}{2})\Big|_{\frac{2}{3}}^{1} = 9\times(\frac{1}{6}) = \frac{11}{2} \end{cases}$$

for A_{ij}. $i \neq j$ $|i-j|=1$

$$A_{12} = A_{21} = -\frac{1}{h^2}\cdot\int_{\frac{1}{3}}^{\frac{2}{3}} (1+x)\, dx = -9\cdot(x+\frac{x^2}{2})\Big|_{\frac{1}{3}}^{\frac{2}{3}} = -\frac{9}{2}.$$

$$A_{23} = A_{32} = -\frac{1}{h^2}\int_{\frac{2}{3}}^{1} (1+x)\cdot dx = -9\cdot(x+\frac{x^2}{2})\Big|_{\frac{2}{3}}^{1} = -\frac{11}{2}$$

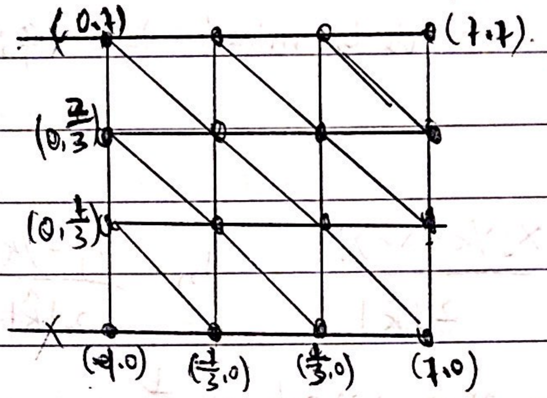
for b_i.

$$\begin{cases} b_1 = 0 \\ b_2 = 0. \\ b_3 = 2\varphi_3(1) = 2. \end{cases}$$

Therefore, $A = \frac{1}{2}\begin{bmatrix} 16 & -9 & 0 \\ -9 & 20 & -11 \\ 0 & -11 & 11 \end{bmatrix}$. $\quad b = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

2. $\begin{cases} -\Delta u(x) = 2. & x \in \Omega. \quad x = (x_1, x_2)^T \\ u(x) = 0. & x \in \partial\Omega. \end{cases}$

(a): Introduce a triangulation $k = \{k\}$, where k is a set of uniform triangles as shown in the figure below:



Use this as mesh T_h, 4 internal nodes.

(b): $V_h := \{v : v \in C^0(\Omega),\ v|_k \in P_1(k),\ \forall k \in T_h,\ v|_{\partial\Omega} = 0\}$.

(c): $V := \{v : \|v\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} < \infty,\ v|_{\partial\Omega} = 0\}$.

$-\int_\Omega \Delta u \cdot v \cdot dx = \int_\Omega 2v \cdot dx \quad \forall \cdot v \in V$.

Green's formula:

$-\int_\Omega \Delta u \cdot v \cdot dx = \int_\Omega \nabla u \cdot \nabla v \cdot dx - \int_{\partial\Omega} n \cdot \nabla u \cdot \overset{0}{\cancel{v}} \cdot ds.$

So. $\int_\Omega \nabla u \cdot \nabla v \cdot dx = \int_\Omega 2v \cdot dx \quad \forall v \in V$.

Thus, the finite element approximation is:

Find $u_h \in V_h$, such that.

$\int_\Omega \nabla u_h \cdot \nabla v \cdot dx = \int_\Omega 2v \cdot dx \quad \forall \cdot v \in V_h$.

(d): compute the stiffness matrix.

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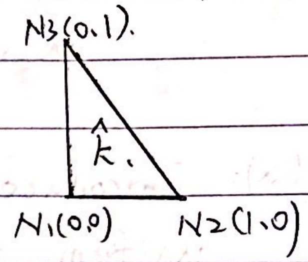
Each function is a linear function on $\hat{k}$, Name nodes of $\hat{k}$ as



$$\begin{cases} N_1 = (x_1, y_1) = (0,0). \\ N_2 = (x_2, y_2) = (1,0) \\ N_3 = (x_3, y_3) = (0,1). \end{cases}$$

$$\varphi_i = a_i + b_i x + c_i y \quad \forall\, i = 1,2,3.$$

the coefficients are determined by $\varphi_i(N_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Thus, we have:

$$a_i = \frac{x_j y_k - x_k y_j}{2|\hat{k}|} \qquad b_i = \frac{y_j - y_k}{2|\hat{k}|} \qquad c_i = \frac{x_k - x_j}{2|\hat{k}|}.$$

Here $|\hat{k}|$ is the area, which is $\frac{1}{2}$,

therefore:

$$a_1 = x_2 y_3 - x_3 y_2 = 1 \qquad b_1 = y_2 - y_3 = -1 \qquad c_1 = x_3 - x_2 = -1$$

$$a_2 = x_3 y_1 - x_1 y_3 = 0 \qquad b_2 = y_3 - y_1 = 1 \qquad c_2 = x_1 - x_3 = 0$$

$$a_3 = x_1 y_2 - x_2 y_1 = 0. \qquad b_3 = y_1 - y_2 = 0 \qquad c_3 = x_2 - x_1 = 1$$

$$\begin{cases} \varphi_1 = 1 - x - y \\ \varphi_2 = x \\ \varphi_3 = y \end{cases}$$

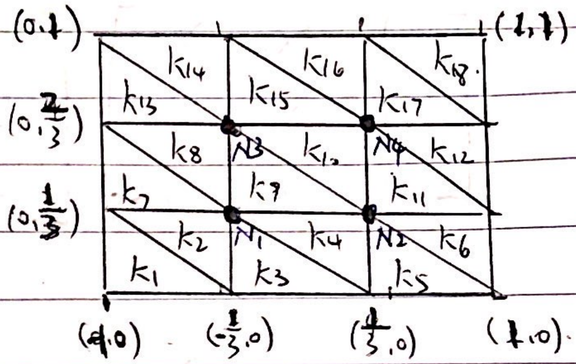
$$A^{\hat{k}}_{ij} = \int_{\hat{k}} \nabla\varphi_i \cdot \nabla\varphi_j \, dx = (b_i b_j + c_i c_j) \int_{\hat{k}} dx$$

$$= (b_i b_j + c_i c_j) \cdot \hat{k} \qquad \forall\, i,j = 1,2,3$$

$$b^{\hat{k}}_i = \int_{\hat{k}} 2\varphi_i \, dx = 2 \cdot \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} \varphi_i \, dx \, dy$$

$$A = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} \frac{2}{6} \\ \frac{2}{6} \\ \frac{2}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

e: Given the mesh in 2a, space V_h in 2b, and weak form in 2c.



internal nodes are N_1, N_2, N_3, N_4,

$A_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx \quad \forall i,j = 1,2,3,4.$

Then, global mesh.

$$A_{11} = A_{11}^{k_2} + A_{11}^{k_3} + A_{11}^{k_4} + A_{11}^{k_7} + A_{11}^{k_8} + A_{11}^{k_9}$$

$A_{ij}^{k} = b_i b_j + c_i c_j \, |k|$ (element).

$$\phi_{i,j}(x,y) = \begin{cases} 1 - \frac{1}{h_1}(x-x_i) - \frac{1}{h_2}(y-y_j) & (x,y) \in k_9 \\ 1 - \frac{1}{h_2}(y-y_j) & (x,y) \in k_8 \\ 1 + \frac{1}{h_1}(x-x_i) & (x,y) \in k_7 \\ 1 + \frac{1}{h_1}(x-x_i) + \frac{1}{h_2}(y-y_i) & (x,y) \in k_2 \\ 1 + \frac{1}{h_2}(y-y_j) & (x,y) \in k_3 \\ 1 - \frac{1}{h_1}(x-x_i) & (x,y) \in k_4 \end{cases}$$

$h_1 = \frac{2}{3} \qquad h_2 = \frac{1}{3}$

$$A_{11} = A_{11}^{k_2} + A_{11}^{k_3} + A_{11}^{k_4} + A_{11}^{k_7} + A_{11}^{k_8} + A_{11}^{k_9}$$

$$= \left[\left(\frac{1}{h_1}\right)^2 + \left(\frac{1}{h_2}\right)^2 + \left(\frac{1}{h_2}\right)^2 + \left(\frac{1}{h_2}\right)^2 + \left(\frac{1}{h_1}\right)^2 + \left(\frac{1}{h_2}\right)^2 + \left(\frac{1}{h_1}\right)^2 + \left(\frac{1}{h_2}\right)^2\right] \cdot k$$

$$= \cancel{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 +} -$$

$$= \left[\left(\frac{3}{2}\right)^2 \times 4 + 3^2 \times 4\right] \times \frac{1}{9} = 5$$

the same for $A_{22}, A_{33}, A_{44} = 5$.

$$A_{12} = A_{12}^{k_4} + A_{12}^{k_9}$$

$$= \left(\begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix}^T \cdot \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}^T \cdot \begin{bmatrix} \frac{3}{2} \\ 3 \end{bmatrix}\right) \times \frac{2}{9} = 1.$$

$A_{13} = A^{k_8} + A^{k_9}.$

3. $-\nabla\cdot(k(x)\nabla u(x)) = f(x)$. $x\in\Omega\subset\mathbb{R}^3$. ~~(Not finished)~~.

$\beta\partial_n u(x) + \gamma u(x) = g(x)$. $x\in\partial\Omega$.

$\partial_n u = \nabla u\cdot\hat{n}$,

(a). State the Lax-Milgram theorem,

Let V be a Hilbert space with inner product $(\cdot,\cdot)$, and let $a(\cdot,\cdot)$ be a coercive continuous bilinear form on V, and let $l(\cdot)$ be a continuous linear form on V. Then, there exist a unique solution $u\in V$ to the abstract variational problem: find $u\in V$ such that

$$a(u,v) = l(v). \quad \forall v\in V.$$

(b) i: case: $\beta=0$, $\gamma=1$. $g=0$. $f\in L_2(\Omega)$.

Then. $u(x)=0$.

In this case, the boundary condition simplifies to $\nabla u\cdot\hat{n}=0$ on $\partial\Omega$

~~(c)~~. The bilinear form can be defined as:

$a(u,v) = \int_\Omega \nabla u\cdot\nabla v\, dx$.

~~Since $\beta=0$, the coercity condition $a(u,v)\geq\alpha\|u\|^2$ is so~~

ii: $\beta=1$, $\gamma=0$. $g=0$. $f\in L_2(\Omega)$.

$\frac{\partial u}{\partial n}=0$ on $\partial\Omega$, this is homogeneous Neuman boundary condition,

$a(u,v) = \int_\Omega \nabla u\cdot\nabla v\, dx$.

(c). Finite Element Formulation:

Find $u_h\in V_h$.

$a(u_h, v_h) = L(v_h)$ for all $v_h\in V_h$.

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