# derivation of weak form for heat transfer equation

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## 1 derivation

Solving time-dependent PDE can use a finite difference method. We introduce $n$ which is a integer counting time steps. $T^n$ means $T$ at time step $n$. Then the heat transfer equation is

$$(\rho c \frac{\partial T}{\partial t})^n = k\nabla^2 T^n + f^n. \tag{1}$$

The time derivative in right hand side $\partial t$ can be approximated by a difference quotient. Here mainly has two methods, Euler forward or Euler backward. The Euler forward method is

$$(\frac{\partial T}{\partial t})^n = \frac{T^{n+1} - T^n}{\Delta t}. \tag{2}$$

Then take equation2 to the equation1:

$$\rho c \frac{T^{n+1} - T^n}{\Delta t} = k\nabla^2 T^n + f^n. \tag{3}$$

Extract $T^{n+1}$, then

$$T^{n+1} = \frac{\Delta t k}{\rho c}\nabla^2 T^n + \frac{\Delta t}{\rho c} f^n + T^n, \tag{4}$$

which is also called explicit Euler. This method to calculate $T^{n+1}$ needs all the information of $T^n$. Then we use FEM, first, multiply by the test function of $v \in \dot{V}$

$$\int_\Omega T^{n+1} v = \int_\Omega \frac{\Delta t k}{\rho c}\nabla^2 T^n v + \int_\Omega \frac{\Delta t}{\rho c} f^n v + \int_\Omega T^n v. \tag{5}$$

For the first item on right hand side,

$$\frac{\Delta t k}{\rho c}\int_\Omega \nabla^2 T^n v = \frac{\Delta t k}{\rho c}(\int_\Omega \nabla \cdot (\nabla T^n v) - \int_\Omega \nabla T^n \cdot \nabla v). \tag{6}$$

and for the first item in right hand side of equation6, based on divergence theorem, we get

$$\frac{\Delta tk}{\rho c}\int_\Omega \nabla^2 T^n v = \frac{\Delta tk}{\rho c}(\int_{\partial\Omega}(\nabla T^n v)\cdot \vec{n} ds - \int_\Omega \nabla T^n \cdot \nabla v). \tag{7}$$

The heat mechanism includes heat conduction, convection and radiation, so

$$-\int_{\partial\Omega}(k\nabla T v)\cdot \vec{n}\ ds = \int_{\partial\Omega} qvds + \int_{\partial\Omega} h(T-T_a)vds + \int_{\partial\Omega}\epsilon\sigma(T^4-T_a^4)vds, \tag{8}$$

where $q$ is the heat source, $h$ is the convection heat transfer coefficient, $T_a$ is the ambient temperature. Take the equation8 to equation7, We get

$$\frac{\Delta tk}{\rho c}\int_\Omega \nabla^2 T^n v = -\frac{\Delta t}{\rho c}(\int_{\partial\Omega} qvds + \int_{\partial\Omega} h(T-T_a)vds + \int_{\partial\Omega}\epsilon\sigma(T^4-T_a^4)vds) - \frac{\Delta tk}{\rho c}\int_\Omega \nabla T^n \cdot \nabla v. \tag{9}$$

Taking the equation9 to equation5, we get

$$\begin{aligned}\int_\Omega T^{n+1}v = -\frac{\Delta t}{\rho c}(\int_{\partial\Omega} qvds + \int_{\partial\Omega} h(T^n-T_a)vds + \int_{\partial\Omega}\epsilon\sigma(T^{n4}-T_a^4)vds)\\ -\frac{\Delta tk}{\rho c}\int_\Omega \nabla T^n\cdot\nabla v + \frac{\Delta t}{\rho c}\int_\Omega f^n v + \int_\Omega T^n v.\end{aligned} \tag{10}$$

Writing in the bilinear and linear form:

$$a(T,v) = \int_\Omega \frac{\rho c}{\Delta t} T v dx, \tag{11}$$

$$L(v) = \int_\Omega f^n v dx + \int_\Omega \frac{\rho c}{\Delta t}T^n v dx - \int_{\partial\Omega} qvds + \int_{\partial\Omega} h(T_a - T^n)vds + \int_{\partial\Omega}\epsilon\sigma(T_a^4 - T^{n4})vds - \int_\Omega k\nabla T\cdot\nabla v dx. \tag{12}$$

Another method is Euler backward, or implicit method

$$(\frac{\partial T}{\partial t})^{n+1} = \frac{T^{n+1}-T^n}{\Delta t}. \tag{13}$$

The same procedure as the above steps for the explicit method, we get

$$\int_\Omega T^{n+1}v - \int_{\partial\Omega}\frac{\Delta tk}{\rho c}(\nabla T^{n+1}v)\cdot n\ ds + \int_\Omega \frac{\Delta tk}{\rho c}\nabla T^{n+1}\cdot\nabla v = \int_\Omega \frac{\Delta t}{\rho c} f^{n+1}v + \int_\Omega T^n v, \tag{14}$$

$$\begin{aligned}\int_\Omega T^{n+1}v + \int_{\partial\Omega}\frac{\Delta t}{\rho c}qvds + \int_{\partial\Omega}\frac{\Delta t}{\rho c}h(T^{n+1}-T_a)vds + \int_{\partial\Omega}\frac{\Delta t}{\rho c}\epsilon\sigma((T^{n+1})^4 - T_a^4)vds + \int_\Omega \frac{\Delta tk}{\rho c}\nabla T^{n+1}\cdot\nabla v\\ = \int_\Omega \frac{\Delta t}{\rho c}f^{n+1}v + \int_\Omega T^n v,\end{aligned} \tag{15}$$

change the position, then we get

$$a(T, v) = L_{n+1}(v), \tag{16}$$

where

$$a(T, v) = \int_{\Omega} \frac{\rho c}{\Delta t} T v dx + \int_{\Omega} k \nabla T \cdot \nabla v dx + \int_{\partial\Omega} hTv ds + \int_{\partial\Omega} \epsilon\sigma T^4 v ds, \tag{17}$$

$$L(v) = \int_{\Omega} f^{n+1} v dx + \int_{\Omega} \frac{\rho c}{\Delta t} T^n v dx - \int_{\partial\Omega} qv ds + \int_{\partial\Omega} hT_a v ds + \int_{\partial\Omega} \epsilon\sigma T_a^4 v ds. \tag{18}$$