derivation of weak form for heat transfer equation

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All of the code used in this research is available on github.com/Yanjun96/thermomechanical.

Solving time-dependent PDE can use a finite difference method. We introduce n which is a integer counting time steps. T^n means T at time step n. Then the heat transfer equation is

$$(\rho c \frac{\partial T}{\partial t})^n = k \nabla^2 T^n + f^n. \tag{1}$$

The time derivative in right hand side ∂t can be approximated by a difference quotient. Here mainly has two methods, Euler forward or Euler backward. The Euler forward method is

$$(\frac{\partial T}{\partial t})^n = \frac{T^{n+1} - T^n}{\Delta t}. \tag{2}$$

Then take equation2 to the equation1:

$$\rho c \frac{T^{n+1} - T^n}{\Delta t} = k \nabla^2 T^n + f^n. \tag{3}$$

Extract T^{n+1}, then

$$T^{n+1} = \frac{\Delta t k}{\rho c} \nabla^2 T^n + \frac{\Delta t}{\rho c} f^n + T^n, \tag{4}$$

which is also called explicit Euler. This method to calculate T^{n+1} needs all the information of T^n. Then we use FEM, first, multiply by the test function of $v \in \dot{V}$

$$\int_\Omega T^{n+1} v = \int_\Omega \frac{\Delta t k}{\rho c} \nabla^2 T^n v + \int_\Omega \frac{\Delta t}{\rho c} f^n v + \int_\Omega T^n v. \tag{5}$$

For the first item on right hand side,

$$\frac{\Delta t k}{\rho c} \int_\Omega \nabla^2 T^n v = \frac{\Delta t k}{\rho c} (\int_\Omega \nabla \cdot (\nabla T^n v) - \int_\Omega \nabla T^n \cdot \nabla v). \tag{6}$$

and for the first item in right hand side of equation6, based on divergence theorem, we get

$$\frac{\Delta t k}{\rho c} \int_\Omega \nabla^2 T^n v = \frac{\Delta t k}{\rho c} \left(\int_{\partial\Omega} (\nabla T^n v) \cdot \vec{n} ds - \int_\Omega \nabla T^n \cdot \nabla v \right). \tag{7}$$

The heat mechanism includes heat conduction, convection and radiation, so

$$-\int_{\partial\Omega} (k\nabla T v) \cdot \vec{n} \ ds = \int_{\partial\Omega} q v ds + \int_{\partial\Omega} h(T - T_a) v ds + \int_{\partial\Omega} \epsilon\sigma(T^4 - T_a^4) v ds, \tag{8}$$

where q is the heat source, h is the convection heat transfer coefficient, T_a is the ambient temperature. Take the equation8 to equation7, We get

$$\frac{\Delta t k}{\rho c} \int_\Omega \nabla^2 T^n v = -\frac{\Delta t}{\rho c} \left(\int_{\partial\Omega} q v ds + \int_{\partial\Omega} h(T - T_a) v ds + \int_{\partial\Omega} \epsilon\sigma(T^4 - T_a^4) v ds \right) - \frac{\Delta t k}{\rho c} \int_\Omega \nabla T^n \cdot \nabla v. \tag{9}$$

Taking the equation9 to equation5, we get

$$\int_\Omega T^{n+1} v = -\frac{\Delta t}{\rho c} \left(\int_{\partial\Omega} q v ds + \int_{\partial\Omega} h(T^n - T_a) v ds + \int_{\partial\Omega} \epsilon\sigma(T^{n4} - T_a^4) v ds \right)$$
$$-\frac{\Delta t k}{\rho c} \int_\Omega \nabla T^n \cdot \nabla v + \frac{\Delta t}{\rho c} \int_\Omega f^n v + \int_\Omega T^n v. \tag{10}$$

Writing in the bilinear and linear form:

$$a(T, v) = \int_\Omega \frac{\rho c}{\Delta t} T v dx, \tag{11}$$

$$L(v) = \int_\Omega f^n v dx + \int_\Omega \frac{\rho c}{\Delta t} T^n v dx - \int_{\partial\Omega} q v ds + \int_{\partial\Omega} h(T_a - T^n) v ds + \int_{\partial\Omega} \epsilon\sigma(T_a^4 - T^{n4}) v ds - \int_\Omega k\nabla T \cdot \nabla v dx. \tag{12}$$

Another method is Euler backward, or implicit method

$$\left(\frac{\partial T}{\partial t}\right)^{n+1} = \frac{T^{n+1} - T^n}{\Delta t}. \tag{13}$$

The same procedure as the above steps for the explicit method, we get

$$\int_\Omega T^{n+1} v - \int_{\partial\Omega} \frac{\Delta t k}{\rho c} (\nabla T^{n+1} v) \cdot n \ ds + \int_\Omega \frac{\Delta t k}{\rho c} \nabla T^{n+1} \cdot \nabla v = \int_\Omega \frac{\Delta t}{\rho c} f^{n+1} v + \int_\Omega T^n v, \tag{14}$$

$$\int_\Omega T^{n+1} v + \int_{\partial\Omega} \frac{\Delta t}{\rho c} q v ds + \int_{\partial\Omega} \frac{\Delta t}{\rho c} h(T^{n+1} - T_a) v ds + \int_{\partial\Omega} \frac{\Delta t}{\rho c} \epsilon\sigma((T^{n+1})^4 - T_a^4) v ds + \int_\Omega \frac{\Delta t k}{\rho c} \nabla T^{n+1} \cdot \nabla v$$
$$= \int_\Omega \frac{\Delta t}{\rho c} f^{n+1} v + \int_\Omega T^n v, \tag{15}$$

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change the position, then we get

$$a(T, v) = L_{n+1}(v), \tag{16}$$

where

$$a(T, v) = \int_\Omega \frac{\rho c}{\Delta t} T v dx + \int_\Omega k \nabla T \cdot \nabla v dx + \int_{\partial \Omega} h T v ds + \int_{\partial \Omega} \epsilon \sigma T^4 v ds, \tag{17}$$

$$L(v) = \int_\Omega f^{n+1} v dx + \int_\Omega \frac{\rho c}{\Delta t} T^n v dx - \int_{\partial \Omega} q v ds + \int_{\partial \Omega} h T_a v ds + \int_{\partial \Omega} \epsilon \sigma T_a^4 v ds. \tag{18}$$