COMP 182 HW 6 Problem 2

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1.1

Let S be the random variable that shows what is sent from the space probe. $S \in \{0, 1\}$
Let R be the random variable that shows what is received by the earth. $R \in \{0, 1\}$
Then, $P(R = 0) = P(R = 0 \cap S = 0) + P(R = 0 \cap S = 1) = P(R = 0|S = 0) \times P(S = 0) + P(R = 0|S = 1) \times P(S = 1) = 0.9 \times (2/3) + 0.2 \times (1/3) = 2/3$

1.2

According to Bayes' Theorem:
$P(S = 0|R = 0) = P(S = 0 \cap R = 0)/P(R = 0) = [P(R = 0|S = 0) \times P(S = 0)]/P(R = 0) = [0.9 \times (2/3)]/(2/3) = 0.9$

2

Since E and F are independent events, we get $P(E \cap F) = P(E) \times P(F) = (1 - P(\overline{E})) \times (1 - P(\overline{E})) = 1 - P(\overline{E}) - P(\overline{F}) + P(\overline{E}) \times P(\overline{F})$

Based on the theorem, $P(E \cap F) = 1 - P(\overline{E} \cup \overline{F}) = 1 - (P(\overline{E}) + P(\overline{F}) - P(\overline{E} \cap \overline{F})) = 1 - P(\overline{E}) - P(\overline{F}) + P(\overline{E} \cap \overline{F})$

Since $1 - P(\overline{E}) - P(\overline{F}) + P(\overline{E}) \times P(\overline{F}) = 1 - P(\overline{E}) - P(\overline{F}) + P(\overline{E} \cap \overline{F})$, we got $P(\overline{E} \cap \overline{F}) = P(\overline{E}) \times P(\overline{F})$. Thus, $\overline{E}, \overline{F}$ are also independent events.

3

3.1

We can write it as $P(A \cap B) = P(B|A) \times P(A) \leq P(A)$ and also $P(A \cap B) = P(A|B) \times P(B) \leq P(B)$. We can see that the probability of the intersection cannot be bigger than the two margin probabilities. Thus, the largest probability will be the smaller probability between $P(A)$ and $P(B)$. We get $P(A \cap B) = P(B) = 1/2$. In the case of the largest probability, $P(A|B) = 1$, which means that the occurrence of A is ensured by the occurrence of B: $B \subset A$. For example, A is the event that it is not raining tomorrow. B is the event that it is sunny tomorrow. If tomorrow is sunny, it is surely not raining, so the probability of tomorrow being both sunny and not raining is equal to the probability that tomorrow is sunny.

For the smallest probability, we use the fact that $P(A \cap B) = P(A) + P(B) - P(A \cup B)$. We know that $P(A \cup B) \leq 1$. Thus, we get the inequality: $P(A) + P(B) - P(A \cap B) = P(A \cup B) \leq 1$. Plug in $P(A)$ and $P(B)$: $2/3 + 1/2 - P(A \cap B) \leq 1$, we got: $P(A \cap B) \geq 1/6$. Thus, the smallest probability for $P(A \cap B)$ is 1/6. This happens when $P(A \cup B) = 1$, which means that either A or B will surely happens. For example, when I throw a die, A is the event that I got a number bigger than 2, B is the event that I got a number smaller or equal to 3. The probability that the number is both bigger than 2 and smaller or equal to 3 is 1/6.

3.2

As said above, the largest probability of $P(A \cup B)$ is 1. This happens when either A or B or both of them surely happens. As the example above, A is the event that I got a number bigger than 2, B is the event that I got a number smaller or equal to 3. The number we get from a die will either be bigger than 2 or be smaller or equal to 3, so $P(A \cup B) = 1$.

For the smallest probability, we use the fact that $P(A \cap B) = P(A) + P(B) - P(A \cup B)$, so $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. In order to get the smallest $P(A \cup B)$, we want to minimize $P(A \cap B)$. As illustrated in the last problem, $P(A \cap B)_{max} = 1/2$. Thus $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 2/3 + 1/2 + 1/2 = 2/3$. Using the example from above, the probability of tomorrow being either sunny or not raining is equal to the probability that tomorrow is not raining ($B \subset A$).

4

We use the formula for Bernoulli trails: $P = \binom{n}{k} p^k (1-p)^{n-k}$

4.1

no success = fails n times: $P = \binom{n}{0} p^0 (1-p)^{n-0} = (1-p)^n$

4.2

P(at least one success) = 1 - P(no success): $P = 1 - (1-p)^n$

4.3

P(at most one success) = P(no success) + P(one success): $P = (1-p)^n + \binom{n}{1}p^1(1-p)^{n-1} = (1-p)^n + np(1-p)^{n-1}$

4.4

P(at least two successes) = 1 - (P(no success) + P(one success)): $P = 1-((1-p)^n+np(1-p)^{n-1})$

5

Based on the definition of expected value:
$E(Z) = \sum_{s \in S} max(X(s), Y(s)) \times P(max(X(s), Y(s)))$

$E(X) + E(Y) = \sum_{s \in S} X(s) \times P(X(s)) + Y(s) \times P(Y(s))$

For each $s \in S$, the term $max(X(s), Y(s)) \times P(max(X(s), Y(s)))$ is either $X(s) \times P(X(s))$ or $Y(s) \times P(Y(s))$.
Since $X(s) \times P(X(s)) \leq X(s) \times P(X(s)) + Y(s) \times P(Y(s))$ and $Y(s) \times P(Y(s)) \leq X(s) \times P(X(s)) + Y(s) \times P(Y(s))$ for all $s \in S$, we get that $E(Z) \leq E(X) + E(Y)$

6

No, they are not independent.
If they are independent, then for all $s \in S$, $P(X(s) \cap P(Y(s))) = P(X(s)) \times P(Y(s))$
We let $s = 2$. The probability of us getting 2 heads when flipping two coins is $X(2) = 1/4$. The probability of us getting 2 tails when flipping two coins is $Y(2) = 1/4$. However, the probability of us getting both 2 heads and 2 tails when flipping two coins is $P(X(2) \cap P(Y(2))) = 0 \neq P(X(2)) \times P(Y(2)) = 1/16$. Thus, the two variables are not independent.

7

First of all, we set up a Bernoulli trail model for this problem, $X \sim B(n = 10, p = 1/6)$.
Then, we know that the variance of Bernoulli trail is equal to the variance of a single trail times the number of trails we take. For a single Bernoulli trail, $V = p(1-p) = 5/36$. For ten Bernoulli trails, its variance equals to $V = np(1-p) = 25/18$

8

Let X and Y be the random variables that count the number of heads and the number of tails that come up when two fair coins are flipped. X and Y are not independent, as proved above.

Let $Z = X + Y$, which counts the number of heads plus tails that come up when two fair coins are flipped. For Z, the sample space is only $Z \in \{2\}$ and the probability distribution is $\{(2, 1)\}$. The sum of heads and tails from flipping a fair coin twice has to be 2. Thus, $V(Z) = \sum_{s \in S}(Z(s) - E(Z))^2 \cdot P(Z(s)) = \sum_{s \in S}(2 - 2)^2 \cdot 1 = 0$

However, for X and Y, the situation is different. $E(X) = E(Y) = 0 \cdot (1/4) + 1 \cdot (1/2) + 2 \cdot (1/4) = 1$
$V(X) = \sum_{s \in S}(X(s) - E(X))^2 \cdot P(X(s)) = (0-1)^2 \cdot (1/4) + (1-1)^2 \cdot (1/2) + (2-1)^2 \cdot (1/4) = 1/2$
$V(Y) = \sum_{s \in S}(Y(s) - E(Y))^2 \cdot P(Y(s)) = (0-1)^2 \cdot (1/4) + (1-1)^2 \cdot (1/2) + (2-1)^2 \cdot (1/4) = 1/2$

Thus, $V(X) + V(Y) = 1 \neq V(X + Y) = 0$

9

Based on the Chebyshev's inequality: $P(|X(s) - E(X)| \geq r) \leq \frac{V}{r^2}$

We want the probability that the number of tails deviates from the mean by more than $\sqrt{n}$, so $r = \sqrt{n}$. We then need the variance.
$V(X) = E(X^2) - E(X)^2 = \sum_{s \in S} X(s)^2 \cdot P(X(s)) - (np)^2 = \sum_{k=1}^{n} k^2 \cdot [\binom{n}{k}(0.4)^k(0.6)^{n-k}] - (0.4n)^2$

Thus, $P(|X(s) - E(X)| \geq \sqrt{n}) \leq \frac{\sum_{k=1}^{n} k^2 \cdot [\binom{n}{k}(0.4)^k(0.6)^{n-k}] - (0.4n)^2}{n} = 0.36$