COMP 182 HW 6 Problem 1

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1

There are in total $2^8 - 2^4$ bit strings of length 10 either start with 000 or end with 111

Firstly, we consider the case when the first three bits are 000. In this case, we will have 7 "free" digits that can either be 0 or 1. Using product rule, there are 2^7 different bit strings starting with 000.
Likewise, there are also 7 "free" digits that can be 0 or 1 if we want to end with 111. Using product rule, there are in total 2^7 different bit strings ending with 111.
However, in the process we count the number of bit strings that both start with 000 and end with 111 twice. When we count the bit strings that start with 000, we include the cases when they also end with 111. When we count the bit strings that end with 111, we include the cases when they also start with 000. Thus, we need to subtract the redundant number of bit strings that both start with 000 and end with 111. In this case, there are in total 4 "free" digits that can either be 0 or 1. Thus, using product rule, there are in total 2^4 different bit strings both starting with 000 and ending with 111.
At last, we got $2^7 + 2^7 - 2^4 = 2^8 - 2^4 = 240$

2

There are $6\binom{8}{4} = 420$ positive integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 13.

Firstly, a positive integer that is less than 1,000,000 has six digits (we allow an integer to start with digit 0). If one of the digit equals to 9, the digit 9 can be at 6 different places.
We thus have 5 "free" digits to consider. The 5 digits have to sum up to $13 - 9 = 4$ (since the digit 9 has been fixed). We recall the "stars and walls" model. In this case, we have 4 stars and $5 - 1 = 4$ walls. We want to know how many different ways to insert those 4 walls into the 4 stars. Using the formula, we got $\binom{4+5-1}{5-1} = \binom{8}{4}$ different ways to arrange the 5 digits to let them sum up to 4. Since the other digit 9 can be at 6 different places, there are in total $6 \times \binom{8}{4}$ positive integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 13.

3

There are in total $\binom{13}{2}+\binom{12}{2}+\binom{11}{2}+\binom{10}{2}+\binom{9}{2}+\binom{8}{2}+\binom{7}{2}+\binom{6}{2}+\binom{5}{2}+\binom{4}{2}+\binom{3}{2}+\binom{2}{2}=364$ solutions that satisfy the inequality.

Firstly, there are in total 12 conditions for the inequality to be fulfilled. In order to let the sum of 3 variables be less than or equal to 11 (and bigger or equal to 0), we can have the 3 variables summing up to 12 different values: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
For each of the value, we use the "stars and walls" model to get the number of solutions. For example, for $x_1+x_2+x_3=11$, there are $\binom{11+3-1}{3-1}=\binom{13}{2}$ solutions. Likewise, we apply the logic to the 12 different values and sum them up: there are $\binom{13}{2}+\binom{12}{2}+\binom{11}{2}+\binom{10}{2}+\binom{9}{2}+\binom{8}{2}+\binom{7}{2}+\binom{6}{2}+\binom{5}{2}+\binom{4}{2}+\binom{3}{2}+\binom{2}{2}$ solutions that satisfy the inequality.

4

We can prove it by proof of contradiction.
We first assume that in any set of n+1 positive integers not exceeding 2n, all pairs of integers are not relatively prime. This means that for every two integer i, j in the set, there exists an common divider $m>1$ such that $i=am, j=bm$ and $i-j=(a-b)m=cm(c\geq 1, m\geq 2)$.
We can simplify the statement: for every pair of integer in the set, their difference $cm\geq 1\times 2=2$. Thus, we need those $n+1$ different positive integers to have a mutual difference greater or equal to 2. We now consider the best case: the set of positive integers are consecutive numbers with gaps 2. With the set of positive integers not exceeding 2n, we can have at most $2n/2=n$ integers that have a difference greater than or equal to 2 from each other integer. However, we want to pick $n+1$ positive integers from this set of n integers. According to the Pigeon Hole, there is at least one integer that is not in the set– that is, there is at least one integer that has a difference 1 with the other integer, which makes them relatively prime according to the definition. As a result, the assumption is false and the original statement is proved to be true.

5

5.1

We can prove it by proof of contradiction.
We first assume that the collection of all subsets with d elements of S is 2-colorable. This means that there exists a way of assigning colors in S such that all subsets with d elements of S have elements of both color. We assume that such a color assignment assigns k elements with the first color, and $2n-1-k$ elements with the second color. Since our collection includes all subsets with d elements of S, and we want to make sure that elements in all subsets are not from the same color, we have to let the number of elements of both color be smaller than the size of our subsets. This is because if we do have $d\geq k$, then there will be a subset of S that has all the elements in the

first color, which makes the collection of subsets not 2-colorable. Thus, we got two inequalities: $k < d, 2k - 1 - k < d$. Simplify the inequalities, we get: $k < d < k - 1$. However, there is no integer that is greater than k and smaller than k-1. Thus, such a color assignment doesn't exist. Our assumption is false: the collection of all subsets with d elements of S is not 2-colorable.

5.2

We may use graph theory to represent the collection of subsets (of set S) $A_1, A_2, ..., A_n$ each containing d=2 elements. In graph theory, each subset containing 2 elements can be represented as an edge that connects 2 nodes in a graph. If a collection is 2-colorable, then the graph represented by the collection is bipartite. Thus, $m(2) = 3$ means that all graphs with less than 3 edges are bipartite.
After reformulating the problem, we can now prove the statement by listing all the situations. For graphs with 2 edges, they can either be a connected graph with three nodes or a disconnected graph with four nodes. In both cases, the graph is bipartite: we can color the nodes so that every edge connects to two nodes with different colors. For example, in the graph {{a,b}, {b,c}}, we can color a and c the first color and b the other color to make it 2-colorable; in the graph {{a,b}, {c,d}}, we can color a and c the first color and b and d the other color. Thus, we have proven that all graph with 2 edges are bipartite.
For graphs with 3 edges, however, a cycle will be formed. For example, graph {{a,b}, {b,c}, {c,a}} is a cycle. More importantly, such a graph is an odd cycle, which is a cycle that has an odd number of nodes. In definition, a graph is bipartite if and only if it doesn't contain an odd cycle. Thus, we know that not all graphs with 3 edges are bipartite.
If we keep adding the number of edges that a graph has, we will see that any graph with > 3 edges can form an odd cycle of three nodes, which make them not bipartite. Thus, graphs with 3 or more than 3 edges are not bipartite but only graphs with 2 edges are. Going back to the problem, it's true that every collection of fewer than 3 sets (graphs of fewer than 3 edges) each containing 2 elements is 2-colorable. Thus, we have proven that $m(2) = 3$.

6

There are $n - p - q + 1$ positive integers not exceeding n that are relatively prime to n.

Firstly, we know that $n = pq$ and p, q are prime numbers. There are in total n positive integers not exceeding n, Among them, n is not relatively prime to n because their $gcd(n, n) = n > 1$. Also, any multiples of p not exceeding n are also not relatively prime to n, because $n = pq$ and $gcd(kp, n) = p$. There are $q - 1$ number of k value that will make kp not exceed qp, so there are $q - 1$ number of positive integers (from $k = 1$ to $k = q - 1$) that are not relatively prime to n. Likewise, for q, we have $p - 1$ number of positive integers that are not relatively prime to n. In total, there are $n - p + 1 - q + 1 - 1 = n - p - q + 1$ positive integers not exceeding n that are relatively prime to n.