CS-E4850 Computer Vision,
Answers to Exercise Round 1

Yangzhe Kong, Student number: 765756

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Exercise 1

a) If we represent point (x, y) on the plane by using homogeneous coordinates, as

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

So then the equation can be written as

$$\mathbf{x}^\top \mathbf{l} = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = ax + by + c$$

This is exactly the same with the equation of a line in the plane.
For the convenience of the proof of b), c), d), $\mathbf{x}^\top \mathbf{l}$ can also be written as $\mathbf{l}^\top \mathbf{x}$ because $\mathbf{l}^\top \mathbf{x} = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = ax + by + c$ gives the same result with $\mathbf{x}^\top \mathbf{l}$

b) Given the fact that if any two of the vectors a, b, c are parallel, $(\mathbf{a} \times \mathbf{b})^\top \mathbf{c}$ is zero. We can say that $(\mathbf{l} \times \mathbf{l}')^\top \mathbf{l} = 0$, which means that $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$ is on the line $\mathbf{l}$, and also $(\mathbf{l} \times \mathbf{l}')^\top \mathbf{l}' = 0$, which means that $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$ is on the line $\mathbf{l}'$. So then we can say that $\mathbf{x}$ is the intersection of 2 lines $\mathbf{l}$ and $\mathbf{l}'$.

c) Similar to b), using the same fact, we can say that $\mathbf{x}^\top (\mathbf{x} \times \mathbf{x}') = 0$, which means that $\mathbf{x}$ is on the line $\mathbf{x} \times \mathbf{x}'$, and $\mathbf{x}'^\top (\mathbf{x} \times \mathbf{x}') = 0$, which means that $\mathbf{x}'$ is on the line $\mathbf{x} \times \mathbf{x}'$. So then we can say that the line through points $\mathbf{x}'$ and $\mathbf{x}$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$.

d) $\mathbf{y}^\top (\mathbf{x} \times \mathbf{x}') = [\alpha \mathbf{x} + (1 - \alpha) \mathbf{x}'] (\mathbf{x} \times \mathbf{x}') = \alpha \mathbf{x}^\top (\mathbf{x} \times \mathbf{x}') + (1 - \alpha) \mathbf{x}'^\top (\mathbf{x} \times \mathbf{x}') = 0$, so we can say $\mathbf{y}$ lies on the line through points $\mathbf{x}$ and $\mathbf{x}'$.

Exercise 2

a) Projective

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \mathbf{Hx}$$

Affine

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Similarity

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} ssin\theta & -ssin\theta & t_x \\ scos\theta & scos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{sR} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Euclidean

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \epsilon sin\theta & -sin\theta & t_x \\ \epsilon cos\theta & cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Where $\epsilon = \pm 1$ Translation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

b) The translation have 2 degrees of freedom.
The Euclidean transformation have 3 degrees of freedom. $(\theta; t_x; t_y)$.
The similarity transformation have 4 degrees of freedom. $(s; ; t_x; t_y)$.
The affine transformation have 6 degrees of freedom. $(a_{11}; a_{12}; a_{21}; a_{22}; t_x; t_y)$.
The projective transformation have 8 degrees of freedom.

c) Because of the property of homogeneity, if we use homogeneous coordinates, it doesn't matter if we scale our projective matrix, we will still get the same points. So if we scale our projective matrix like this,

$$\begin{bmatrix} h_{11}/h_{33} & h_{12}/h_{33} & h_{13}/h_{33} \\ h_{21}/h_{33} & h_{22}/h_{33} & h_{23}/h_{33} \\ h_{31}/h_{33} & h_{32}/h_{33} & 1 \end{bmatrix}$$

we can tell that in this matrix there are only **8** independent variables, so our projective matrix has **8** degrees of freedom, not **9**.

Exercise 3

a) Suppose $\mathbf{x}$ lies on $\mathbf{l}$, so the transformed point $\mathbf{x}' = \mathbf{H}\mathbf{x}$ should be on $\mathbf{l}'$. So we can write

$$\mathbf{l}'^{\top}\mathbf{x}' = \mathbf{l}'^{\top}\mathbf{H}\mathbf{x} = 0$$

We can then derive that $\mathbf{l}' = \mathbf{H}^{-\top}\mathbf{l}$ because $(\mathbf{H}^{-\top}\mathbf{l})^{\top}\mathbf{H}\mathbf{x} = \mathbf{l}^{\top}\mathbf{H}^{-1}\mathbf{H}\mathbf{x} = \mathbf{l}^{\top}\mathbf{x} = 0$ using $\mathbf{l}^{\top}\mathbf{x} = 0$. So the line transformation is $\mathbf{l}' = \mathbf{H}^{-\top}\mathbf{l}$.

b) Because of the property of homogeneity, scaling doesn't affect the result of transformation. So if we scale $\mathbf{l_1}^{\top}$ by s_1, $\mathbf{l_2}^{\top}$ by s_2, $\mathbf{x_1}^{\top}$ by k_1 and $\mathbf{x_2}^{\top}$ by k_2, the transformation of this invariant should be expressed as

$$\begin{aligned}
\mathbf{I}' &= \frac{[(s_1\mathbf{H}^{-\top}\mathbf{l}_1)^{\top}k_1\mathbf{H}\mathbf{x_1}][(s_2\mathbf{H}^{-\top}\mathbf{l}_2)^{\top}k_2\mathbf{H}\mathbf{x_2}]}{[(s_1\mathbf{H}^{-\top}\mathbf{l}_1)^{\top}k_2\mathbf{H}\mathbf{x_2}][(s_2\mathbf{H}^{-\top}\mathbf{l}_2)^{\top}k_1\mathbf{H}\mathbf{x_1}]} \\
&= \frac{s_1 s_2 k_1 k_2(\mathbf{l_1}^{\top}\mathbf{H^{-1}}\mathbf{H}\mathbf{x_1})(\mathbf{l_2}^{\top}\mathbf{H^{-1}}\mathbf{H}\mathbf{x_2})}{s_1 s_2 k_1 k_2(\mathbf{l_1}^{\top}\mathbf{H^{-1}}\mathbf{H}\mathbf{x_2})(\mathbf{l_2}^{\top}\mathbf{H^{-1}}\mathbf{H}\mathbf{x_1})} \\
&= \frac{(\mathbf{l_1}^{\top}\mathbf{x_1})(\mathbf{l_2}^{\top}\mathbf{x_2})}{(\mathbf{l_1}^{\top}\mathbf{x_2})(\mathbf{l_2}^{\top}\mathbf{x_1})} = \mathbf{I}
\end{aligned}$$

With fewer number of points or lines, when we calculate $\mathbf{I}'$, the scalars of nominator and denominator cannot be canceled out so it wont be invariant.