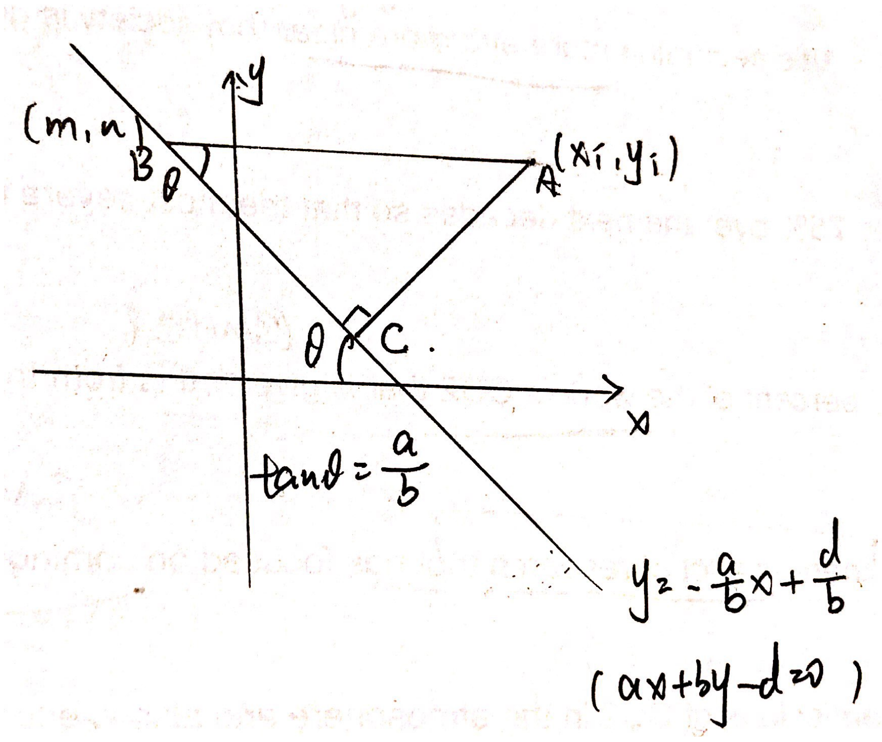
CS-E4850 Computer Vision, Answers to Exercise Round 5

Yangzhe Kong, Student number: 765756

October 10, 2019

Exercise 1



a) Let's assume that a and b are positive. Then The line $ax + by - d = 0$ can also be expressed as $y = -\frac{a}{b}x + \frac{d}{b}$ whose angle to the x axis have the relationship $tan\theta = \frac{a}{b}$. Let's denote the intersection point between the line $ax + by - d = 0$ and the line through the point $A(x_i, y_i)$ which is also parallel with x axis as $B(m, y_i)$. Then from the properties of parallel lines we can derive that $\angle ABC = \theta$. From this we can derive that the distance $|AC| = |AB| \times sin\theta = |x_i - m| \times sin\theta$.
We know that point B is on the line $ax + by - d = 0$ so we can get that $m = \frac{d - by_i}{a}$.

Also because $tan\theta = \frac{a}{b}$ and $a^2 + b^2 = 1$ we can have $sin\theta = a$. Substituting m and $sin\theta$ into the equation of $|AC|$ we can get $|AC| = |x_i - m| \times sin\theta = |x_i - \frac{d-by_1}{a}| = |d - ax_i - by_i| = |ax_i + by_i - d|$. Loosing the assumption of the positivity of a and b won't change the same result.

b) The partial derivative of d is

$$\frac{\delta E}{\delta d} = \sum_{i=1}^{n} -2(ax_i + by_i - d) = 0$$

Then we can get

$$\begin{aligned}
\sum_{i=1}^{n} 2d =& \sum_{i=1}^{n} 2(ax_i + by_i) \\
nd =& \sum_{i=1}^{n} (ax_i + by_i) \\
d =& a\frac{\sum_{i=1}^{n} x_i}{n} + b\frac{\sum_{i=1}^{n} y_i}{n} \\
d =& a\overline{x_i} + b\overline{y_i}
\end{aligned}$$

c) Substitute d into E gives us

$$\begin{aligned}
E =& \sum_{i=1}^{n} (ax_i + by_i - a\overline{x_i} - b\overline{y_i}) \\
=& \sum_{i=1}^{n} (a(x_i - \overline{x_i}) + b(y_i - \overline{y_i})) \\
=& \left\| \begin{bmatrix} x_1 - \overline{x} & y_1 - \overline{y} \\ x_2 - \overline{x} & y_2 - \overline{y} \\ \vdots & \vdots \\ x_n - \overline{x} & y_n - \overline{y} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right\|^2 \\
=& \left\| U(a\,b)^\top \right\| \\
=& (U(a\,b)^\top)^\top U(a\,b)^\top \\
=& (a\,b)U^\top U(a\,b)^\top
\end{aligned}$$

d) In order to find the $(a\,b)^\top$, we take the derivative of the equation $E = (a\,b)U^\top U(a\,b)^\top$ which we would like to minimize and set it to zero

$$\frac{dE}{d(a\,b)^\top} = (U^\top U)(a\,b)^\top = 0$$

. This question is equivalent to prove that the least square solution x for $Ax = 0$ is the eigenvector of the matrix $A^\top A$ with the smallest eigenvalue under the condition

of $\|x\|^2 = 1$.
We can separate this question into 2 cases.

1) x is the eigenvector of $A^\top A$.
   We can have $A^\top Ax = \lambda x$, which means that

$$\begin{aligned}\|Ax\|^2 =&(Ax)^\top(Ax)\\ =&x^\top A^\top Ax\\ =&x^\top\lambda x\\ =&\lambda\end{aligned}$$

   Since we take the eigenvector associated with the smallest eigenvalue, x is truly the solution.

2) x is not the eigenvectors of $A^\top A$.
   We can take the SVD decomposition of A and get

$$A = U\Lambda V^\top$$

   So then

$$\begin{aligned}\|Ax\|^2 =&(Ax)^\top(Ax)\\ =&x^\top A^\top Ax\\ =&x^\top V\Lambda^\top U^\top U\Lambda V^\top x\\ =&xV\Lambda^\top\Lambda V^\top x\end{aligned}$$

   Because of the properties of SVD decomposition, we can get

$$\Lambda^\top\Lambda = \begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdots & \\ & & & \lambda_n\end{bmatrix}$$

   Also because in SVD decomposition $V = [v_1 \quad v_2 \quad \cdots \quad v_n]$ is a combination of n-dimensional orthonormal bases. We can express x in terms of V:

$$x = [v_1 \quad v_2 \quad \cdots \quad v_n]\begin{bmatrix}\alpha_1\\ \alpha_2\\ \cdots\\ \alpha_n\end{bmatrix}$$

We can substitute these two equations into $\left\|Ax\right\|^2$ and we can derive that:

$$\begin{aligned}
\left\|Ax\right\|^2 =& xV\Lambda^\top \Lambda V^\top x \\
=& \begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdots & \\ & & & \lambda_n \end{bmatrix} \\
& \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{bmatrix} \\
=& \begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{bmatrix} \\
=& \alpha_1^2\lambda_1^2 + \alpha_2^2\lambda_2^2 + \cdots + \alpha_n^2\lambda_n^2
\end{aligned}$$

Without losing universality we can sort λ_i with descent order so that

$$\left\|Ax\right\|^2 \geq \lambda_n^2(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2)$$

Also because $\left\|x\right\|^2 = 1$ which means that $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = 1$, thus

$$\left\|Ax\right\|^2 \geq \lambda_n^2$$

$$\left\|Ax\right\|^2$$

defined here is for sure larger than the one in 1), then the argument in the question proves to be correct.

Exercise 2

The code for implementation can be found in **RANSAC.m**. Basic code is shown below.

```
function [w,b]=RANSAC(x,y,N,t)
[~,n]=size(x);
inlier_index_max=[];
for i=1:N
R = randi(n,1,2);
x_sampled=[x(R(1)) x(R(2))];
y_sampled=[y(R(1)) y(R(2))];
p = polyfit(x_sampled,y_sampled,1);
distance=abs((1/sqrt(p(1)^2+1))*y+(1/sqrt(p(1)^2+1))*+
```

```
x*(-p(1))-p(2)*(1/sqrt(p(1)^2+1)));
inlier_index=find(distance<t);
if (length(inlier_index)+2>length(inlier_index_max))
    inlier_index_max=[inlier_index,R];
end
end
x_resampled=[];
y_resampled=[];
for i=1:length(inlier_index_max)
    x_resampled=[x_resampled x(inlier_index_max(i))];
    y_resampled=[y_resampled y(inlier_index_max(i))];
end
p = polyfit(x_resampled,y_resampled,1);
w=p(1);b=p(2);
end
end
```

The image contains the points and the fitted line using RANSAC is shown below.

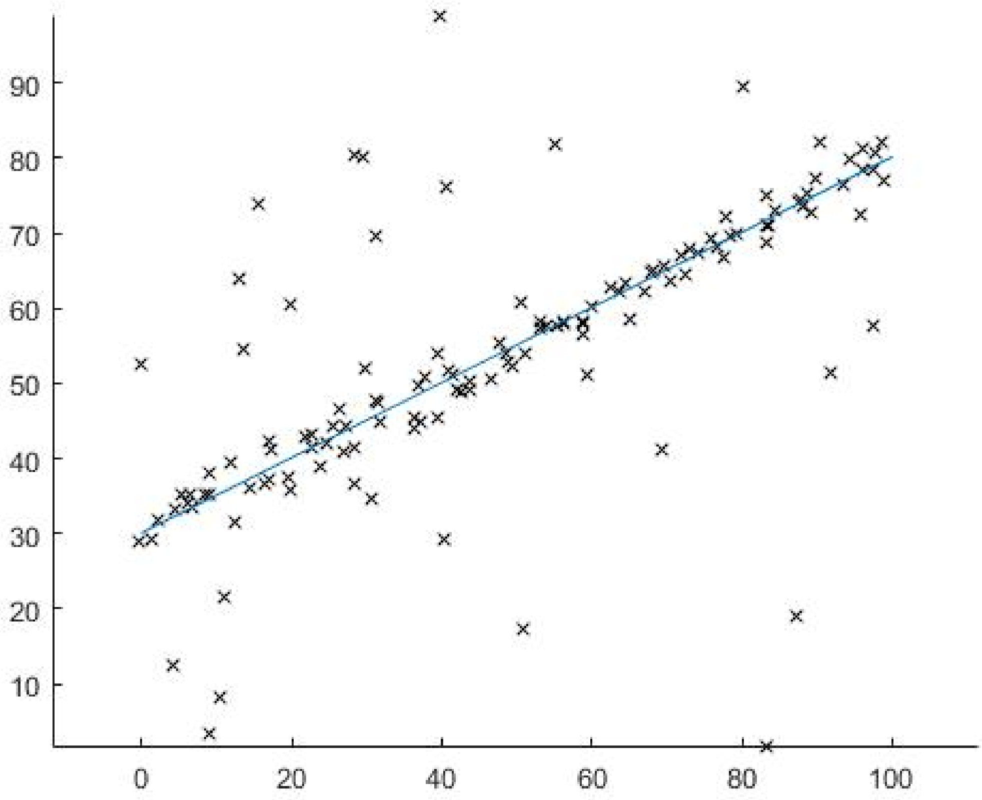


Figure 1: Illustration of RANSAC