CS-E4850 Computer Vision, Answers to Exercise Round 6

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Exercise 1.

Least-squares fitting for affine transformations.

a)

$$E = \sum_{i=1}^{n} \|x_i' - Mx_i - t\|^2 = \sum_{i=1}^{n} \left\| \begin{bmatrix} x_i' \\ y_i' \end{bmatrix} - \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} \right\|^2$$

$$= \sum_{i=1}^{n} [(x_i' - M_1 x_i - M_2 y_i - t_1)^2 + (y_i' - M_3 x_i - M_4 y_i - t_2)^2]$$

We can take the derivative for each parameter and we can derive that

$$\frac{dE}{dM_1} = \sum_{i=1}^{n} 2(x_i' - M_1 x_i - M_2 y_i - t_1)(-x_i)$$

$$\frac{dE}{dM_2} = \sum_{i=1}^{n} 2(x_i' - M_1 x_i - M_2 y_i - t_1)(-y_i)$$

$$\frac{dE}{dM_3} = \sum_{i=1}^{n} 2(y_i' - M_3 x_i - M_4 y_i - t_2)(-x_i)$$

$$\frac{dE}{dM_4} = \sum_{i=1}^{n} 2(y_i' - M_3 x_i - M_4 y_i - t_2)(-y_i)$$

$$\frac{dE}{dt_1} = \sum_{i=1}^{n} 2(x_i' - M_1 x_i - M_2 y_i - t_1)(-1)$$

$$\frac{dE}{dt_2} = \sum_{i=1}^{n} 2(y_i' - M_3 x_i - M_4 y_i - t_2)(-1)$$

b) We can derive the form $\mathbf{Sh} = \mathbf{u}$ by setting the derivatives to 0 and rearranging their formulas

$$\begin{bmatrix} \sum_{i=1}^2 x_i^2 & \sum_{i=1}^2 x_i y_i & 0 & 0 & \sum_{i=1}^2 x_i & 0 \\ \sum_{i=1}^2 x_i y_i & \sum_{i=1}^2 y_i^2 & 0 & 0 & \sum_{i=1}^2 y_i & 0 \\ 0 & 0 & \sum_{i=1}^2 x_i^2 & \sum_{i=1}^2 x_i y_i & 0 & \sum_{i=1}^2 x_i \\ 0 & 0 & \sum_{i=1}^2 x_i y_i & \sum_{i=1}^2 y_i^2 & 0 & \sum_{i=1}^2 y_i \\ \sum_{i=1}^2 x_i & \sum_{i=1}^2 y_i & 0 & 0 & n & 0 \\ 0 & 0 & \sum_{i=1}^2 x_i & \sum_{i=1}^2 y_i 0 & n & \end{bmatrix}$$

and $h = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ t_1 \\ t_2 \end{bmatrix}$, $u = \begin{bmatrix} \sum_{i=1}^n x_i x' \\ \sum_{i=1}^n y_i x' \\ \sum_{i=1}^n x_i y' \\ \sum_{i=1}^n y_i y' \\ \sum_{i=1}^n x_i' \\ \sum_{i=1}^n y_i' \end{bmatrix}$

c) Using the three matches we are given, we can compute that:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 3 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{bmatrix}$$

so we can derive that $\mathbf{h} = \mathbf{S^{-1}u} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

So the affine transformation matrix is that $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Exercise 2

a) We can compute that $v = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$, $v' = \begin{pmatrix} x'_2 - x'_1 \\ y'_2 - y'_1 \end{pmatrix} = s \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & -cos\theta \end{pmatrix} \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$
We can recover the angle θ using the definition of dot product

$$\begin{aligned} cos\theta &= \frac{v' \cdot v}{\|v'\| \, \|v\|} \\ &= \frac{(x'_2 - x'_1)(x_2 - x_1) + (y'_2 - y'_1)(y_2 - y_1)}{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \end{aligned}$$

So we have $\theta = cos^{-1}(\frac{(x'_2-x'_1)(x_2-x_1)+(y'_2-y'_1)(y_2-y_1)}{\sqrt{(x'_2-x'_1)^2+(y'_2-y'_1)^2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}})$

b) $s = \frac{\|v'\|}{\|v\|} = \frac{\sqrt{(x'_2-x'_1)^2+(y'_2-y'_1)^2}}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$

c) From equation 1 we can get that $x' = s(cos\theta - sin\theta) + t_x$ and $y' = s(sin\theta + cos\theta) + t_y$. Thus using formulas derived in a) and b) we can compute t_x and t_y:

$$t_x = x' - scos\theta + ssin\theta$$

$$t_y = y' - ssin\theta - scos\theta$$

d) Using the formulas derived above, we can get:

$$v = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}, v' = \begin{pmatrix} x'_2 - x'_1 \\ y'_2 - y'_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Since $v' \cdot v = 0$, we have

$$\begin{aligned} \theta &= cos^{-1}(\frac{v' \cdot v}{\|v'\| \, \|v\|}) \\ &= cos^{-1}(0) \\ &= \frac{\pi}{2} \end{aligned}$$

Also we have $s = \frac{\|v'\|}{\|v\|} = \sqrt{\frac{2}{1/2}} = \sqrt{4} = 2$ So finally we can get

$$t_x = x' - scos\theta + ssin\theta = 0 - 2 \times 0 \times \frac{1}{2} + 2 \times 1 \times 0 = 0$$

$$t_y = y' - ssin\theta - scos\theta = 0 - 2 \times 1 \times \frac{1}{2} + 2 \times 0 \times 0 = -1$$

Therefore the transformation is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = 2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$