CS-E4850 Computer Vision, Answers to Exercise Round 2

Yangzhe Kong, Student number: 765756

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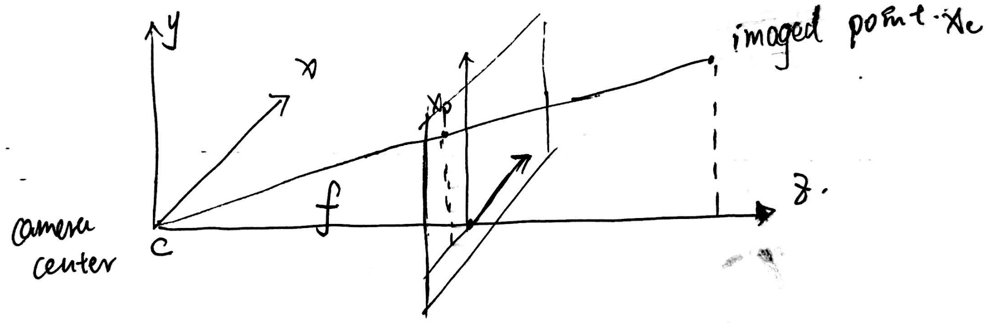
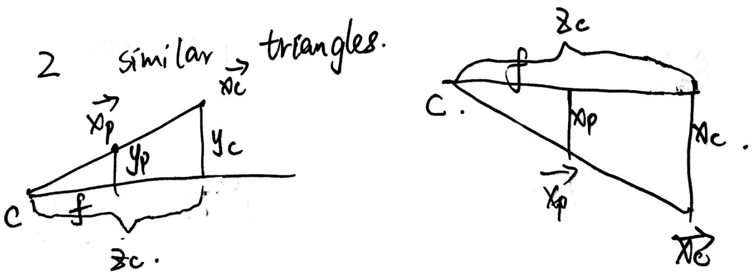
Exercise 1





a)

Just as the image above shown, if we use similar triangles, we can get:

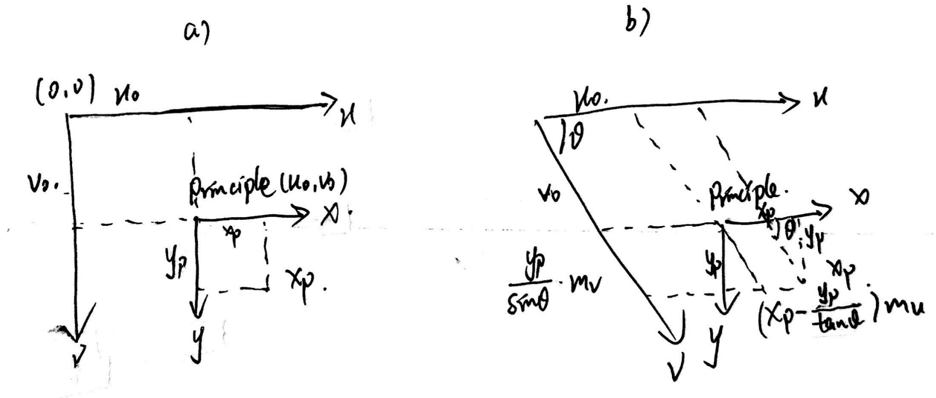
$$\frac{y_p}{y_c} = \frac{f}{z_c}$$

$$\frac{x_p}{x_c} = \frac{f}{z_c}$$

And then we get

$$x_p = \frac{fx_c}{z_c}, y_p = \frac{fy_c}{z_c}$$

Exercise 2



a)

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} m_u x_p + u_0 \\ m_v y_p + v_0 \end{bmatrix}$$

b)

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} m_u(x_p - \frac{y_p}{tan\theta}) + u_0 \\ m_v \frac{y_p}{sin\theta} + v_0 \end{bmatrix}$$

Exercise 3

Based on what we've got from the first 2 exercises, we have

$$\mathbf{x}_c = [x_c \quad y_c \quad z_c]^\top$$

$$x_p = \frac{fx_c}{z_c}, y_p = \frac{fy_c}{z_c}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} m_u x_p + u_0 \\ m_v y_p + v_0 \end{bmatrix}$$

So then we get

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f m_u & 0 & u_0 \\ 0 & f m_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} f m_u & 0 & u_0 \\ 0 & f m_v & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 4

If we express the point in world coordinates as

$$\mathbf{x_w} = [X \quad Y \quad Z \quad 1]^\top$$

we can have

$$\mathbf{x_c} = [\mathbf{R}|\mathbf{t}]\mathbf{x_w}$$

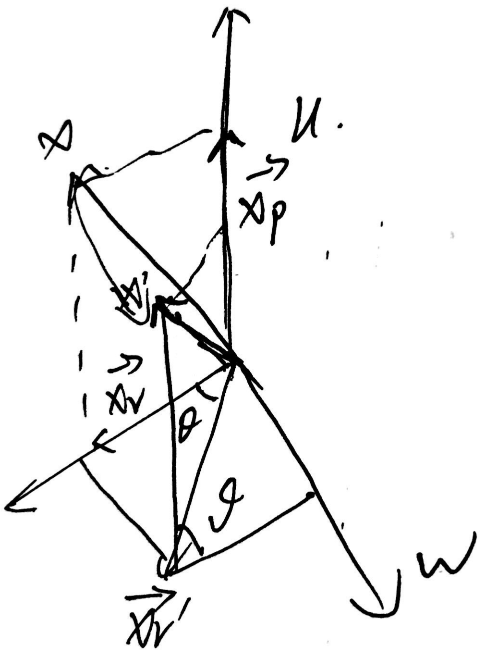
Also according to exercise 3, we can get the pixel coordinates as

$$\mathbf{x_p} = \mathbf{K}\mathbf{x_c}$$

So then

$$\mathbf{P} = [\mathbf{K}[\mathbf{R}|\mathbf{t}]$$

Exercise 5



a) First we decompose $\mathbf{X}$ into 2 components: x_p, perpendicular to axis u and x_r parallel with axis u.

$$\mathbf{x} = \mathbf{x_p} + \mathbf{x_r}$$
$$\mathbf{x_r} = (\mathbf{u} \cdot \mathbf{x})\mathbf{u}$$

Then we draw a vector $\mathbf{w}$ that is perpendicular to both $\mathbf{u}$ and $\mathbf{x}$. so we can get

$$\mathbf{w} = \mathbf{x} \times \mathbf{u}$$

Also because $\mathbf{u}$ is a unit vector so $\mathbf{w}$ has the same length with $\mathbf{x}$. So then for the rotated vector $\mathbf{x}'$, its perpendicular component $\mathbf{x_r'}$ to the axis $\mathbf{u}$ can be expressed as

$$\mathbf{x_r'} = \mathbf{x_r} cos\theta + \mathbf{w} sin\theta$$

.

So finally we can get

$$\begin{aligned}
\mathbf{x}' =& \mathbf{x_p} + \mathbf{x_r} cos\theta + \mathbf{w} sin\theta \\
=& (\mathbf{u} \cdot \mathbf{x})\mathbf{u} * cos\theta + \mathbf{x} \times \mathbf{u} * sin\theta + \mathbf{x} - (\mathbf{u} \cdot \mathbf{x})\mathbf{u} \\
=& cos\theta \mathbf{x} + sin\theta \mathbf{u} \times \mathbf{x} + (1 - cos\theta)(\mathbf{u} \cdot \mathbf{x})\mathbf{x}
\end{aligned}$$

b) First let's take a look at $\mathbf{u} \times \mathbf{x}$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

so $\mathbf{u} \times \mathbf{x} = \begin{bmatrix} u_2x_3 - x_2v_3 \\ u_3x_1 - x_3v_1 \\ u_1x_2 - x_1v_2 \end{bmatrix} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{u}_\times \mathbf{x}$ So then we use the notation $\mathbf{u}_\times$ for convenience. And then let's take a look at $(\mathbf{u} \cdot \mathbf{x})\mathbf{u}$, it's actually equals to $\mathbf{u}(\mathbf{u}^\top \mathbf{x})$.
And then $\mathbf{u}\mathbf{u}^\top = \begin{bmatrix} u_1^2 & u_1u_2 & u_1u_3 \\ u_1u_2 & u_2^2 & u_2u_3 \\ u_1U-3 & u_2u_3 & u_3^2 \end{bmatrix}$.
This actually relates to $\mathbf{u}_\top$ because

$$\mathbf{u}_\times{}^2 = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 1 \end{bmatrix} = \begin{bmatrix} -u_3^2 - u_2^2 & u_1u_2 & u_1u_3 \\ u_1u_2 & -u_3^2 - u_1^2 & -u_1 \\ u_1u_3 & u_2u_3 & -u_2^2 - u_1^2 \end{bmatrix}$$

And $\mathbf{u}$ is a unit vector so $u_1^2 + u_2^2 + u_3^2 = 1$.
We then get

$$\mathbf{u}\mathbf{u}^\top = \mathbf{u}_\times{}^2 + \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix.
So finally we have

$$\begin{aligned} \mathbf{R}\mathbf{x} =& cos\theta\mathbf{x} + sin\theta\mathbf{u}_\times\mathbf{x} + (1 - cos\theta)(\mathbf{u}_\times{}^2 + \mathbf{I})\mathbf{x} \\ =& (cos\theta + sin\theta\mathbf{u}_\times + \mathbf{u}_\times{}^2 + \mathbf{I} - cos\theta\mathbf{u}_\times{}^2 - cos\theta)\mathbf{x} \\ =& (\mathbf{I} + sin\theta\mathbf{u}_\times + \mathbf{u}_\times{}^2(1 - cos\theta))\mathbf{x} \end{aligned}$$

$$\mathbf{R} = \mathbf{I} + sin\theta\mathbf{u}_\times + \mathbf{u}_\times{}^2(1 - cos\theta)$$