Review Session – Statistical Inference

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Point Estimation

- **General definition**

- **Point estimation** is used to consider a distribution $f\ (x;\theta)$ with known type f but unknown parameter value θ.

- A **point estimator** $\hat{\theta}$ of θ is any (reasonable) statistic that is used to estimate θ.

Point Estimation

- **Def 0.2.** A point estimator $\hat{\theta}$ of θ is said to be unbiased if

$$E(\hat{\theta}) = \theta.$$

- It does not systematically overestimate or underestimate the true value

- Otherwise, it is biased and the bias of θ is defined as

$$B(\hat{\theta}) = E(\hat{\theta}) - \theta.$$

Point Estimation

- Given a random sample $X_1, \ldots, X_n$ from a population with unknown variance σ^2,
- The sample mean

$$\bar{X} = \frac{1}{n}\sum X_i$$

is an unbiased estimator used for μ

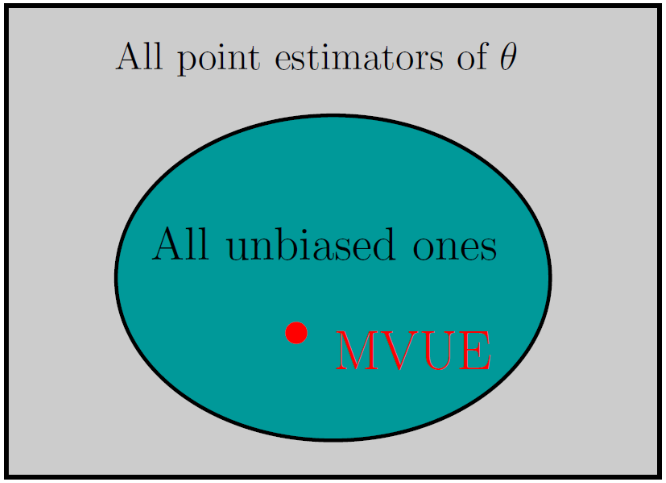
- The sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

- is the most common unbiased estimator used for σ^2

Point Estimation

- Between two unbiased estimators (of some parameter), the one with smaller variance is better.
- **Def 0.3.** The unbiased estimator $\hat{\theta}^*$ of θ that has the smallest variance is called a minimum variance unbiased estimator (MVUE).



- *Theorem* 0.2. For normal populations, $\bar{X}$ is a MVUE for μ.

Confidence Intervals

Confidence intervals – unknown μ, known σ

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- *Theorem* 0.1. Assume $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ where μ is unknown, but σ^2 is known.

For any given $0 < \alpha < 1$, we have

Margin of error: $m = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

- The confidence interval at the confidence level $1-\alpha$ for μ

$$\boxed{\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}}$$

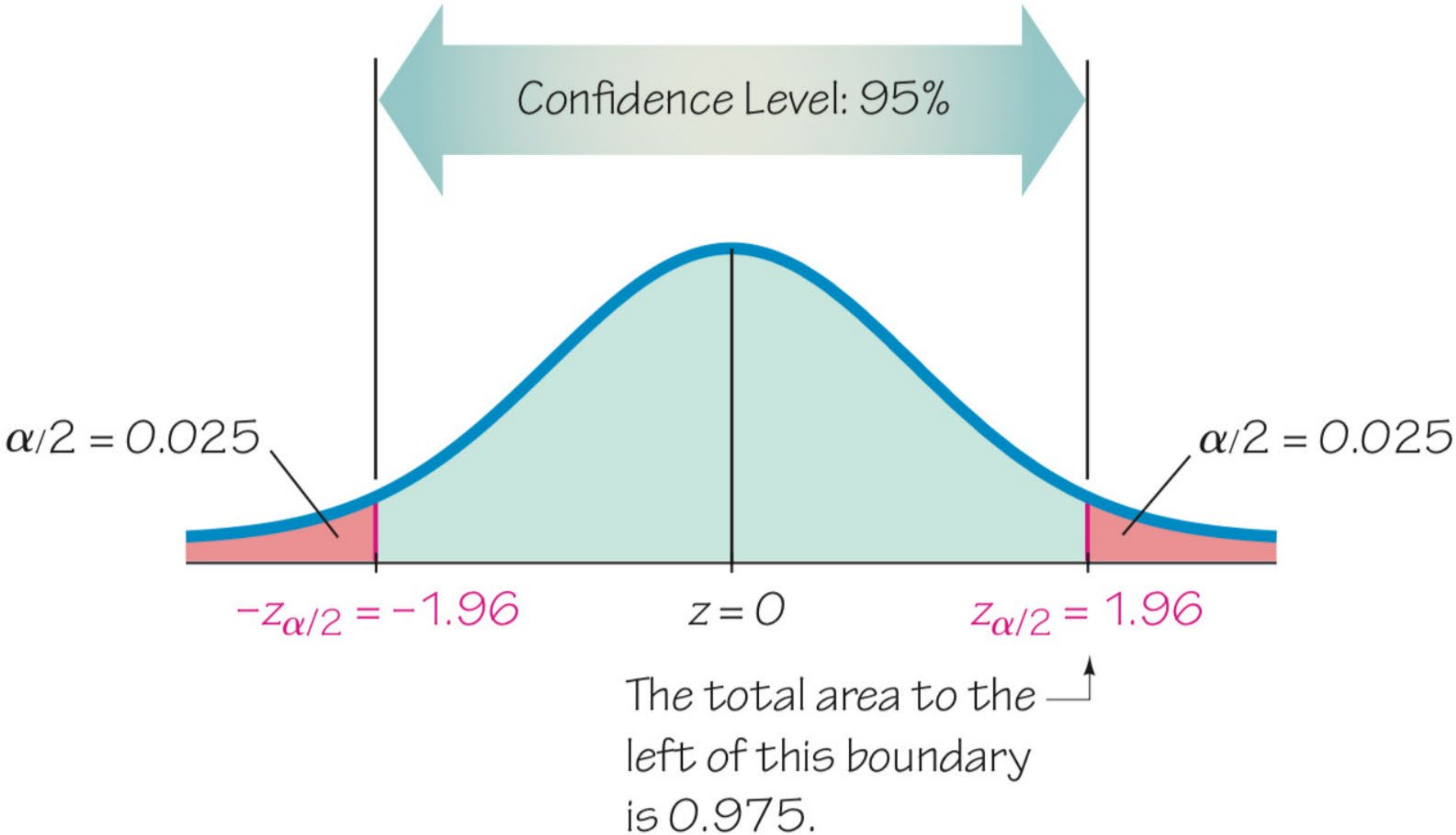


Figure 7-3 **Finding $z_{\alpha/2}$ for a 95% Confidence Level**

Confidence intervals – unknown μ, known σ

- **Example 0.2** (Continuation of the brown egg example).
- Another sample from the same population
  - Same mean $\bar{x}$ = 65.5 but a larger size n = 48
  - A 95% confidence interval is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 65.5 \pm 1.96 \cdot \frac{2}{\sqrt{48}} = 65.5 \pm 0.55.$$

- How large should the sample size be in order for the margin of error to be 0.2 (at level 95%)?

$$n = \left(z_{\alpha/2}\frac{\sigma}{m}\right)^2 = \left(1.96 \cdot \frac{2}{0.2}\right)^2 = 384.2.$$

The smallest sample size thus is 385.

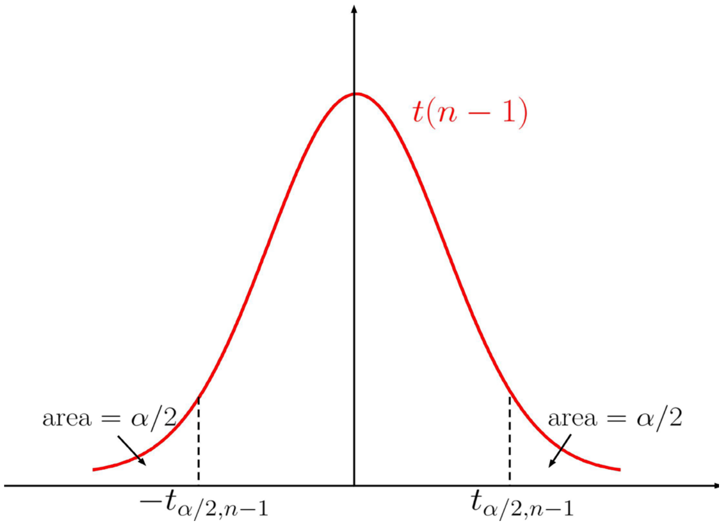
Confidence intervals – unknown μ, unknown σ

Confidence intervals – unknown μ, unknown σ

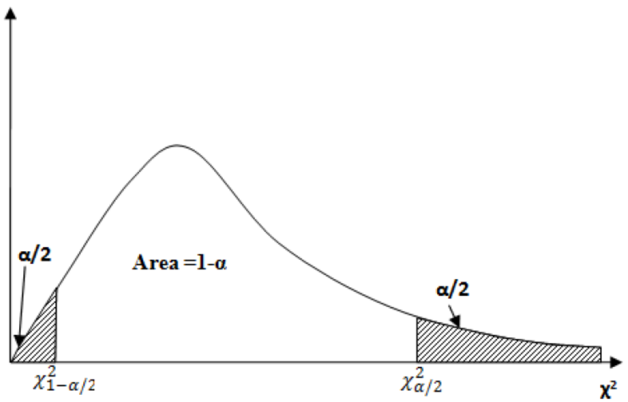
❖ Assuming a normal population $N(\mu, \sigma^2)$, with both μ, σ^2 unknown, we can still construct a $1 - \alpha$ confidence intervals for

μ: $$\bar{x} \pm t_{\alpha/2,n-1} \frac{s}{\sqrt{n}}.$$

σ^2: $$\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2,\, n-1}},\ \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,\, n-1}} \right)$$



Student's *t*-distribution

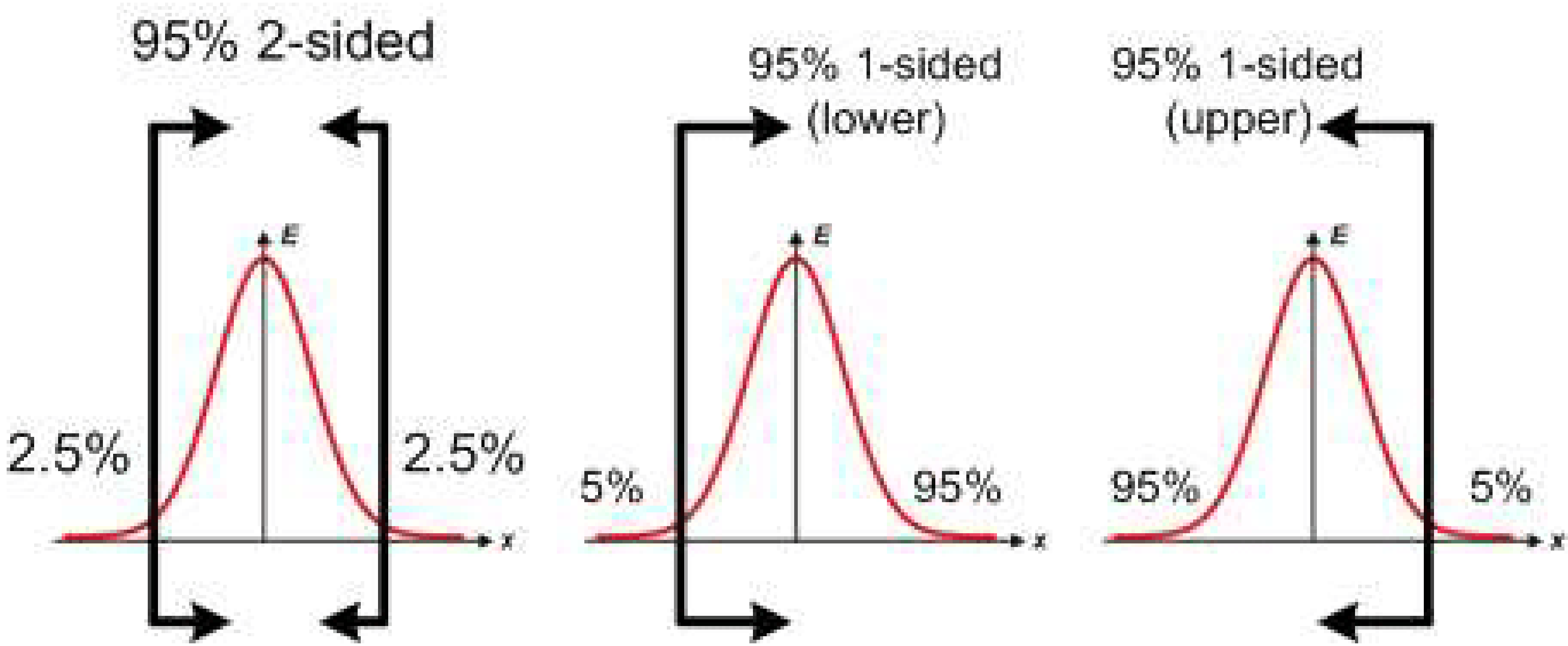


chi-squared (χ^2) distribution

One-Sided Confidence Bounds

One-Sided Confidence Bounds

- Compare 1-sided CI with 2-sided CI



One-Sided Confidence Bounds for σ^2

- Assuming a random sample $X_1, X_2, X_3, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with unknown μ but known σ^2. Then
- A $1-\alpha$ lower confidence bound for μ is

$$\mu > \bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}}$$

- A $1-\alpha$ upper confidence bound for μ is

$$\mu < \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}$$

Remark. For each confidence bound $m = z_\alpha \frac{\sigma}{\sqrt{n}}$

One-Sided Confidence Bounds

Similarly, the one-sided confidence intervals for σ^2 are

A $1-\alpha$ lower confidence bound for σ^2 is

$$\sigma^2 > \frac{(n-1)s^2}{\chi^2_{\alpha, n-1}}$$

A $1-\alpha$ upper confidence bound for σ^2 is

$$0 < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha, n-1}}$$

Hypothesis testing

Hypothesis testing

When σ^2 is *known*, a level α test for μ is

- $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$: Two-sided test

$$\text{Reject } H_0 \text{ if and only if } |\bar{x} - \mu_0| > z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

- $H_0 : \mu = \mu_0$ vs $H_1 : \mu < \mu_0$: One-sided test

$$\text{Reject } H_0 \text{ if and only if } \bar{x} - \mu_0 < -z_{\alpha} \frac{\sigma}{\sqrt{n}}$$

- $H_0 : \mu = \mu_0$ vs $H_1 : \mu > \mu_0$: One-sided test

$$\text{Reject } H_0 \text{ if and only if } \bar{x} - \mu_0 > z_{\alpha} \frac{\sigma}{\sqrt{n}}$$

Hypothesis testing

Test errors

There are two kinds of test errors depending on whether H_0 is true or not.

		Decision	
		Retain H_0	Reject H_0
H_0	true	Correct decision	Type I error
	false	Type II error	Correct decision

Type I error: Rejecting the null hypothesis H_0 when it is true
type II error: Failing to reject the null hypothesis H_0 when it is false

Hypothesis testing

- Because our decision is based on RVs, we can associate probabilities with the *type I* and *type II* errors
  - The probability of making a type I error is denoted by α
  - The probability of making a type II error is denoted by β

Hypothesis Test - Two-Sided Test

Two-sided Type I & II Error

Remark. For a two-sided test such as

$$H_0 : \mu = \mu_0 \qquad \text{vs} \qquad H_a : \mu \neq \mu_0$$

with corresponding decision rule

$$|\,\bar{x} - \mu_0| > c$$

the two equations (for determining n, c) become

$$\alpha = P\,(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ true}) = P\,(|\bar{X} - \mu_0| > c \mid \mu = \mu_0)$$

$$\beta = P\,(\text{Fail to reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ false}) = P\,(|\bar{X} - \mu_0| < c \mid \mu = \mu')$$

Two-sided Type I Error - Example

- Refer to the in-class example notes

1. Decision Rule: $|\bar{x}-65|>1$

$\bar{X} \sim (\mu, (\frac{\sigma}{\sqrt{n}})^2)$

2. Hypothesis: $H_0: \mu = 65$, $H_1: \mu \neq 65$

3. Calculate standard deviation of $\bar{X} \sim N(65, (\frac{\sigma}{\sqrt{n}})^2 = (\frac{2}{\sqrt{12}})^2)$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{2}{\sqrt{12}} = \sqrt{\frac{1}{3}}$$

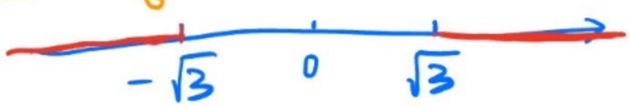
4. Convert the decision rule to the Standard $N(0,1)$

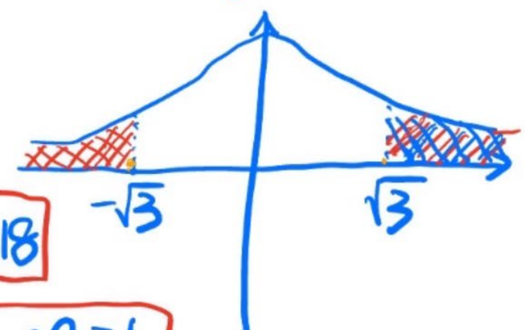
$$|\bar{x}-65|>1 \Rightarrow \left|\frac{\bar{x}-65}{\sqrt{\frac{1}{3}}}\right| > \frac{1}{\sqrt{\frac{1}{3}}} \Rightarrow |Z| > \sqrt{3}$$

$$Z = \frac{\bar{X}-\mu_{\bar{x}}}{\sigma_{\bar{x}}}$$

$$Z = \frac{X-\mu}{\sigma}$$

$Z < -\sqrt{3}$ and $Z > \sqrt{3}$





$$P(Z>\sqrt{3}) = 1 - P(Z<\sqrt{3})$$

$$= 1 - P(Z<1.732) = 1 - \Phi(Z=1.732) = \boxed{0.0418}$$

$$\alpha = 2P(Z>\sqrt{3}) = 2 \cdot 0.0418 = \boxed{0.0836}$$

Two-sided Type II Error - Example

❖ Refer to the in-class example notes

$$\begin{cases} H_0: \mu = 65 \\ H_1: \mu \neq 65 \end{cases}$$

Decision Rule

$|\bar{x} - \mu| \boxed{\geq} c$

Acceptance Region

$$\rightarrow P(\mu=64) = P(\underbrace{|\bar{x}-65| \boxed{<} c}\,|\, \mu = 64) \leftarrow$$

Type II error: The probability that we wrongly accept H_0 ($|\bar{x}-65|<c$) when H_0 is false ($\mu=64$)

$$= P(-c < \bar{x} - 65 < c \,|\, \mu = 64)$$

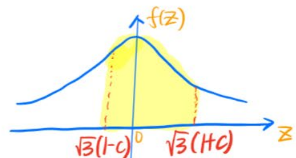
$$= P(65-c < \bar{x} < 65+c \,|\, \mu = 64), \quad \bar{x} \sim N(64, (\sqrt{\tfrac{1}{3}})^2)$$

Standardization:

$$= P\left(\frac{65-c-\boxed{64}}{\sqrt{\frac{1}{3}}} < \boxed{\frac{\bar{x}-\boxed{64}}{\sqrt{\frac{1}{3}}}} < \frac{65+c-\boxed{64}}{\sqrt{\frac{1}{3}}} \,\middle|\, \mu = 64\right)$$

$$= P(\underline{\sqrt{3}(1-c)} < Z < \underline{\sqrt{3}(1+c)})$$

$$= \Phi(\sqrt{3}(1+c)) - \Phi(\sqrt{3}(1-c))$$



The lower bound could either be larger than 0 or less than 0, depending on 1−c.

Hypothesis Test - 1-Sided Test

1-sided Type-I error

- 1-sided Type-I error
- $H_0 : \mu = \mu_0$ vs $H_a : \mu < \mu_0$

$$\begin{aligned}
\alpha &= P(\text{Reject } H_0 \mid H_0 \text{ true}) \\
&= P(\bar{X} < \mu_0 - c \mid \mu = \mu_0) \\
&= P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} < -\frac{c}{\sigma/\sqrt{n}} \,\middle|\, \mu = \mu_0 \right) \\
&\boxed{= P\left(Z < -\frac{c}{\sigma/\sqrt{n}} \right) \quad \longrightarrow \frac{c}{\sigma/\sqrt{n}} = z_\alpha}
\end{aligned}$$

$$\bar{x} < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}, \quad \text{or equivalently, } \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} < -z_\alpha$$

1-sided Type-II error

- Find n for 1-sided Type-II error

$$H_0: \mu = \mu_0 \qquad \text{vs} \qquad H_a: \mu < \mu_0$$

- Choose sample size n to achieve type-II error probability β at an alternative value $\mu = \mu'$:

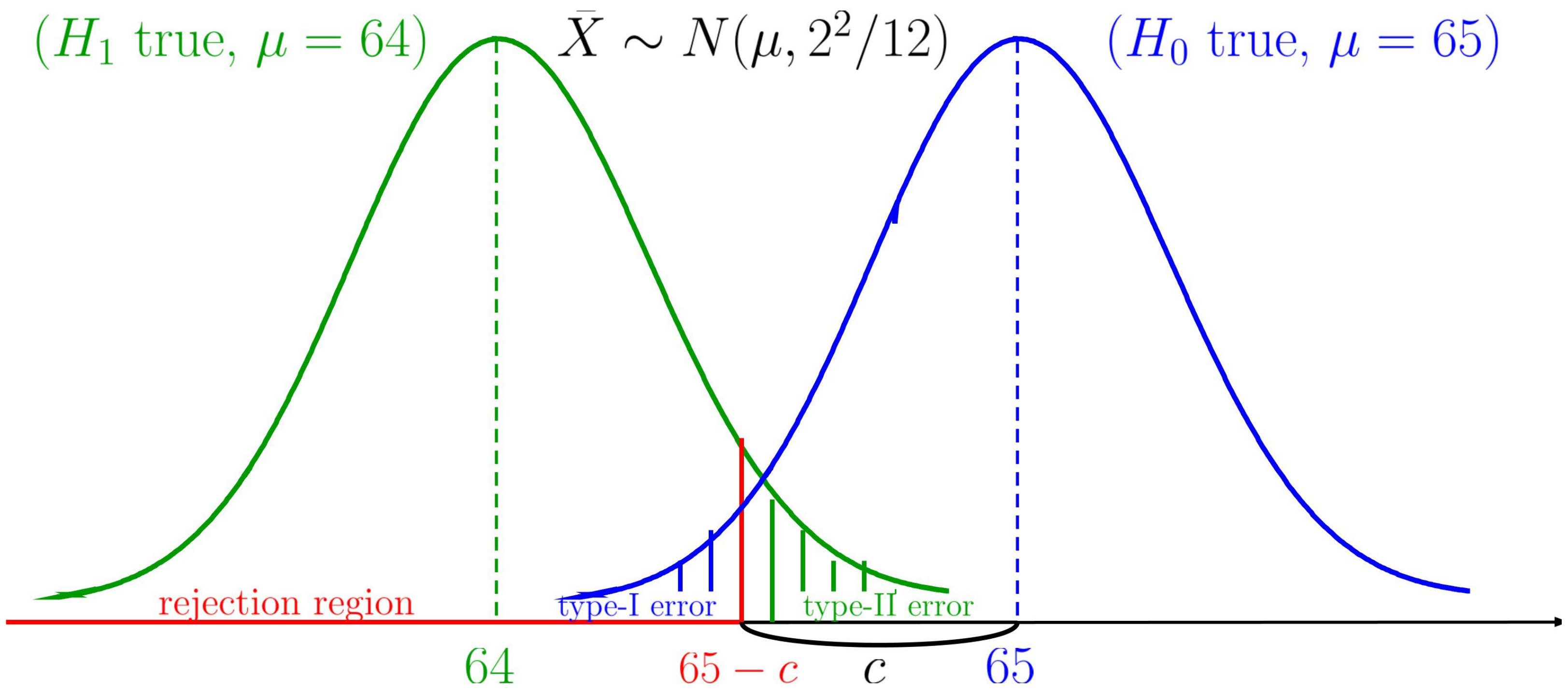
$$\begin{aligned}
\beta &= P(\text{Fail to reject } H_0 \mid H_0 \text{ false}) \\
&= P(\bar{X} > \mu_0 - c \mid \mu = \mu') \\
&= P\left(\frac{\bar{X} - \mu'}{\sigma/\sqrt{n}} > \frac{\mu_0 - c - \mu'}{\sigma/\sqrt{n}} \,\middle|\, \mu = \mu'\right) \\
&= P\left(Z > -z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)
\end{aligned}$$

This yields that

$$z_\beta = -z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}, \text{ and thus, } n = \left(\frac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu'}\right)^2.$$

1-sided Type-II error

$$H_0 : \mu = 65 \;\text{ vs }\; H_1 : \mu < 65$$



Decision rule: $\bar{x} < 65 - c$

Finding the sample size for given α and β

Hypothesis testing

When α ← typically 5% **and** $\beta(\mu')$ ← typically 20% **is given:**

To achieve a type-II error probability of β at an alternative value μ', the required sample size is

- for the two-sided test ($H_0 : \mu = \mu_0$ vs $H_a : \mu \neq \mu_0$):

$$n \approx \left(\frac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_0 - \mu'} \right)^2$$

- for both one-sided tests ($H_0 : \mu = \mu_0$ vs $H_a : \mu < \mu_0$):

$$n = \left(\frac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu'} \right)^2$$

- Other examples by HW problems