XI. Confidence Intervals

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- Sec 7.1: Basic properties of confidence intervals

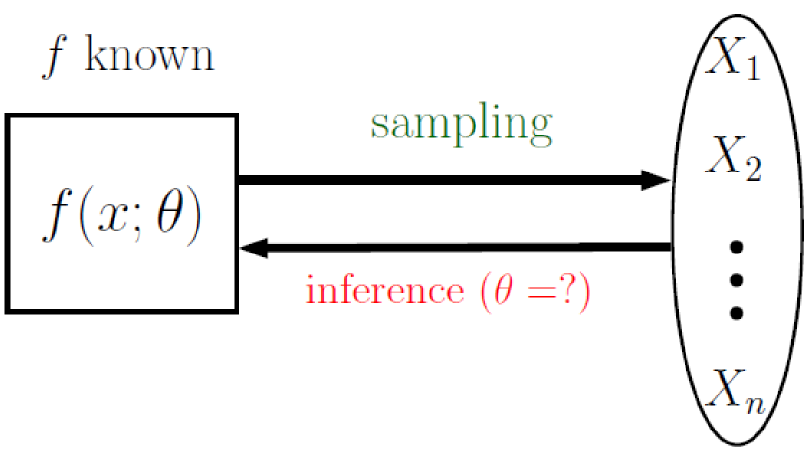
- Sec 7.3: Intervals based on a normal population distribution

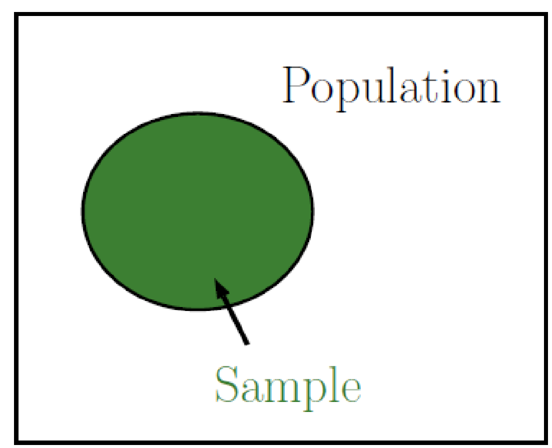
- Sec 7.4 Confidence intervals for the variance of a normal population

Introduction

- Last time we started considering the new setting in which we only know the distribution type, but not the values of its parameters.

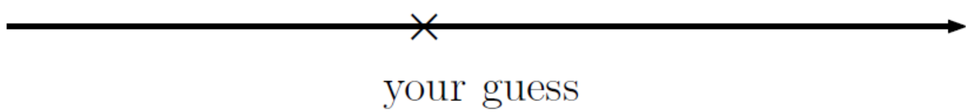
- The new goal is to use a random sample to infer about the unknown population parameter. This is called statistical inference.





Confidence intervals

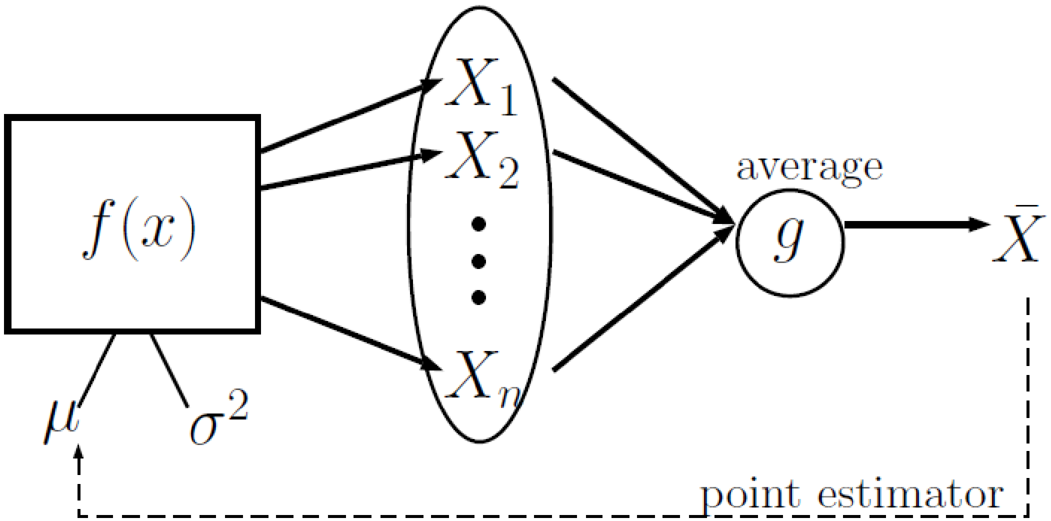
- We also mentioned three different kinds of inference tasks
  - ✓ **Point estimation**: What is a single (best) guess of the value of θ?



  - **Interval estimation**: Can you find an interval to capture the value of θ?
  - **Hypothesis testing**: It is claimed that $\theta = \theta_0$ (θ_0 represents a specific number). How do you test the hypothesis based on a random sample from the population?

Confidence intervals

- Recall that mathematically, a **point estimator** $\hat{\theta}$ of θ is a (reasonable) statistic used to estimate θ.

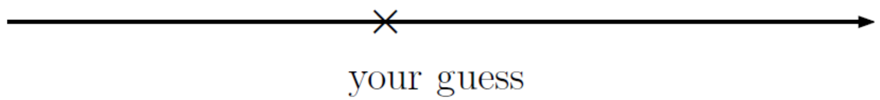


- For any specific realization of the random sample, the corresponding value of $\hat{\theta}$ is called a point estimate of θ.

Confidence Intervals

Confidence intervals

- Limitations with point estimation:



  - Point estimates are rarely exactly correct (even when point estimators that are unbiased and have least variance are used).

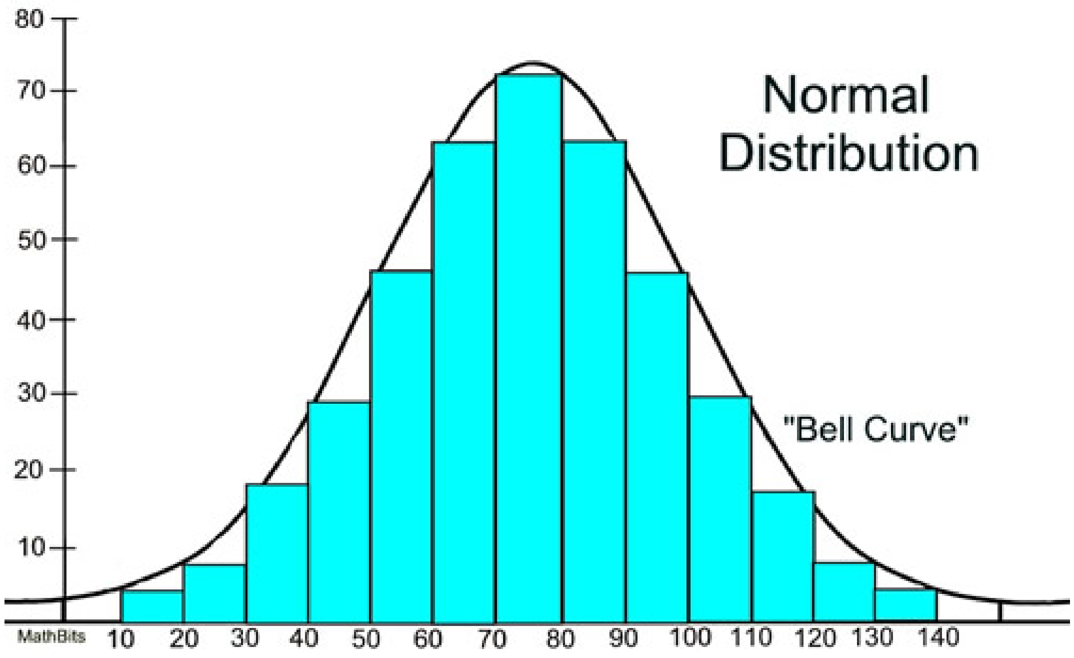
- For example, for a random sample from the $N(\mu, \sigma^2)$ population, the point estimator $\bar{X}$ of μ is a MVUE. For any small $c > 0$, the probability that $\bar{X}$ is within a distance of c from μ is

  - $P(\mu - c < \bar{X} < \mu + c) \approx 2cf(\mu)$.

  - Point estimates provide no error information.

Confidence intervals

- $P\,(\mu - c < \bar{X} < \mu + c) \approx 2cf\,(\mu).$



Confidence intervals

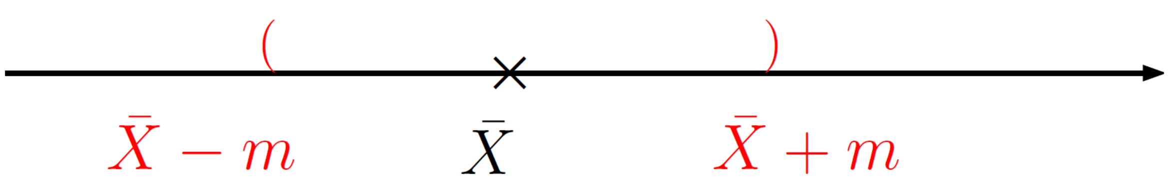
- **Question:** Can we make the "coverage probability" much higher than 0?
  - The answer is yes (by using an interval around X). One extreme case is

$$P(\mu \in (\bar{X} - \infty, \bar{X} + \infty) = 1)$$

<u>but it is useless.</u>

- A more favorable solution is to find a "<u>short</u>" interval with "<u>high</u>" coverage probability:

$$P(\mu \in (\bar{X} - m, \bar{X} + m)) = 1 - \alpha \qquad \text{(for some small } \alpha\text{)}.$$



Confidence intervals

- Rewrite as

$$P(\mu \in (\bar{X} - m, \bar{X} + m) = 1 - \alpha$$

- In the equation,
  - μ: population mean (unknown parameter to be estimated)
  - $\bar{X}$: sample mean (statistic)
  - m: half width (fixed scalar, to be found)
  - $1 - \alpha$: coverage probability (specified by user)
  - $(\bar{X} - m, \bar{X} + m)$: interval estimator (random)
- **Task**: Given α, find m.

Confidence intervals

- *Theorem* 0.1. Assume $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ where μ is unknown, but σ^2 is known. For any given $0 < \alpha < 1$, we have

$$m = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Confidence intervals

Proof.

$$P(\mu \in (\bar{X} - m, \bar{X} + m) = 1 - \alpha$$

is equivalent to

$$P(-m < \bar{X} - \mu < m) = 1 - \alpha$$

standardize $(\bar{X} - \mu)$ to a standard NRV Z:

$$P\left(-\frac{m}{\sigma/\sqrt{n}} < Z < \frac{m}{\sigma/\sqrt{n}}\right) = 1 - \alpha.$$

This implies that

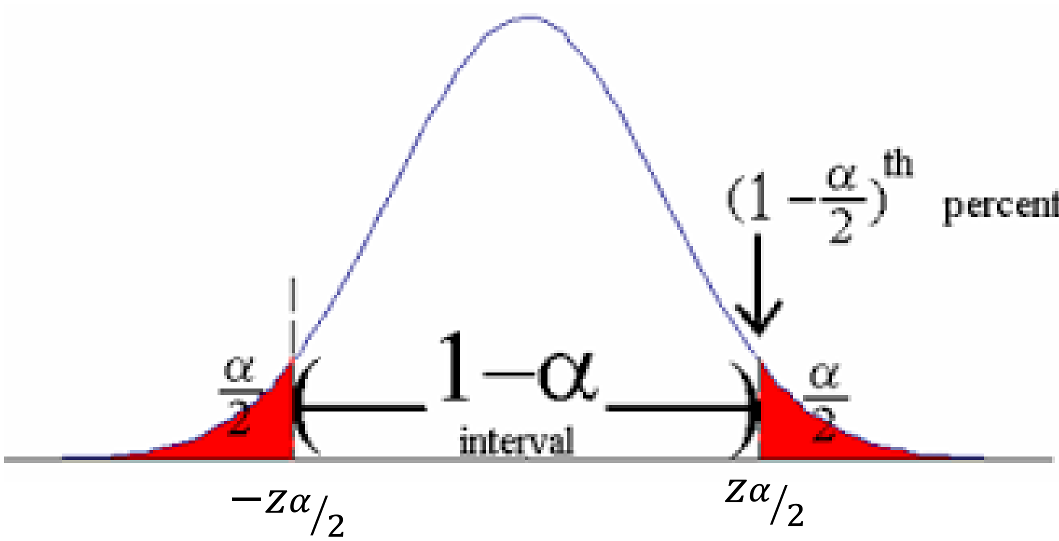
$$\frac{m}{\sigma/\sqrt{n}} = z_{\alpha/2},$$

Confidence intervals

Proof.

This implies that

$$\frac{m}{\sigma/\sqrt{n}} = z_{\alpha/2},$$



and accordingly,

$$m = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

Interval estimator

- We have just obtained that

$$P\left(\mu \in \left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)\right) = 1 - \alpha.$$

- **Def 0.1.** We call the interval estimator

$$\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \equiv \bar{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

- a $1 - \alpha$ random interval for μ. The quantity $m = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ is called the margin of error of the point estimator $\bar{X}$

Remark. If $\alpha = 0.05$ (i.e., $1 - \alpha = 0.95$), then $m = 1.96\,\frac{\sigma}{\sqrt{n}}$

Confidence interval

- **Def 0.2.** For any specific sample $X_1 = x_1, \ldots, X_n = x_n$ (along with the observed value $\bar{x}$ of $\bar{X}$), the <u>interval estimate</u>

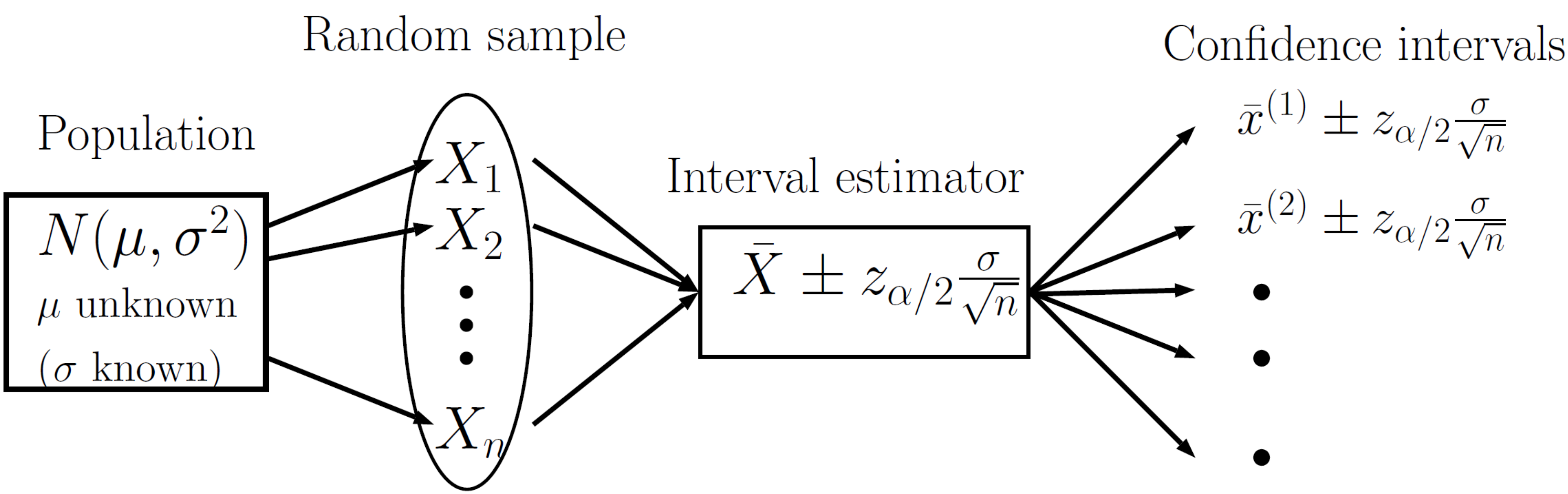
$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

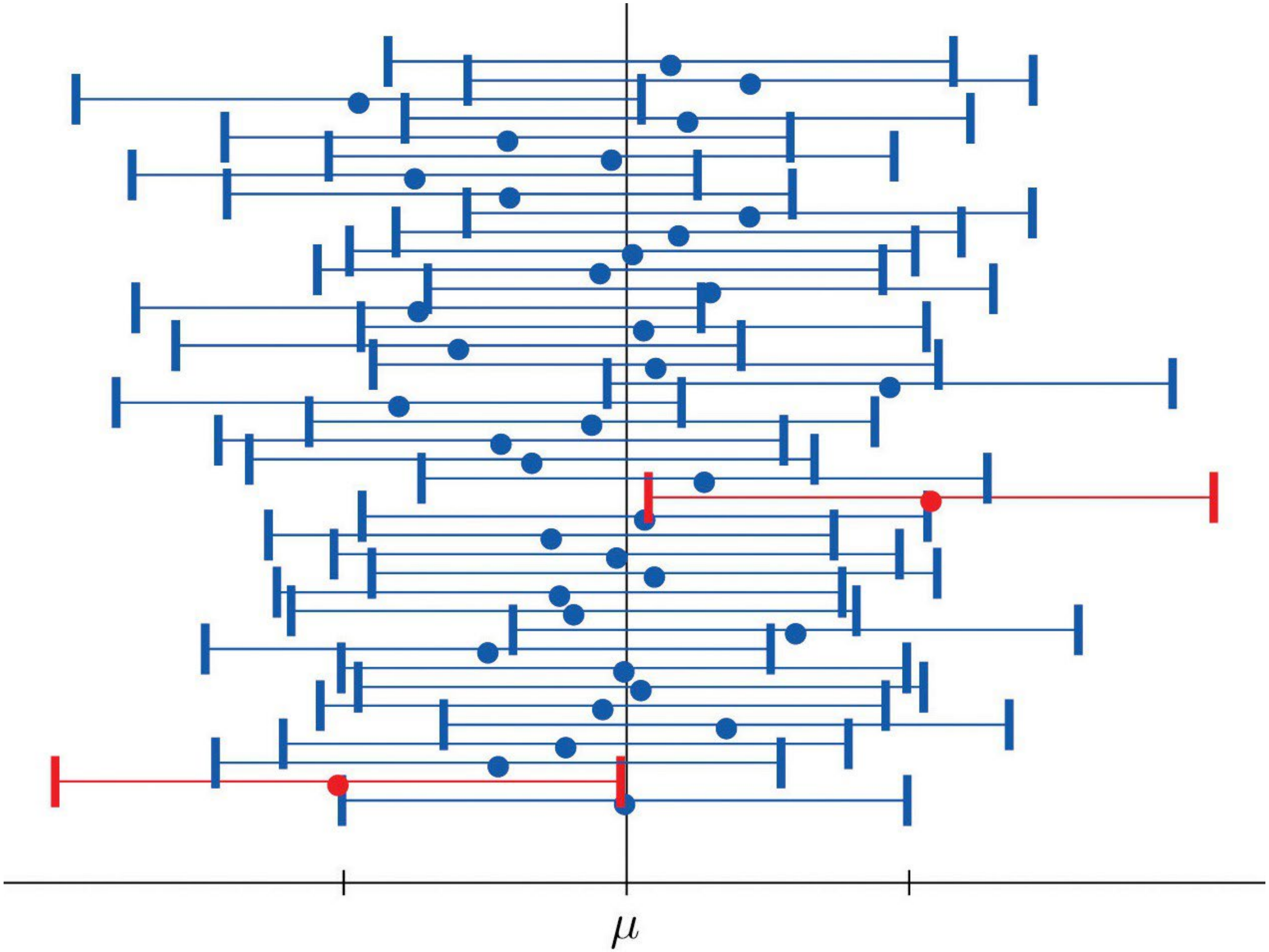
is called a 1− α confidence interval for μ. In this setting, 1− α is called the confidence level.

Confidence intervals

- **Example 0.1.** Recall the brown egg example where $n = 12$, $\bar{x} = 65.5$ and $\sigma = 2$.
- A 95% confidence interval is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 65.5 \pm 1.96 \cdot \frac{2}{\sqrt{12}} = 65.5 \pm 1.1 = (64.4, 66.6).$$



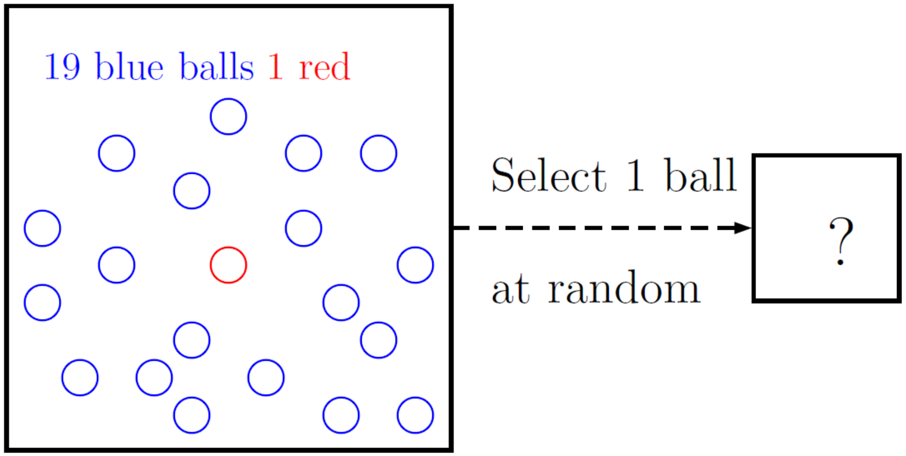

μ

Confidence intervals

- Interpretations of confidence intervals
- We can say that
  - (64.4, 66.6) is a 95% confidence interval for μ, or
  - We are 95% confident that the true μ is contained by this interval (i.e., between 64.4 and 66.6 grams).
- We cannot say that
  - The probability that μ is contained by this interval is 0.95,
- as both μ and this interval are fixed and there is only one truth: "contain" or "not contain". We just do not know which one is true (when μ is un-known).

Confidence intervals

❖ Confidence is not probability!



- Probability describes, the chance of selecting a blue ball before you actually do it (or if you do it many times)
- Confidence is, after you selected one ball, how certain you believe the ball you got is blue (without looking at it).

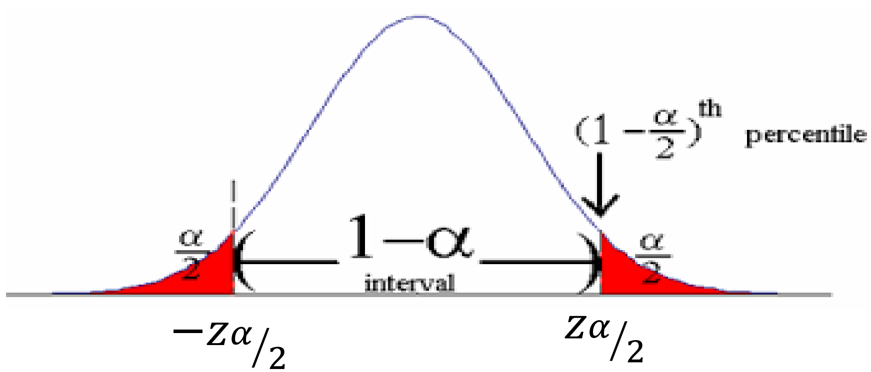
Confidence intervals

- Relationship between m and n, α

(m: margin of error, n: sample size, $1-\alpha$: confidence level)

$$m = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

- The larger the sample size *n*, the smaller the margin of error *m* (the shorter the confidence interval);
- The larger the confidence level $1-\alpha$, the bigger the margin of error *m* (the wider the confidence interval).



Confidence intervals (Previous example)

- **Example 0.1.** Recall the brown egg example where $n = 12$, $\bar{X} = 65.5$ and $\sigma = 2$.
- A 95% confidence interval is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 65.5 \pm 1.96 \cdot \frac{2}{\sqrt{12}} = 65.5 \pm 1.1 = (64.4, 66.6).$$

Confidence intervals

- **Example 0.2** (Continuation of the brown egg example).
- Another sample from the same population
  - Same mean $\bar{x}$ = 65.5 but a larger size n = 48
  - A 95% confidence interval is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 65.5 \pm 1.96 \cdot \frac{2}{\sqrt{48}} = 65.5 \pm 0.55.$$

- How large should the sample size be in order for the margin of error to be 0.2 (at level 95%)?

$$n = \left(z_{\alpha/2}\frac{\sigma}{m}\right)^2 = \left(1.96 \cdot \frac{2}{0.2}\right)^2 = 384.2.$$

The smallest sample size thus is 385.

- **Example 0.3** (Continuation of the brown egg example)**.** Using the same sample, a 99% confidence interval is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 65.5 \pm \boxed{2.576} \cdot \frac{2}{\sqrt{12}} = 65.5 \pm 1.5 = (64.0, 67.0),$$

- and a 90% confidence interval is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 65.5 \pm \boxed{1.645} \cdot \frac{2}{\sqrt{12}} = 65.5 \pm 0.95$$

Remark. 99% CI > (longer than) 95% CI > 90% CI

What if we also do not know σ?

CI on the Mean of an Unknown PDF, Variance Unknown

- If the PDF and the standard deviation σ are unknown and the sample size is not large ($n < 30$)
  - Then a different distribution from the normal must be employed to construct the CI
  - However, we have to assume the PDF as normal

CI on the Mean of an Unknown PDF, Variance Unknown

- Assuming a normal population $N(\mu, \sigma^2)$, with both μ, σ^2 unknown, we can still construct a $1 - \alpha$ confidence intervals for

  (1) μ

  (2) σ^2

- We present the details next.

CI on the Mean of an Unknown PDF, Variance Unknown

Confidence interval for μ (when σ is unknown)

- Recall when σ was assumed to be known, to derive a $1 - \alpha$ confidence interval for μ, we started with

$$P(\bar{X} - m < \mu < \bar{X} + m) = 1 - \alpha$$

- and got (after rearranging terms)
  - $P(-m < \bar{X} - \mu < m) = 1 - \alpha.$
- In order to solve for m, we then standardized $\bar{X} \sim N(\mu, \sigma^2/n)$:

$$P\left(-\frac{m}{\sigma/\sqrt{n}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{m}{\sigma/\sqrt{n}}\right) = 1 - \alpha.$$

CI on the Mean of an Unknown PDF, Variance Unknown

- When σ is unknown, we can use its estimator S in place of σ: Dividing all sides of the inequalities in the equation

  - $P(-m < \bar{X} - \mu < m) = 1 - \alpha.$

- by $\frac{S}{\sqrt{n}}$ gives that

$$P\left(-\frac{m}{S/\sqrt{n}} < \frac{\bar{X}-\mu}{S/\sqrt{n}} < \frac{m}{S/\sqrt{n}}\right) = 1 - \alpha$$

- To determine m, we need to know the distribution of the middle quantity. It turns out that it follows **a** ***t*** **distribution with** $n - 1$ **degrees of freedom**:

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t(n-1) = t_{n-1}.$$

CI on the Mean of an Unknown PDF, Variance Unknown

Student's *t* distributions

Def 0.3. The t distribution with ν degrees of freedom is a continuous distribution whose pdf has the following form

$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad -\infty < x < \infty.$$

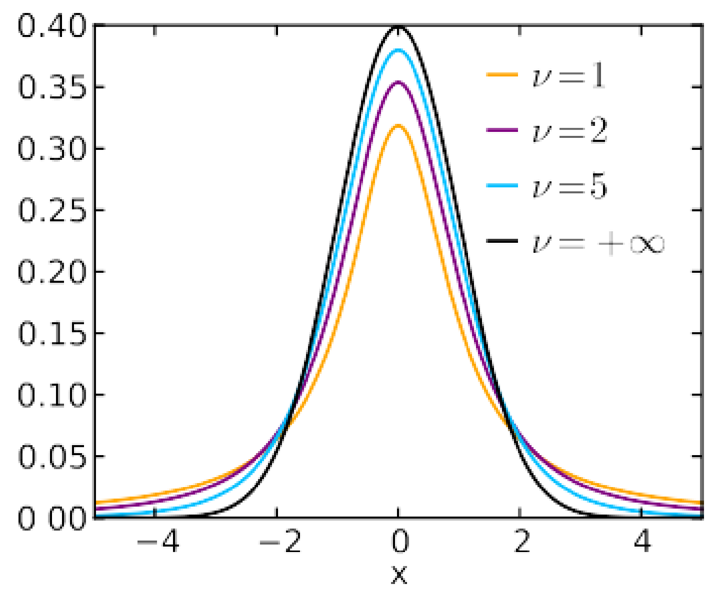
Properties:

(1) The graphs are also symmetric, unimodal and bell-shaped.

(2) $\mathrm{E}(X) = 0$.

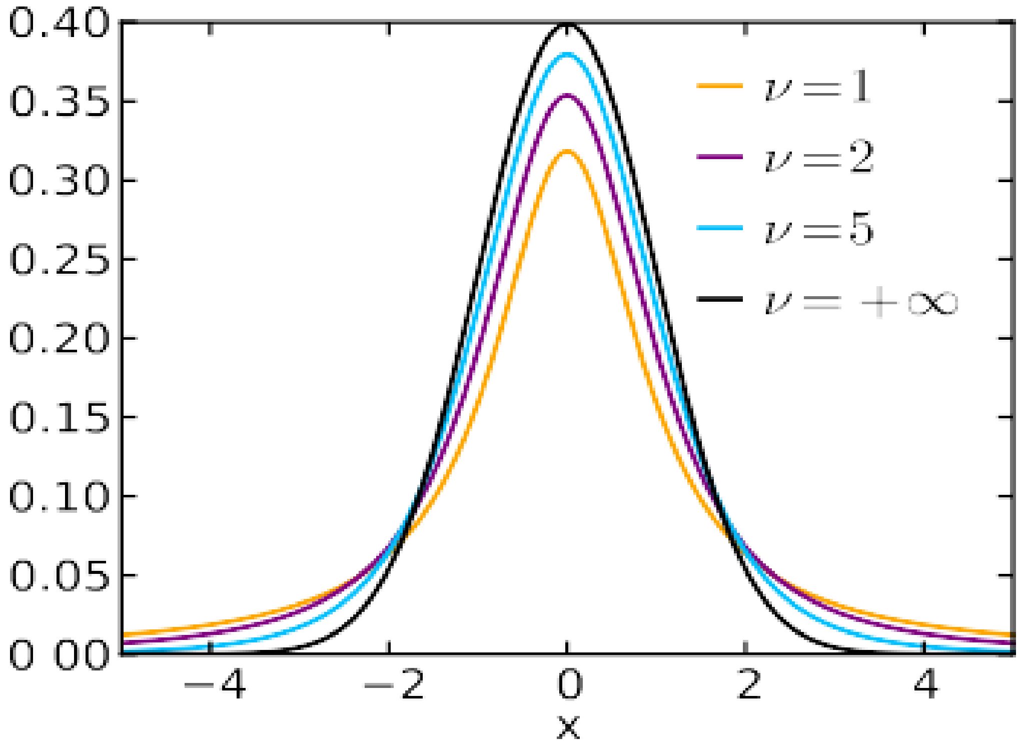
(3) Var(X) = $\frac{v}{v-2}$ (when $v > 2$).

(4) $t(v) \to N(0,1)$ *as* $v \to +\infty$.



CI on the Mean of an Unknown PDF, Variance Unknown

- The t-distribution, particularly when degrees of freedom are high, closely **resembles** the normal distribution but has heavier tails.



CI on the Mean of an Unknown PDF, Variance Unknown

Confidence interval for μ (when σ unknown)

Theorem **0.2.** A $1-\alpha$ confidence interval for μ in the case of a normal population

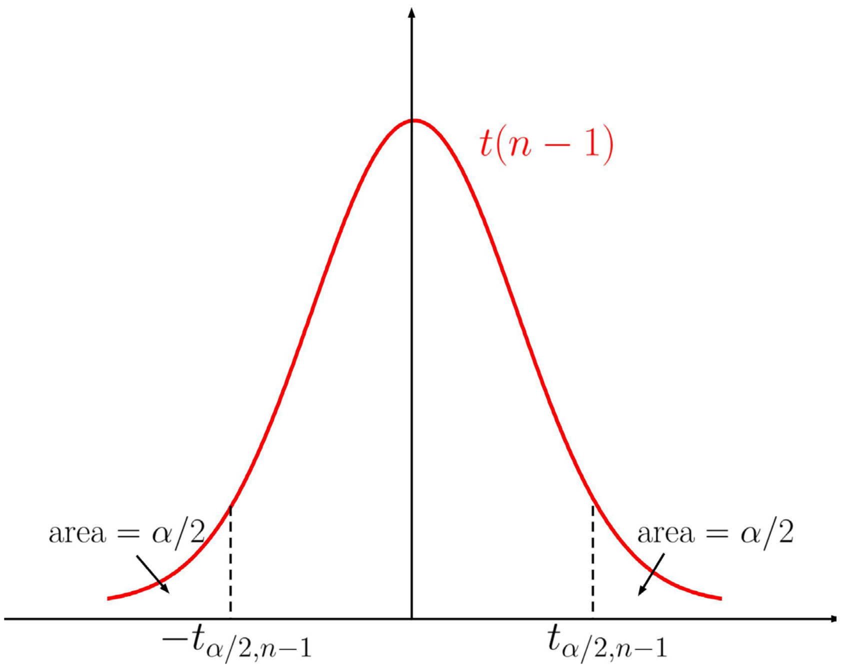
$$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2),$$

where σ is unknown, is

$$\bar{x} \pm t_{\alpha/2, n-1} \frac{s}{\sqrt{n}}.$$

Remark. Compare with:

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$ (when σ known).



(Use the t table to find the t critical Value $t_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

CI on the Mean of an Unknown PDF, Variance Unknown

Example 0.4. In the brown egg example, we selected a sample of 12 eggs (in a carton) and obtained that $\bar{x} = 65.5$ and $s^2 = 4.69$. Assuming normal population (with unknown variance), we obtain a 95% confidence interval

$$\bar{x} \pm t_{\alpha/2,n-1} \frac{s}{\sqrt{n}} = 65.5 \pm t_{0.025,11} \frac{\sqrt{4.69}}{\sqrt{12}} = 65.5 \pm 2.201 \sqrt{\frac{4.69}{12}} = 65.5 \pm 1.4.$$

Remark. Previously, when $\sigma = 2$ was used, we obtained the following 95% confidence interval

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 65.5 \pm 1.96 \cdot \frac{2}{\sqrt{12}} = 65.5 \pm 1.1,$$

which is shorter. Why?

Confidence interval on Variance

Confidence Interval on Variance of NPDF

- Assume the same setting of a random sample from a normal population:
  - Let $X_1, X_2, X_3, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$(both are unknown), and let S^2 be the sample variance
  - We already know that S^2 is an unbiased estimator of σ^2

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

- We can further use S^2 to construct a $1-\alpha$ confidence interval for σ^2.

Confidence Interval on Variance of NPDF

- *Theorem* 0.3. A $1-\alpha$ confidence interval for σ^2 in the case of a normal population $X_1, X_2, X_3, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ is

$$\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2,\,n-1}}, \ \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,\,n-1}} \right)$$

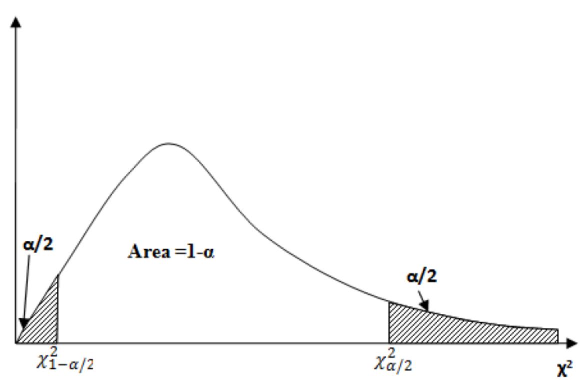
$$\frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2},n-1}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}$$

Where $\chi^2_{\alpha/2,\,n-1}$ and $\chi^2_{1-\alpha/2,\,n-1}$ denote the critical values associated to the chi-square (χ^2) distribution with n-1 degrees of freedom

For any $X \sim \chi^2(n-1)$,

$$P(X > \chi^2_{\alpha/2,\,n-1}) = \alpha/2$$

$$P(X > \chi^2_{1-\alpha/2,\,n-1}) = 1 - \alpha/2.$$



Confidence Interval on Variance of NPDF

In the brown egg example, suppose we did not know the true value of σ^2. Let us find a 95% confidence interval for σ^2 based on the specific sample we have been using: $n = 12, s^2 = 4.69$.

We need to find the two χ^2 critical values (by using table):

- $\chi^2_{\alpha/2,\, n-1} = \chi^2_{.025,\, 11} = 21.92;$
- $\chi^2_{1-\alpha/2,\, n-1} = \chi^2_{.975,\, 11} = 3.82.$

Therefore, a 95% confidence interval for σ^2 is

$$\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2,\, n-1}},\ \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,\, n-1}}\right) = \left(\frac{11 \cdot 4.69}{21.92},\ \frac{11 \cdot 4.69}{3.82}\right) = (2.35, 13.51).$$

One-Sided Confidence Bounds

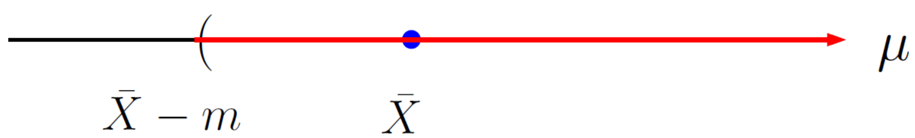
One-Sided Confidence Bounds

One-sided confidence intervals

Sometimes there is a need for only one-sided confidence intervals:

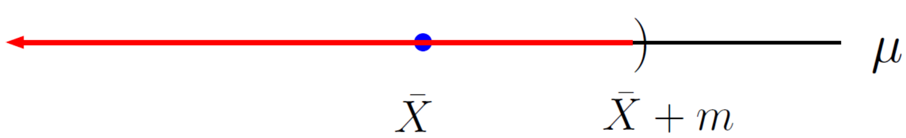
- **Lower confidence bound**

$$1 - \alpha = P(\mu > \bar{X} - m)$$



- **Upper confidence bound**

$$1 - \alpha = P(\mu < \bar{X} + m)$$



One-Sided Confidence Bounds

- It is possible to obtain one-sided confidence bounds for μ by setting either $l = -\infty$ or $u = \infty$ and replacing $z_{\alpha/2}$ by z_{α}
- A 100(1 – α)% upper-confidence bound for μ is

$$\mu \leq u = \bar{x} + z_{\alpha} \frac{\sigma}{\sqrt{n}}$$

- A 100(1 – α)% lower-confidence bound for μ is

$$\bar{x} - z_{\alpha} \frac{\sigma}{\sqrt{n}} = l \leq \mu$$

One-Sided Confidence Bounds

- Theorem 0.4. Assuming a random sample $X_1, X_2, X_3, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with unknown μ but known σ^2. Then
- A $1-\alpha$ lower confidence bound for μ is

$$\mu > \bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}}$$

- A $1-\alpha$ upper confidence bound for μ is

$$\mu < \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}$$

Remark. For each confidence bound $m = z_\alpha \frac{\sigma}{\sqrt{n}}$

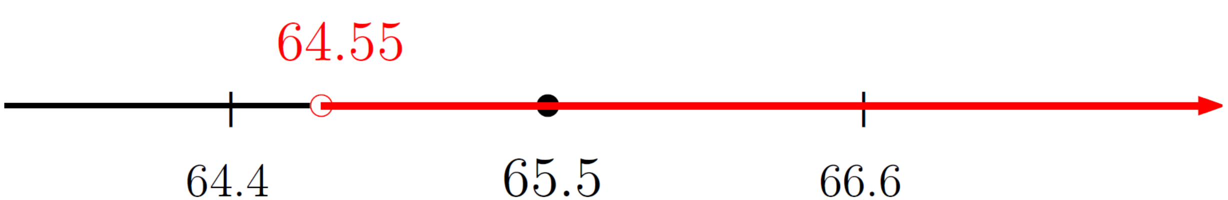
One-Sided Confidence Bounds

- **Example 0.5.** In the brown egg example (where $\bar{x} = 65.5$, $\sigma = 2$).
- A 95% upper confidence bound for μ is

$$\mu < \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}} = 65.5 + z_{.05} \frac{2}{\sqrt{12}} = 65.5 + 1.645 \frac{2}{\sqrt{12}} = 66.45.$$

- Similarly, a 95% lower confidence bound for μ is

$$\mu > \bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}} = 65.5 - 1.645 \frac{2}{\sqrt{12}} = 64.55.$$



One-Sided Confidence Bounds

- *Remark*. When σ is unknown, the one-sided confidence intervals for μ can be obtained by using the t distribution instead:
- A $1 - \alpha$ lower confidence bound for μ is

$$\mu < \bar{x} + t_{\alpha,n-1}\frac{s}{\sqrt{n}}$$

- A $1 - \alpha$ upper confidence bound for μ is

$$\mu > \bar{x} - t_{\alpha,n-1}\frac{s}{\sqrt{n}}$$

One-Sided Confidence Bounds

Similarly, the one-sided confidence intervals for σ^2 are

A $1-\alpha$ lower confidence bound for σ^2 is

$$\sigma^2 > \frac{(n-1)s^2}{\chi^2_{\alpha,\,n-1}}$$

A $1-\alpha$ upper confidence bound for σ^2 is

$$0 < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha,\,n-1}}$$

Summary

- Assume a random sample from a distribution, $X_1, X_2, X_3, \ldots, X_n \overset{iid}{\sim} f(x)$ with unknown parameter θ.
  - Basic concepts
    - Interval estimator: a random interval of the form $\hat{\theta} \pm m = (\hat{\theta} - m, \hat{\theta} + m)$, where m is called the margin of error.
    - A desired property of an interval estimator is the high *coverage probability* :

$$P\,(\theta - m < \theta < \theta + m)\ = 1 - \alpha$$

Summary

- Assume a random sample from a distribution, $X_1, X_2, X_3, \dots, X_n \overset{iid}{\sim} f(x)$ with unknown parameter θ.
  - Basic concepts
    - Interval estimator: a random interval of the form $\hat{\theta} \pm m = (\hat{\theta} - m, \hat{\theta} + m)$, where m is called the margin of error.
    - A desired property of an interval estimator is the high *coverage probability* :

$$P\left(\hat{\theta} - m < \theta < \hat{\theta} + m\right) = 1 - \alpha$$

Summary

– For any specific sample, the corresponding specific interval is called a **confidence interval** for θ (at level $1-\alpha$)

Important results

❖ For a normal population $N(\mu, \sigma^2)$ with unknown μ but *known* σ^2, a $1-\alpha$ confidence interval for μ is

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Pay attention to how the margin of error $m = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

depends on the sample size *n* and confidence level $1 - \alpha$.

Summary

For a normal population $N(\mu, \sigma^2)$ with both μ, σ^2 *unknown*, a $1-\alpha$ confidence interval for μ is

$$\bar{x} \pm t_{\alpha/2,\, n-1} \frac{s}{\sqrt{n}}$$

In this case, a $1-\alpha$ confidence interval for σ^2 is

$$\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2,\, n-1}},\ \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,\, n-1}} \right)$$