II. Counting & Probability

Instructor: **Yanlin Qi**

Institute of Transportation Studies
Department of Statistics
University of California, Davis

Counting –
Multiplication Rule

Linking Sample Space with Counting

- How to know the number of possible outcomes in the Sample Space for a given random experiment?

Identifying Sample Space

Identify the sample space for a given random experiment

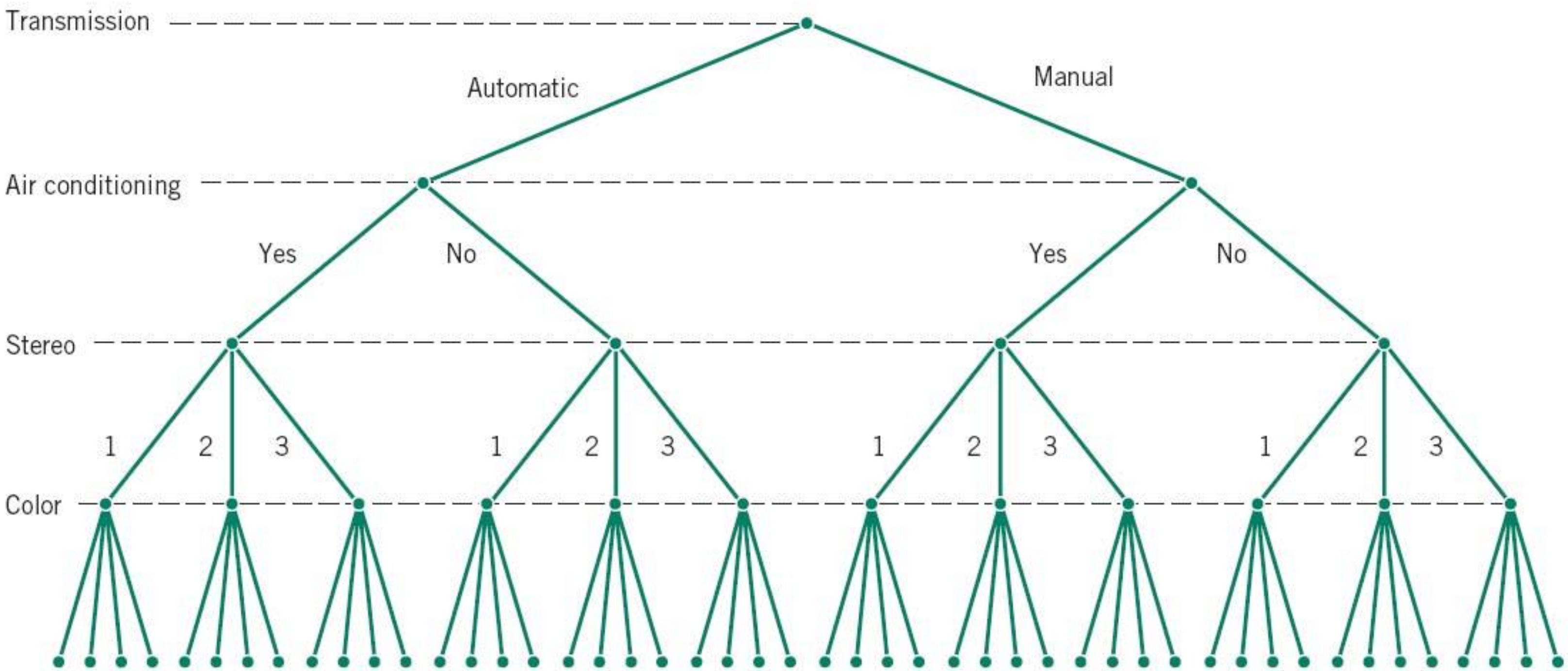

Figure 2-6 Tree diagram for different types of vehicles with 48 outcomes in the sample space.

$$Number\ of\ possible\ outsomes = 2 \times 2 \times 3 \times 4 = 48$$

Multiplication Rule

- If an operation can be described as **a sequence of k steps**, and
  - If the number of ways of completing step 1 is n_1, and
  - If the number of ways of completing step 2 is n_2, and
  - If the number of ways of completing step 3 is n_3, and
  - so forth

  The total number of ways of completing the operation is

$$n_1 \times n_2 \times \ldots \times n_k$$

- Example:
  - If you have 2 hats and 3 shirts you can wear them in

3×2 ways

Example 2: Multiplication Rule

- You are doing a diet and subscribe to a diet website. They send you 13 different recipes for soups, entrees, and desserts as follows:
  - 3 soups
  - 6 entrées
  - 4 desserts
- How many possible different meals could you have (exactly one for each type)?
  Ans. $Total = 3 \times 6 \times 4 = 72$

Counting –
Permutations and Combinations

Permutations & Combinations

- Problems requiring Permutations and Combinations
  - **Arranging Objects**
    - Calculating the number of ways to arrange any set where the arrangement creates different outcomes
      - books on a shelf, people in a line, letters in a word
  - **Scheduling and Task Allocation**
      - Assigning tasks, arranging schedules, or determining the order of operations
  - **Statistical Design and Analysis**
      - How many groups or orderings need to be considered while designing experiments?
  - **Combinatorial Designs**
      - How many ways can a network be tested by checking paths between pairs of nodes?

Permutations

Permutation

- Used to

  count the Number of **ordered sequence** of the elements of a set, consider a set of elements as *S={a, b, c}*
  - *Number of ways to pick elements from a set*

- A **permutation** of the elements is an ordered sequence of the elements
  - S= {abc, acb, bac, bca, cab, cba}
- The number of permutations of n different elements is

$$n! = n\times(n-1)\times(n-2)\times\ldots\times2\times1$$

  - Note: 0!=1!=1
  -

Example 1: Counting Possible Orders

You have 3 candies:

- one red
- one yellow
- one green

Pick 3, one candy at a time (How many **orders**?)

Formula

Number	First	Second	Third
1	red	yellow	green
2	red	green	yellow
3	yellow	red	green
4	yellow	green	red
5	green	red	yellow
6	green	yellow	red

3 x 2 x 1 = 6 = 3!

If there were 5 pieces of candy to be picked up:

The number of orders: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Example 1: Permutations

4 candies:

- **1 red**
- **1 yellow**
- **1 green**
- **1 brown**

Pick 2, one candy at a time (How many orders?)

Example 1: Permutations

You have 4 candies, one red, one yellow, one green and one brown
Pick exactly 2, one candy at a time (How many orders?)

Number	First	Second
1	red	yellow
2	red	green
3	red	brown
4	yellow	red
5	yellow	green
6	yellow	brown
7	green	red
8	green	yellow
9	green	brown
10	brown	red
11	brown	yellow
12	brown	green

Permutations of Subsets!

Permutations on Subsets

- The count of permutations of r-element subsets selected from a set of *n* different elements is:

$$P(n,r) = P_n^r$$ **(Call n permute r)**

$$= n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

$$= \frac{n!}{(n-r)!}$$

Example 5: How many four-letter words can be formed with letters A, B, C, D, E, and F?

- Ans:

$${}_6P_4 = 6!/(6-4)! = 6\times5\times4\times3\times2\times1/(2\times1) = 6\times5\times4\times3 = 360$$

Example 1: Permutations

4 candies:

- **1 red**
- **1 yellow**
- **1 green**
- **1 brown**

Pick 2, one candy at a time
(How many orders?)

Number	First	Second
1	red	yellow
2	red	green
3	red	brown
4	yellow	red
5	yellow	green
6	yellow	brown
7	green	red
8	green	yellow
9	green	brown
10	brown	red
11	brown	yellow
12	brown	green

$$ {}_{n}P_{r} = \frac{n!}{(n-r)!} $$

$$ {}_{4}P_{2} = \frac{4!}{(4-2)!} = \frac{4 \text{ x } 3 \text{ x } 2 \text{ x } 1}{2 \text{ x } 1} = 12 $$

Permutations on Subsets

The number of permutations of n different elements is

$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$

The number of permutations of subsets of *r* elements selected from a set of *n* different elements is:

$$P(n,r) = P_n^r = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

$$= \frac{n!}{(n-r)!}$$

Disregarding the arrangement of the remaining (n-r) elements

Permutation of Similar Objects

- The number of permutation of *n* objects in r different types, where n$=n_1+n_2+\ldots+n_r$
- which n_i *is the number of objects in the ith type*:

$$\frac{n!}{n_1!\,n_2!\,n_3!\ldots n_r!}$$

- Example 9: **How many words** can we build with the letters of the word "Mississippi"?

Ans. M:1, I:4, S:4, and P:2

$$\frac{11!}{1!4!4!2!} = 34650$$

Only the order of different letters matters!

Combinations

Combination

- How many subset of *r* elements can be selected form a set of *n* elements when order does not matter?

$$C(n,r) = nCr = C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- Example 10: How many ways we can choose 4 student from a group of 9, To send to another session?

$$\binom{9}{4} = \frac{9!}{4!5!}$$

Example 1: Combinations: Pick two pieces

Number	First	Second
1	red	yellow
2	red	green
3	red	brown
x	yellow	red
4	yellow	green
5	yellow	brown
x	green	red
x	green	yellow
6	green	brown
x	brown	red
x	brown	yellow
x	brown	green

$$ {}_{n}C_{r} = \frac{n!}{(n-r)!\,r!} $$

$$ {}_{4}C_{2} = \frac{4!}{(4-2)!\,2!} = \frac{4 \text{ x } 3 \text{ x } 2 \text{ x } 1}{(2 \text{ x } 1)(2 \text{ x } 1)} = 6 $$

Example 3

- A group of 15 engineers is to be divided into three distinct project teams of 5 engineers each.
- In how many ways can this division be made?
- Combinations for each project
  - Projects A, B, C
  - For A, choose 5 from 15, order does not matter: $\binom{15}{5}$
  - For B, choose 5 from 15-5, order does not matter: $\binom{15}{5}$
  - For C, choose 5 from 15-5-5, order does not matter: $\binom{5}{5}$
    - $\binom{5}{5} = \frac{5!}{5!(5-5)!} = 1$, only one possible choice
  - $\binom{15}{5} * \binom{15}{5} * \binom{5}{5} = \frac{15!}{5!10!} \cdot \frac{10!}{5!5!} \cdot \frac{5!}{5!0!} = \frac{15!}{5!5!5!}$
- Any other solution method by treating them as permutation of similar objects?

Permutations of Similar Objects

- Example | Hospital Schedule
  - A hospital operating room needs to schedule **three knee** surgeries and **two hip** surgeries in a day. We denote a knee and hip surgery as k and h, respectively.
  - The number of possible sequences of three knee and two hip surgeries is

$$\frac{5!}{2!\,3!} = 10$$

  - The 10 sequences are easily summarized:

    {kkkhh, kkhkh, kkhhk, khkkh,

    khkhk, khhkk, hkkkh, hkkhk,

    hkhkk, hhkkk}

Counting Techniques - Review

Counting: Determine the number of possible outcomes in the sample space/various events

- Permutations and Combinations:
  - Advanced counting techniques
  - Permutations: **Order matters** (e.g., arranging books on a shelf).
    - Formula: $P(n, r) = \frac{n!}{(n-r)!}$
  - Combinations: **Order does not matter** (e.g., selecting a committee from a group).
    - Formula: $C(n, r) = \frac{n!}{r!(n-r)!}$

Probability

Probability

- Probability is used to quantify
  the **likelihood**, or **chance**
  that an outcome of
  a **random** experiment will occur
- This is quantified by assigning a number from the interval [0,1] to the outcome (or a percentage from 0 to 100%).
  - **Higher numbers** indicate that the outcome is **more likely** than lower numbers.
  - A **0** indicates that an outcome will **not occur**.
  - A **1** indicates that an outcome will **occur with certainty**.

Example

- A student working at the trader joe's conducted a **visual inspection** on a lot of bananas that was received last night. The student compiled the information about **the number of defects** found on each banana and created the following table:

Number of defects	Proportion of products
0	0.40
1	0.20
2	0.15
3	0.10
4	0.05
5 or more	0.10

- If one banana is selected randomly, what is the probability that it has no defects?
  - What is the probability that is has three or more defects

More Formal Concepts

Axioms of Probability

- Probability of event is **a number** that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:
- If S is the sample space, and E is any event:
  - $P(S) = 1$
  - $0 \leq P(E) \leq 1$
  - For two events E_1 and E_2 with $E_1 \cap E_2 = \varnothing$
    $P(E_1 \cup E_2) = P(E_1) + P(E_2)$
- Axioms imply that:
  - $P(\varnothing) = 0$
  - $P(E') = 1 - P(E)$
  - if $E_1 \subseteq E_2$ then $P(E_1) \leq P(E_2)$

Example

- $S = \{a, b, c, d, e\}$
- *Events:* $A = \{a, b, \underline{c}\}$ and $B = \{\underline{c}, d, e\}$
- Determine:
  - $P(A)$
  - $P(B)$
  - $P(A')$
  - $P(A \cap B)$
  - $P(A \cup B)$

Exercise 2-50

- $S = \{a, b, c, d, e\}$ (all outcomes are equally likely)
- $A = \{a, b, c\}$ and $B = \{c, d, e\}$
- Determine:
  - $P(A)$ **0.60**
  - $P(B)$ **0.60**
  - $P(A')$ **$1 - P(A) = 0.40$**
  - $P(A \cap B)$ **0.20**
  - $P(A \cup B)$ **$P(A) + P(B) = 0.60 + 0.60 = 1.20$?!?**

    $P(A) + P(B) - P(A \cap B) = 0.60 + 0.60 - 0.20 = 1.00$

Addition Rules

Probability of A or B

- $P(A \cup B) = P(A)+P(B) - P(A \cap B)$
- Note that "A or B" includes:

  1) Event A occurs but Event B does not
  2) Event A does not occur but Event B does
  3) Both Events A and B occur

Addition Rules

- Probability of a **union**:

$$P(A \cup B) = P(A) + P(B) - \mathbf{P(A \cap B)}$$

- Then, if two events are **mutually exclusive**:

$$P(A \cup B) = P(A) + P(B)$$

- For three or more events:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$$
$$- P(B \cap C) - P(C \cap A) + \mathbf{P(A \cap B \cap C)}$$

- If a collection of events is such that, for all pairs:

$$E_i \cap E_j = \varnothing \quad \text{then:}$$

$$P(E_1 \cup E_2 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + P(E_k)$$

Other sets rules

- Complement

$$(E')' = E$$

- Distributive law for set operations

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

- DeMorgan's laws

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

- For any events A and B

$$P(B) = P(A' \cap B) + P(A \cap B)$$

Example

- If $P(A) = 0.2$, $P(B) = 0.3$, and $P(A \cap B) = 0.1$, determine:
  - $P(A')$
  - $P(A \cup B)$
  - $P(A' \cap B)$
  - $P(A \cap B')$
  - $P[(A \cup B)']$
  - $P(A' \cup B)$

Example

- If $P(A) = 0.2$, $P(B) = 0.3$, and $P(A \cap B) = 0.1$, determine:
  - $P(A')$ **$1 - P(A) = 1 - 0.2 = 0.8$**
  - $P(A \cup B)$ **$P(A) + P(B) - P(A \cap B) = 0.2 + 0.3 - 0.1 = 0.4$**
  - $P(A' \cap B)$ **$P(B) = P(A' \cap B) + P(A \cap B) \rightarrow P(A' \cap B) = 0.3 - 0.1 = 0.2$**
  - $P(A \cap B')$ **$P(A) = P(A \cap B) + P(A \cap B') \rightarrow P(A \cap B') = 0.2 - 0.1 = 0.1$**
  - $P[(A \cup B)']$ **$1 - P(A \cup B) = 1 - 0.4 = 0.6$**
  - $P(A' \cup B)$ **$P(A') + P(B) - P(A' \cap B) = 0.8 + 0.2 - 0.1 = 0.9$**

Probability –
Independent & Non-independent Events

Independent Events

- Events A and B are independent if the probability of event B happening is unaffected by whether event A happens

- Event A =

  {a fair coin comes up heads on the first toss} and
- Event B =

  {a fair coin comes up heads on the second toss}

- Are independent events

If events A and B are **independent**, then

Probability of A and B

- P(A and B) = P(AB)= P(A) x P(B)

Only they are <u>INDEPENDENT</u>!

Coin Examples

1. Flip a coin twice,
   - What is the probability of a Head in the First Flip Or the Second Flip?
2. Toss a die and flip a coin
   - What is the probability that you get a 6 or a head?
3. Toss a die three times
   - What is the probability that you get **at least** one 6?

Non-Independent Events

- Event A = {It will rain tomorrow in Houston}
- Event B = {It will rain tomorrow in Galveston}
- Since Houston and Galveston are less than 50 miles apart, **Events A and B are dependent**
- When it rains in Houston, it will most likely rain in Galveston too.

Non-Independent!!

Conditional probabilities!!