ECI-114 Probabilistic Systems Analysis for Civil Engineers, 2024, Summer I
Department of Civil and Environmental Engineering
University of California Davis

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Homework due: July 20th, 2024

Homework # 3: Joint Probability Distributions (100 pts)

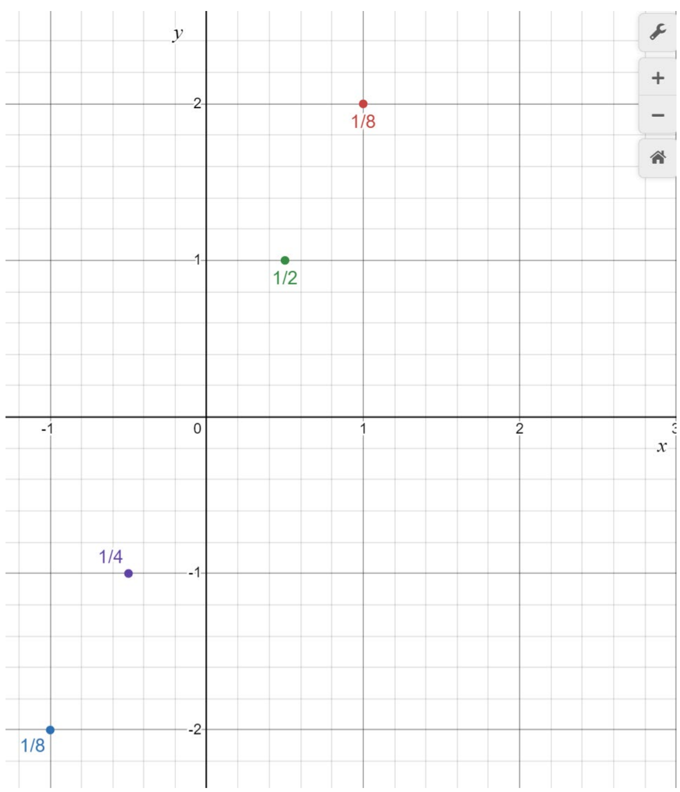
The objective of this assignment is to familiarize yourself with the concepts discussed in class.
Remember to be as explicit as possible in your development of the solutions.

Note: Please be organized and clear. If we cannot understand your work, we will grade accordingly. For the problems that you only provide a solution without the process we will mark down the grade.

For this assignment you need to:

1) 40 *points, 4 each.* You are given the following joint distribution function, where X and Y are two random variables.

X	Y	P(x,y)
1	2	1/8
-1	-2	1/8
.5	1	1/2
-.5	-1	1/4



a) Show that the function satisfies the properties of a joint probability mass function.

$f(x_i, y_i) \geq 0$ for all pairs of (x,y)
1/8+1/8+1/2+1/4=1
$f(x_i, y_i)$ represents the probability $P(X = x_i \text{ and } Y = y_i)$

b) Find P (X<0.5, Y<1.8)
(x,y) = (-1,-2) or (-0.5, -1)
P (X<0.5, Y<1.8)=f(-1,-2)+f(-0.5, -1)=1/8+1/4=3/8

c) Find P (X<0.5)
(x,y) = (-1, -2) or (-0.5,-1)
P (X<0.5)=f(-1,-2)+f(-0.5, -1)=1/8+1/4=3/8

d) Find P (Y<1.95)
(x,y) = (-1, -2) or (0.5,1) or(-0.5,-1)
P (Y<1.95)=f(-1,-2)+f(0.5,1)+f(-0.5, -1)=1/8+1/2+1/4=7/8

e) Find P (X> 1.5, Y<3)
No pair of (x,y) satisfies the condition.
P (X> 1.5, Y<3)=0

f) Find E(X), E(Y), V(X), V(Y)
E(X)=1*1/8+(-1)*1/8+(0.5)*1/2+(-0.5)*1/4=1/8=0.125
E(Y)=2*1/8+(-2)*1/8+1*1/2+(-1)*1/4=1/4=0.25
V(X)=(1-1/8)^2*1/8+(-1-1/8)^2*1/8+(0.5-1/8)^2*1/2+(-0.5-1/8)^2*1/4=0.421875
V(Y)=(2-1/4)^2*1/8+(-2-1/4)^2*1/8+(1-1/4)^2*1/2+(-1-1/4)^2*1/4=1.6875

g) Marginal probability of the random variable Y.

$$f_Y(y) = \sum_x f_{XY}(x, y)$$

$$f_Y(-2) = f_{XY}(-1, -2) = \frac{1}{8}$$

$$f_Y(-1) = f_{XY}(-0.5, -1) = \frac{1}{4}$$

$$f_Y(1) = f_{XY}(0.5,1) = \frac{1}{2}$$

$$f_Y(2) = f_{XY}(1,2) = \frac{1}{8}$$

h) The conditional probability of X given that Y= 1.

$$f_{X|Y=1}(x) = \frac{f_{XY}(x, 1)}{f_Y(1)}$$

When x=0.5, $f_{X|Y=1}(0.5) = \frac{f_{XY}(0.5,1)}{f_Y(1)} = 1$, when $x \neq 0.5$, $f_{X|Y=1}(x) = \frac{f_{XY}(x,1)}{f_Y(1)} = 0$.
i) E(X|y=1) = 0.5*1=0.5
j) Check if X and Y are independent.

$$f_X(0.5) = f_{XY}(0.5,1) = \frac{1}{2}$$

$$f_X(0.5) \neq f_{X|Y=1}(0.5)$$

So X and Y are NOT independent.

2) ***25 points, 5 each*****. A company only produce 1-pound bag of almond, and each bag could be classified in Grade A, Grade B, or Grade C, depending on the quality (A=Best, B= Medium, C= Low). The**

probability that a bag is of Grade C is 0.01, of Grade B is 0.04, and of Grade A is 0.95. Suppose that you have 3 bags of almond produced by this company, and the quality of each of the bags is independent. Consider random variables I and J as the number of Grade A and Grade B bags out of the three, respectively. Find

a) P(i,j)

$P(0,0) = (0.01)^3 = 0.000001$ (All 3 bags are C)

$P(0,1) = {}_3C_1(0.01)^2(0.04) = 0.000012$ (2 bags are C and one bag is B; choose one out of three as the 'B' using ${}_3C_1$; or equivalently, choose two out of three as the 'C's; similar for some other cases below)

$P(0,2) = {}_3C_2(0.04)^2(0.01) = 0.000048$ (1 bag is C and 2 are B; choose two out of three as the 'B's using ${}_3C_2$)

$P(0,3) = (0.04)^3 = 0.000064$ (All 3 bags are B)

$P(1,0) = {}_3C_1(0.95)(0.01)^2 = 0.000285$ (1 bag is A and 2 are C; choose one out of three as the 'A' using ${}_3C_1$)

$P(1,1) = {}_3P_3(0.95)(0.04)(0.01) = 0.002280$ (1 bag for each of A, B, C; order these three in ${}_3P_3$ ways)

$P(1,2) = {}_3C_1(0.95)(0.04)^2 = 0.004560$ (1 bag is A and two bags are B; choose one out of three as the 'A' using ${}_3C_1$)

$P(2,0) = {}_3C_2(0.95)^2(0.01) = 0.027075$ (2 bags are A and 1 is C; choose two out of three as the 'A's using ${}_3C_2$)

$P(2,1) = {}_3C_1(0.95)^2(0.04) = 0.108300$ (2 bags are A and 1 is B; choose one out of three as the 'B' using ${}_3C_1$)

$P(3,0) = (0.95)^3 = 0.857375$ (3 bags are A)

$P(i,j) = 0$ for all other cases not listed above.

(In fact, there is a general formulation for P(i,j), using the multinomial coefficient,

$$P(i,j) = \frac{n!}{i!\,j!\,(n-i-j)!}(0.95)^i(0.04)^j(0.01)^{n-i-j},$$

where n=3)

b) P(i)

$$P(i) = \sum_j P(i,j)$$

$P(0) = P(0,0) + P(0,1) + P(0,2) + P(0,3) = 0.000001 + 0.000012 + 0.000048 + 0.000064 = 0.000125$

$P(1) = P(1,0) + P(1,1) + P(1,2) = 0.000285 + 0.002280 + 0.004560 = 0.007125$

$P(2) = P(2,0) + P(2,1) = 0.027075 + 0.108300 = 0.135375$

$P(3) = P(3,0) = 0.857375$

c) E(I) $= \sum_i iP(i) = 0 \times P(0) + 1 \times P(1) + 2 \times P(2) + 3 \times P(3) = 0.007125 + 2 * 0.135375 + 3 * 0.857375 = 2.85$

d) P(j|i=1)= $\frac{P(1,j)}{P(i=1)}$

$$P(j=0|i=1) = \frac{P(1,0)}{P(1)} = \frac{0.000285}{0.007125} = 0.04$$
$$P(j=1|i=1) = \frac{P(1,1)}{P(1)} = \frac{0.002280}{0.007125} = 0.32$$
$$P(j=2|i=1) = \frac{P(1,2)}{P(1)} = \frac{0.004560}{0.007125} = 0.64$$

$$P(j|i=1) = 0 \; if \; j \neq 0{,}1{,}2$$

e) E(J|I=1)= $\sum_{j=0,1,2} j \cdot P(j|i=1) = 0.32 + 2 * 0.64 = 1.6$

3) ***20 points, 4 each*****. Two random variables X and Y have a probability function equal to cxy. The variable X is a positive continuous variable which at most could take the value of 3. Similarly, the variable Y is also a positive continuous variable which at most could take the value of 3.**

a) Find the value of c that makes the probability function of joint pdf.

$$\int_0^3 \int_0^3 cxy\,dx\,dy = \int_0^3 \left(\frac{cyx^2}{2}\right)\Big|_0^3 dy = \int_0^3 \frac{9}{2}cy\,dy = \frac{9}{4}cy^2\Big|_0^3 = \frac{81}{4}c = 1 \overset{\square}{\Rightarrow} c = \frac{4}{81}$$

(or c=0.04938)

b) Find E(X) and E(Y)

$$E(X) = \int_0^3 \int_0^3 x \cdot cxy\,dx\,dy = \frac{81}{2}c = 2$$
$$E(Y) = \int_0^3 \int_0^3 y \cdot cxy\,dx\,dy = \frac{81}{2}c = 2$$

c) Find $P(X>1.6, 1<Y<2.5)$

$$\mathrm{P}(\mathrm{X} > 1.6, 1 < \mathrm{Y} < 2.5) = \int_1^{2.5} \int_{1.6}^3 cxydx\,dy = 0.4174$$

d) Find the marginal probability of the random variable X

$$f_X(x) = \int_0^3 cxydy = \frac{9c}{2}x = \frac{2}{9}x, 0 \le x \le 3$$

e) Find $P(Y>2|\, X = 1.5)$

$$f_X(x) = \int_0^3 cxy\,dx = \frac{2}{9}x, 0 \le \mathrm{x} \le 3$$
$$P(Y>2|X=1.5) = \int_2^3 \square \left(\frac{f_{XY}(1.5,y)}{f_Y(1.5)}\right) dy = \int_2^3 \frac{1.5cy}{\frac{2}{9} * 1.5} dy = \int_2^3 \frac{2}{9}ydy = \frac{5}{9} = 0.556$$

4) ***10 points*****. Determine the value of c and the covariance and correlation for the joint pmf P(x,y)=c(x+y) for x = 1,2,3 and y = 1,2,3.**

$$\sum_{\substack{x=1,2,3\\y=1,2,3}} c(x+y) = 1$$

$$c(1+1+1+2+1+3+2+1+2+2+2+3+3+1+3+2+3+3) = 1 \overset{\square}{\Rightarrow} c = \frac{1}{36}$$

$$\mu_X = \sum_{\substack{x=1,2,3\\y=1,2,3}} x f_{XY}(x,y) = \sum_{\substack{x=1,2,3\\y=1,2,3}} cx(x+y)$$
$$= 1\big(c(1+1)+c(1+2)+c(1+3)\big) + 2\big(c(2+1)+c(2+2)+c(2+3)\big)$$
$$+ 3\big(c(3+1)+c(3+2)+c(3+3)\big) = 78c = \frac{78}{36} = \frac{13}{6}$$

Similarly, $\mu_Y = \frac{13}{6}$.

$$\sigma_X^2 = \left(\sum_{\substack{x=1,2,3\\y=1,2,3}} x^2 f_{XY}(x,y)\right) - (\mu_X)^2 = \frac{23}{36}$$

(or use $\sigma_X^2 = \sum_{\substack{x=1,2,3\\y=1,2,3}} (x-\mu_X)^2 f_{XY}(x,y)$ to yield the same result.)

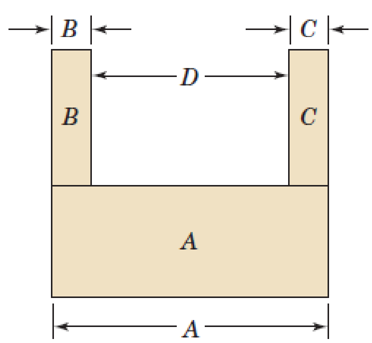
Similarly, $\sigma_Y^2 = \frac{23}{36}$.

$$E(XY) = (1*1)c(1+1) + (1*2)c(1+2) + (1*3)c(1+3) + (2*1)c(2+1) + (2*2)c(2+2)$$
$$+ (2*3)c(2+3) + (3*1)c(3+1) + (3*2)c(3+2) + (3*3)c(3+3) = 168c$$
$$= \frac{168}{36} = \frac{14}{3}$$

$$\sigma_{XY} = cov(X,Y) = E(XY) - \mu_X\mu_Y = \frac{14}{3} - \left(\frac{13}{6}\right)^2 = -\frac{1}{36} = -0.02778$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = -\frac{\frac{1}{36}}{\frac{23}{36}} = -\frac{1}{23} = -0.04348$$

5) 5 points. A U-shaped component is to be formed from the three parts A, B, and C. The thickness of A is normally distributed with a mean of 10 mm and a standard deviation of 0.1 mm. The thickness of parts B and C is normally distributed with a mean of 2 mm and a standard deviation of 0.05 mm. Assume that all dimensions are independent. Determine the mean and standard deviation of the length of the gap D.



D=A-B-C, Linear function

$$E(D) = E(A) - E(B) - E(C) = 10 - 2 - 2 = 6\ mm.$$
$$\sigma_D = \sqrt{V(D)} = \sqrt{1^2V(A) + (-1)^2V(B) + (-1)^2V(C)} = \sqrt{V(A)+V(B)+V(C)}$$
$$= \sqrt{0.1^2 + 0.05^2 + 0.05^2} = 0.1225$$

6) 20 points, *10 each*. Suppose that X is a continuous random variable with probability distribution

$$f_X(x) = \frac{x}{18}, 0 \le x \le 6.$$

(a) Determine the probability distribution of the random variable Y = 2X + 10.

Y=h(X)=2X+10. The inverse solution for h(.) is x=u(y)=(y-10)/2.
The Jacobian of u is J=u'(y)=dx/dy=1/2.
h(0)=10, h(6)=22.

$f_Y(y) = |J| \cdot f_X(x) = |J| \cdot f_X(u(y)) = \frac{1}{2} \cdot \frac{(y-10)}{2\times 18} = \frac{y-10}{72}, 10 \le y \le 22.$

(b) Determine the expected value of Y.

$$E(Y) = \int_{10}^{22} y \cdot \frac{y-10}{72} dy = \left(\frac{y^3}{216} - \frac{5}{72}y^2\right)\Big|_{10}^{22} = 18$$

(Q.6 does not count towards your homework scores.)