Probabilistic Systems Analysis for Civil Engineers

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Today

1. Independent & Dependent Events
2. Conditional Probability
3. Independence
4. Bayes' Theorem

Independent events

- Events A and B are independent if the probability of event B happening is unaffected by whether event A happens

Example

- Event A =

  {a fair coin comes up heads on the first toss} and
- Event B =

  {a fair coin comes up heads on the second toss}

- Are independent events

Dependent Events

- Event A = {It will rain tomorrow in Houston}
- Event B = {It will rain tomorrow in Galveston}
- Since Houston and Galveston are less than 50 miles apart, **Events A and B are dependent**
- When it rains in Houston, it will most likely rain in Galveston too.

Dependent Events

Conditional probabilities

1. Conditional Probabilities

P(draw two aces from a deck of 52)

- By what we have done before:
- Example 1: P(Two Aces)

  = P(Ace on 1st Draw) x P(Ace on 2nd Draw)

  = 1/13 x 1/13

  = 1/169

- Is this right?

No: There are only 4 Aces in the Deck

- The events are **not** independent:
- If the first card drawn is an ace, the probability of getting an ace on the second draw is smaller:
- There are only 3 aces left to find in the remaining 51 cards

Conditional Probability

- Once the first card is drawn as an Ace, we are interested in the *conditional probability* that the second card is an ace:

- Example 2: P(Ace on 2nd draw | Ace on 1st draw)

- This value is 3/51 since there are 3 aces left in the remaining deck of 51

General Formula

- If Events A and B are **not** independent, we can write:

- P(A **and** B) = P(A) x P(B|A)

See later
Multiplication rule

- So P(draw 2 Aces)

  = (4/52)(3/51)

  = 12/2652 = 1/221.

P(Ace of Diamonds and a Black Card)

- Draw two cards from the deck. What is the probability of getting the Ace of Diamonds and a Black (spade/club) Card?

- Remember, you can get the Ace of Diamonds in either the first draw or the second.

Two Cases

- Case A: the first card is the Ace of Diamonds.
  P(Case A)
  = P(A of D) x P(Black|A of D)
  = 1/52 x 26/51 = 1/102

- Case B: the second card is the Ace of Diamonds.
  P(Case B)
  = P(Black) x P(A of D|Black)
  = 26/52 x 1/51 = 1/102

Result

- Thus, P(Ace of Diamonds **and** Black)

= P(Case A **or** Case B)

= P(Case A) + P(Case B) – P(Case A **and** Case B)

= 1/102 + 1/102 - 0

= 1/51

Conditional Probability

- Sometimes probabilities need to be reevaluated as additional information becomes available
- The probability of an **event** $\boldsymbol{B}$ under the knowledge that the outcome will be in **event** $\boldsymbol{A}$ is denoted as

$$P(B \mid A)$$

- This is called the conditional probability of B given A

Joint Probability

- **Joint Probability:** The probability of two (or more) events occurring simultaneously
- Denoted as $P(A \cap B)$ where A and B are events.

Example 1: Coin Toss

Events:

- A: First coin lands on heads.
- B: Second coin lands on heads.

Example 2: Weather and Traffic

- Events:
  - A: It rains.
  - B: There is heavy traffic.

Conditional Probability

- The conditional probability of an event B given an event A, denoted as $P(B \mid A)$, is

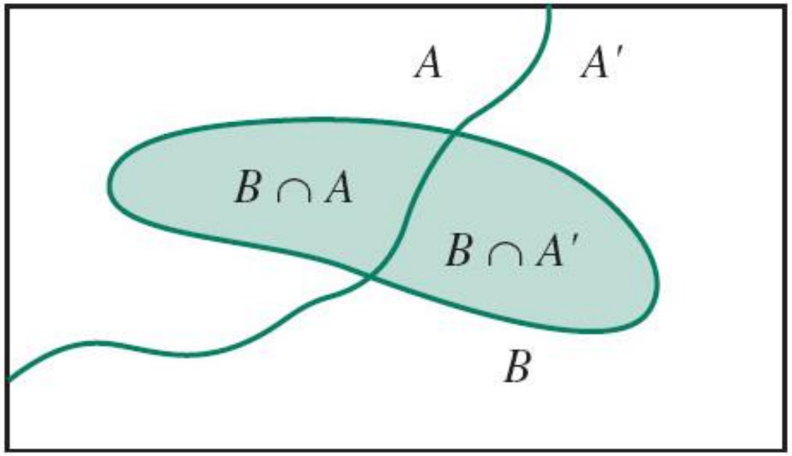
$$P(B \mid A) = P(B \cap A) \,/\, P(A)$$

- Therefore, $P(B \mid A)$ can be interpreted as the **relative** frequency of event $\boldsymbol{B}$ among the trials that produce an outcome in event $\boldsymbol{A}$
- It is like **scaling down** to a **smaller sample space**

Total Probability Rule

- Sometimes, the probability of an event can be recovered by summing up a series of conditional probabilities
- For any event B, we can write B as the union of the part of B in A and the part of B in A':

$$B = (B \cap A) \cup (B \cap A')$$

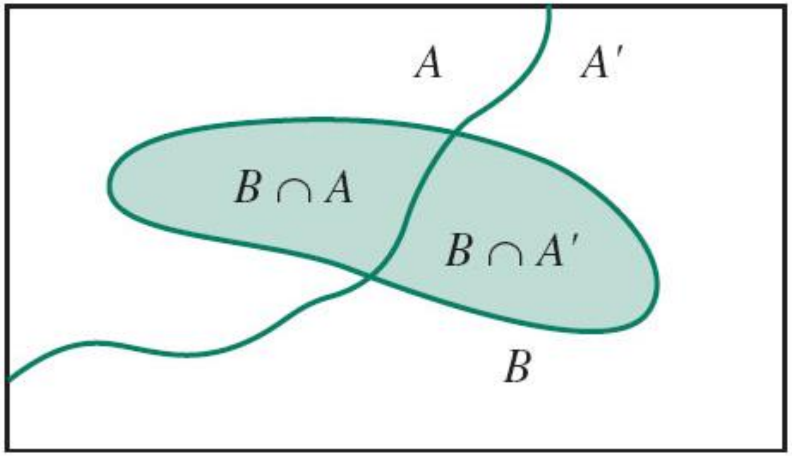


Total Probability Rule for Two Events

- For two events we have:

$$P(B) = P(B \cap A) + P(B \cap A')$$
$$= P(B | A)\, P(A) + P(B | A')\, P(A')$$

Recall: $P(B | A) = P(B \cap A) / P(A)$



Total Probability Rule for Multiple Events

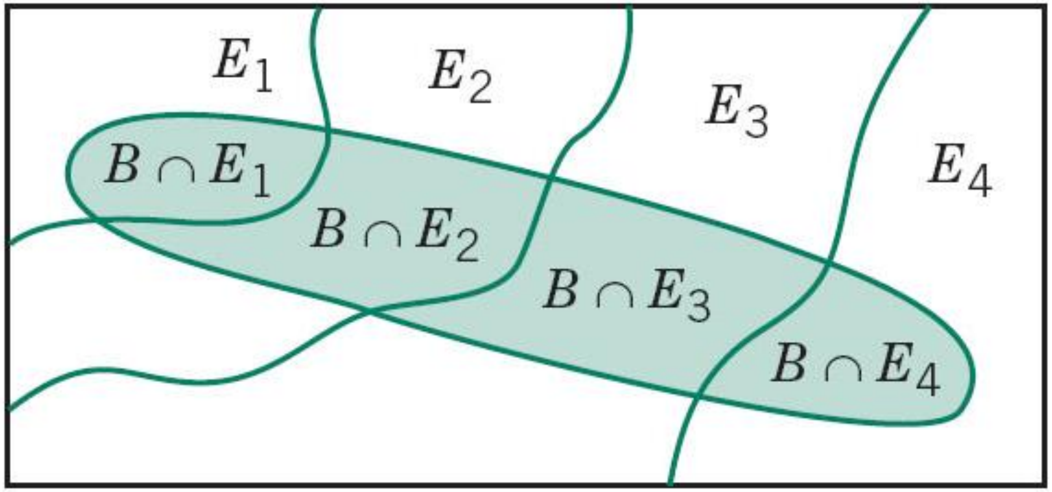
- For many mutually exclusive and exhaustive events we have:

$$P(B) = P(B \cap E1) + P(B \cap \mathrm{E2}) + \ldots + P(B \cap \mathrm{Ek})$$

$$P(B) = P(B \mid E1)\, P(E1) + P(B \mid E2)\, P(E2) + \ldots$$

$$P(B \mid Ek)\, P(Ek)$$

  - Exhaustive events: $E1 \cup E2 \cup \ldots \cup Ek = S$



Summary of Rules

- Multiplication Rule

$$P(A \cap B) = P(B \mid A)\, P(A) = P(A \mid B)\, P(B)$$

- Total Probability Rule
  - Assume E1, E2, … Ek, are mutually exclusive and exhaustive sets:

$$P(B) = P(A \cap E1) + P(A \cap \mathrm{E2}) + \ldots + P(A \cap \mathrm{Ek})$$

$$P(B) = P(B \mid E1)\, P(E1) + P(B \mid E2)\, P(E2) + \ldots$$

$$P(B \mid Ek)\, P(Ek)$$

Example 3

- 100 samples of a cast aluminum part are summarized as:

		length	
		excellent (B)	good (C)
surface finish	excellent (A)	80	2
	good	10	8

- Determine:
  - P(A) **0.82**
  - P(B) **0.90**
  - P(A | B) **80/90 = 0.889**
  - P(B | A) **80/82 = 0.9756**
  - P(A | C) **2/10 = 0.20**

Example 4

- A batch of 350 steel bars contains 8 that are defective
  - 2 are selected, at random, without replacement
- What is the probability that … :
  - … both are defective?
    - $P(D_1 \cap D_2) = 8/350 \times 7/349 = 0.000458$
  - … the second one selected is defective given that the first one was defective?
    - $P(D_2 | D_1) = P(D_2 \cap D_1) / P(D_1) = 0.000458/(8/350) = 0.020057 = (7/349)$
  - … both are acceptable?
    - $P(D_1' \cap D_2') = 342/350 \times 341/349 = 0.954744$

Example 4 – cont.

- The probability that both are acceptable can also be found as follows, by applying De Morgan's rule:
  - $P(D_1' \cap D_2') = P[(D_1 \cup D_2)'] = 1 - P(D_1 \cup D_2)$
    $= 1 - [P(D_1) + P(D_2) - P(D_1 \cap D_2)]$
    $= 1 - [8/350 + \underline{8/350} - 0.000458] = 0.954744$
- Here, we are looking at the event **$(D_1$ OR $D_2)$**, so **$P(D_2)$** depends on the outcome of the first selection, which can be either **defective** or **acceptable**
- Thus, we use the **Total Probability Rule**, to obtain:
  - $P(D_2) = P(D_2 \mid D_1)\ P(D_1) + P(D_2 \mid D_1')\ P(D_1')$
    $= 7/349 \times 8/350 + 8/349 \times 342/350 = \underline{8/350}$

Multiplication Rule

- When the probability of the intersection is needed:

$$P(A \cap B) = P(B|A)\,P(A) = P(A|B)\,P(B)$$

- Example 5:
  - A concrete batch passes compressive tests with $P(A) = 0.90$;
  - A second concrete batch is known to pass the tests if the first does, with $P(B|A) = 0.95$.
- What is the probability $P(A \cap B)$ that both pass the tests?
  - Ans.: $P(A \cap B) = P(B|A)\,P(A) = 0.95 \cdot 0.90 = 0.855$

Example 6

- A manufactured piece of steel fails when subjected to various tension levels, with the following probability

Probability of Failure	Tension Level	Probability of Tension Level
0.20	High	0.20
0.01	Not High	0.80

- Let F = failure, and H = piece has high tension level
- What is the probability of failure of the piece?

Ans.: $P(F) = P(F/H)\,P(H) + P(F/H')\,P(H')$

$= 0.20 \cdot 0.20 + 0.01 \cdot 0.80 = 0.048$

Example 7

- Another manufactured piece fails when subjected to various compression levels, with the following probability

Probability of Failure	Stress Level	Probability of Tension Level
0.20	High	0.30
0.02	Medium	0.30
0.002	Low	0.40

- Let H, M, L = piece has high/medium/low compression level
- What is the probability of failure of the piece?

Ans.: $P(F) = P(F/H)\,P(H) + P(F/M)\,P(M) + P(F/L)\,P(L)$

$$= 0.20 \cdot 0.30 + 0.02 \cdot 0.30 + 0.002 \cdot 0.40$$

$$= 0.0668$$

Example 8

- Building failures are due to either:
  - Natural actions N (87%) and include
    - Earthquakes E (56%)
    - Wind W (27%); and
    - Snow S (17%)
  - Man-made causes (13%) and include:
    - Construction errors C (73%); or
    - Design errors D (27%)
- What is the probability of failure due to construction errors?
  - $P(F) = P(C/M)\,P(M) = 0.73 \times 0.13 = 0.0949$
- What is the probability of failure due to wind or snow?
  - $P(F) = [P(W/N) + P(S/N)]\,P(N) = [0.27 + 0.17] \times 0.87 = 0.3828$

2. Independence

Independence

- In some cases, we might have that

$$P(A \mid B) = P(A)$$

  - e.g., the probability of failure of a building in the US given that an earthquake occurred in Italy

- In this special and important case, we have

$$P(A \cap B) = P(A \mid B)\, P(B) = P(A)\, P(B)$$

- Then, the two events are said to be independent

Independence (Two Events)

- Two events are independent if any one of the following equivalent statements is true:
  - $P(A \mid B) = P(A)$
  - $P(B \mid A) = P(B)$
  - $P(A \cap B) = P(A)\,P(B)$

- The concept of independence is fundamental and should not be confused with the concept of mutual exclusivity ($P(A \cap B) = 0$)

Independence (Multiple Events)

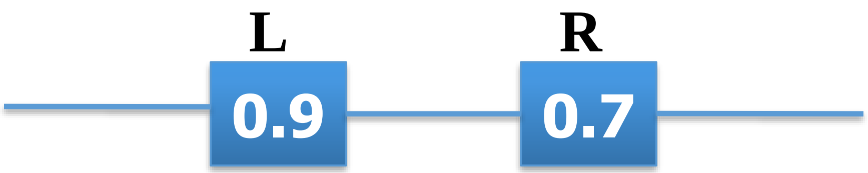
- The events $E_1, E_2, \ldots E_n$ are independent if and only if for any subset of these events $E_{i1}, E_{i2}, \ldots E_{ik}$:

$$P(E_{i1} \cap E_{i2} \cap \ldots \cap E_{ik}) = P(E_{i1})\, P(E_{i2}) \ldots P(E_{ik})$$

Note: The knowledge that the events are independent usually comes from a fundamental understanding of the random experiment

Example 9 – Series Systems

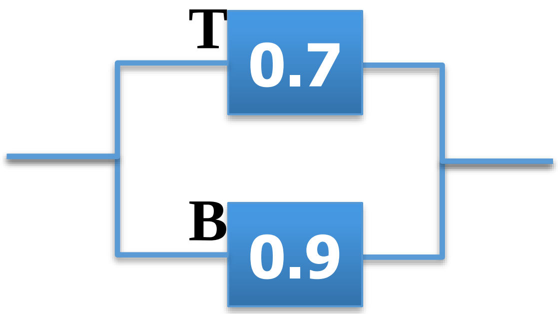
- The following system works only if all elements work
- The $P(L)$ and $P(R)$ that each element works are shown



  - Assume elements fail independently
- What is the probability $P(W)$ that the system works?
  - Ans.: $P(W) = P(L \cap R) = P(L)\,P(R) = 0.90 \cdot 0.70 = 0.63$

Example 10 – Parallel Systems

- The following system fails only if all elements fail
- The $P(T)$ and $P(B)$ that each element works are shown



  - Assume elements fail independently
- What is the probability $P(W)$ that the system works?

Ans.:

$P(W') = P(T' \cap B') = P(T')\,P(B') = (1-0.70)(1-0.90) = 0.03$

$P(W) = 1 - P(W') = 0.97$

Series and Parallel Systems Elements with Independent Failure E_i

- Series system
  - It works only if all elements work
  - Probability of working:

$$P(E_1' \cap E_2' \cap \ldots \cap E_k') = P(E_1')P(E_2')\ldots P(E_k')$$

  - Probability of failure:

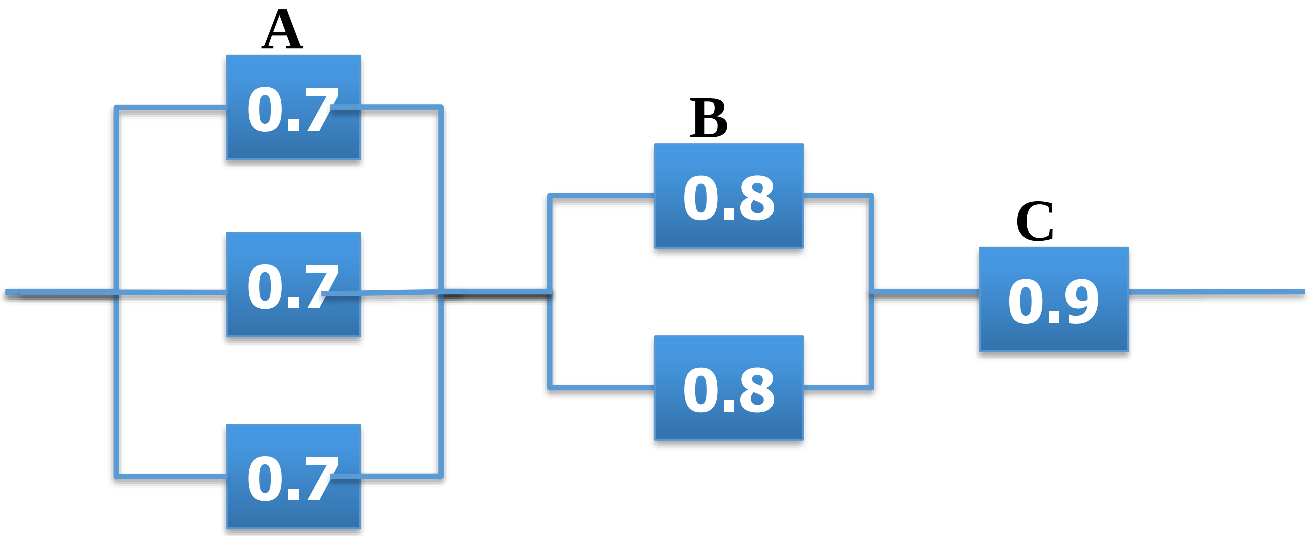
$$1 - P(\text{working})$$

  - The least reliable element has the biggest effect on the system reliability, because it is the one that will most likely fail first
- Parallel system
  - It fails only if all elements fail
  - Probability of failure:

$$P(E_1 \cap E_2 \cap \ldots \cap E_k) = P(E_1)P(E_2)\ldots P(E_k)$$

  - Probability of working: $1 - P(\text{failure})$
  - The most reliable element has the biggest effect on the system reliability, because it is the one that will most likely fail last

Example 11



- What is the probability $P(W)$ that the system works?
  - Ans.: $P(W) = P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$
    $= [1\text{-}P(A')]\, [1\text{-}P(B')]\, P(C)$
    $= (1\text{-}0.3^3)\, (1\text{-}0.2^2)\, (0.9) = 0.973 \cdot 0.96 \cdot 0.9 = 0.84$

3. Bayes´Theorem

Bayes' Theorem

- After a random experiment generates an outcome, we are naturally interested in the probability that a condition was present given an outcome

Bayes' Theorem

- The mathematical background is very simple

$$P(A \cap B) = \boxed{P(A \mid B)\, P(B)} = P(B \cap A) = \boxed{P(B \mid A)\, P(A)}$$

- So we can write:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- We can solve for $P(A \mid B)$ in terms of $P(B \mid A)$

Why We Need Bayes' Rule

- Purpose of Bayes' Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- **Updating Probabilities with New Evidence:**
  1. Bayes' Rule is essential for updating the probability of a hypothesis A given new data B.
  2. It allows us to incorporate new information systematically and revise our beliefs accordingly.
- **Some Practical Applications:**
  1. **Medical Diagnosis:**
     1. Given the symptoms (evidence B), determine the probability of a particular disease (hypothesis A).
  2. **Spam Filtering:**
     1. Calculate the probability that an email is spam (hypothesis A) based on the presence of certain keywords (evidence B).

Why We Need Bayes' Rule

- Purpose of Bayes' Rule $$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- **Connecting Conditional Probabilities**:

- Often, we have access to P(B|A) (likelihood of observing the evidence given the hypothesis)
- And need to find P(A|B) (probability of the hypothesis given the evidence).
- Bayes' Rule provides this connection

Example 13

- A manufactured piece of steel fails when subjected to various tension levels, with the following probability

Probability of Failure	Tension Level	Probability of Tension Level
0.20	High	0.20
0.01	Low	0.80

- Let F = {piece failure}, and H = {piece has high tension level}
- What is the probability that a high tension is present when a failure occurs?
  - Ans.: $P(H|F) = P(F|H)\, P(H) / P(F)$

    (Remember: $P(F) = P(F/H)\, P(H) + P(F/H')\, P(H')$)

    - then

$$= 0.20 \cdot 0.20 / 0.048 = 0.83$$

Bayes' Theorem

- From the Total Probability Rule
  - e.g., $P(F) = P(F/H)\,P(H) + P(F/H')\,P(H')$
- So we can obtain the final form of Bayes' Theorem

$$P(E_1|B) = \frac{P(B|E_1)P(E_1)}{P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k)}$$

Example 14

- It is known that Zika affects 0.01% of the population in Latin-America (LA)
- Different laboratories are working on developing medical tests for early detection
- The new tests are not 100% precise, but are effective
  - They can detect ZIKA in 980 cases out of 1000
  - It can detect the lack of illness in 950 cases out of 1000
- During, Spring Break a faculty in the C&E Department went to LA, and once coming back, took the test, and the results were positive
- What is the probability that the faculty has ZIKA?

Example 14 – cont.

- Let D = {Has the illness}, S = {the test is positive}
- We need the $P(D|S)$
- The probability of a positive test without ZIKA is:

$$P(S|D') = 0.05$$

- From Bayes' Theorem, the probability that the faculty has ZIKA given that the test is positive is:

$$P(D|S) = P(S|D)P(D) / [P(S|D)P(D) + P(S|D')P(D')]$$
$$= 0.98 \cdot 0.0001/[0.98 \cdot 0.0001 + 0.05 \cdot 0.9999]$$
$$= 0.001956$$

Example 15

- Bayesian techniques are used to quickly diagnose cyber-security problems
- Cyber-security breach/hacks are associated with three types of techniques:
  - Phishing, with probability: $P(P) = 0.1$
  - Social engineering, with probability: $P(S) = 0.6$
  - Other, with probability: $P(O) = 0.3$
- The conditional probabilities of breaks/hacks are:
  - Phishing, with probability: $P(H|P) = 0.9$
  - Social Engineering, with probability: $P(H|S) = 0.2$
  - Other, with probability: $P(H|O) = 0.5$

Example 15 – cont.

- If a security breach/hack **has been observed**, what is the **most likely cause**?

  i.e., max[P(P | H), P(S | H), P(O | H)] ?

- For the Total Probability Rule:

  $$P(H) = P(H/P)\,P(P) + P(H/S)\,P(S) + P(H/O)\,P(O)$$

  $$= 0.9 \cdot 0.1 + 0.2 \cdot 0.6 + 0.5 \cdot 0.3 = 0.36$$

- Then, for Bayes' Theorem
  - $P(P/H) = P(H/P)\,P(P) / P(H) = 0.9 \cdot 0.1 / 0.36 = 0.250$
  - $P(S/H) = P(H/S)\,P(S) / P(H) = 0.2 \cdot 0.6 / 0.36 = 0.333$
  - $P(O/H) = P(H/O)\,P(O) / P(H) = 0.5 \cdot 0.3 / 0.36 = 0.417$

They add up to 1

Summary

- Reviewed of conditional probability
  - Showed the important of the total probability rule
- Described the multiplication rule to estimate the P(intersection of events)
- Discussed the importance of independence (independent events)
  - Different than mutual exclusivity
- Baye’s Theorem

Next Class

- Please read sections 3-1, …, 3-4 and 4-1, …, 4-4
  - Counting techniques
  - Discrete and Continuous Random Variables
  - Probability Distributions
    - Probability Mass Function
    - Probability Density Function
  - Cumulative Distribution Function
  - Mean and Variance