Answer 1

1.a) Naming the left circle "A", the right one "B", and the bottom one "C" we have $A \cap B$ here, because the Venn diagram represents "A and B" occurrence.

1.b) In this case, everything occurs except C or the intersection of A and B. Therefore, by applying DeMorgan's law we have :

$[C \cup (A \cap B)]' = C' \cap (A' \cup B')$

1.c) This represents the event that either B or C occurs. Therefore, we have: $B \cup C$

1.d) This represents the event that A and B occur but not C, which means: $A \cap B \cap C'$

Answer 2

S={r,s,t,u,v,w,x,y} , A={r,s,t} , B={u,v,w,x}

2.a) If each outcome is equally likely, and since there are 8 possible outcomes, then the probability of each outcome is $\frac{1}{8}$.
P (A) = probability that either of three possible outcomes occurs

$$\rightarrow \; P(A) = \frac{3}{8}$$

2.b) Considering explanation for part "2.a" , we have :
P (B) = probability that either of four possible outcomes occurs

$$\rightarrow \; P(B) = \frac{4}{8} = \frac{1}{2}$$

2.c) Considering explanation for part "2.a" , we have :
$P(A') =$ probability that either of five possible outcomes occurs

$$\rightarrow \; P(A') = \frac{5}{8}$$

2.d) In this case, A and B are mutually exclusive since A and B share no outcomes, so the probability of A and B occurring is 0 .

$$\rightarrow \; P(A \cap B) = 0$$

2.e) Since A and B are mutually exclusive, the total number of outcomes in A ∪ B is simply the sum of the outcomes in each, which makes it 7 of the 8 outcomes in A ∪ B.
P (A ∪ B) = probability that either of the 7 possible outcomes occurs

$$\rightarrow \; P(A \cup B) = \frac{7}{8}$$

Answer 3

S = { r,s,t,u,v } , P(r) = 0.01 , P(s) = 0.6 , P(t) = 0.05 , P(u) = 0.04 , P(v) = 0.3 , A = {r,u} , B = {s,t,v } , C = {r,s,t}

3.a) This is simply the probability of u or r occurring :
$P(A) = P(r) + P(u) = 0.01 + 0.04 = 0.05$

3.b) This is simply the probability of s, t, or v occurring :

$P(B) = P(s) + P(t) + P(v) = 0.6 + 0.05 + 0.3 = 0.95$

3.c) This is the probability of anything except for $\{r, s, t\}$ occurring, which is the same as the probability that $\{u, v\}$ occurs :

$P(C') = P(u) + P(v) = 0.04 + 0.3 = 0.34$

3.d) $A \cap C = \{r, u\} \cap \{r, s, t\} = \{r\}$
$P(A \cap C) = P(r)$
$P(A \cap C) = 0.01$

3.e) $A \cup B = \{r, u\} \cup \{s, t, v\} = \{r, s, t, u, v\}$
$P(A \cup B) = P(r) + P(s) + P(t) + P(u) + P(v)$
$P(A \cup B) = 1$

Answer 4

4.a) There are 688 + 216 = 904 beams meeting high strength standards out of 1000. Therefore, the probability of the beam meeting high standards is :
$904/1000 = 90.4\%$

4.b) There are 688 beams meeting high strength standards and that have no detected flaws out of 1000. Therefore, the probability of the beam meeting high strength standards and having no detected flaws is :
$688/1000 = 68.8\%$

4.c) There are 21 beams that don't meet high strength standards and have flaws detected out of 216 + 21 = 237 beams with flaws detected. Therefore, the probability of the beam not meeting high standard given that it has detected flaws is :
$21/237 = 8.86\%$

4.d) $\frac{P(NHS \cap FD)}{P(FD)} = P(NHS|FD)$

Answer 5

5.a) In this case, each bridge's inspection is a mutually exclusive event. This means that one outcome of the random experiment would be if all of the bridges are classified as

above specification. Since there are 4 bridges and two possible classifications for each, this means there will be 8 elements in the sample space.
To make things simple, the following diagram will show the sample space where A signifies the bridge is above specification, and B signifies the bridge is below specification. The position of A or B identifies whether the first, second, third, or fourth bridge is above or below specification. For example, AABA means that all bridges are above specification except for the third:

$$S = \begin{Bmatrix} AAAA, & AAAB, & AABA, & AABB, \\ ABAA, & ABAB, & ABBA, & ABBB, \\ BAAA, & BAAB, & BABA, & BABB, \\ BBAA, & BBAB, & BBBA, & BBBB \end{Bmatrix}$$

5.b) Each dart's outcome is mutually exclusive to the next. Following the same logic as the first part of this problem, '1' signifies the dart lands on center section, '0' signifies the dart lands on outer section:

$$S = \begin{Bmatrix} 000, & 001, & 010, & 100, \\ 101, & 110, & 011, & 111 \end{Bmatrix}$$

5.c) Each potato chip classification is mutually exclusive to the next. Following the same logic as the first part of this problem, "H" signifies high quality, "M" signifies mediocre, "l" signifies large, "m" signifies medium, " s" signifies small:

$$S = \begin{Bmatrix} Hl, & Hm, & Hs, \\ Ml, & Mm, & Ms \end{Bmatrix}$$

Answer 6

P (A|B) = 0.2 , P (B) = 0.8 , P (A) = 0.64

6.a) By definition:

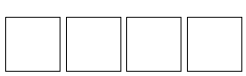
$$P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow P(A \cap B) = P(B).P(A|B) = (0.8)(0.2) = 0.16$$

6.b) By definition:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \rightarrow P(B|A) = \frac{0.16}{0.64} = 0.25$$

Answer 7

7.a) Let's label the universities as A, B, C, D, E, F, G. And there are 4 slots available:



This number can be found through a series of logical steps:

- The first grant can be given to either of the 7 universities, that makes 7 ways:

G
F
E
D
C
B
A

The final different number of ways in which this step could be awarded at this point is at least 7.

• The second grant can be given to either of the 6 other universities since one of them has been awarded already. For example, if A was awarded first, then there are 6 other universities that can be awarded the second grant:

G
F
E
D
C
B

A			

Likewise if any of the 7 universities was given the first grant, there would be one less possible University that would get the second grant. This increases the number by ×6 ways, the number at this point is: 7×6

• The third grant can be given to either of the 5 other universities since two of them has been awarded already. For example, if A was awarded first and B was awarded second, then there are 5 other universities that can be awarded the third grant:

G
F
E
D
C

A	B		

Likewise if any of the 7 universities were given the first two grants, there would be two less possible universities that would get the third grant. This increases the number by ×5 ways, the number at this point is: $7 \times 6 \times 5$

- The fourth grant can be given to either of the 4 other universities since three of them has been awarded already. For example, if A was awarded first, B was awarded second, and C was awarded third, then there are 4 other universities that can be awarded the fourth grant:

G
F
E
D

A	B	C	

Likewise if any of the 7 universities were given the first three grants, there would be three less possible universities that would get the fourth grant. This increases the number by ×4 ways, the number at this point is: $7 \times 6 \times 5 \times 4$

The total number is therefore: $7 \times 6 \times 5 \times 4 = 840$. Indeed, this is one way to visualize the expression:

$$ {}_7P_4 = \frac{7!}{(7-4)!} $$

which identifies the number of ways 7 different items can be placed in 4 different containers.

7.b) Since each of the events are equally likely, we can count the number of ways that the awards could not be given to University No.2 and subtract from the total number of permutations to then divide by the total number of permutations. Imagine doing the same steps as above but discounting University B from the start, this would leave us with ${}_6P_4$:

$$ {}_6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 360 $$

Which means that there are 840 - 360 = 480 ways in which University No. 2 could be given the award out of 840 possible permutations, which means that the probability of University No. 2 being awarded a grant is:

$$ \frac{480}{840} = 57.14\% $$

7.c)

<u>Analyzing the conditional probabilities:</u> Assigning E as the event that university 5 is chosen and F as the event that university 6 is chosen makes:

$$ P(E \cap F) = P(F|E) \cdot P(E) $$

Where $P(F|E) = \frac{{}_6P_3 - {}_5P_3}{{}_6P_3} = \frac{60}{120}$ as per the previous analysis. This makes the probability :

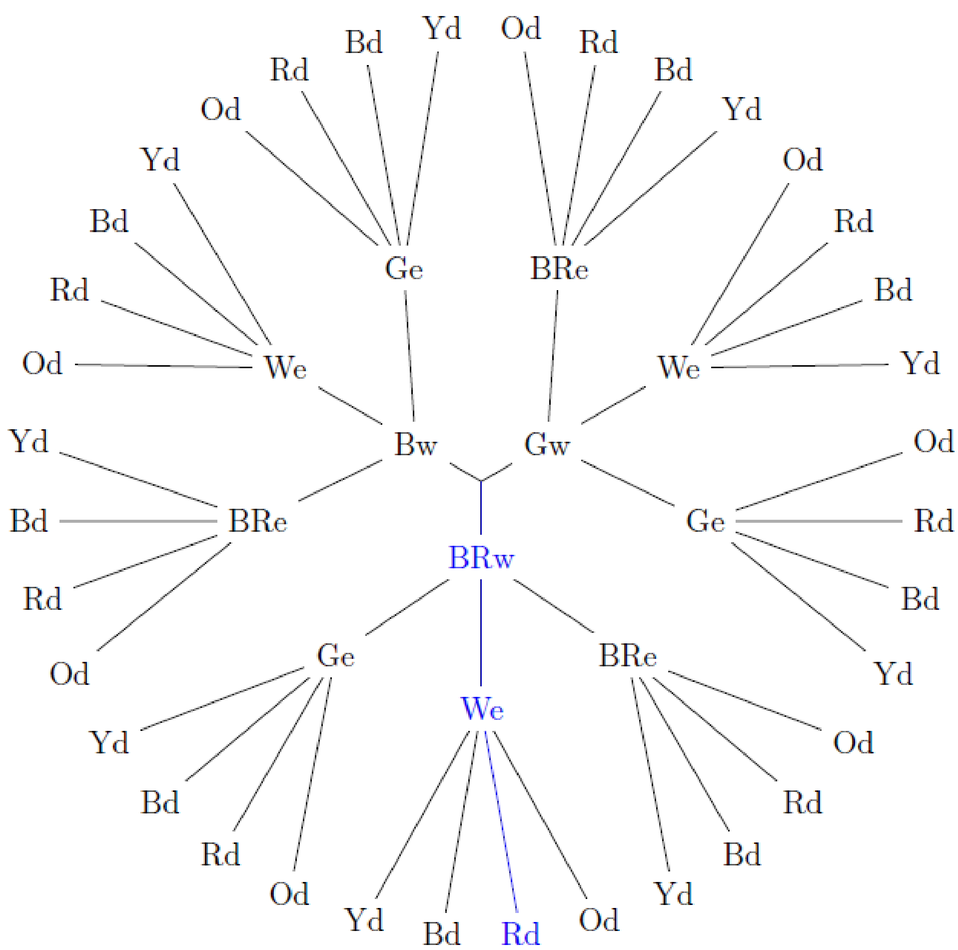
$$P(E \cap F) = \frac{60}{120}\frac{480}{840} = \frac{240}{840} = 28.57\%$$

Using counting methods: Randomness makes each of the events equally likely, which means that to find the probability, we can count the number of ways that the awards could not be given to Universities 5 and/or 6 and subtract from the total number of permutations to then divide by the total number of permutations. The number of ways that University 5 or 6 is not chosen is no different to the number of ways University 2 is not chosen as the previous part of the problem but twice = 720. But this overcounts the number of ways that 5 and 6 are not chosen, which is $_5P_4=120$. Therefore, the number of ways that 5 and 6 can be chosen is: $840 - 720 + 120 = 240$. This makes the probability :

$$\frac{240}{840} = 28.57\%$$

Answer 8

8.a) we can use some symbols for the different options: Gw = Green windows, Bw = Blue windows, BRw = Brown windows, We = White exterior, Ge = Gray exterior, BRe = Brown exterior, Yd = Yellow doors, Bd = Blue doors, Rd = Red doors, Od = Orange doors
Here's the diagram:



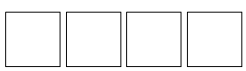
Each branch of this diagram symbolizes one set of choices, starting from the center and moving down to BRw symbolizes we chose brown windows. Moving down again symbolyzes we chose white exterior, moving down and a little to the right to Rd symbolizes we chose red doors. Counting the number of branches by simply counting the number of elements in the outermost circle can give us the number of choices which is 36.

8.b) The counting technique to use is simple multiplication, as opposed to the previous questions where permutations were used. This is because the choices for a category do not affect the choices for the next category, i.e. choosing green windows does not affect the color you will choose for the fa̧ade as opposed to the grant problem where University A getting the grant made it impossible for the next grant to go to A again. Therefore, the size of the sample space is: $3 \times 3 \times 4 = 36$.

Answer 9

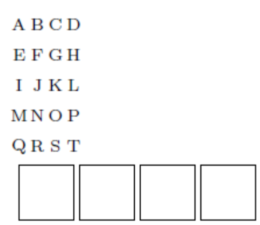
9.a) This is similar to question 7 of this assignments, with one important difference. The positions are <u>not</u> different from each other, this means that the number of configurations for the first team will be some fraction of ${}_{20}P_4$. This can be found through a number of steps:

Let's first label the students as A, B, C,...,T. And there are 4 slots available:



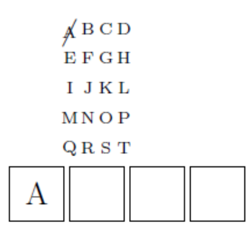
Then we can begin counting:

- The first spot can be given to any of the 20 students, that makes 20 ways:



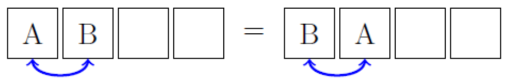
The different number of people that can be the first member of the first team is 20.

- The second member can be either of the 19 other students since one of them is already a member. For example, if A is a member, then there are 19 other students that can be the second member:



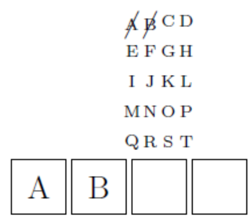
Likewise if any of the 20 students were chosen as the first member, only 19 could be the second member. This increases the number by $\times 19$ ways, the number at this point is: 20×19

- Now, the number we have at this point is over-counting the fact that there is no functional difference as to which student becomes part of the team first. For example, if A gets chosen as the first member and then B gets chosen as the second member is no different than if B was chosen as the first member and A was chosen as the second member.



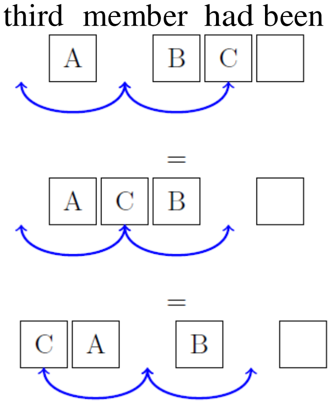
This is because the order of choice does not alter the way the team looks, i.e. a team composed of A and B is the same as a team composed of B and A. This decreases the number by 1/2 , the number at this point is: $(20 \times 19) / 2$

- The third member can be either of the 18 students who are not cur- rently members of the team. For example, if A and B are members, then there are 18 other students that can be the third member



Likewise if any 2 of the 20 students were already members, there would be 2 less possible students to be the third member. This increases the number by $\times 18$ ways, the number at this point is: $(20 \times 19 \times 18) / 2$

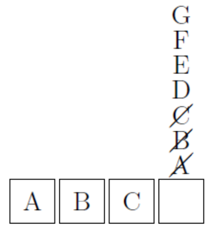
- The same as two steps ago, the number we have at this point is over- counting the fact that there is no difference if the third member had been chosen first or second instead:



This decreases the number by $1/3$, the number at this point is: $(20 \times 19 \times 18) / (2 \times 3)$

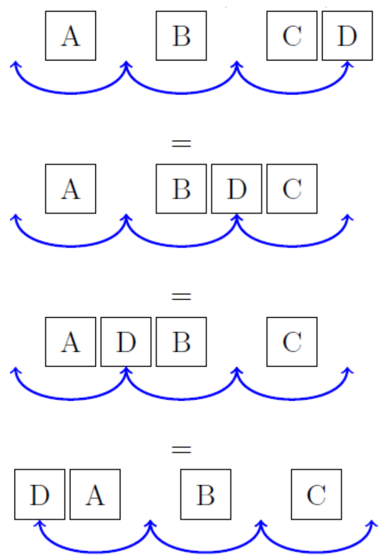
- The fourth member can be either of the 17 students that have not been chosen yet.

For example, if A, B, and C are already in the team, there are 17 students that can fill the fourth position:



Likewise if 3 of the 20 students were already members, 17 others could be the fourth member of the first team. This increases the number by ×17 ways, the number at this point is :
(20 × 19 × 18 ×17) / (2×3)

- The same as two steps ago, the number we have at this point is over- counting the fact that there is no difference if the fourth member had been chosen first, second, or third instead:



This increases the number by 1/4, the number at this point is :
(20 × 19 × 18 ×17) / (2×3×4)

The total number of ways the first team could be assembled is therefore:
(20 × 19 × 18 ×17) / (2×3×4) = 4845

Indeed, this is one way to visualize the expression "20 choose 4":

$$ {}_{20}C_4 = \binom{20}{4} = \frac{20!}{(4!)(16!)} $$

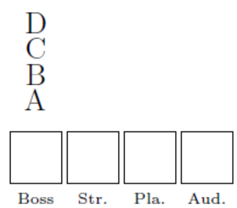
which identifies the number of ways 20 different items can be placed in 4 identical containers.

By the same logic previously applied, there are ${}_{16}C_4$ ways to assemble the second team because there are only 16 available students left. Likewise ${}_{12}C_4$ ways to assemble the third team, ${}_{8}C_4$ ways to assemble the fourth team, and ${}_{4}C_4$ ways to assemble the third team. This multiplies to:

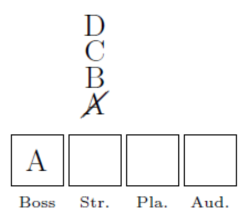
$$\frac{20!}{16!\,4!}\times\frac{16!}{12!\,4!}\times\frac{12!}{8!\,4!}\times\frac{8!}{4!\,4!}\times\frac{4!}{0!\,4!}=\frac{20!}{4!\,4!\,4!\,4!\,4!}=\binom{20}{4,4,4,4,4}=305{,}554{,}235{,}000$$

9.b) Given that each team has 4 students and there are 4 roles to go around, a number of steps can be taken to find the number of ways the team can be formed via a number of steps. I will label the students as A, B, C, and D; and the roles will be represented as positions from the left. The left most position is the boss, the second from the left is the strategist, the third from the left is the planner, and the last one is the auditor:

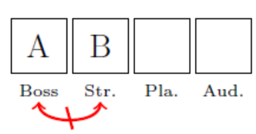
- Either of the 4 students already in the team can be the boss:



- Having chosen a student to be Boss, there are only 3 choices for strategist. For example, if A is chosen as the boss; only B, C, or D can be the strategist:



- A note of caution: While it was necessary to divide the number of configurations by 2 in problem 7, this is not possible here. The reason why is that a configuration in which A is the boss and B is the strategist is not equal to a configuration in which B is the boss and A is the strategist. In other words, the "containers" are non-identical:



- Since the boss and the strategist positions have been taken, two students can be the planner.

- Since there is only the position of auditor left and one student left, that position is determined at that point.

Put together, there are $4 \times 3 \times 2 \times 1 = 4! = 24$ ways each team can be assembled. In total, the five teams can be assembled $(4!)^5 = 7,962,624$ ways.

9.c) If we had known that the teams and roles would have been assembled, it would have become the case that each of the 20 positions are different from each other, i.e. being the boss of team 1 is distinct from being the boss of team 3 or any other position. Knowing this, we would know that given 20 different positions for 20 different students causes, that makes ${}_{20}P_{20} = 20! = 2,432,902,008,176,640,000$ ways to assign them. Note that this result is equal to:

$$\binom{20}{4,4,4,4,4} \times (4!)^5 = \frac{20!}{4!\,4!\,4!\,4!\,4!} \times (4!)^5 = 20!$$

Answer 10

10.a) There are a total of 5 steps to be made in making this choice:

- Choosing one of the 6 colors.
- Choosing to have the sunroof or not (2 choices.)
- Choosing to have the enhanced navigation system or not (2 choices.)
- Choosing to have the spoiler or not (2 choices.)
- Choosing to have the extended battery pack or not (2 choices.)

This makes $6 \times 2 \times 2 \times 2 \times 2 = 96$ ways to select options for the new model.

10.b.i) There are a total of 3 steps to make a choice in this package:

- Choosing to have the sunroof or not (2 choices.)
- Choosing to have the enhanced navigation system or not (2 choices.)
- Choosing one of the 6 colors.

This makes $2 \times 2 \times 6 = 24$ ways to select options for the “extended battery” package.

10.b.ii) There is only 1 step to make a choice in this package:

- Choosing one of the 4 colors.

This makes 4 ways to select options for the “basic edition” package.

10.b.iii) There are a total of 3 steps to make a choice in this package:

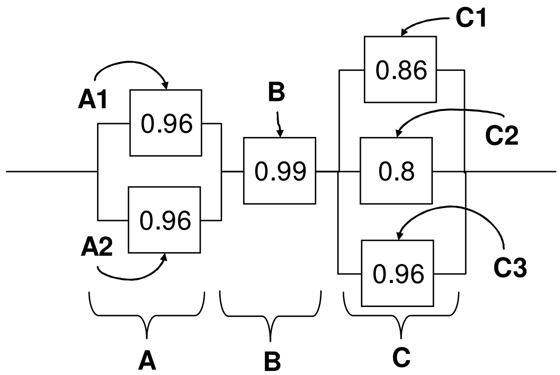
- Choosing one of the 6 colors.
- Choosing to have the sunroof or not (2 choices.)
- Choosing to have the enhanced navigation system or not (2 choices.)

This makes $6 \times 2 \times 2 = 24$ ways to select options for the “intermediate” package.

In total there are a $24 + 4 + 24 = 52$ ways that vehicles can be ordered now.

Answer 11

First let's break the circuit into three "columns" of devices that are connected "in series", and the devices are numbered starting with the value 1 at the top of each column as shown in the figure below.



Reliability of A: A is composed of 2 devices connected in parallel so if either A1 **OR** A2 function, the circuit functions: P(A) = P(A1 ∪ A2), but to solve this we are going to use the multiplication rule which applies to the intersection of independent events $\{P(E_1 \cap E_2 \cap \ldots E_N) = P(E_1)\, P(E_2) \ldots P(E_N)\}$ so we have to change P(A) = P(A1 ∪ A2) from a union of two events into an intersection of two events.

$$P(A) = P(A1 \cup A2) = 1 - P(A1' \cap A2') = 1 - (1\text{-}0.96)(1\text{-}0.96) = 1\text{-}(0.04)^2 = 0.9984$$

$$P(B) = 0.99$$

Reliability of C: C is composed of 3 devices connected in parallel, so we will apply the same logic we used to calculated the reliability of A.

$$\begin{aligned} P(C) = P(C1 \cup C2 \cup C3) = 1 - P(C1' \cap C2' \cap C3') &= 1\text{-}(1\text{-}0.86)(1\text{-}0.8)(1\text{-}0.96) \\ &= 1\text{-}(0.14)(0.2)(.04) \\ &= 0.99888 \end{aligned}$$

Circuit Reliability: Since A, B and C are connected in series we can use the multiplication rule for independent events (e.g. the reliability of each system)

$$P(\text{Circuit}) = P(A)P(B)P(C) = 0.9984 \times 0.99 \times 0.99888 = 0.9873$$

Answer 12

12.a) The criteria for a probability mass function is : $f(x_i) \geq 0$, $\sum_{i=1}^{n} f(x_i) = 1$
Considering values of the given table we have :

- All P(x) are positive.
- $\sum_{i=1}^{7} P(x_i) = 0.5 + 0.01 + 0.04 + 0.03 + 0.23 + 0.1 + 0.09 = 1$

12.b) Pr ($X = -2$ or $X = 2$) = $0.01 + 0.1 = 0.11$
12.c) Pr ($X \leq 4$) = $0.5 + 0.01 + 0.04 + 0.03 + 0.23 + 0.1 + 0.09 = 1$
12.d) Pr ($X \geq 0$ or $X = -3$) = $0.03 + 0.23 + 0.1 + 0.09 + 0.5 = 0.95$
12.e) Pr ($-3 < X < -1$) = Pr ($X = -2$) = 0.01