1 PDF & CDF

The thickness x of a protective coating applied to a conductor designed to work in corrosive conditions follows a uniform distribution over the interval $[20, 40]$ microns. Find the mean, standard deviation, and cumulative distribution function of the thickness of the protective coating. Find also the probability that the coating is less than 35 microns thick.

Solution

Over the interval $[20, 40]$, the probability density function $f(x)$ is given by the formula

$$f(x) = \begin{cases} 0.05, & 20 \le x \le 40 \\ 0, & \text{otherwise} \end{cases}$$

Using the formulae developed for the mean and variance gives

$$E(X) = \frac{a+b}{2} = \frac{20+40}{2} = 30\,\mu\text{m}$$

and

$$\sigma = \sqrt{V(X)} = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(40-20)^2}{12}} = \sqrt{\frac{400}{12}} \approx 5.77\,\mu\text{m}$$

The cumulative distribution function is given by

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

Hence, choosing appropriate ranges for x, the cumulative distribution function is obtained as:

$$F(x) = \begin{cases} 0, & x < 20 \\ \frac{x-20}{20}, & 20 \le x \le 40 \\ 1, & x > 40 \end{cases}$$

Hence the probability that the coating is less than 35 microns thick is

$$F(x < 35) = \frac{35-20}{20} = 0.75$$

2 Poisson Process

In a hospital ER, patients arrive at an average rate of 6 patients per hour.

Problem 1: Probability of Exactly 8 Patients Arriving in the Next Hour

Given:

- $\lambda = 6$ patients per hour (average rate)
- $k = 8$ patients (number of occurrences we're calculating the probability for)

Solution Using Poisson Distribution
1. Poisson Probability Formula:

$$P(Y = k; \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

2. Substitute the values into the formula:

$$P(Y = 8; \lambda = 6) = \frac{6^8 \cdot e^{-6}}{8!}$$

3. Calculate the individual components:

$$6^8 = 1679616$$
$$e^{-6} \approx 0.002478752$$
$$8! = 40320$$

4. Combine these into the formula:

$$P(Y = 8; \lambda = 6) = \frac{1679616 \cdot 0.002478752}{40320}$$

5. Simplify the fraction:

$$P(Y = 8; \lambda = 6) = \frac{4168.41660832}{40320} \approx 0.1033$$

So, the probability that exactly 8 patients will arrive at the ER in the next hour is approximately 0.1033 or 10.33%.

Problem 2: Probability That the Next Patient Will Arrive Within the Next 5 Minutes

Given:

- $\lambda = 6$ patients per hour
- Time interval = 5 minutes = $\frac{5}{60}$ hours = $\frac{1}{12}$ hours

Solution Using Poisson Distribution

1. Convert the rate to the given time interval:

$$\lambda' = 6 \times \frac{1}{12} = 0.5 \text{ patients per } \frac{1}{12} \text{ hour}$$

2. Poisson Probability Formula:

$$P(Y = k; \lambda') = \frac{\lambda'^k e^{-\lambda'}}{k!}$$

3. Find the probability of at least one patient arriving in 5 minutes:

$$\text{Probability of no patients arriving } (k = 0): \quad P(Y = 0; \lambda' = 0.5) = \frac{0.5^0 e^{-0.5}}{0!} = e^{-0.5} \approx 0.6065$$

4. Find the complement:

$$P(Y \geq 1; \lambda' = 0.5) = 1 - P(Y = 0; \lambda' = 0.5)$$

$$P(Y \geq 1; \lambda' = 0.5) = 1 - 0.6065 = 0.3935$$

So, the probability that the next patient will arrive within the next 5 minutes is approximately 0.3935 or 39.35%.

Solution Using Exponential Distribution

1. Exponential Distribution PDF:

$$f(z; \lambda) = \lambda e^{-\lambda z}$$

where:

- $\lambda = 6$ (rate of patients per hour)
- z is the time interval in hours

2. Convert the time interval:

$$z = 5 \text{ minutes} = \frac{1}{12} \text{ hours}$$

3. Exponential Distribution CDF:

$$P(Z \leq z; \lambda) = 1 - e^{-\lambda z}$$

4. Substitute the values into the CDF formula:

$$P(Z \leq \frac{1}{12}; \lambda = 6) = 1 - e^{-6 \times \frac{1}{12}}$$

$$P(Z \leq \frac{1}{12}; \lambda = 6) = 1 - e^{-0.5}$$

5. Calculate the exponential term:

$$e^{-0.5} \approx 0.6065$$

6. Find the probability:

$$P(Z \leq \frac{1}{12}; \lambda = 6) = 1 - 0.6065 = 0.3935$$

So, the probability that the next patient will arrive within the next 5 minutes using the exponential distribution is also approximately 0.3935 or 39.35%.

3 Standardizing the Normal Distribution

If we want to find the probability that a randomly selected adult man is between 165 cm and 185 cm, we would integrate the PDF over this interval. However, in practice, we often use the standard normal distribution (Z-distribution) to simplify this calculation.

To use standard normal tables, we standardize the normal distribution using the Z-score formula:

$$Z = \frac{X - \mu}{\sigma}$$

For $X = 165$:

$$Z = \frac{165 - 175}{10} = -1$$

For $X = 185$:

$$Z = \frac{185 - 175}{10} = 1$$

We then look up these Z-scores in the standard normal distribution table:

$$P(-1 \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq -1) = \Phi(Z = 1) - \Phi(Z = -1)$$

Using Z-tables or a calculator, we find:

$$P(Z \leq 1) = \Phi(Z = 1) = \approx 0.8413$$
$$P(Z \leq -1) = \Phi(Z = -1) = \approx 0.1587$$

Therefore:

$$P(165 \leq X \leq 185) = 0.8413 - 0.1587 = 0.6826$$

So, the probability that a randomly selected adult man is between 165 cm and 185 cm is approximately 0.6826 (or 68.26%).