What Is The Fastest Possible Slide?

Path of Fastest Descent With and Without Friction

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Section I: Aim, Introduction & Outline:

Aim: What are the fastest possible slides?

The aim of this exploration is to find the fastest possible slide. Who will not recall on the child's experience and the sensation of moving down very fast along a slide or a roller coaster? Now, towards the end of my HL IB math, I attempt to answer mathematically how to design the fast slide? What are the considerations going into this question? And how can I treat this mathematically?

Introduction:

Exploring this question, I came across this video on social media, there is a screenshot of the clip below. I feel very well prepared to analyze this problem.

Figure 1: Balls rolling down different curves ("The Brachistochrone" 00:22:36)

When coming across this I was fairly curious about what's happening in the image, and after further research I discovered that they are conducting an investigation about how different length and shape of curves can affect the travel time of a ball. I was eager to learn more about this, since I had just finished an experiment in physics about the dependence of the velocity on the frictional force of a free-falling ball.

I was intrigued by how varying the curves will give unexpected results in the time taken for the ball travel from one point to another. I wanted to discover that is the best optimum cover which will get me the shortest time. After doing some further exploration I came across the Brachistochrone (which in Greek means the shortest time) problem which is similar to what I wanted to pursue as my investigation (Spiegel 375) ("The Brachistochrone").

The problem can be treated by the so-called Euler LaGrange method. When researching the topic further I noted that the Euler LaGrange method has many applications in math and physics and, in particular in mechanics, quantum mechanics and electrodynamic. As I intend to study physics at university, I got even more interested in this topic.

Outline:

In Section II I define the problem of the fastest descent mathematically. I need to introduce here the integral for the length of a curve (not part of the HL curriculum). Next, in order to obtain a good appreciation of the problem, I solve the key equation for time of descent (II.3) numerically and calculate the time for straight line slide, a parabolic slide and a circular slide by integration.

In Section III I will apply Euler LaGrange method in order to minimize the descent time equation. The result will be a differential equation that I will solve by integration. This solution is the cycloid curve. Next, I will fix the integration constants such as to create a cycloid curve between points (0,1) and (1,0) as in Section II. I can now calculate the descent time for the cycloid slides numerically and make comparisons with the other slides, finalizing our treatment of the fastest descent problem without friction.

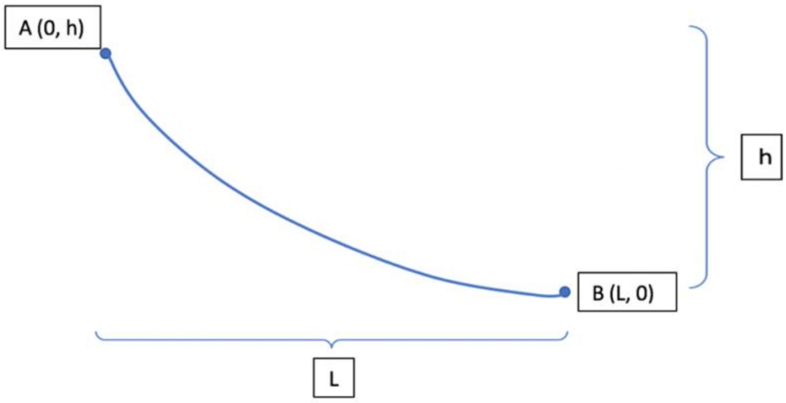
In Section IV I make the time of descent problem more realistically by introducing friction which alters the treatment of the problem dramatically. I first attempt to apply Euler LaGrange methodology analytically to this more realistic problem. I do however not succeed as the resulting differential equation becomes fiercely complex. Then, I take a numerical approach akin to that in prior sections, and I compare the results with the no friction case previously calculated.

Section II: Precise Formulation of the Problem of Fastest Descent without Friction

Section II.1 - Defining the Problem

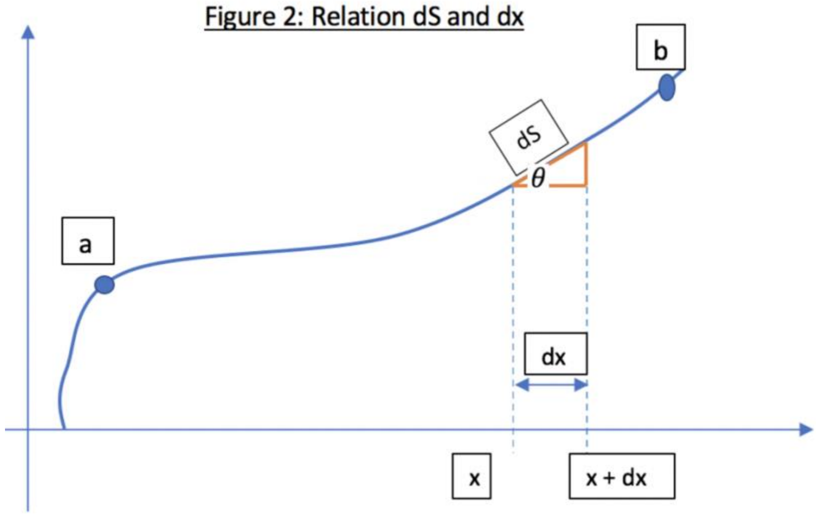
A particle slides frictionless from point A to point B under the influence of gravity. The problem is to choose the path such as to minimize the time taken from A to B. There are two factors to take into consideration. Firstly, the length of the path and secondly the velocity of the particle along the path.

Figure 1: The Formulation of the Curve of Descent



First, I discuss the length along a curve, that is how an infinitesimal length element dS depends on the function defining the curve. Where the variable h is the difference in height from A to B, and L is the horizontal distance between the two points.

Figure 2: Relation dS and dx



The tangent line in x is given by:

$$\tan\theta = y'(x)$$

From the figure: $dS = dx/cos\theta$

Using Pythagorean identity *(IBO 7)*: $1 + \tan^2\theta = 1/\cos^2\theta$,

hence $1 + \left(y'(x)\right)^2 = 1/\cos^2\theta$

therefore, $dS = dx/cos\theta = dx\sqrt{1 + \left(y'(x)\right)^2}$ (II.1)

Next I discuss the velocity component. I will use the conservation of energy to find the velocity along the curve assuming that there is no heat loss.

Initially the particle is at rest (v = 0) and has potential energy of *mgh*, where *g* is the gravitational constant $9.81ms^{-2}$ *(IBO 2)*, and *m* is the particle's mass.

Assume that after time t fell to a new height of y and will have aqcuired a velocity v.

Then by conservation of energy:

$$mgh = mgy + (1/2)mv^2$$

hence, $v = \sqrt{2g\,(h - y)}$ (II.2)

This is an interesting result, because velocity on the curve only depends on its height.

The standard formula for time is distance / velocity. So, $dt = dS/v$, for the infinitesimal distance dS.

Hence, using equations (II.1) and (II.2), the total time is

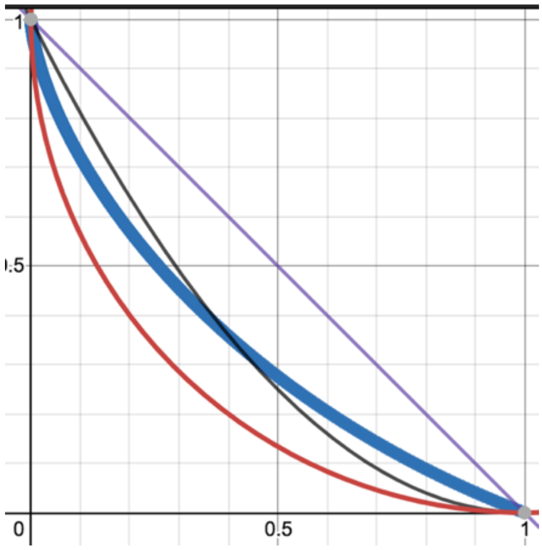
$$T = \int dt = \int dS/v = \int \frac{\sqrt{1+\left(y'(x)\right)^2}}{\sqrt{2g\,(h-y(x))}}dx \qquad (II.3)$$

Now I can define more precisely the problem of fastest descent. I need to minimize the time equation (III.3) by varying the function y(x). Note that this problem is much more complicated than finding optima for a function y(x) which only requires differentiation of y(x). In this case the whole function has to be varied over an interval.

Section II.2: Numerical Approaches for straight line, parabolic and circular curves:

Let's illustrate the use of equation (II.3) using paths as shown in figure 3. I choose a straight-line descent, a circular descent and a parabolic descent (freely falling objects under the influence of gravity follow a parabolic curve). I chose the circular descent because Galileo believed this to be the fastest curve; I chose the parabolic descent is because free moving objects in a gravitational field have parabolic curves. I choose a height and distance of 1 for convenience.

Figure 3: Paths for height = 1 and distance = 1 according to straight line (purple), parabola (black), circular (red), and cycloid (blue) *(Desmos)*. Cycloid curve is discussed in Section III.2



I calculate the integral (II.3) explicitly for the straight line (purple), for the parabola and circle segment are calculated numerically using the *(Integral Calculator)*.

For the straight line I have y(x) = x – 1 and y'(x) = 1. Using equation (II.3) I find:

$$T = \frac{1}{\sqrt{2g}}\int_0^1 \frac{\sqrt{1+(-1)^2}}{\sqrt{1-1+x}}dx = \frac{1}{\sqrt{2g}}\int_0^1 \frac{\sqrt{2}}{\sqrt{x}}dx = 2/\sqrt{g} \approx 0.6386 \quad (II.4)$$

(using g = 9.81ms^{-2}).

Using conventional equations of motions from classical physics *(IBO 5)*, for an object sliding from a 45° slope. The length of the slope $S = \sqrt{1^2+1^2} = \sqrt{2}$.

Using Newton's Law *(IBO 5)*, the acceleration $a = g\sin(45°) = \frac{1}{2}\sqrt{2}g$,
and thus, using *v = at (IBO 5)*, the velocity v(t) = $\frac{1}{2}\sqrt{2}gt$

The path travelled by the object is $S(t) = (\frac{1}{2})\frac{1}{2}\sqrt{2}gt^2 = \frac{\sqrt{2}}{4}gt^2$ *(IBO 5)*.

Upon arrival of the particle, $\sqrt{2} = \frac{\sqrt{2}}{4}gt^2$, and solving for T = $2/\sqrt{g} \approx$ 0.6386 sec (using g = 9.81ms^{-2}). This agrees with the integral calculation as expected.

Now I am going to numerically calculate using the online integral calculator (*Integral Calculator*) the time integrals (II.3) for the parabola and circular curve as shown in figure 3. For the parabola path; y(x) = (x-1)^2 and y'(x) = 2(x-1). So, integral (II.3) for the parabola curve becomes (*Integral Calculator*):

$$T = \frac{1}{\sqrt{2g}}\int_0^1 \frac{\sqrt{1+4(x-1)^2}}{\sqrt{1-(x-1)^2}}dx \approx 2.63518358/\sqrt{2g} \approx 0.5949$$

For the circular path, $(x-1)^2 + (y-1)^2 = 1$; So, y(x) = $1 - \sqrt{1-(x-1)^2}$,
hence y'(x) = $\frac{(x-1)}{\sqrt{1-(x-1)^2}}$.
And so, integral (II.3) becomes (*Integral Calculator*):

$$T = \frac{1}{\sqrt{2g}}\int_0^1 (1-(x-1)^2)^{-3/4}\,dx \approx \frac{2.6221}{\sqrt{2g}} \approx 0.5920$$

Table 1: Time of Descent for curve shown in figure 3, with the exception of the cycloid curve (blue) which will be discussed in Section III because it involves a lot more analysis.

Descent	Time (seconds)
Straight line	0.6386
Parabolic Curve	0.5949
Circular Segment	0.5920

The numerical results are summarized in table 1. The parabolic and circular path has a larger path length than the straight line, however they have shorter times. This is due to their initial fall to be more vertical, meaning that the gravitational acceleration along the path will be stronger. So, they will pick up speed faster, and this appears to be more than enough to compensate the longer path length. The circular path is not necessarily the solution to our problem, the calculations above are more of a trial and error procedure to try find solution. The numerical calculations above are obviously no substitute to proper optimization of equation (II.3). I still have to show what the real optimum curve is. This is what I will do in the next section.

Section III: Solving The Fastest Descent Problem Analytically and Numerically (Without Friction)

Section III.1: Analytical Solution of the Fastest Descent (Without Friction)

For this section III.1 I could have cited solutions that can be find in literature e.g. *(Spiegel 375) ("The Brachistochrone")* but I have to go through this exercise otherwise I cannot treat our problem including friction later. I follow here loosely the approach of Khan academy *("The Brachistochrone").* Recall that the problem was to find the optimal curve for equation (II.3), that I repeat here:

$$T = \frac{1}{\sqrt{2g}} \int \frac{\sqrt{1+(y')^2}}{\sqrt{(h-y)}} dx \qquad (III.1)$$

Minimizing such an equation with respect to y(x) is a problem that than can be handled by the so-called Euler-LaGrange formulation *[See reference ("The Brachistochrone") (Spiegel 375)]*. It states that for a function F(y, y', x) the integral $I = \int_A^B F(y, y', x)dx$ is extreme when

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad \text{This is the Euler LaGrange equation.}$$

where $\frac{\partial F}{\partial y'}$ and $\frac{\partial F}{\partial y}$ are the derivatives of F with respect to y' and y. Moreover, for the special case of $\frac{\partial F}{\partial x} = 0$ the so-called Beltrami identity applies:

$$F - y'\frac{\partial F}{\partial y'} = C \qquad (\text{III. 2a})$$

where C is a constant. I apply this theory to the problem, that is I take F(y, y') to be

$$F(y, y') = \frac{\sqrt{1+(y')^2}}{\sqrt{(h-y)}} \qquad (III.2)$$

Then the optimum can be found by solving equation (III.2a)
More information and the precise derivation of the Euler LaGrange formulation can be found in references *("The Brachistochrone") (Spiegel 375).*
Using this Euler-LaGrange formulation:

$$F - y'\frac{\partial F}{\partial y'} = \frac{\sqrt{1+(y')^2}}{\sqrt{(h-y)}} - \frac{(y')^2}{\sqrt{(1+y'^2)}\sqrt{(h-y)}} = C$$

After cross multiplication and basic algebra

$$C = \frac{1}{\sqrt{(h-y)(1+y'^2)}} \qquad (III.3)$$

Re-arranging the equation:

$$1 + y'^2 = \frac{1}{C^2(h-y)} = \frac{C_1}{h-y}$$

I redefined the constant as $C_1 = 1/C^2$. Solving for y'

$$y' = \frac{dy}{dx} = \sqrt{\frac{C_1 - (h-y)}{h-y}}$$

This differential equation can be written as

$$dx = \sqrt{\frac{(h-y)}{C_1 - (h-y)}}dy \quad \rightarrow \quad x = \int \sqrt{\frac{(h-y)}{C_1 - (h-y)}}dy \qquad (III.4)$$

I make the substitution $y = h - C_1 \sin^2\theta$, so $dy = -C_1 2\, sin\,\theta\, cos\,\theta\; d\theta$ (III.5)
and,

$$x = \int \sqrt{\frac{C_1 \sin^2\theta}{C_1 - C_1\sin^2\theta}}(-C_1 2\, sin\,\theta\, cos\,\theta\,)d\theta = -2C_1 \int \sin^2\theta\; d\theta$$

Using double angle identities (IBO 7): $2\, sin^2\,\theta = 1 - cos2\theta$, I obtain

$$x = -C_1 \int (1 - cos2\theta)d\theta = \frac{-C_1}{2}\int (1 - cos\emptyset)d\emptyset$$

where $\emptyset = 2\theta$ and $d\emptyset = 2d\theta$. The integration is straight forward:

$$x = -\tfrac{C_1}{2}\,(\emptyset - sin\emptyset) + K \qquad (III.6),$$

where K is an integration constant. Using equation (III.5) and the double angle identities (IBO 7) I obtain

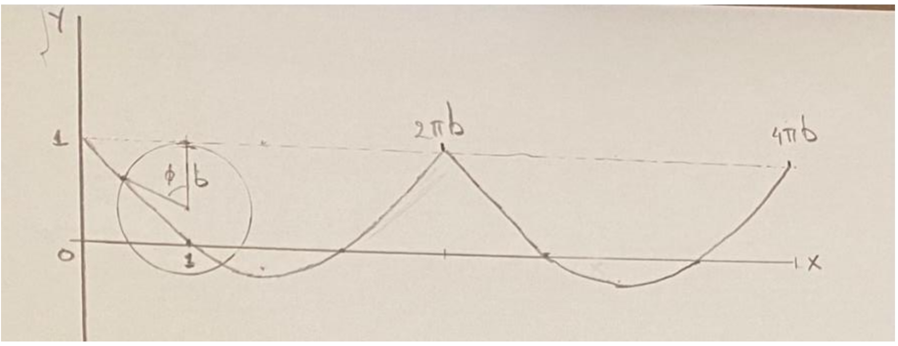
$$y = h - \frac{C_1}{2}(1 - cos\emptyset) \qquad (III.7)$$

The integration constant is determined by the boundary conditions. I choose K = 0 because I want x = 0 when $\emptyset = 0$. I define the new constant b = $\frac{-C_1}{2}$, then rewriting equations (III.6 and III.7).

$$x = b\,(\emptyset - sin\emptyset)$$
$$y = h + b(1 - cos\emptyset) \qquad (III.8)$$

The curve in the x-y plane represented by equations (III.6 and III.7) is cycloid curve (Spiegel 382). Therefore, I proved that the curve of fastest descent (minimizing T) is the cycloid curve. This is the case because the curve (III.8) is obtained by using the Euler LaGrange formulism. The cycloid curve is the solution of equation (III.2a) for the time integral T (III.1).

Figure 5: Cycloid curve can be constructed by tracking a fixed point on rolling a circle. (similar to using the kid's toy called Spirograph)



Now, I know that this curve is a stationary path of the equation (IV.1), it is either a minimum or a maximum.

Section III.2: Numerical Solution for Cycloid Slide (Without Friction)

Having the formulas for the cycloid curve (III.8), I can numerically calculate the descent time. I use h = 1 and L = 1 (see figure 1), as I have done for some other paths in Section II.

From equation (III.1)

$$T = \frac{1}{\sqrt{2g}} \int \frac{\sqrt{1 + (y')^2}}{\sqrt{(1 - y)}} dx \qquad (III.9)$$

y' is given by equation (III.4)

$$y' = \frac{dy}{dx} = \sqrt{\frac{C_1 - (1 - y)}{1 - y}}$$

Where C_1 = -2b as defined in (III.8). I can solve for the descent time T, by substituting the dx of (III.4) into equation (III.9) and changing the integration variable from x to y (a trick to calculate the integral).

$$T = \frac{1}{\sqrt{2g}} \int \frac{\sqrt{1 + (y')^2}}{\sqrt{(1 - y)}} dx = \frac{1}{\sqrt{2g}} \int \frac{\sqrt{1 + \left(\frac{-2b - (1 - y)}{1 - y}\right)^2}}{\sqrt{(1 - y)}} \frac{1}{\sqrt{\frac{-2b - 1 - (1 - y)}{1 - y}}} dy$$

$$= \sqrt{\frac{-2b}{2g}} \int_0^1 \frac{1}{\sqrt{1 - y}\sqrt{-2b - (1 - y)}} dy \qquad (III.10)$$

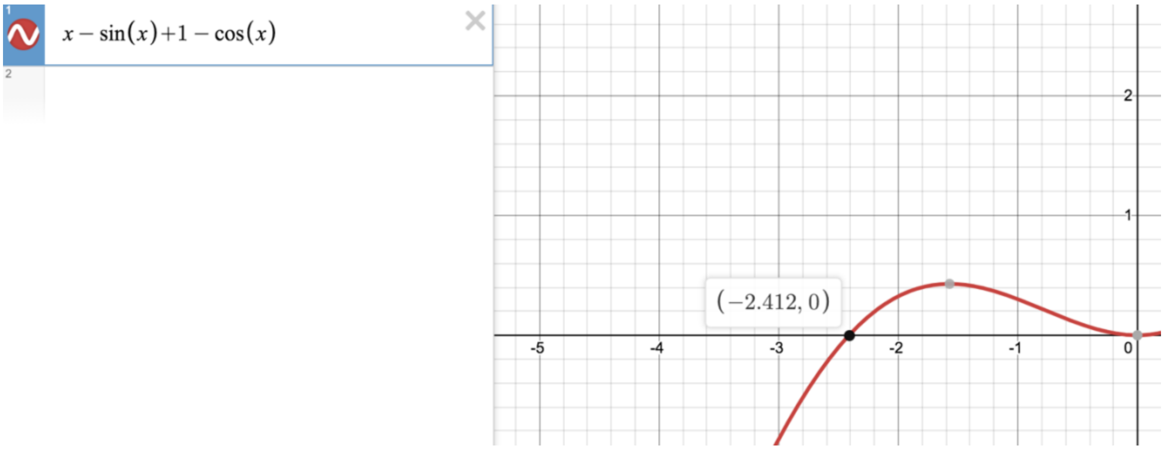
Now I need to calculate of the radius, b, of the circle defining the cycloid.
Let $\emptyset_1$ be the angle at (x,y) = (1,0). Using equation (III.8):

$$\begin{cases} 1 = b(\emptyset_1 - sin\emptyset_1) \\ 0 = 1 + b(1 - cos\emptyset_1) \end{cases} \qquad (III.11)$$

Eliminating b results in:

$$\emptyset_1 - sin\emptyset_1 + 1 - cos\emptyset_1 = 0$$

Plotting this (equation) on the online calculator (Desmos) gives $\emptyset_1$:



And also solving equation (III.11) for b gives

$$\begin{cases} \emptyset_1 = -2.412 \\ b = -0.5729 \end{cases} \qquad (III.12)$$

Using this value for b in equation (III.10) gives *(Integral Calculator):*

$$T = \frac{1.07042}{\sqrt{2g}} \int_0^1 \frac{1}{\sqrt{1-y}\sqrt{0.1458+y}} dy \approx \frac{(1.07042)(2.412)}{\sqrt{2g}} \approx 0.5829\, sec \quad (III.13)$$

In the last step I use g = 9.81ms^{-2}.

Now I consolidate this result with the numerical results obtained in chapter IV.

Table 2: Descent Times for Different Curves as Shown in Figure 2

Type of Curve:	**Descent Time (seconds)**
Straight Line	0.6386
Parabolic	0.5949
Circular	0.5920
Cycloid	0.5829

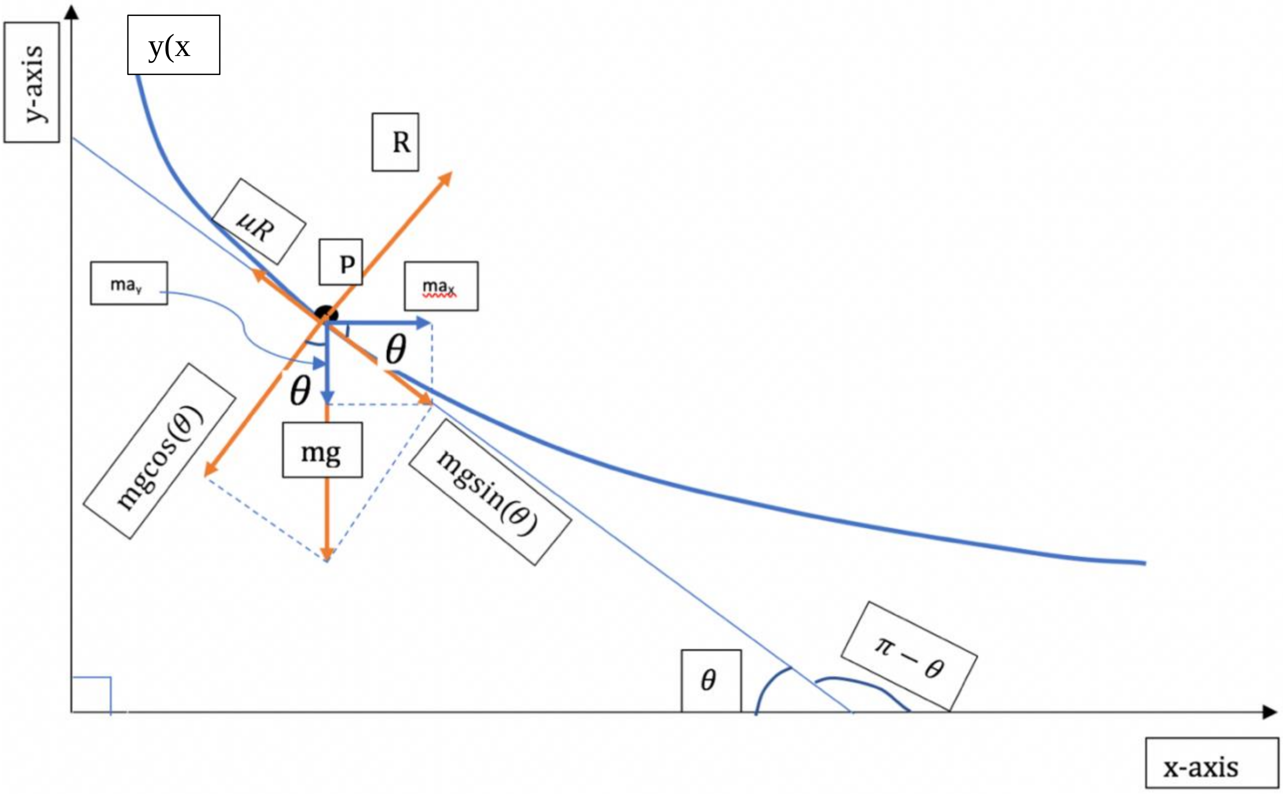
It is no surprise that the time for the cycloid descent is shortest, since the Euler-LaGrange optimization method ensures that the time of the fastest descent is for the cycloid curve.

Section IV: Problem of Fastest Descent (including Friction)

Section IV.1: Formulation of the problem using Newtonian Mechanics

This section is basically an application of HL physics IB curriculum, nothing else beyond this being used.
I want to derive a formula for the time of descent including friction. So, basically, a formula similar to (II.3), that is $T = \int dS/v$, but with v affected by friction. Equations (II.2) for v was easy to derive because I could use energy conversation. In the case of friction, however, energy will be dissipated in the form of heat and so energy conservation cannot be used. So, I have to go back to the basics of Newton's second law *(F = ma) (IBO 5)*. The situation is shown below in figure 7. I have to find an expression for v including friction.

Figure 7: Forces acting on point P sliding down a curve



The gravitational force *mg (IBO 5)* at point *P* can be decomposed as a vector in components tangent to the curve at point *P* and a component perpendicular to this. With reference to figure 7, *R* is the normal force $R = mgcos\theta$ *(IBO 5)*, perpendicular to the curve at point *P*, g is the gravitational constant $g = 9.81ms^{-2}$ *(IBO 2)*. The frictional force along the curve is given by μR *(IBO 5)* where μ is the coefficient of friction. "*a*" is the acceleration of the particle in point *P*, along the curve that is tangent to the curve at point P. The acceleration a can be decomposed into a horizontal (a_x) and vertical component (a_y) *(IBO 5)*.

With reference to figure 7 I have for the forces along the curve:

$ma = mgsin\theta - \mu mgcos\theta$ and so, for the acceleration along the curve I have:

$$a = gsin\theta - \mu gcos\theta \qquad (IV.1)$$

Acceleration can be written as $a = dv/dt$, Hence by integration $v = a\ t$ $(v(0) = 0)$

To find an expression for t I consider the movement on the vertical axis, $v_y = - a_y t$
Integrating this gives $y = -\frac{1}{2}a_y t^2 + h$ (IBO 5) where y (0) = h, the initial height (these are standard equations of motion in physics). With respect to figure 7, $a_y = a\ sin\theta$ (IBO 5). I can now solve for the time t.

$$t = \sqrt{\frac{2(h-y)}{a\ sin\theta}} \qquad IV.2)$$

The square root can only be taken if acceleration a > 0.
I can now calculate the expression for velocity v along the curve (IBO 5).

$$v = at = \sqrt{2(h-y)}\sqrt{\frac{a}{sin\theta}} = \sqrt{2(h-y)}\sqrt{\frac{gsin\theta - \mu gcos\theta}{sin\theta}}$$

For the last step I use equation (IV.1). This expression for v can be simplified to

$$v = \sqrt{2g(h-y)}\sqrt{1 - \mu/tan\theta}$$

With reference to figure 7, I can say $tan(\pi - \theta) = y'(x)$, hence $tan\theta = -\ y'(x)$.
Thus,

$$v = \sqrt{2g(h-y)}\sqrt{1 + \frac{\mu}{y'(x)}}$$

And so, the time of descent T is,

$$T = \int dt = \int dS/v = \frac{1}{\sqrt{2g}}\int \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{(h - y(x))}\sqrt{1 + \frac{\mu}{y'(x)}}}dx \qquad (IV.3)$$

<u>Does this formula make sense?</u>

Firstly, note that when $\mu = 0$ I recover the original formula for time T equation (II.6).
Secondly, note that when the slope is very steep e.g. y'(x) = -10, then whatever μ is, its effect is divided by 10. This is what I expect because when the slope is steep the normal force (R) is small, and the frictional forces becomes less significant. The contrary applies when y'(x) is small and the curve is flatter.

To further corroborate the reasonableness of equation (IV.3), consider the example of the straight line for h =1, and y(x) = 1-x, where y'(x) = -1.
Using equation (IV.3) the time for this straight-line scenario gives:

$$T = \frac{1}{\sqrt{2g}}\int_0^1 \frac{\sqrt{2}}{\sqrt{1-(1-x)}\sqrt{1-\mu}}dx = \frac{1}{\sqrt{2g(1-\mu)}}\int_0^1 \frac{\sqrt{2}}{\sqrt{x}}dx \qquad (IV.4)$$

Note that this equal to equation (II.4), except for the factor of $(1-\mu)$. This is what I expect, which can be seen as follows. For $\theta = 45°, \sin\theta = cos\,\theta = \frac{\sqrt{2}}{2}$, and so the acceleration along the curves becomes using equation (IV.1) a = $\frac{\sqrt{2}}{2} g(1-\mu)$. So, the effect of the frictional force is to reduce the gravitational force by a factor of $(1-\mu)$, as expected.

Section IV.2: Applying Euler LaGrange To Find the Curve of Fastest Descent Including Friction

This section is analogues to section III.I, except that this time friction is included. To find the curve of fastest descent I consider a function F(y,y') analogous as in equation (III.2), including friction this F(y,y') becomes:

$$F(y,y') = \frac{\sqrt{1+(y')^2}}{\sqrt{(h-y)}\sqrt{1+\frac{\mu}{y'}}} \qquad (VI.5)$$

Applying Euler-LaGrange means solving equation (III.2a) for y(x). I repeat here (II.2a): (Spiegel 375)

$$F - y'\frac{\partial F}{\partial y'} = constant$$

First, I calculate the partial derivative of F with respect to y':

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{h-y}\sqrt{1+(y')^2}(y')}\left[\frac{\mu(1+(y')^2)}{2(y')}\left(1+\frac{\mu}{(y')}\right)^{-3/2} + (y')^2\left(1+\frac{\mu}{(y')}\right)^{-1/2}\right]$$

So,

$$F - y'\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{h-y}\sqrt{1+(y')^2}}\left[\frac{1+(y')^2}{\sqrt{1+\frac{\mu}{y'}}} - \frac{\mu(1+(y')^2)}{2(y')}\left(1+\frac{\mu}{(y')}\right)^{-\frac{3}{2}} - (y')^2\left(1+\frac{\mu}{(y')}\right)^{-\frac{1}{2}}\right]$$
$$= constant. \qquad (IV.6)$$

Comparing this formula to its non-friction equivalent equation (III.1) I see that only the factors in front of the large brackets are the same. Note that If μ = 0 the term in the large brackets simplifies to 1+(y')^2 – (y')^2 = 1, and so equation (IV.6) reduces to our original non friction (μ=0) equation (III.1). This shows that I did not make an obvious calculation error.

This differential equation is very complicated to solve, and I have absolutely no idea on how to get about this. It is highly likely that the solution is not a cycloid function as found in section III.1. So, I am not able to minimize equation (IV.3) for the time of descent T. Therefore, I leave things as it is here, and I will continue with numerical studies of the problem.

Section IV.3: Numerical Approaches (Including Friction)

Analogous to section III.2 I will now do some numerical calculations of a number of descents (linear, parabolic, circular, cycloid), including friction this time.

For illustrational purposes I use the frictional coefficient 0.257 respectively corresponding to magnesium on cast iron (Fuller 44-45). At the end of this section it will become clear why I have chosen this particular value of μ.

The straight-line descent was done algebraically at the end of section IV.1, and I repeat here formula (IV.4)

$$T = \frac{1}{\sqrt{2g(1-\mu)}}\int_0^1 \frac{\sqrt{2}}{\sqrt{x}}dx = \frac{2\sqrt{2}}{\sqrt{2g(1-\mu)}}$$

Time T for $\mu = 0.257$ gives *3.28134/*$\sqrt{2g} = 0.7408$ (*Integral Calculator*).

I apply formula (IV.3) for a parabolic descent. In case h=1, y(x) = (x-1)^2, and y'(x) =2(x-1) hence,

$$T = \frac{1}{\sqrt{2g}}\int_0^1 \frac{\sqrt{1+4(x-1)^2}}{\sqrt{1-(x-1)^2}\sqrt{1+\frac{\mu}{2(x-1)}}}dx$$

Time T for $\mu = 0.257$ gives 3.15083/$\sqrt{2g} = \ 0.7113$ (*Integral Calculator*).

Note in the above formula that $1+\frac{\mu}{2(x-1)}$ will become negative for some value of x. This is the point where the acceleration a (equation IV.1) flips sign, which is at the point when the frictional force overcomes the gravitational force. Therefore, I need to replace $1+\frac{\mu}{2(x-1)}$ by its absolute value in the integral. At x close to 1 the numerical calculation crashes so I change the integration interval from $\int_0^1 \quad = \int_0^{0.99} \quad$.

For the circular descent I have h=1, $y(x) = \sqrt{1-(x-1)^2}$, and $y'(x) = \frac{x-1}{\sqrt{1-(x-1)^2}}$.
So, with IV.3:

$$T = \frac{1}{\sqrt{2g}}\int_0^1 \frac{\sqrt{1+\frac{(x-1)^2}{1-(x-1)^2}}}{\sqrt{1-(1-\sqrt{1-(x-1)^2}}}\frac{1}{\sqrt{1+\frac{\mu(x-1)}{\sqrt{1-(x-1)^2}}}}dx$$

$$= \frac{1}{\sqrt{2g}}\int_0^1 \frac{(1-(x-1)^2)^{-3/4}}{\sqrt{1+\frac{\mu(x-1)}{\sqrt{1-(x-1)^2}}}}dx$$

For $\mu = 0.257$, I find T = 2.98456/$\sqrt{2g} = 0.6738$ (*Integral Calculator*).I had to change the integration intervals from $\int_0^1$ to $\int_{0.01}^1$, because of the same numerical issue as above.

<u>Next, I discuss the time of descent along a cycloid curve.</u> I repeat here the equations of the cycloid (III.8) (for h = 1):

$$x = b\ (\emptyset - sin\emptyset) \quad and \quad y = 1 + b(1 - cos\emptyset)$$

Where b = -0.5729 (equation III.12). For the integral (IV.3), I need to transform the integration variable x into $\emptyset$.

$$dx = b(1 - cos\emptyset)d\emptyset \text{ and } dy = bsin\emptyset d\emptyset \tag{IV.7}$$

Next, I need an expression in terms of $\emptyset$ for y'(x) = dy/dx:

$$\frac{dy}{d\emptyset} = \frac{dy}{dx}\frac{dx}{d\emptyset}$$

Using equation (IV.7) this gives:

$$bsin\emptyset = \frac{dy}{dx}b(1 - cos\emptyset)$$

Hence,

$$\frac{dy}{dx} = \frac{sin\emptyset}{(1 - cos\emptyset)}$$

The integration boundaries go from $\int_0^1$ to $\int_0^{\emptyset_1}$, where $\emptyset_1 = -2.412$ Recall that b = -0.5729 (see equation III.12).

So, the time of descent T (equation IV.3) for the cycloid curve is

$$T = \frac{1}{\sqrt{2g}}\int_0^{-2.412} \frac{\sqrt{1 + \left(\frac{sin\emptyset}{1 - cos\emptyset}\right)^2}}{\sqrt{1 - (1 + b(1 - cos\emptyset)}} \frac{b(1 - cos\emptyset)}{\sqrt{1 + \mu\frac{(1 - cos\emptyset)}{sin\emptyset}}}\ d\emptyset$$

For $\mu = 0.257$, I find $T = 2.99088/\sqrt{2g} = 0.6752$ (*Integral Calculator*).

<u>Table 2: The time of descent with and without friction</u>

<u>Descent</u>	<u>Time (seconds) without friction</u>	<u>Time (seconds) with friction ($\mu = 0.257$)</u>
Straight line	0.6386	0.7408
Parabolic Curve	0.5949	0.7113
Circular Segment	0.5920	0.6738
Cycloid Curve	0.5829	0.6752

With respect to table 2. For the case without friction it has been proven (Section III.1) that the cycloid slide is the curve of fastest descent, so it is no surprise that the numerical results in the table reflect this.
In the more realistic case of a slide including friction I had already noticed (section IV.2) that it was unlikely that the cycloid would always be the fastest curve. I see from table 2 that this is indeed not the case! For the given frictional coefficient, the circular slide is faster. For different friction coefficients this result might be different.

Conclusions & Summary:

What is the fastest slide? If friction is not existent or very small the fastest possible descent is along a cycloid curve. This curve does not have the shortest length, but its strong initial pick up in speed (due to steepness) compensates for this making it the fastest descent.

However, when friction along the slide is large, the cycloid curve is not necessarily the curve of fastest descent anymore, as shown above.

As an aside I believe that the circular slide is the most interesting. It is fast, if not the fastest (dependant on friction). Moreover, the initial pickup in velocity is high, and towards the end of the slide the velocity will strongly reduce, ejecting the sliding person horizontally from the slide. The latter is obviously safer as a cycloid slide will eject the person towards the ground risking injury.

Summary:

I studied the problem of fastest descent along a slide/curve. After some numerical studies I introduced the Euler-Lagrange equation which can be used to solve this type of optimization problems.

The resulting curve turned out to be a cycloid. Next, I solved the time integral numerically using the cycloid function in order to verify whether this descent is indeed the fastest of the straight line, parabolic and circular descents. This indeed happens to be the case.

Next, I moved to the more complicated problem of finding the path of fastest descent including friction between the sliding object and the slide. I derive from Newton's equations a formula for the time of descent. I do not succeed, however, minimizing it using Euler-LaGrange because of the complexity of the resulting differential equation. I finalise presenting numerical studies for the descent time including friction. I find that for certain values of the friction the cycloid is not necessarily the curve of fastest descent anymore.

Reflection & Evaluation:

During this exploration, I have learned various new concepts that are beyond the HL curriculum, such as the line element (dS), integration along a curve, concept of partial derivatives and functions with multiple variables and a new class of optimization problems using the Euler-LaGrange equation. As I intend to study physics at university, this exploration permitted me to study the Euler-Lagrange methodology which has many applications in physics that I am yet to discover.

Further investigation:

Imagine I want to design a fast roller coaster. In this case air resistance will be an additional complication. I studied air resistance of a free-falling object in my HL physics IA, which permits me to sketch how to treat this problem. In figure 7 I need to add an additional force $-Kv^2$ (with K as a constant) along the curve. This introduces another parameter K into the problem, moreover the mass m does not fall out of equation (IV.1), leaving us with three parameters μ, m and K. Now acceleration a = dv/dt is a differential equation dependant of v^2 which is not easily solved as in section (IV.1). The problem can be solved by programming a numerical simulation. In my physics IA I wrote an algorithm to deal with a free fall with air resistance. Here, the problem is more complex. But it would roughly work as follows. Divide the x-axis into very small intervals Δx. Over a small interval of Δx the slide is a can be considered as a straight line. I start at x = 0, v_1 can be calculated using the equation for dv/dt, and $\Delta S_1 = \sqrt{1+(y'(x))^2}$ Δx, hence $\Delta T_1 = \Delta S_1 / v_1$ with this information I can calculate v_2, and $\Delta T_2 = \Delta S_2 / v_2$, etc, etc... The time of descent is then simply the sum of the ΔT_i's. This method is similar to solving ordinary differential equations numerically as described in the HL math textbook (Rondie 546).

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