Basics in fluid plasma equations

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`Courses/Cores/C5_Numerical_methods_and_simulation_codes/Smets/`

General form of a PDE

$$a\partial^2_{x^2}u + b\partial_x\partial_y u + c\partial^2_{y^2}u + d\partial_x u + e\partial_y u + fu = 0 \qquad (1)$$

Eq. (1) is

- hyperbolic if and only if $b^2 - 4ac > 0$.
- parabolic if and only if $b^2 - 4ac = 0$.
- elliptic if and only if $b^2 - 4ac < 0$.

Remark : If any of the coefficients $a, \dots f$ depend on x and/or y, the PDE is then said to be **local**.

Remark : The kind of a PDE does not depend on the base.
$\rightarrow$ Not that easy to identify as is !

Classification of PDE

- Elliptic equations

$\rightarrow$ Usually governing stationnary and bounded problems, defined on a domain Ω for which the boundary conditions (defined on $\partial\Omega$) are usually of Dirichlet- or Neumann-type.

The very classical elliptic equation is the Poisson equation :

$$\begin{cases} \Delta u = -f & \text{on } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

Remark : Solving such an equation generally means invert a very large sparse matrix.

Remark : There is many libraries that solve these kinds of problems.

Classification of PDE

- Parabolic equations

Parabolic equations are governing the time evolution of systems where diffusion (or any kind of dissipation) processes are at play.

These problems are usually defined in a bounded domain Ω where the conditions on $\partial\Omega$ are Dirichlet- or Neumann-type (and eventually non-stationnary), but also involve some **initial conditions**.

The heat equation with Dirichlet conditions is generally the most classical case of parabolic equation :

$$\left\{ \begin{array}{lcll} \partial_t T & = & D\partial^2_{x^2} T & \text{on } \Omega \text{ with } D > 0 \\ T & = & T_0 & \text{on } \partial\Omega \text{ and } T(x,0) = f(x) \text{ on } \Omega \end{array} \right. \tag{3}$$

$\rightarrow$ it has an analytic solution (using Green's functions).

Classification of PDE

- Hyperbolic equations

These equations are associated to dissipation-less wave propagation.

A first form of hyperbolic equation is

$$\partial^2_{t^2} u - A^2 \partial^2_{x^2} u = 0 \tag{4}$$

$\rightarrow$ fluctuation $u(x,t)$ moving at speed $+A$ or $-A$.

Note : The equation is linear if A is a constant or non-linear for $A = A(x,t)$

Note : We use both "fluctuations" and "waves", as the former ones can be Fourier decomposed as a linear combination of the latter ones.

Classification of PDE

The other form of hyperbolic equation is

$$\partial_t u + A\partial_x u = 0 \tag{5}$$

$\rightarrow$ fluctuations only moving at speed $+A$.

Property : In the multidimensional case, that is involving $n > 1$ spatial coordinates, $\mathbf{A}$ turns to be a $n \times n$ matrix. In such a case, the equation is hyperbolic if the matrix $\mathbf{A}$ is diagonalizable.

$\rightarrow$ we will only use this form as its derivative straightforwardly gives the former one.

General form of PDE for fluid plasma

The fluid treatment of a plasma generally involves both **advection** and **diffusion** processes

$$\partial_t u + A\partial_x u = D\partial^2_{x^2} u \tag{6}$$

$\rightarrow$ the numerical treatment of the time integration of such an equation oftenly involves a separations of these processes.
$\rightarrow$ then need to identify a strategy for scheduling these processes.
$\rightarrow$ numerical integration can be more an **art** than a **science**...

Remark : Advection generally steepens the structures while diffusion smooth them.

Caracteristics

Let's introduce the total derivative of u along a path $X(t)$:

$$d_t u|_X = d_t u(X(t), t) = \partial_t u + \partial_x u \times d_t x|_X \tag{7}$$

$\rightarrow$ along $X(t)$, u is a fonction of t only.
$\rightarrow$ defining $X(t)$ as the solution of $d_t x = A$, Eq. (5) writes

$$\partial_t u + A\partial_x u = 0 = d_t u|_X \tag{8}$$

The solution of Eq. (5) reduces to the solution of the Ordinary Differential Equation (ODE) $d_t u|_X = 0$.

The curves $X(t)$ solution of this ODE (each of them depending on the initial condition $x(t = 0) = x_0$ for the solutions of the ODE) are called the **caracteristics** of the hyperbolic equation.

Caracteristics

The promising method of caracteristics is then essentially a reformulation of the equation using a change of variable.

$\rightarrow$ this approach leads to the set of coupled differential equations.

$$\begin{cases} d_t x|_X & = & A \\ d_t u|_X & = & 0 \end{cases} \tag{9}$$

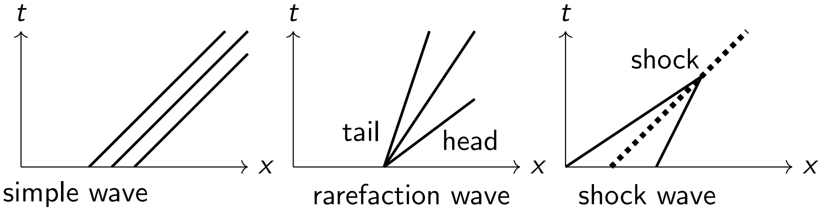
hmmm... are we in better shape with this formulation ?

- this set of equation is coupled and can generally not be decoupled in a straightforward way.
- singularities will arise where caracteristics will cross because of the steppening of $u(x, t)$.

For a constant speed A, the solution of $d_t x|_X = A$ is $X(t) = X_0 + At$ for $X(t = 0) = X_0$.

Caracteristics

With the initial condition $u(x, t = 0) = u_0(x)$, the general solution is then $u(x, t) = u_0(x - At)$ and the caracteristics are straight lines of slope A^{-1} in the (x, t) plan.



- propagative wave advects the structure at speed A.
- Expansion wave or rarefaction wave : distance between head and tail increases with time.
- Shock waves and contact discontinuities : steppening of a structure until infinite derivative.

Shocks and discontinuities

Property : For a shock wave travelling from left to right, pressure, density and wave speed are higher on the left than on the right.

Note : For the Euler equation (and fluid in general), the only discontinuity that can exist is a contact discontinuity. For a plasma, there is several types : rotational, contact and tangential discontinuities.

Property : For a contact discontinuity, only density and pressure can be discontinuous, but the pressure is continuous.

Remark : In a contact discontinuity, the discontinuity is co-moving with the fluid, hence preserving the pressure balance condition.

Remark : In a plasma, the magnetic field and associated magnetic pressure has to be considered in the pressure balance condition.

Sonic point

For both rarefaction and compression waves, the slopes of the caracteristics can be of any sign.

$\rightarrow$ a singular situation appears in regions where both $A > 0$ and $A < 0$ are observed :

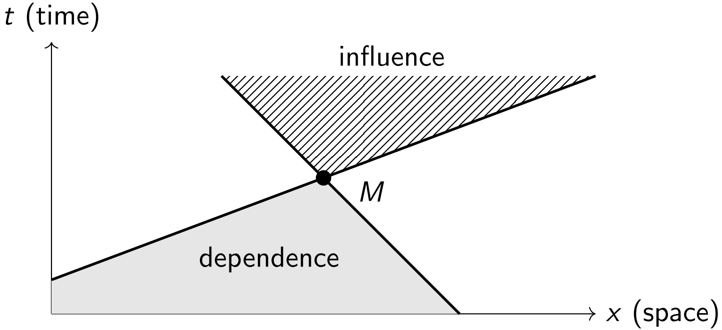
- for a rarefaction wave, there is a point where u is getting null and diverges out from this point.
- for a compression wave, there is a point where u is getting very large because of a compression from both left and right neighbours.

Definition : The point where A is changing sign is called a **sonic point** or a **critical point**. We prefer sonic point as "critical" can be used in other contexts.

Dependence and influence domains

Dependence and influence domains are bounded by the caracteristic curves.

$\rightarrow$ the number of caracteristics depends on the number of equations $\equiv$ number of unknowns for a well-posed problem.



Dependence and influence domains

Definition : The dependence domain associated to (x, t) is the region of phase space $\chi(\tau)$ so that $u(x, t)$ depends on all the $u(\chi, \tau)$.

The choice of the scheme defines a numerical domain of dependence.

Definition : The influence domain associated to (x, t) is the region of phase space $\chi(\tau)$ so that $u(x, t)$ influences all the points $u(\chi, \tau)$.

Numerical dependence domain and **physical dependence domain** are related.

Conservative form

The conservative form of Eq. (5) is

$$\partial_t u + \partial_x f(u) = 0 \tag{10}$$

- This form is called **consevative form**.
- The non-linear fonction $f(u)$ is called the **flux function**.

Property : At a sonic point, the flux function $f(u)$ has a null derivative (with respect to x).

Assuming that the solution $u(x,t) \in C^1$ the x-derivative of the flux function can be calculated from the Jacobian matrix of f with respect to u. The resulting **non-conservative** form is

$$\partial_t u + A(u)\partial_x u = 0 \tag{11}$$

with $A = \partial_u f$ and still $u(x, t = 0) = u_0(x)$.

Conservative form

For the multi-dimensional case,

$$\partial_t \mathbf{u} + \mathbf{A}(\mathbf{u}) \,.\, \partial_{\mathbf{r}} \mathbf{u} = 0 \tag{12}$$

The matrix A is the Jacobian of f, hence defined as

$$A_{ij} = \partial_{u_j} f_i(u) \tag{13}$$

It can be shown that for $u \in \mathbb{R}$, A is diagonalizable and all the eigen values are **real**. Hence, $\mathbf{A}$ can write

$$\mathbf{A} = \mathbf{Q} \,.\, \mathbf{\Lambda} \,.\, \mathbf{Q}^{-1} \tag{14}$$

where $\mathbf{Q}$ is the transition matrix and $\mathbf{\Lambda}$ is a diagonal matrix with λ_i being the eigen values of $\mathbf{A}$.
with $\mathbf{v} = \mathbf{Q}^{-1} \,.\, \mathbf{u}$,

$$\partial_t \mathbf{v} + \mathbf{\Lambda} \,.\, \partial_{\mathbf{r}} \mathbf{v} = 0 \tag{15}$$

One-dimensional Euler equation

Notation : For a given fluid, ρ is the mass density, $\rho\mathbf{v}$ is the momentum density and ρe (e being the internal energy per mass unit) is the internal energy density

Definition : The Euler equations are the equations describing an inviscid flow described by the three moments, ρ, $\rho\mathbf{v}$ and ρe.

$\rightarrow$ the term "moment" is justified as these 3 quantities are the moments of order 0, 1 and 2 of the distribution functions.

Remark : The flow is inviscid as any viscous term would turn this system to be parabolic and its solution to be regularized by viscosity.

One-dimensional Euler equation

The hierarchy of fluid equations is a set of coupled equations where the n^{th} of this equation involves the $n+1$ order moment.

$\rightarrow$ to be well-posed, this system needs a "closure" equation : in the Euler equation, this is the equation of state for the pressure $p = p(\rho, e)$.

Definition : This equation is a closure equation as the moment of order 2 is given by the lower order moments of order 0 and 1.

$\rightarrow$ the most classical closure equations are the isothermal closure ($\mathrm{d}p/p = \mathrm{d}\rho/\rho$) and adiabatic closure ($\mathrm{d}p/p = \gamma \mathrm{d}\rho/\rho$)

Note : Aside from the closure equation, one needs the relation between e and p. It depends on the number of internal degrees of liberty, that is to the adiabatic index γ. In the case of the Euler equation, $\rho e = p/(\gamma - 1)$

Conservative variables vs. primitive variables

The Euler equations can be written in different forms, depending on the choice of the variables to describe the fluid (or plasma) : the **conservative variables** or the **primitive variables**.

Definition : The conservative variables are the ones used in the conservative form of an hyperbolic system. For the Euler equations, they are ρ, $\rho\mathbf{v}$ and ρe.

Definition : The primitive variables are the ones usually describing a physical system, that is ρ , $\mathbf{v}$ and p for a fluid.

Notation : We note e_T the total energy per mass unit, that is $e_T = e + \frac{1}{2}\mathbf{v}.\mathbf{v} = e + \frac{1}{2}v^2$ hence including the energy flow.

Fluxes

Notation : We note V the control volume, $\mathrm{d}S$ the surface element of this control volume and **n** the associated unit vector normal to this surface and directed **outward** from this control volume.

The change of mass density ρ, momentum density $\rho\mathbf{v}$ and internal energy density ρe is feeded by the flux of these quantities across $\mathrm{d}S$:

- the flux of mass density (for the mass transport) is $\rho\mathbf{v}.\mathbf{n}\mathrm{d}S$
- the flux of momentum density (for the momentum transport) is $(\rho\mathbf{v}\mathbf{v} + p\mathbf{1}).\mathbf{n}\mathrm{d}S$
- the flux of energy density (for the internal energy transport) is $(\rho e_T\mathbf{v} + p\mathbf{v}).\mathbf{n}\mathrm{d}S$

Integral form of the Euler equations

The temporal change between arbitrary times t_1 and t_2 of a conservative variable inside the control volume V equals the associated flux of this quantity through its surface $\mathbf{n}\,\mathrm{d}S$.
The mass conservation equation is then

$$\int_V [\rho(\mathbf{r}, t_2) - \rho(\mathbf{r}, t_1)]\mathrm{d}V + \int_{t_1}^{t_2} \oint_{\partial V} \rho\mathbf{v}.\mathbf{n}\mathrm{d}S\mathrm{d}t = 0 \tag{16}$$

the momentum equation is

$$\int_V [\rho\mathbf{v}(\mathbf{r}, t_2) - \rho\mathbf{v}(\mathbf{r}, t_1)]\mathrm{d}V + \int_{t_1}^{t_2} \oint_{\partial V} [\rho\mathbf{v}\mathbf{v} + p\mathbf{1}).\mathbf{n}\mathrm{d}S\mathrm{d}t = 0 \tag{17}$$

and the energy equation is

$$\int_V [\rho e_T(\mathbf{r}, t_2) - \rho e_T(\mathbf{r}, t_1)]\mathrm{d}V + \int_{t_1}^{t_2} \oint_{\partial V} [\rho e_T\mathbf{v}\mathbf{v} + p\mathbf{v}).\mathbf{n}\mathrm{d}S\mathrm{d}t = 0 \tag{18}$$

Differential form of the Euler equations

Eq. (16) can be divided by $t_2 - t_1$

$$\int_V \frac{\rho(\mathbf{r}, t_2) - \rho(\mathbf{r}, t_1)}{t_2 - t_1} \mathrm{d}V + \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \oint_{\partial V} \rho \mathbf{v}.\mathbf{n} \mathrm{d}S \mathrm{d}t = 0 \quad (19)$$

In the limit $t_1 \to t_2$ and making use of the Gauss theorem,

$$\int_V \partial_t \rho(\mathbf{r}, t) \mathrm{d}V + \int_V \boldsymbol{\nabla}.\rho \mathbf{v}(\mathbf{r}, t) \mathrm{d}V = 0 \quad (20)$$

As Eq. (27) has to verified whatever the control volume, the integrand has to be zero, so the mass conservation writes

$$\partial_t \rho + \boldsymbol{\nabla}.\rho \mathbf{v} = 0 \quad (21)$$

Differential form of the Euler equations

The momentum conservation is then

$$\partial_t(\rho \mathbf{v}) + \boldsymbol{\nabla}.(\rho \mathbf{v}\mathbf{v} + p\mathbf{1}) = 0 \tag{22}$$

and the energy conservation is

$$\partial_t(\rho e_T) + \boldsymbol{\nabla}.(\rho e_T \mathbf{v} + p\mathbf{v}) = 0 \tag{23}$$

Remark : One needs to keep in mind that the differential form of this equation needs the hypothesis $\rho(\mathbf{r}, t) \in C^1(x, t)$ and $\mathbf{v} \in (\mathbf{r}, t) C^1(x, t)$. This is of importance in the cases of shocks or discontinuities. It is hence clear that the differential form of the Euler equations make sense if and only if the spatial and temporal derivatives are defines...

Viscosity

In he second and third Euler equations, the term $-p\mathbf{1}$ is replaced by the stress tensor $\mathbf{T}$ defined as

$$\mathbf{T} = -p\mathbf{1} + \mu[\nabla \mathbf{v} + \mathbf{v}\nabla - \frac{2}{3}(\nabla . \mathbf{v})\mathbf{1}] \tag{24}$$

In this equation, we hence need to introduce a dynamical viscosity μ.

As a consequence, the divergence of this flux contains second order derivatives of the velocity, so that the 2^{nd} Euler equation turns to be parabolic.

Gravity

Gravity is generally introduced through the gravitational potential Φ which general definition is

$$\Phi(\mathbf{r}) = -G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}\mathbf{r}' \tag{25}$$

In the case of the Earth (which mass is $M_\oplus$)

$$\Phi(\mathbf{r}) = -G \frac{M_\oplus}{r} \tag{26}$$

The second Euler equation then writes

$$\partial_t(\rho \mathbf{v}) + \boldsymbol{\nabla}.(\rho \mathbf{v}\mathbf{v} + p\mathbf{1}) = -\rho \boldsymbol{\nabla} \Phi \tag{27}$$

The third one is unchanged with the total energy

$$e_T = e + \frac{1}{2} v^2 + \Phi \tag{28}$$

Eq. (27) hence has a "true" source term (that is not a flux).

Magnetic field

The time evolution of the magnetic field is given by the Maxwell-Faraday Eq. in ideal MHD (the simlpler case)

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} \tag{29}$$

so that Maxwell-Faraday writes (in conservative form)

$$\partial_t \mathbf{B} + \boldsymbol{\nabla}.(\mathbf{vB} - \mathbf{Bv}) = 0 \tag{30}$$

with

$$\mathbf{T}_B = \frac{1}{\mu_0}[\mathbf{BB} - \frac{B^2}{2}\mathbf{1}] \quad \text{and} \quad e_T = e + \frac{1}{2}v^2 + \frac{1}{2}\frac{B^2}{\rho} \tag{31}$$

$$\partial_t(\rho\mathbf{v}) + \boldsymbol{\nabla}.(\rho\mathbf{vv} + p\mathbf{1} - \mathbf{T}_B) = 0 \tag{32}$$

$$\partial_t(\rho e_T) + \boldsymbol{\nabla}.(\rho e_T\mathbf{v} + p\mathbf{v} - \mathbf{T}_B.\mathbf{v}) = 0 \tag{33}$$

Radiation

The radiative energy flux (depending on the spectral intensity I_ν) is

$$\mathbf{F}_{\mathrm{rad}} = \int_{\mathbb{R}^+} \mathrm{d}\nu \oint_\Omega \mathrm{d}\omega \, \mathbf{n} I_\nu(\mathbf{r}, \mathbf{n}, \nu) \tag{34}$$

The radiative pressure is

$$\mathbf{P}_{\mathrm{rad}} = \frac{1}{c} \int_{\mathbb{R}^+} \mathrm{d}\nu \oint_\Omega \mathrm{d}\omega \, \mathbf{nn} I_\nu(\mathbf{r}, \mathbf{n}, \nu) \tag{35}$$

The second Euler equation is then

$$\partial_t(\rho \mathbf{v}) + \boldsymbol{\nabla}.(\rho \mathbf{vv} + p\mathbf{1} + \mathbf{P}_{\mathrm{rad}}) = 0 \tag{36}$$

and the third one is

$$\partial_t(\rho e_T) + \boldsymbol{\nabla}.[(\rho e_T \mathbf{v} + p)\mathbf{v} + \mathbf{F}_{\mathrm{rad}}] = 0 \tag{37}$$