CentraleSupelec **Numerical Methods**

Workshop #02: Pressure losses in a square cross-section channel

Key concepts

- Resolution of an elliptic PDE such as a steady diffusion problem
- Iterative methods for solving linear systems of equations

1 Steady flow in a square cross section channel

We look for the pressure losses of a square duct in the laminar steady and fully-developed regime. The height of the square duct is denoted L. The velocity field has only one component $u(x, y)$ in the streamwise direction, where x and y are the coordinates that are normal to the streamwise axis z. The pressure gradient $\frac{\mathrm{d}p}{\mathrm{d}z}$ is constant inside the duct, and is linked to the velocity field through the Navier-Stokes equation which, under our assumptions, is limited to:

$$0 = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\mathrm{d}p}{\mathrm{d}z} \tag{1}$$

where μ is the dynamic viscosity of the fluid. The two-dimensional velocity profile $u(x, y)$ in $\Omega = [0, L]^2$ is thus given by the following equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\beta \text{ for } x \in \mathrm{int}(\Omega), \tag{2}$$

$$u = 0 \text{ for } x \in \partial\Omega \tag{3}$$

where $\beta = \frac{\mathrm{d}p}{\mathrm{d}z}/\mu$. For a given bulk velocity, the velocity field imposes the pressure loss. Conversely, for a given pressure drop, the velocity field and thus the bulk velocity is given by Eq. 3. Our objective is to solve the Poisson equation Eq. 3.
In the following, we will consider a domain of size $L = 2$ mm, and the fluid will be water with dynamic viscosity $\mu = 10^{-3}$ kg/m/s and density $\rho = 10^3$ kg/m^3.

2 Approximation of the Laplacian operator

Before solving the Poisson equation, we first investigate how to compute numerically the Laplacian operator on the square domain. To this aim, we consider the following velocity in the domain $[0, L]^2$:

$$u(x, y) = \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right) \tag{4}$$

a) Implement a finite difference approximation of the Laplacian of u.

b) Measure the order of convergence of your approximation of the Laplacian operator. **This is the first error of your Elliptic solver.**

3 Solution of the Poisson equation

Now that you now how to discretize a Laplacian operator, we will aim at finding the solution of the Poisson equation.

3.1 Implementing the solver

We first consider a pressure loss $\frac{\mathrm{d}p}{\mathrm{d}z} = 1000$ Pa/m

c) Implement the Jacobi method to solve Eq. 3 for $N_x = 20$. The exit of the iterative procedure will be driven by an error control at each iteration with respect to Eq. 3, which is known as the residual norm. **This is the second error of your Elliptic solver.**

d) Implement the Gauss-Siedel and SOR methods and compare the three methods with respect to the number of iterations and computational time.

e) Perform a convergence study of your solver for the most efficient method: change the number of mesh points to see the impact on the bulk velocity $U = \frac{1}{L^2}\int_{[0,L]^2} u(x,y)\mathrm{d}x\mathrm{d}y$. Monitor the number of iterations required to get convergence and see how it is impacted by the mesh refinement.