

We now focus on the following time-dependent PDE:

$$\mathcal{L}u(x, t) := \partial_t u(x, t) - \nabla \cdot (a(x) \nabla u(x, t)) = f(x, t), \quad x \in \Omega, \quad t \in [0, T] \quad (1)$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t \in [0, T] \quad (2)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega \quad (3)$$

where the function $f \sim \mathcal{N}(\bar{f}, K)$. The solution u is thus also random.

We will set up a prior on the solution u to the above problem. To do so we first let $v_h \in S_h$ be some approximation of the initial condition $u_0(x)$ in the FEM space S_h . To be more specific we will assume that $v_h(x) = \Phi(x)^* \gamma := \sum_{i=1}^J \phi_i(x) \gamma_i$. Note that $\Phi(x) := (\phi_1(x), \dots, \phi_J(x))^T$. We take the prior on u to be:

$$u \sim \mathcal{N}(m_0, V_0) \quad (4)$$

where $m_0(x, t) := v_h(x) = \Phi(x)^* \gamma$ (m_0 is constant in time). The prior covariance operator V_0 is defined as:

$$(V_0 g)(x, t) = \int_{\Omega} \int_0^T k_{x,t,y,s}^{(0)} g(y, s) ds dy \quad (5)$$

where $k_{x,t,y,s}^{(0)}$ is defined as follows:

$$k_{x,t,y,s}^{(0)} := \sum_{i=1}^J \lambda_j \phi_j(x) \phi_j(y) k^{(0)}(t, s) \quad (6)$$

We now introduce a uniform time grid:

$$t_n = n\delta, \quad n = 0, 1, \dots, N$$

where δ is the spacing between consecutive times and $N = \frac{T}{\delta}$ (assume that N is an integer). The time kernel $k^{(0)}(t, s)$ will be taken to be:

$$k^{(0)}(t, s) := \sum_{i=0}^{N-1} l^{(i)}(t) l^{(i)}(s) \quad (7)$$

Where the functions $\{l^{(i)}\}_{i=0}^{N-1}$ are defined as follows:

$$l^{(i)}(t) = \begin{cases} (t - t_i) \mathbb{1}_{(t_i, t_{i+1}]}(t) + \delta \mathbb{1}_{(t_{i+1}, t_N]}(t), & i = 0, \dots, N-2 \\ (t - t_{N-1}) \mathbb{1}_{(t_{N-1}, t_N]}(t), & i = N-1 \end{cases} \quad (8)$$

i.e. for $i = 0, \dots, N-2$ we have:

$$l^{(i)}(t) = \begin{cases} 0, & t \leq t_i \\ t - t_i, & t_i < t \leq t_{i+1} \\ \delta, & t > t_{i+1} \end{cases} \quad (9)$$

while for $i = N-1$ we have:

$$l^{(i)}(t) = l^{(N-1)}(t) = \begin{cases} 0, & t \leq t_{N-1} \\ t - t_{N-1}, & t > t_{N-1} \end{cases} \quad (10)$$

note: we are working only with times in the interval $[0, T]$ here.

We now introduce the following information operators $\mathcal{I}_s := (I_1(s), \dots, I_J(s))^T$ where:

$$I_i(s)g := \int_{\Omega} \phi_i(x) g(x, s) dx \quad (11)$$

To update our belief in the distribution of u we will condition on the following events: $\mathcal{I}_{t_i} \mathcal{L}u = \mathcal{I}_{t_i} f =: F^{(i)}$ sequentially for $i = 1, \dots, N$. Letting $\tilde{A}_t := \mathcal{I}_t \mathcal{L}$ we seek, for a fixed realisation of f (and hence of the $\{F^{(i)}\}$), the following conditional distributions:

$$u | \{\tilde{A}_{t_1} u = F^{(1)}, \dots, \tilde{A}_{t_p} u = F^{(p)}, f\} \sim \mathcal{N}(m_p, V_p) \quad (12)$$

for $p \in \{1, \dots, N\}$. We make the following claim:

Proposition 0.1. With the prior specified as above we have that m_p and V_p are given as follows:

$$m_p(x, t) := \Phi(x)^* \gamma + \sum_{i=1}^p l^{(i-1)}(t) \Phi(x)^* \mathbf{c}^{(i)} \quad (13)$$

$$(V_p g)(x, t) := \int_{\Omega} \int_0^T k_{x,t,y,s}^{(p)} g(y, s) ds dy \quad (14)$$

$$\mathbf{c}^{(i)} := Q^{-1} \left[F^{(i)} - A \gamma_{i-1} \right] \text{ for } i = 1, \dots, p \quad (15)$$

$$k_{x,t,y,s}^{(p)} := \sum_{j=1}^J \lambda_j \phi_j(x) \phi_j(y) k^{(p)}(t, s) \quad (16)$$

$$k^{(p)}(t, s) := \sum_{i=p}^{N-1} l^{(i)}(t) l^{(i)}(s) \quad (17)$$

and where the $\{\gamma_i\}_{i=0}^p$ are defined recursively by:

$$\gamma_0 := \gamma, \quad (18)$$

$$\gamma_i := Q^{-1} \left[M \gamma_{i-1} + \delta F^{(i)} \right] \text{ for } i \geq 1 \quad (19)$$

The matrices M and A are the Galerkin Mass and Stiffness matrices respectively, i.e. $M_{ij} := \int_{\Omega} \phi_i(x) \phi_j(x) dx$ and $A_{ij} := \int_{\Omega} a(x) \nabla \phi_i(x) \cdot \nabla \phi_j(x) dx$. The matrix $Q := (M + \delta A)$. Further, we have that evaluating the conditional mean m_p at time t_p yields the following:

$$m_p(x, t_p) = \Phi(x)^* \gamma_p \quad (20)$$

Thus, we can see that this choice of prior yields (for a fixed realisation of f) what the classical Backward-Euler Galerkin method yields.

Proof: We proceed via proof by induction.

For $p = 1$ it follows that the distribution of $u | \{\tilde{A}_{\delta} u = F^{(1)}, f\}$ is Gaussian $\mathcal{N}(m_1, V_1)$ by considering the following joint distribution:

$$\begin{pmatrix} u \\ \tilde{A}_{\delta} u \end{pmatrix} = \begin{pmatrix} I \\ \tilde{A}_{\delta} \end{pmatrix} u \sim \mathcal{N} \left(\begin{pmatrix} m_0 \\ \tilde{A}_{\delta} m_0 \end{pmatrix}, \begin{pmatrix} V_0 & V_0 \tilde{A}_{\delta}^* \\ \tilde{A}_{\delta} V_0 & \tilde{A}_{\delta} V_0 \tilde{A}_{\delta}^* \end{pmatrix} \right)$$

It follows that the conditional distribution is Gaussian and the mean and covariance are given by:

$$m_1 = m_0 + V_0 \tilde{A}_{\delta}^* (\tilde{A}_{\delta} V_0 \tilde{A}_{\delta}^*)^{-1} (F^{(1)} - \tilde{A}_{\delta} m_0) \quad (21)$$

$$V_1 = V_0 - V_0 \tilde{A}_{\delta}^* (\tilde{A}_{\delta} V_0 \tilde{A}_{\delta}^*)^{-1} \tilde{A}_{\delta} V_0 \quad (22)$$

To make progress we must now start computing various terms needed for our mean and covariance update rules. We start with $V_0 \tilde{A}_{\delta}^*$. We have:

$$V_0 \tilde{A}_{\delta}^* = V_0 \mathcal{L}^* (I_1(\delta)^*, \dots, I_J(\delta)^*)$$

We can thus see that we need to be able to compute terms of the form $V_0 \mathcal{L}^* I_i(\delta)^* = V_0 (I_i(\delta) \mathcal{L})^*$. Now since the operator $I_i(\delta) \mathcal{L}$ takes in a function on $\Omega \times [0, T]$ and outputs a real number its adjoint should take in a real number and output a function on $\Omega \times [0, T]$. This adjoint should satisfy the following relation:

$$\alpha (I_i(\delta) \mathcal{L} g) = \int_{\Omega} \int_0^T ((I_i(\delta) \mathcal{L})^* \alpha)(x, t) g(x, t) dt dx \quad \forall g, \quad \forall \alpha \in \mathbb{R} \quad (23)$$

Using this we can now compute:

$$\begin{aligned} (V_0 (I_i(\delta) \mathcal{L})^* \alpha)(x, s) &= \int_{\Omega} \int_0^T k_{x,s,y,w}^{(0)} ((I_i(\delta) \mathcal{L})^* \alpha)(y, w) dw dy \\ &= \alpha (I_i(\delta) \mathcal{L} k_{x,s,\cdot,\cdot}^{(0)}) \\ &= \alpha \int_{\Omega} \phi_i(y) (\mathcal{L} k_{x,s,\cdot,\cdot}^{(0)})(y, \delta) dy \end{aligned}$$

We now work out $(\mathcal{L}k_{x,s,\cdot,\cdot}^{(0)})(y, \delta)$ taking care to remember that x, s are fixed and so \mathcal{L} acts on the variables y, δ :

$$\begin{aligned} (\mathcal{L}k_{x,s,\cdot,\cdot}^{(0)})(y, \delta) &= \partial_2 k^{(0)}(s, \delta) \sum_{j=1}^J \lambda_j \phi_j(x) \phi_j(y) - k^{(0)}(s, \delta) \nabla_y \cdot \left(a(y) \nabla_y \sum_{j=1}^J (\lambda_j \phi_j(x) \phi_j(y)) \right) \\ &= \partial_2 k^{(0)}(s, \delta) \sum_{j=1}^J \lambda_j \phi_j(x) \phi_j(y) - k^{(0)}(s, \delta) \sum_{j=1}^J \lambda_j \phi_j(x) \nabla_y \cdot (a(y) \nabla_y \phi_j(y)) \end{aligned}$$

So we can now compute:

$$\begin{aligned} (V_0(I_i(\delta)\mathcal{L})^*\alpha)(x, s) &= \alpha \int_{\Omega} \phi_i(y) \partial_2 k^{(0)}(s, \delta) \sum_{j=1}^J \lambda_j \phi_j(x) \phi_j(y) dy - \alpha \int_{\Omega} \phi_i(y) k^{(0)}(s, \delta) \sum_{j=1}^J \lambda_j \phi_j(x) \nabla_y \cdot (a(y) \nabla_y \phi_j(y)) dy \\ &= \alpha \partial_2 k^{(0)}(s, \delta) \sum_{j=1}^J \lambda_j \phi_j(x) M_{ji} + \alpha k^{(0)}(s, \delta) \sum_{j=1}^J \lambda_j \phi_j(x) A_{ji} \end{aligned}$$

Using this result we can deduce that:

$$(V_0 \tilde{A}_{\delta}^* \mathbf{v})(x, s) = \partial_2 k^{(0)}(s, \delta) \Phi(x)^* \Lambda M \mathbf{v} + k^{(0)}(s, \delta) \Phi(x)^* \Lambda A \mathbf{v} \quad (24)$$

for any $\mathbf{v} \in \mathbb{R}^J$, where $\Lambda = \text{diag}\{\lambda_i\}_{i=1}^J$.

For our time kernel we can deduce:

$$\partial_1 k^{(0)}(t, s) = \sum_{i=0}^{N-1} l^{(i)'}(t) l^{(i)}(s) \quad (25)$$

$$\partial_2 k^{(0)}(t, s) = \sum_{i=0}^{N-1} l^{(i)}(t) l^{(i)'}(s) \quad (26)$$

$$\partial_1 \partial_2 k^{(0)}(t, s) = \sum_{i=0}^{N-1} l^{(i)'}(t) l^{(i)'}(s) \quad (27)$$

$$(28)$$

We also have:

$$l^{(i)'}(t) = \mathbb{1}_{(t_i, t_{i+1}]}(t) \quad (29)$$

for all $i = 0, \dots, N-1$. (Note: this includes even the case of $i = N-1$.) Noting that the kernel $k^{(0)}$ is symmetric we have:

$$\begin{aligned} k^{(0)}(s, \delta) &= k^{(0)}(\delta, s) \\ &= \sum_{i=0}^{N-1} l^{(i)}(\delta) l^{(i)}(s) \\ &= \sum_{i=0}^{N-1} \delta \delta_{i,0} l^{(i)}(s) \\ &= \delta l^{(0)}(s) \end{aligned}$$

and

$$\begin{aligned} \partial_2 k^{(0)}(s, \delta) &= \partial_1 k^{(0)}(\delta, s) \\ &= \sum_{i=0}^{N-1} l^{(i)'}(\delta) l^{(i)}(s) \\ &= \sum_{i=0}^{N-1} \delta_{i,0} l^{(i)}(s) \\ &= l^{(0)}(s) \end{aligned}$$

where we have used the following properties of the functions $\{l^{(i)}\}$ which can easily be shown:

$$l^{(i)}(t_j) = \delta \cdot \delta_{i,j-1} \quad (30)$$

$$l^{(i)'}(t_j) = \delta_{i,j-1} \quad (31)$$

for $i = 0, \dots, N-1$ and $j = 1, \dots, N$.

We can now simplify (24) to:

$$\begin{aligned} (V_0 \tilde{A}_\delta^* \mathbf{v})(x, s) &= l^{(0)}(s) \Phi(x)^* \Lambda M \mathbf{v} + \delta l^{(0)}(s) \Phi(x)^* \Lambda A \mathbf{v} \\ &= l^{(0)}(s) \Phi(x)^* \Lambda (M + \delta A) \mathbf{v} \\ &= l^{(0)}(s) \Phi(x)^* \Lambda Q \mathbf{v} \end{aligned} \quad (32)$$

We now move onto computing:

$$\begin{aligned} \tilde{A}_\delta V_0 \tilde{A}_\delta^* &= \mathcal{I}_\delta \mathcal{L} V_0 \mathcal{L}^* \mathcal{I}_\delta^* \\ &= \begin{pmatrix} I_1(\delta) \\ \vdots \\ I_J(\delta) \end{pmatrix} \mathcal{L} V_0 \mathcal{L}^* \begin{pmatrix} I_1(\delta)^* & \dots & I_J(\delta)^* \end{pmatrix} \end{aligned}$$

This operator has ij -th entry which is given by:

$$\begin{aligned} (\tilde{A}_\delta V_0 \tilde{A}_\delta^*)_{ij} \alpha &= I_i(\delta) \mathcal{L} V_0 \mathcal{L}^* I_j(\delta)^* \alpha \\ &= \int_{\Omega} \phi_i(x) [(\mathcal{L} V_0 (I_j(\delta) \mathcal{L})^* \alpha)(x, \delta)] dx \\ &= \int_{\Omega} \phi_i(x) \left[\alpha \partial_1 \partial_2 k^{(0)}(\delta, \delta) \sum_{l=1}^J \lambda_l \phi_l(x) M_{lj} + \alpha \partial_1 k^{(0)}(\delta, \delta) \sum_{l=1}^J \lambda_l \phi_l(x) A_{lj} \right. \\ &\quad \left. - \alpha \partial_2 k^{(0)}(\delta, \delta) \sum_{l=1}^J \lambda_l M_{lj} \nabla \cdot (a(x) \nabla \phi_l(x)) - \alpha k^{(0)}(\delta, \delta) \sum_{l=1}^J \lambda_l A_{lj} \nabla \cdot (a(x) \nabla \phi_l(x)) \right] dx \\ &= \alpha \partial_1 \partial_2 k^{(0)}(\delta, \delta) \sum_{l=1}^J \lambda_l M_{il} M_{lj} + \alpha \partial_1 k^{(0)}(\delta, \delta) \sum_{l=1}^J \lambda_l M_{il} A_{lj} \\ &\quad + \alpha \partial_2 k^{(0)}(\delta, \delta) \sum_{l=1}^J \lambda_l M_{lj} A_{il} + \alpha k^{(0)}(\delta, \delta) \sum_{l=1}^J \lambda_l A_{il} A_{lj} \\ &= \alpha \partial_1 \partial_2 k^{(0)}(\delta, \delta) (M \Lambda M)_{ij} + \alpha \partial_1 k^{(0)}(\delta, \delta) (M \Lambda A)_{ij} + \alpha \partial_2 k^{(0)}(\delta, \delta) (A \Lambda M)_{ij} + \alpha k^{(0)}(\delta, \delta) (A \Lambda A)_{ij} \end{aligned}$$

We can thus conclude that $\tilde{A}_\delta V_0 \tilde{A}_\delta^*$ is the $J \times J$ matrix given by:

$$\tilde{A}_\delta V_0 \tilde{A}_\delta^* = \partial_1 \partial_2 k^{(0)}(\delta, \delta) M \Lambda M + \partial_1 k^{(0)}(\delta, \delta) M \Lambda A + \partial_2 k^{(0)}(\delta, \delta) A \Lambda M + k^{(0)}(\delta, \delta) A \Lambda A \quad (33)$$

Using our time kernel and its derivatives we can easily conclude:

$$\begin{aligned} k^{(0)}(\delta, \delta) &= \delta^2 \\ \partial_1 k^{(0)}(\delta, \delta) &= \delta \\ \partial_2 k^{(0)}(\delta, \delta) &= \delta \\ \partial_1 \partial_2 k^{(0)}(\delta, \delta) &= 1 \end{aligned}$$

And thus:

$$\begin{aligned} \tilde{A}_\delta V_0 \tilde{A}_\delta^* &= M \Lambda M + \delta M \Lambda A + \delta A \Lambda M + \delta^2 A \Lambda A \\ &= M \Lambda (M + \delta A) + \delta A \Lambda (M + \delta A) \\ &= (M \Lambda + \delta A \Lambda) (M + \delta A) \\ &= (M + \delta A) \Lambda (M + \delta A) = Q \Lambda Q \end{aligned}$$

We can now make progress with the mean update equation. We first work out the following term using (32):

$$\begin{aligned}
(V_0 \tilde{A}_\delta^* (\tilde{A}_\delta V_0 \tilde{A}_\delta^*)^{-1} F^{(1)})(x, s) &= l^{(0)}(s) \Phi(x)^* \Lambda Q (Q \Lambda Q)^{-1} F^{(1)} \\
&= l^{(0)}(s) \Phi(x)^* \Lambda Q Q^{-1} \Lambda^{-1} Q^{-1} F^{(1)} \\
&= l^{(0)}(s) \Phi(x)^* Q^{-1} F^{(1)}
\end{aligned}$$

For the other term involving m_0 in the mean update equation we must first work out $\tilde{A}_\delta m_0 = \mathcal{I}_\delta \mathcal{L} m_0$. To do this we compute:

$$\begin{aligned}
(\mathcal{L} m_0)(x, t) &= \partial_t m_0 - \nabla \cdot (a(x) \nabla m_0(x, t)) \\
&= -\nabla(a(x) \nabla \Phi(x)^* \gamma) \\
&= -\sum_{j=1}^J \gamma_j \nabla \cdot (a(x) \nabla \phi_j(x))
\end{aligned}$$

Thus, the i -th entry of $\tilde{A}_\delta m_0$ can be computed as:

$$\begin{aligned}
(\tilde{A}_\delta m_0)_i &= I_i(\delta) \mathcal{L} m_0 \\
&= \int_{\Omega} \phi_i(x) \left(-\sum_{j=1}^J \gamma_j \nabla \cdot (a(x) \nabla \phi_j(x)) \right) dx \\
&= \sum_{j=1}^J \gamma_j A_{ij} = (A \gamma)_i
\end{aligned}$$

So $\tilde{A}_\delta m_0 = A \gamma$. We can now compute the second term involving m_0 as follows (again using (32)):

$$\begin{aligned}
((V_0 \tilde{A}_\delta^* (\tilde{A}_\delta V_0 \tilde{A}_\delta^*)^{-1} \tilde{A}_\delta) m_0)(x, s) &= (V_0 \tilde{A}_\delta^* (Q \Lambda Q)^{-1} A \gamma)(x, s) \\
&= l^{(0)}(s) \Phi(x)^* \Lambda Q Q^{-1} \Lambda^{-1} Q^{-1} A \gamma \\
&= l^{(0)}(s) \Phi(x)^* Q^{-1} A \gamma
\end{aligned}$$

Using (21) we can now compute:

$$\begin{aligned}
m_1(x, s) &= \Phi(x)^* \gamma + l^{(0)}(s) \Phi(x)^* Q^{-1} F^{(1)} - l^{(0)}(s) \Phi(x)^* Q^{-1} A \gamma \\
&= \Phi(x)^* \gamma + l^{(0)}(s) \Phi(x)^* Q^{-1} [F^{(1)} - A \gamma] \\
&= \Phi(x)^* \gamma + l^{(0)}(s) \Phi(x)^* c^{(1)}
\end{aligned}$$

From this we can note:

$$\begin{aligned}
m_1(x, t_1) &= \Phi(x)^* \gamma + \delta \Phi(x)^* c^{(1)} \\
&= \Phi(x)^* \gamma + \delta \Phi(x)^* Q^{-1} [F^{(1)} - A \gamma] \\
&= \Phi(x)^* Q^{-1} [Q \gamma + \delta F^{(1)} - \delta A \gamma] \\
&= \Phi(x)^* Q^{-1} [M \gamma + \delta F^{(1)}] \\
&= \Phi(x)^* \gamma_1
\end{aligned}$$

We now move on to computing the covariance V_1 . To do this we must first work out $\tilde{A}_\delta V_0$. Computing this involves determining how $I_j(\delta) \mathcal{L} V_0$ acts on functions g for $j = 1, \dots, J$. We have:

$$I_j(\delta) \mathcal{L} V_0 g = \int_{\Omega} \phi_j(x) (\mathcal{L} V_0 g)(x, \delta) dx$$

Now recalling that $V_0 g(x, \delta) = \int_{\Omega} \int_0^T k_{x,\delta,y,s}^{(0)} g(y, s) ds dy$ we deduce:

$$\begin{aligned} (\mathcal{L}V_0 g)(x, \delta) &= \int_{\Omega} \int_0^T (\mathcal{L}k_{\cdot,\cdot,y,s}^{(0)})(x, \delta) g(y, s) ds dy \\ &= \int_{\Omega} \int_0^T \left(\partial_1 k^{(0)}(\delta, s) \sum_{i=1}^J \lambda_i \phi_i(x) \phi_i(y) - k^{(0)}(\delta, s) \sum_{i=1}^J \lambda_i \nabla_x \cdot (a(x) \nabla_x \phi_i(x)) \phi_i(y) \right) g(y, s) ds dy \end{aligned}$$

We can now perform the integration to obtain:

$$\begin{aligned} I_j(\delta) \mathcal{L}V_0 g &= \int_{\Omega} \phi_j(x) \left(\int_{\Omega} \int_0^T \left(\partial_1 k^{(0)}(\delta, s) \sum_{i=1}^J \lambda_i \phi_i(x) \phi_i(y) - k^{(0)}(\delta, s) \sum_{i=1}^J \lambda_i \nabla_x \cdot (a(x) \nabla_x \phi_i(x)) \phi_i(y) \right) g(y, s) ds dy \right) dx \\ &= \int_{\Omega} \int_0^T \left(\partial_1 k^{(0)}(\delta, s) \sum_{i=1}^J \lambda_i M_{ij} \phi_i(y) g(y, s) + k^{(0)}(\delta, s) \sum_{i=1}^J \lambda_i A_{ij} \phi_i(y) g(y, s) \right) ds dy \\ &= \sum_{i=1}^J \lambda_i \int_{\Omega} \int_0^T (\partial_1 k^{(0)}(\delta, s) M_{ij} + k^{(0)}(\delta, s) A_{ij}) \phi_i(y) g(y, s) ds dy \\ &= \sum_{i=1}^J \lambda_i \left[\int_0^T M_{ij} \partial_1 k^{(0)}(\delta, s) (I_i(s) g) ds + \int_0^T A_{ij} k^{(0)}(\delta, s) (I_i(s) g) ds \right] \\ &= \sum_{i=1}^J \lambda_i \left[M_{ij} \left(\int_0^T \partial_1 k^{(0)}(\delta, s) (\mathcal{I}_s g) ds \right)_i + A_{ij} \left(\int_0^T k^{(0)}(\delta, s) (\mathcal{I}_s g) ds \right)_i \right] \\ &= \left(M \Lambda \int_0^T \partial_1 k^{(0)}(\delta, s) (\mathcal{I}_s g) ds + A \Lambda \int_0^T k^{(0)}(\delta, s) (\mathcal{I}_s g) ds \right)_j \end{aligned}$$

Thus we can deduce:

$$\tilde{A}_{\delta} V_0 g = M \Lambda \int_0^T \partial_1 k^{(0)}(\delta, s) (\mathcal{I}_s g) ds + A \Lambda \int_0^T k^{(0)}(\delta, s) (\mathcal{I}_s g) ds \quad (34)$$

We now utilise the specific form of the time kernel to simplify this to:

$$\tilde{A}_{\delta} V_0 g = M \Lambda \int_0^T l^{(0)}(s) (\mathcal{I}_s g) ds + \delta A \Lambda \int_0^T l^{(0)}(s) (\mathcal{I}_s g) ds \quad (35)$$

$$= (M + \delta A) \Lambda \int_0^T l^{(0)}(s) (\mathcal{I}_s g) ds \quad (36)$$

$$= Q \Lambda \nu_g^{(0)} \quad (37)$$

where $\nu_g^{(i)} := \int_0^T l^{(i)}(s) (\mathcal{I}_s g) ds$ for $i = 0, \dots, N-1$.

We can use this to now compute:

$$\begin{aligned} (V_0 \tilde{A}_{\delta}^* (\tilde{A}_{\delta} V_0 \tilde{A}_{\delta}^*)^{-1} \tilde{A}_{\delta} V_0 g)(x, t) &= (V_0 \tilde{A}_{\delta}^* (Q \Lambda Q)^{-1} Q \Lambda \nu_g^{(0)})(x, t) \\ &= l^{(0)}(t) \Phi(x)^* \Lambda Q Q^{-1} \Lambda^{-1} Q^{-1} Q \Lambda \nu_g^{(0)} \\ &= l^{(0)}(t) \Phi(x)^* \Lambda \nu_g^{(0)} \end{aligned}$$

One can also easily show that the action of V_0 on functions can be rewritten as follows:

$$(V_0 g)(x, t) = \sum_{i=0}^{N-1} l^{(i)}(t) \Phi(x)^* \Lambda \nu_g^{(i)} \quad (38)$$

Thus, we can conclude using (22) that:

$$\begin{aligned} (V_1 g)(x, t) &= \sum_{i=1}^{N-1} l^{(i)}(t) \Phi(x)^* \Lambda \mathbf{v}_g^{(i)} \\ &= \int_{\Omega} \int_0^T k_{x,t,y,s}^{(1)} g(y, s) ds dy \end{aligned} \quad (39)$$

where:

$$k_{x,t,y,s}^{(1)} := \sum_{j=1}^J \lambda_j \phi_j(x) \phi_j(y) k^{(1)}(t, s) \quad (40)$$

$$k^{(1)}(t, s) := \sum_{i=1}^{N-1} l^{(i)}(t) l^{(i)}(s) \quad (41)$$

We thus see that the result holds for $p = 1$. We now proceed with the inductive step. Assume the result holds for some $p < N$. As we did in the proof of the case $p = 1$ we can easily deduce that

$$u | \{ \tilde{A}_{t_1} u = F^{(1)}, \dots, \tilde{A}_{t_{p+1}} u = F^{(p+1)}, f \} \sim \mathcal{N}(m_{p+1}, V_{p+1}) \quad (42)$$

where we have the update equations:

$$m_{p+1} = m_p + V_p \tilde{A}_{t_{p+1}}^* (\tilde{A}_{t_{p+1}} V_p \tilde{A}_{t_{p+1}}^*)^{-1} (F^{(p+1)} - \tilde{A}_{t_{p+1}} m_p) \quad (43)$$

$$V_{p+1} = V_p - V_p \tilde{A}_{t_{p+1}}^* (\tilde{A}_{t_{p+1}} V_p \tilde{A}_{t_{p+1}}^*)^{-1} \tilde{A}_{t_{p+1}} V_p \quad (44)$$

From the time kernel $k^{(p)}(t, s)$ we can work out:

$$\partial_1 k^{(p)}(t, s) = \sum_{i=p}^{N-1} l^{(i)'}(t) l^{(i)}(s) \quad (45)$$

$$\partial_2 k^{(p)}(t, s) = \sum_{i=p}^{N-1} l^{(i)}(t) l^{(i)'}(s) \quad (46)$$

$$\partial_1 \partial_2 k^{(p)}(t, s) = \sum_{i=p}^{N-1} l^{(i)'}(t) l^{(i)'}(s) \quad (47)$$

$$(48)$$

Using (30) and (31) we have

$$k^{(p)}(s, t_{p+1}) = \delta l^{(p)}(s) \quad (49)$$

$$\partial_2 k^{(p)}(s, t_{p+1}) = l^{(p)}(s) \quad (50)$$

These results will help with figuring out the analogue of (32). Since V_p is of the same form as V_0 we can deduce:

$$(V_p \tilde{A}_{t_{p+1}}^* \mathbf{v})(x, s) = \partial_2 k^{(p)}(s, t_{p+1}) \Phi(x)^* \Lambda M \mathbf{v} + k^{(p)}(s, t_{p+1}) \Phi(x)^* \Lambda A \mathbf{v} \quad (51)$$

$$= l^{(p)}(s) \Phi(x)^* \Lambda M \mathbf{v} + \delta l^{(p)}(s) \Phi(x)^* \Lambda A \mathbf{v} \quad (52)$$

$$= l^{(p)}(s) \Phi(x)^* \Lambda Q \mathbf{v} \quad (53)$$

Using (30) and (31) we can also conclude the following:

$$\begin{aligned} k^{(p)}(t_{p+1}, t_{p+1}) &= \delta^2 \\ \partial_1 k^{(p)}(t_{p+1}, t_{p+1}) &= \delta \\ \partial_2 k^{(p)}(t_{p+1}, t_{p+1}) &= \delta \\ \partial_1 \partial_2 k^{(p)}(t_{p+1}, t_{p+1}) &= 1 \end{aligned}$$

Thus, just as in the proof for $p = 1$ we have:

$$\begin{aligned}\tilde{A}_{t_{p+1}} V_p \tilde{A}_{t_{p+1}}^* &= \partial_1 \partial_2 k^{(p)}(t_{p+1}, t_{p+1}) M \Lambda M + \partial_1 k^{(p)}(t_{p+1}, t_{p+1}) M \Lambda A + \partial_2 k^{(p)}(t_{p+1}, t_{p+1}) A \Lambda M + k^{(p)}(\delta, \delta) A \Lambda A \\ &= Q \Lambda Q\end{aligned}$$

We can now compute:

$$\begin{aligned}\left(V_p \tilde{A}_{t_{p+1}}^* (\tilde{A}_{t_{p+1}} V_p \tilde{A}_{t_{p+1}}^*)^{-1} F^{(p+1)} \right) (x, s) &= l^{(p)}(s) \Phi(x)^* \Lambda Q (Q \Lambda Q)^{-1} F^{(p+1)} \\ &= l^{(p)}(s) \Phi(x)^* \Lambda Q Q^{-1} \Lambda^{-1} Q^{-1} F^{(p+1)} \\ &= l^{(p)}(s) \Phi(x)^* Q^{-1} F^{(p+1)}\end{aligned}$$

In order to finish with the mean update we must now compute $\tilde{A}_{t_{p+1}} m_p = \mathcal{I}_{t_{p+1}} \mathcal{L} m_p$. Recalling that

$$m_p(x, s) = \Phi(x)^* \gamma + \sum_{i=1}^p l^{(i-1)}(s) \Phi(x)^* \mathbf{c}^{(i)}$$

we can work out:

$$(\mathcal{L} m_p)(x, s) = \sum_{i=1}^p l^{(i-1)'}(s) \Phi(x)^* \mathbf{c}^{(i)} - \sum_{j=1}^J \left(\gamma + \sum_{i=1}^p l^{(i-1)}(s) \mathbf{c}^{(i)} \right)_j \nabla \cdot (a(x) \nabla \phi_j(x))$$

Using this, together with (30) and (31), we can work out:

$$\begin{aligned}(\tilde{A}_{t_{p+1}} m_p)_k &= I_k(t_{p+1}) \mathcal{L} m_p \\ &= \int_{\Omega} \phi_k(x) (\mathcal{L} m_p)(x, t_{p+1}) dx \\ &= - \sum_{j=1}^J \left(\gamma + \sum_{i=1}^p \delta \mathbf{c}^{(i)} \right)_j \int_{\Omega} \phi_k(x) \nabla \cdot (a(x) \nabla \phi_j(x)) dx \\ &= \sum_{j=1}^J \left(\gamma + \sum_{i=1}^p \delta \mathbf{c}^{(i)} \right)_j A_{kj} \\ &= \left[A \left(\gamma + \sum_{i=1}^p \delta \mathbf{c}^{(i)} \right) \right]_k\end{aligned}$$

i.e. we have:

$$\tilde{A}_{t_{p+1}} m_p = A \left(\gamma + \sum_{i=1}^p \delta \mathbf{c}^{(i)} \right) \quad (54)$$

We now claim that $\gamma + \sum_{i=1}^p \delta \mathbf{c}^{(i)} = \gamma_p$. This can proven by induction. It is true for $p = 1$ (see proof that $m_1(x, t_1) = \Phi(x)^* \gamma_1$). Assume its true for p . For $p + 1$ we have:

$$\begin{aligned}\gamma + \sum_{i=1}^{p+1} \delta \mathbf{c}^{(i)} &= \gamma + \sum_{i=1}^p \delta \mathbf{c}^{(i)} + \delta \mathbf{c}^{(p+1)} \\ &= \gamma_p + \delta \mathbf{c}^{(p+1)} \\ &= \gamma_p + \delta Q^{-1} \left[F^{(p+1)} - A \gamma_p \right] \\ &= Q^{-1} \left[(Q - \delta A) \gamma_p + \delta F^{(p+1)} \right] \\ &= Q^{-1} \left[M \gamma_p + \delta F^{(p+1)} \right] \\ &= \gamma_{p+1}\end{aligned}$$

as claimed. Thus, $\tilde{A}_{t_{p+1}} m_p = A \gamma_p$ and we can now compute:

$$\begin{aligned}\left(V_p \tilde{A}_{t_{p+1}}^* (\tilde{A}_{t_{p+1}} V_p \tilde{A}_{t_{p+1}}^*)^{-1} \tilde{A}_{t_{p+1}} m_p \right) (x, s) &= l^{(p)}(s) \Phi(x)^* \Lambda Q (Q \Lambda Q)^{-1} A \gamma_p \\ &= l^{(p)}(s) \Phi(x)^* \Lambda Q Q^{-1} \Lambda^{-1} Q^{-1} A \gamma_p \\ &= l^{(p)}(s) \Phi(x)^* Q^{-1} A \gamma_p\end{aligned}$$

It now follows from (43) that we have:

$$\begin{aligned}
m_{p+1}(x, s) &= \Phi(x)^* \gamma + \sum_{i=1}^p l^{(i-1)}(s) \Phi(x)^* \mathbf{c}^{(i)} + l^{(p)}(s) \Phi(x)^* Q^{-1} F^{(p+1)} - l^{(p)}(s) \Phi(x)^* Q^{-1} A \gamma_p \\
&= \Phi(x)^* \gamma + \sum_{i=1}^p l^{(i-1)}(s) \Phi(x)^* \mathbf{c}^{(i)} + l^{(p)}(s) \Phi(x)^* Q^{-1} [F^{(p+1)} - A \gamma_p] \\
&= \Phi(x)^* \gamma + \sum_{i=1}^p l^{(i-1)}(s) \Phi(x)^* \mathbf{c}^{(i)} + l^{(p)}(s) \Phi(x)^* \mathbf{c}^{(p+1)} \\
&= \Phi(x)^* \gamma + \sum_{i=1}^{p+1} l^{(i-1)}(s) \Phi(x)^* \mathbf{c}^{(i)}
\end{aligned}$$

Having obtained this formula we can now evaluate the mean at $s = t_{p+1}$ to deduce:

$$m_{p+1}(x, t_{p+1}) = \Phi(x)^* \left[\gamma + \sum_{i=1}^{p+1} \delta \mathbf{c}^{(i)} \right] = \Phi(x)^* \gamma_{p+1}$$

by the above claim.

We now move on to the covariance update (44). The piece missing before we can proceed is the computation of $\tilde{A}_{t_{p+1}} V_p$. Following the proof of the case $p = 1$ we can write:

$$\tilde{A}_{t_{p+1}} V_p g = M \Lambda \int_0^T \partial_1 k^{(p)}(t_{p+1}, s) (\mathcal{I}_s g) ds + A \Lambda \int_0^T k^{(p)}(t_{p+1}, s) (\mathcal{I}_s g) ds \quad (55)$$

$$= M \Lambda \int_0^T l^{(p)}(s) (\mathcal{I}_s g) ds + \delta A \Lambda \int_0^T l^{(p)}(s) (\mathcal{I}_s g) ds \quad (56)$$

$$= (M + \delta A) \Lambda \int_0^T l^{(p)}(s) (\mathcal{I}_s g) ds \quad (57)$$

$$= Q \Lambda \boldsymbol{\nu}_g^{(p)} \quad (58)$$

And thus, we can compute:

$$\begin{aligned}
(V_p \tilde{A}_{t_{p+1}}^* (\tilde{A}_{t_{p+1}} V_p \tilde{A}_{t_{p+1}}^*)^{-1} \tilde{A}_{t_{p+1}} V_p g)(x, t) &= (V_p \tilde{A}_{t_{p+1}}^* (Q \Lambda Q)^{-1} Q \Lambda \boldsymbol{\nu}_g^{(p)})(x, t) \\
&= l^{(p)}(t) \Phi(x)^* \Lambda Q Q^{-1} \Lambda^{-1} Q^{-1} Q \Lambda \boldsymbol{\nu}_g^{(p)} \\
&= l^{(p)}(t) \Phi(x)^* \Lambda \boldsymbol{\nu}_g^{(p)}
\end{aligned}$$

Thus, using (44) we can deduce:

$$\begin{aligned}
(V_{p+1} g)(x, t) &= \sum_{i=p+1}^{N-1} l^{(i)}(t) \Phi(x)^* \Lambda \boldsymbol{\nu}_g^{(i)} \\
&= \int_{\Omega} \int_0^T k_{x,t,y,s}^{(p+1)} g(y, s) ds dy
\end{aligned}$$

where:

$$\begin{aligned}
k_{x,t,y,s}^{(p+1)} &:= \sum_{j=1}^J \lambda_j \phi_j(x) \phi_j(y) k^{(p+1)}(t, s) \\
k^{(p+1)}(t, s) &:= \sum_{i=p+1}^{N-1} l^{(i)}(t) l^{(i)}(s)
\end{aligned}$$

Thus, the result is true for $p = 1$. So by induction it is true for $p \in \{1, \dots, N\}$. ■