We now focus on the following time-dependent PDE:

$$\mathcal{L}u(x,t) := \partial_t u(x,t) - \nabla \cdot (a(x)\nabla u(x,t)) = f(x,t), \quad x \in \Omega, \ t \in [0,T] \tag{1}$$
$$u(x,t) = 0, \quad x \in \partial\Omega, \ t \in [0,T] \tag{2}$$
$$u(x,0) = u_0(x), \quad x \in \Omega \tag{3}$$

We will set up a prior on the solution u to the above problem. To do so we first let $v_h \in S_h$ be some approximation of the initial condition $u_0(x)$ in the FEM space S_h. To be more specific we will assume that $v_h(x) = \Phi(x)^*\boldsymbol{\gamma} := \sum_{i=1}^J \phi_i(x)\gamma_i$. Note that $\Phi(x) := (\phi_1(x), \dots, \phi_J(x))^T$. We take the prior on u to be:

$$u \sim \mathcal{N}(m_0, V_0) \tag{4}$$

where $m_0(x,t) := v_h(x) = \Phi(x)^*\boldsymbol{\gamma}$ (m_0 is constant in time). The prior covariance operator V_0 is defined as:

$$(V_0 g)(x,t) = \int_\Omega \int_0^T k^{(0)}_{x,t,y,s} g(y,s) \mathrm{d}s \mathrm{d}y \tag{5}$$

where $k^{(0)}_{x,t,y,s}$ is defined as follows:

$$k^{(0)}_{x,t,y,s} := \sum_{i=1}^J \lambda_j \phi_j(x)\phi_j(y) k^{(0)}(t,s) \tag{6}$$

We now introduce a uniform time grid:

$$t_n = n\delta, \ \ n = 0, 1, \dots, N$$

where δ is the spacing between consecutive times and $N = \frac{T}{\delta}$ (*assume that N is an integer*). The time kernel $k^{(0)}(t,s)$ will be taken to be:

$$k^{(0)}(t,s) := \sum_{i=0}^{N-1} (t-t_i)(s-t_i)\mathbb{1}_{(t_i,t_{i+1}]^2}(t,s) \tag{7}$$

We now introduce the following operators $\mathcal{I}_s := (I_1(s), \dots, I_J(s))^T$ where:

$$I_i(s)g := \int_\Omega \phi_i(x) g(x,s) \mathrm{d}x \tag{8}$$

To move from $t = t_0 = 0$ to $t = t_1 = \delta$ we condition on observing $\mathcal{I}_\delta \mathcal{L} u = \mathcal{I}_\delta f =: F^{(1)}$. Let $\tilde{A}_\delta := \mathcal{I}_\delta \mathcal{L}$. For a fixed realisation of f (and so of $F^{(1)}$) we thus seek the following conditional distribution:

$$u | \{\tilde{A}_\delta u = F^{(1)}, f\} \sim \mathcal{N}(m_1, V_1) \tag{9}$$

That this distribution is itself Gaussian follows from considering the following joint distribution:

$$\begin{pmatrix} u \\ \tilde{A}_\delta u \end{pmatrix} = \begin{pmatrix} I \\ \tilde{A}_\delta \end{pmatrix} u \sim \mathcal{N}\left(\begin{pmatrix} m_o \\ \tilde{A}_\delta m_0 \end{pmatrix}, \begin{pmatrix} V_0 & V_0 \tilde{A}_\delta^* \\ \tilde{A}_\delta V_0 & \tilde{A}_\delta V_0 \tilde{A}_\delta^* \end{pmatrix} \right)$$

It follows that the conditional distribution is Gaussian and the mean and covariance are given by:

$$m_1 = m_0 + V_0 \tilde{A}_\delta^* (\tilde{A}_\delta V_0 \tilde{A}_\delta^*)^{-1} (F^{(1)} - \tilde{A}_\delta m_0) \tag{10}$$
$$V_1 = V_0 - V_0 \tilde{A}_\delta^* (\tilde{A}_\delta V_0 \tilde{A}_\delta^*)^{-1} \tilde{A}_\delta V_0 \tag{11}$$

We now rewrite the mean update equation as follows:

$$m_1 = \left(1 - V_0 \tilde{A}_\delta^* (\tilde{A}_\delta V_0 \tilde{A}_\delta^*)^{-1} \tilde{A}_\delta \right) m_0 + V_0 \tilde{A}_\delta^* (\tilde{A}_\delta V_0 \tilde{A}_\delta^*)^{-1} F^{(1)} \tag{12}$$

To make progress we must now start computing various terms needed for our mean and covariance update rules. We start with $V_0 \tilde{A}_\delta^*$. We have:

$$V_0 \tilde{A}_\delta^* = V_0 \mathcal{L}^* (I_1(\delta)^*, \dots, I_J(\delta)^*)$$

We can thus see that we need to be able to compute terms of the form $V_0\mathcal{L}^* I_i(\delta)^* = V_0(I_i(\delta)\mathcal{L})^*$. Now since the operator $I_i(\delta)\mathcal{L}$ takes in a function on $\Omega \times [0, T]$ and outputs a real number its adjoint should take in a real number and output a function on $\Omega \times [0, T]$. This adjoint should satisfy the following relation:

$$\alpha(I_i(\delta)\mathcal{L}g) = \int_\Omega \int_0^T ((I_i(\delta)\mathcal{L})^*\alpha)(x,t)g(x,t)\mathrm{d}t\mathrm{d}x \ \forall g, \ \forall \alpha \in \mathbb{R} \tag{13}$$

Using this we can now compute:

$$\begin{aligned}
(V_0(I_i(\delta)\mathcal{L})^*\alpha)(x,s) &= \int_\Omega \int_0^T k^{(0)}_{x,s,y,w}((I_i(\delta)\mathcal{L})^*\alpha)(y,w)\mathrm{d}w\mathrm{d}y \\
&= \alpha(I_i(\delta)\mathcal{L}k^{(0)}_{x,s,\cdot,\cdot}) \\
&= \alpha \int_\Omega \phi_i(y)(\mathcal{L}k^{(0)}_{x,s,\cdot,\cdot})(y,\delta)\mathrm{d}y
\end{aligned}$$

We now work out $(\mathcal{L}k^{(0)}_{x,s,\cdot,\cdot})(y,\delta)$ taking care to remember that x, s are fixed and so $\mathcal{L}$ acts on the variables y, δ:

$$\begin{aligned}
(\mathcal{L}k^{(0)}_{x,s,\cdot,\cdot})(y,\delta) &= \partial_2 k^{(0)}(s,\delta)\sum_{j=1}^J \lambda_j\phi_j(x)\phi_j(y) - k^{(0)}(s,\delta)\nabla_y \cdot \left(a(y)\nabla_y \sum_{j=1}^J(\lambda_j\phi_j(x)\phi_j(y)\right) \\
&= \partial_2 k^{(0)}(s,\delta)\sum_{j=1}^J \lambda_j\phi_j(x)\phi_j(y) - k^{(0)}(s,\delta)\sum_{j=1}^J \lambda_j\phi_j(x)\nabla_y \cdot (a(y)\nabla_y\phi_j(y))
\end{aligned}$$

So we can now compute:

$$\begin{aligned}
(V_0(I_i(\delta)\mathcal{L})^*\alpha)(x,s) &= \alpha\int_\Omega \phi_i(y)\partial_2 k^{(0)}(s,\delta)\sum_{j=1}^J \lambda_j\phi_j(x)\phi_j(y)\mathrm{d}y - \alpha\int_\Omega \phi_i(y)k^{(0)}(s,\delta)\sum_{j=1}^J \lambda_j\phi_j(x)\nabla_y \cdot (a(y)\nabla_y\phi_j(y))\,\mathrm{d}y \\
&= \alpha\partial_2 k^{(0)}(s,\delta)\sum_{j=1}^J \lambda_j\phi_j(x)M_{ji} + \alpha k^{(0)}(s,\delta)\sum_{j=1}^J \lambda_j\phi_j(x)A_{ji}
\end{aligned}$$

where M is the Galerkin mass matrix and A is the Galerkin stiffness matrix, i.e. the matrices with entries given by:

$$M_{ij} := \int_\Omega \phi_i(x)\phi_j(x)\mathrm{d}x \tag{14}$$

$$A_{ij} := \int_\Omega a(x)\nabla\phi_i(x)\nabla\phi_j(x)\mathrm{d}x \tag{15}$$

Using this result we can deduce that:

$$(V_0\tilde{A}^*_\delta \boldsymbol{v})(x,s) = \partial_2 k^{(0)}(s,\delta)\Phi(x)^*\Lambda M\boldsymbol{v} + k^{(0)}(s,\delta)\Phi(x)^*\Lambda A\boldsymbol{v} \tag{16}$$

for any $\boldsymbol{v} \in \mathbb{R}^J$, where $\Lambda = \mathrm{diag}\{\lambda_i\}_{i=1}^J$.

We now move onto computing:

$$\begin{aligned}
\tilde{A}_\delta V_0 \tilde{A}^*_\delta &= \mathcal{I}_\delta \mathcal{L} V_0 \mathcal{L}^* \mathcal{I}^*_\delta \\
&= \begin{pmatrix} I_1(\delta) \\ \vdots \\ I_J(\delta) \end{pmatrix} \mathcal{L}V_0\mathcal{L}^* \begin{pmatrix} I_1(\delta)^* & \dots & I_J(\delta)^* \end{pmatrix}
\end{aligned}$$

This operator has ij-th entry which is given by:

$$\begin{aligned}
(\tilde{A}_\delta V_0 \tilde{A}_\delta^*)_{ij}\alpha &= I_i(\delta)\mathcal{L}V_0\mathcal{L}^* I_j(\delta)^*\alpha \\
&= \int_\Omega \phi_i(x)\left[(\mathcal{L}V_0(I_j(\delta)\mathcal{L})^*\alpha)(x,\delta)\right]\mathrm{d}x \\
&= \int_\Omega \phi_i(x)\Bigg[\alpha\partial_1\partial_2 k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l\phi_l(x)M_{lj} + \alpha\partial_1 k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l\phi_l(x)A_{lj} \\
&- \alpha\partial_2 k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l M_{lj}\nabla\cdot(a(x)\nabla\phi_l(x)) - \alpha k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l A_{lj}\nabla\cdot(a(x)\nabla\phi_l(x))\Bigg]\mathrm{d}x \\
&= \alpha\partial_1\partial_2 k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l M_{il}M_{lj} + \alpha\partial_1 k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l M_{il}A_{lj} \\
&+ \alpha\partial_2 k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l M_{lj}A_{il} + \alpha k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l A_{il}A_{lj} \\
&= \alpha\partial_1\partial_2 k^{(0)}(\delta,\delta)(M\Lambda M)_{ij} + \alpha\partial_1 k^{(0)}(\delta,\delta)(M\Lambda A)_{ij} + \alpha\partial_2 k^{(0)}(\delta,\delta)(A\Lambda M)_{ij} + \alpha k^{(0)}(\delta,\delta)(A\Lambda A)_{ij}
\end{aligned}$$

We can thus conclude that $\tilde{A}_\delta V_0 \tilde{A}_\delta^*$ is the $J \times J$ matrix given by:

$$\tilde{A}_\delta V_0 \tilde{A}_\delta^* = \partial_1\partial_2 k^{(0)}(\delta,\delta)M\Lambda M + \partial_1 k^{(0)}(\delta,\delta)M\Lambda A + \partial_2 k^{(0)}(\delta,\delta)A\Lambda M + k^{(0)}(\delta,\delta)A\Lambda A \tag{17}$$

We now give the various derivatives for our time kernel which are required (**!!these should be checked rigorously!!**):

$$\begin{aligned}
\partial_1 k^{(0)}(t,s) &= \sum_{i=0}^{N-1}(s-t_i)\mathbb{1}_{(t_i,t_{i+1}]^2}(t,s) \\
\partial_2 k^{(0)}(t,s) &= \sum_{i=0}^{N-1}(t-t_i)\mathbb{1}_{(t_i,t_{i+1}]^2}(t,s) \\
\partial_1\partial_2 k^{(0)}(t,s) &= \sum_{i=0}^{N-1}\mathbb{1}_{(t_i,t_{i+1}]^2}(t,s)
\end{aligned}$$

We thus have:

$$\begin{aligned}
k^{(0)}(\delta,\delta) &= \delta^2 \\
\partial_1 k^{(0)}(\delta,\delta) &= \delta \\
\partial_2 k^{(0)}(\delta,\delta) &= \delta \\
\partial_1\partial_2 k^{(0)}(\delta,\delta) &= 1
\end{aligned}$$

So we have:

$$\begin{aligned}
\tilde{A}_\delta V_0 \tilde{A}_\delta^* &= M\Lambda M + \delta M\Lambda A + \delta A\Lambda M + \delta^2 A\Lambda A \\
&= M\Lambda(M+\delta A) + \delta A\Lambda(M+\delta A) \\
&= (M\Lambda + \delta A\Lambda)(M+\delta A) \\
&= (M+\delta A)\Lambda(M+\delta A) = Q\Lambda Q
\end{aligned}$$

where we have defined $Q := M + \delta A$. To make progress we now simplify equation (16) using the derivatives noting the following:

$$\begin{aligned}
k^{(0)}(s,\delta) &= s\delta\mathbb{1}_{(0,\delta]}(s) \\
\partial_2 k^{(0)}(s,\delta) &= s\mathbb{1}_{(0,\delta]}(s)
\end{aligned}$$

We thus have:

$$
\begin{aligned}
(V_0\tilde{A}_\delta^*\boldsymbol{v})(x,s) &= s\mathbb{1}_{(0,\delta]}(s)\Phi(x)^*\Lambda M\boldsymbol{v} + s\delta\mathbb{1}_{(0,\delta]}(s)\Phi(x)^*\Lambda A\boldsymbol{v} \\
&= s\mathbb{1}_{(0,\delta]}(s)\Phi(x)^*\Lambda(M+\delta A)\mathbf{v} \\
&= s\mathbb{1}_{(0,\delta]}(s)\Phi(x)^*\Lambda Q\mathbf{v}
\end{aligned} \tag{18}
$$

We can now make progress with the mean update equation. We first work out the following term using (18):

$$
\begin{aligned}
(V_0\tilde{A}_\delta^*(\tilde{A}_\delta V_0\tilde{A}_\delta^*)^{-1}F^{(1)})(x,s) &= s\mathbb{1}_{(0,\delta]}(s)\Phi(x)^*\Lambda Q(Q\Lambda Q)^{-1}F^{(1)} \\
&= s\mathbb{1}_{(0,\delta]}(s)\Phi(x)^*\Lambda Q Q^{-1}\Lambda^{-1}Q^{-1}F^{(1)} \\
&= s\mathbb{1}_{(0,\delta]}(s)\Phi(x)^*Q^{-1}F^{(1)}
\end{aligned}
$$

For the other term involving m_0 in the mean update equation we must first work out $\tilde{A}_\delta m_0 = \mathcal{I}_\delta\mathcal{L}m_0$. To do this we compute:

$$
\begin{aligned}
(\mathcal{L}m_0)(x,t) &= \partial_t m_0 - \nabla\cdot(a(x)\nabla m_0(x,t)) \\
&= -\nabla(a(x)\nabla\Phi(x)^*\boldsymbol{\gamma}) \\
&= -\sum_{j=1}^{J}\gamma_j\nabla\cdot(a(x)\nabla\phi_j(x))
\end{aligned}
$$

Thus, the *i-th* entry of $\tilde{A}_\delta m_0$ can be computed as:

$$
\begin{aligned}
(\tilde{A}_\delta m_0)_i &= I_i(\delta)\mathcal{L}m_0 \\
&= \int_\Omega \phi_i(x)\left(-\sum_{j=1}^{J}\gamma_j\nabla\cdot(a(x)\nabla\phi_j(x))\right)\mathrm{d}x \\
&= \sum_{j=1}^{J}\gamma_j A_{ij} = (A\boldsymbol{\gamma})_i
\end{aligned}
$$

So $\tilde{A}_\delta m_0 = A\boldsymbol{\gamma}$. We can now compute the second term involving m_0 as follows (again using (18)):

$$
\begin{aligned}
((V_0\tilde{A}_\delta^*(\tilde{A}_\delta V_0\tilde{A}_\delta^*)^{-1}\tilde{A}_\delta)m_0)(x,s) &= (V_0\tilde{A}_\delta^*(Q\Lambda Q)^{-1}A\boldsymbol{\gamma})(x,s) \\
&= s\mathbb{1}_{(0,\delta]}(s)\Phi(x)^*\Lambda Q Q^{-1}\Lambda^{-1}Q^{-1}A\boldsymbol{\gamma} \\
&= s\mathbb{1}_{(0,\delta]}(s)\Phi(x)^*Q^{-1}A\boldsymbol{\gamma}
\end{aligned}
$$

Thus we can compute:

$$
\begin{aligned}
(1-(V_0\tilde{A}_\delta^*(\tilde{A}_\delta V_0\tilde{A}_\delta^*)^{-1}\tilde{A}_\delta)m_0)(x,s) &= \Phi(x)^*\boldsymbol{\gamma} - s\mathbb{1}_{(0,\delta]}(s)\Phi(x)^*Q^{-1}A\boldsymbol{\gamma} \\
&= \Phi(x)^*Q^{-1}[Q - s\mathbb{1}_{(0,\delta]}(s)A]\boldsymbol{\gamma} \\
&= \Phi(x)^*Q^{-1}[M+\delta A - s\mathbb{1}_{(0,\delta]}(s)A]\boldsymbol{\gamma} \\
&= \Phi(x)^*Q^{-1}[M+(\delta - s\mathbb{1}_{(0,\delta]}(s))A]\boldsymbol{\gamma}
\end{aligned}
$$

Putting this all together we obtain:

$$
m_1(x,s) = \Phi(x)^*Q^{-1}[M+(\delta - s\mathbb{1}_{(0,\delta]}(s))A]\boldsymbol{\gamma} + s\mathbb{1}_{(0,\delta]}(s)\Phi(x)^*Q^{-1}F^{(1)} \tag{19}
$$

Thus we see that performing this mean update and then evaluating at the time $t_1 = \delta$ we obtain the following:

$$
m_1(x,t_1) = m_1(x,\delta) = \Phi(x)^*\left(Q^{-1}M\boldsymbol{\gamma} + \delta Q^{-1}F^{(1)}\right) \tag{20}
$$

Note that this is the same as the update equation for the coefficients in the backward-Euler Galerkin method:

$$
\boldsymbol{\gamma} \longmapsto (M+\delta A)^{-1}M\boldsymbol{\gamma} + \delta(M+\delta A)^{-1}F^{(1)} \tag{21}
$$

We can now move on to computing the covariance V_1. We start by computing $\tilde{A}_\delta V_0$. Computing this involves determining how $I_j(\delta)\mathcal{L}V_0$ acts on functions g for $j = 1, \dots, J$. We have:

$$I_j(\delta)\mathcal{L}V_0 g = \int_\Omega \phi_j(x)(\mathcal{L}V_0 g)(x,\delta)\mathrm{d}x$$

Now recalling that $V_0 g(x,\delta) = \int_\Omega \int_0^T k^{(0)}_{x,\delta,y,s} g(y,s)\mathrm{d}s\mathrm{d}y$ we deduce:

$$\begin{aligned}
(\mathcal{L}V_0 g)(x,\delta) &= \int_\Omega \int_0^T (\mathcal{L}k^{(0)}_{\cdot,\cdot,y,s})(x,\delta) g(y,s)\mathrm{d}s\mathrm{d}y \\
&= \int_\Omega \int_0^T \left(\partial_1 k^{(0)}(\delta,s) \sum_{i=1}^J \lambda_i \phi_i(x)\phi_i(y) - k^{(0)}(\delta,s) \sum_{i=1}^J \lambda_i \nabla_x \cdot (a(x)\nabla_x \phi_i(x))\phi_i(y) \right) g(y,s)\mathrm{d}s\mathrm{d}y
\end{aligned}$$

We can now perform the integration to obtain:

$$\begin{aligned}
I_j(\delta)\mathcal{L}V_0 g &= \int_\Omega \phi_j(x) \left(\int_\Omega \int_0^T \left(\partial_1 k^{(0)}(\delta,s) \sum_{i=1}^J \lambda_i \phi_i(x)\phi_i(y) - k^{(0)}(\delta,s) \sum_{i=1}^J \lambda_i \nabla_x \cdot (a(x)\nabla_x \phi_i(x))\phi_i(y) \right) g(y,s)\mathrm{d}s\mathrm{d}y \right) \mathrm{d}x \\
&= \int_\Omega \int_0^T \left(\partial_1 k^{(0)}(\delta,s) \sum_{i=1}^J \lambda_i M_{ij}\phi_i(y) g(y,s) + k^{(0)}(\delta,s) \sum_{i=1}^J \lambda_i A_{ij}\phi_i(y) g(y,s) \right) \mathrm{d}s\mathrm{d}y \\
&= \sum_{i=1}^J \lambda_i \int_\Omega \int_0^T (\partial_1 k^{(0)}(\delta,s) M_{ij} + k^{(0)}(\delta,s) A_{ij})\phi_i(y) g(y,s)\mathrm{d}s\mathrm{d}y \\
&= \sum_{i=1}^J \lambda_i \left[\int_0^T M_{ij}\partial_1 k^{(0)}(\delta,s)(I_i(s)g)\mathrm{d}s + \int_0^T A_{ij} k^{(0)}(\delta,s)(I_i(s)g)\mathrm{d}s \right] \\
&= \sum_{i=1}^J \lambda_i \left[M_{ij} \left(\int_0^T \partial_1 k^{(0)}(\delta,s)(\mathcal{I}_s g)\mathrm{d}s \right)_i + A_{ij} \left(\int_0^T k^{(0)}(\delta,s)(\mathcal{I}_s g)\mathrm{d}s \right)_i \right] \\
&= \left(M\Lambda \int_0^T \partial_1 k^{(0)}(\delta,s)(\mathcal{I}_s g)\mathrm{d}s + A\Lambda \int_0^T k^{(0)}(\delta,s)(\mathcal{I}_s g)\mathrm{d}s \right)_j
\end{aligned}$$

Thus we can deduce:

$$\tilde{A}_\delta V_0 g = M\Lambda \int_0^T \partial_1 k^{(0)}(\delta,s)(\mathcal{I}_s g)\mathrm{d}s + A\Lambda \int_0^T k^{(0)}(\delta,s)(\mathcal{I}_s g)\mathrm{d}s \tag{22}$$

We now utilise the specific form of the time kernel:

$$\begin{aligned}
k^{(0)}(\delta,s) &= s\delta \mathbb{1}_{(0,\delta]}(s) \\
\partial_1 k^{(0)}(\delta,s) &= s \mathbb{1}_{(0,\delta]}(s)
\end{aligned}$$

to further deduce:

$$\begin{aligned}
\tilde{A}_\delta V_0 g &= M\Lambda \int_0^T s\mathbb{1}_{(0,\delta]}(s)(\mathcal{I}_s g)\mathrm{d}s + A\Lambda \int_0^T s\delta \mathbb{1}_{(0,\delta]}(s)(\mathcal{I}_s g)\mathrm{d}s \\
&= Q\Lambda \boldsymbol{\nu}_g^{(0)}
\end{aligned}$$

where:

$$\boldsymbol{\nu}_g^{(i)} := \int_{t_i}^{t_{i+1}} (s - t_i)(\mathcal{I}_s g)\mathrm{d}s \text{ for } i = 0, \dots, N-1 \tag{23}$$

Having worked this out we can now compute the second term in the formula for V_1 as follows:

$$\begin{aligned}
(V_0 \tilde{A}_\delta^* (\tilde{A}_\delta V_0 \tilde{A}_\delta^*)^{-1} \tilde{A}_\delta V_0 g)(x,t) &= (V_0 \tilde{A}_\delta^* (Q\Lambda Q)^{-1} Q\Lambda \boldsymbol{\nu}_g^{(0)})(x,t) \\
&= t\mathbb{1}_{(0,\delta]}(t)\Phi(x)^* \Lambda Q Q^{-1} \Lambda^{-1} Q^{-1} Q \Lambda \boldsymbol{\nu}_g^{(0)} \\
&= t\mathbb{1}_{(0,\delta]}(t)\Phi(x)^* \Lambda \boldsymbol{\nu}_g^{(0)}
\end{aligned}$$

One can also easily show that the action of V_0 on functions can be rewritten as follows:

$$(V_0 g)(x,t) = \sum_{i=0}^{N-1} (t-t_i)\mathbb{1}_{(t_i,t_{i+1}]}(t)\Phi(x)^*\Lambda \boldsymbol{\nu}_g^{(i)} \tag{24}$$

and so we can deduce that V_1 is given by:

$$(V_1 g)(x,t) = \sum_{i=1}^{N-1} (t-t_i)\mathbb{1}_{(t_i,t_{i+1}]}(t)\Phi(x)^*\Lambda \boldsymbol{\nu}_g^{(i)}$$

This can be rewritten as follows:

$$(V_1 g)(x,t) = \int_\Omega \int_0^T k^{(1)}_{x,t,y,s} g(y,s)\mathrm{d}s\mathrm{d}y \tag{25}$$

where the kernel is now:

$$k^{(1)}_{x,t,y,s} := \sum_{i=1}^{J} \lambda_j \phi_j(x)\phi_j(y) k^{(1)}(t,s) \tag{26}$$

with time kernel:

$$k^{(1)}(t,s) := \sum_{i=1}^{N-1} (t-t_i)(s-t_i)\mathbb{1}_{(t_i,t_{i+1}]^2}(t,s) \tag{27}$$

We can thus see that the new covariance operator is of the same form as the prior covariance; the only difference is the time kernel has changed by simply removing the term corresponding to the first time interval $(t_0, t_1]$. This observation will allow us to very easily move to the next time step $t = t_2 = 2\delta$. To do so we now condition on observing $\mathcal{I}_{2\delta}\mathcal{L}u = \mathcal{I}_{2\delta} f =: F^{(2)}$. As before let $\tilde{A}_{2\delta} = \mathcal{I}_{2\delta}\mathcal{L}$. For the same fixed realisation of f we seek the following conditional distribution:

$$u|\{\tilde{A}_\delta u = F^{(1)}, \tilde{A}_{2\delta} u = F^{(2)}, f\} \sim \mathcal{N}(m_2, V_2) \tag{28}$$

where we have the update equations:

$$m_2 = m_1 + V_1\tilde{A}^*_{2\delta}(\tilde{A}_{2\delta}V_1\tilde{A}^*_{2\delta})^{-1}(F^{(2)} - \tilde{A}_{2\delta}m_1) \tag{29}$$

$$V_2 = V_1 - V_1\tilde{A}^*_{2\delta}(\tilde{A}_{2\delta}V_1\tilde{A}^*_{2\delta})^{-1}\tilde{A}_{2\delta}V_1 \tag{30}$$

Since the covariance operator V_1 has the same form as V_0 but with a different time kernel most of the computations from before are the same. We give the results, and note where there are differences. First we have the analogue of (16):

$$(V_1\tilde{A}^*_{2\delta}\boldsymbol{v})(x,s) = \partial_2 k^{(1)}(s,2\delta)\Phi(x)^*\Lambda M\boldsymbol{v} + k^{(1)}(s,2\delta)\Phi(x)^*\Lambda A\boldsymbol{v} \tag{31}$$

The analogue of (17) is given by:

$$\tilde{A}_{2\delta}V_1\tilde{A}^*_{2\delta} = \partial_1\partial_2 k^{(1)}(2\delta,2\delta)M\Lambda M + \partial_1 k^{(1)}(2\delta,2\delta)M\Lambda A + \partial_2 k^{(1)}(2\delta,2\delta)A\Lambda M + k^{(1)}(2\delta,2\delta)A\Lambda A \tag{32}$$

The various derivatives for our new time kernel follow almost exactly the same as before:

$$\partial_1 k^{(1)}(t,s) = \sum_{i=1}^{N-1} (s-t_i)\mathbb{1}_{(t_i,t_{i+1}]^2}(t,s)$$

$$\partial_2 k^{(1)}(t,s) = \sum_{i=1}^{N-1} (t-t_i)\mathbb{1}_{(t_i,t_{i+1}]^2}(t,s)$$

$$\partial_1\partial_2 k^{(1)}(t,s) = \sum_{i=1}^{N-1} \mathbb{1}_{(t_i,t_{i+1}]^2}(t,s)$$

We thus have:

$$\begin{aligned}
k^{(1)}(\delta,\delta) &= \delta^2 \\
\partial_1 k^{(1)}(\delta,\delta) &= \delta \\
\partial_2 k^{(1)}(\delta,\delta) &= \delta \\
\partial_1\partial_2 k^{(1)}(\delta,\delta) &= 1 \\
k^{(1)}(s,2\delta) &= \delta(s-\delta)\mathbb{1}_{(\delta,2\delta]}(s) \\
\partial_2 k^{(1)}(s,2\delta) &= (s-\delta)\mathbb{1}_{(\delta,2\delta]}(s)
\end{aligned}$$

Using this we can now simplify (31) to:

$$(V_1\tilde{A}^*_{2\delta}\boldsymbol{v})(x,s) = (s-\delta)\mathbb{1}_{(\delta,2\delta]}(s)\Phi(x)^*\Lambda Q\boldsymbol{v} \tag{33}$$

and (32) to:

$$\tilde{A}_{2\delta}V_1\tilde{A}^*_{2\delta} = Q\Lambda Q \tag{34}$$

We now need to compute the term $\tilde{A}_{2\delta}m_1$. Doing this will involve first computing $\mathcal{L}m_1$. Rewriting m_1 as follows will help with this:

$$\begin{aligned}
m_1(x,s) &= \Phi(x)^*Q^{-1}\left[M+\delta A\right]\boldsymbol{\gamma} + s\mathbb{1}_{(0,\delta]}(s)\Phi(x)^*Q^{-1}\left[F^{(1)}-A\boldsymbol{\gamma}\right] \\
&= \Phi(x)^*\boldsymbol{\gamma} + s\mathbb{1}_{(0,\delta]}(s)\Phi(x)^*\boldsymbol{c}^{(1)}
\end{aligned}$$

where $\boldsymbol{c}^{(1)} := Q^{-1}\left[F^{(1)}-A\boldsymbol{\gamma}\right]$. We can now compute more easily:

$$(\mathcal{L}m_1)(x,s) = \mathbb{1}_{(0,\delta]}(s)\Phi(x)^*\boldsymbol{c}^{(1)} - \sum_{j=1}^{J}(\boldsymbol{\gamma}+s\mathbb{1}_{(0,\delta]}(s)\boldsymbol{c}^{(1)})_j\nabla\cdot(a(x)\nabla\phi_j(x))$$

This allows us to work out the i-*th* entry of $\tilde{A}_{2\delta}m_1$ as follows:

$$\begin{aligned}
(\tilde{A}_{2\delta}m_1)_i &= I_i(2\delta)\mathcal{L}m_1 \\
&= \int_\Omega \phi_i(x)(\mathcal{L}m_1)(x,2\delta)\mathrm{d}x \\
&= -\int_\Omega \phi_i(x)\sum_{j=1}^{J}\gamma_j\nabla\cdot(a(x)\nabla\phi_j(x))\mathrm{d}x \\
&= \sum_{j=1}^{J}\gamma_j A_{ij} \\
&= (A\boldsymbol{\gamma})_i
\end{aligned}$$

So $\tilde{A}_{2\delta}m_1 = A\boldsymbol{\gamma}$. These results enable us to work out the following:

$$\begin{aligned}
(V_1\tilde{A}^*_{2\delta}(\tilde{A}_{2\delta}V_1\tilde{A}^*_{2\delta})^{-1}F^{(2)})(x,s) &= (s-t_1)\mathbb{1}_{(t_1,t_2]}(s)\Phi(x)^*Q^{-1}F^{(2)} \\
(V_1\tilde{A}^*_{2\delta}(\tilde{A}_{2\delta}V_1\tilde{A}^*_{2\delta})^{-1}\tilde{A}_{2\delta}m_1)(x,s) &= (s-t_1)\mathbb{1}_{(t_1,t_2]}(s)\Phi(x)^*Q^{-1}A\boldsymbol{\gamma}
\end{aligned}$$

Combining these results yields the new mean:

$$\begin{aligned}
m_2(x,s) &= m_1(x,s) + (s-t_1)\mathbb{1}_{(t_1,t_2]}\Phi(x)^*Q^{-1}\left[F^{(2)}-A\boldsymbol{\gamma}\right] \\
&= m_1(x,s) + (s-t_1)\mathbb{1}_{(t_1,t_2]}(s)\Phi(x)^*\boldsymbol{c}^{(2)} \\
&= \Phi(x)^*\boldsymbol{\gamma} + (s-t_0)\mathbb{1}_{(t_0,t_1]}(s)\Phi(x)^*\boldsymbol{c}^{(1)} + (s-t_1)\mathbb{1}_{(t_1,t_2]}(s)\Phi(x)^*\boldsymbol{c}^{(2)} \\
&= \Phi(x)^*\boldsymbol{\gamma} + \sum_{i=1}^{2}(s-t_{i-1})\mathbb{1}_{(t_{i-1},t_i]}(s)\Phi(x)^*\boldsymbol{c}^{(i)}
\end{aligned}$$

Evaluating this new mean at time $t = t_2 = 2\delta$ we obtain:

$$\begin{aligned} m_2(x,t_2) = m_2(x,2\delta) &= \Phi(x)^*\boldsymbol{\gamma} + (2\delta-\delta)\Phi(x)^*\boldsymbol{c}^{(2)} \\ &= \Phi(x)^*\boldsymbol{\gamma} + \delta\Phi(x)^*Q^{-1}\left[F^{(2)} - A\boldsymbol{\gamma}\right] \\ &= \Phi(x)^*Q^{-1}\left[Q\gamma + \delta F^{(2)} - \delta A\boldsymbol{\gamma}\right] \\ &= \Phi(x)^*Q^{-1}\left[(Q-\delta A)\boldsymbol{\gamma} + \delta F^{(2)}\right] \\ &= \Phi(x)^*Q^{-1}\left[M\boldsymbol{\gamma} + \delta F^{(2)}\right] \end{aligned}$$

We now move onto computing V_2. Since V_1 is of the same form as V_0 with a different time kernel, the computations for V_1 can be reused almost identically to compute V_2. We have:

$$\begin{aligned} \tilde{A}_{2\delta}V_1 g &= M\Lambda\int_0^T \partial_1 k^{(1)}(2\delta,s)(\mathcal{I}_s g)\mathrm{d}s + A\Lambda\int_0^T k^{(1)}(2\delta,s)(\mathcal{I}_s g)\mathrm{d}s \\ &= M\Lambda\int_0^T (s-t_1)\mathbb{1}_{(t_1,t_2]}(s)(\mathcal{I}_s g)\mathrm{d}s + \delta A\Lambda\int_0^T (s-t_1)\mathbb{1}_{(t_1,t_2]}(s)(\mathcal{I}_s g)\mathrm{d}s \\ &= M\Lambda\int_{t_1}^{t_2} (s-t_1)(\mathcal{I}_s g)\mathrm{d}s + \delta A\Lambda\int_{t_1}^{t_2} (s-t_1)(\mathcal{I}_s g)\mathrm{d}s \\ &= M\Lambda\boldsymbol{\nu}_g^{(1)} + \delta A\Lambda\boldsymbol{\nu}_g^{(1)} \\ &= Q\Lambda\boldsymbol{\nu}_g^{(1)} \end{aligned}$$

Having worked this out we can now compute the second term in the formula for V_2 as follows:

$$\begin{aligned} (V_1\tilde{A}_{2\delta}^*(\tilde{A}_{2\delta}V_1\tilde{A}_{2\delta}^*)^{-1}\tilde{A}_{2\delta}V_1 g)(x,t) &= (V_1\tilde{A}_{2\delta}^*(Q\Lambda Q)^{-1}Q\Lambda\boldsymbol{\nu}_g^{(1)})(x,t) \\ &= (t-t_1)\mathbb{1}_{(t_1,t_2]}(t)\Phi(x)^*\Lambda Q Q^{-1}\Lambda^{-1}Q^{-1}Q\Lambda\boldsymbol{\nu}_g^{(1)} \\ &= (t-t_1)\mathbb{1}_{(t_1,t_2]}(t)\Phi(x)^*\Lambda\boldsymbol{\nu}_g^{(1)} \end{aligned}$$

Thus, we can conclude that V_2 acts on functions g as follows:

$$(V_2 g)(x,t) = \sum_{i=2}^{N-1}(t-t_i)\mathbb{1}_{(t_i,t_{i+1}]}(t)\Phi(x)^*\Lambda\boldsymbol{\nu}_g^{(i)} = \int_\Omega\int_0^T k^{(2)}_{x,t,y,s}g(y,s)\mathrm{d}s\mathrm{d}y \tag{35}$$

where the kernel is now:

$$k^{(2)}_{x,t,y,s} := \sum_{i=2}^{J}\lambda_j\phi_j(x)\phi_j(y)k^{(2)}(t,s) \tag{36}$$

with time kernel:

$$k^{(2)}(t,s) := \sum_{i=2}^{N-1}(t-t_i)(s-t_i)\mathbb{1}_{(t_i,t_{i+1}]^2}(t,s) \tag{37}$$

We again can see that the form of the covariance operator remains the same as the previous one. We can now hypothesize a formula for conditioning up to time t_p for $p \in \{1,\ldots,N\}$:

$$u|\{\tilde{A}_{t_1}u = F^{(1)},\ldots,\tilde{A}_{t_p}u = F^{(p)}, f\} \sim \mathcal{N}(m_p, V_p) \tag{38}$$

where $F^{(p)} := \mathcal{I}_{t_p} f = \mathcal{I}_{p\delta} f$ and where the mean and covariance are given by:

$$m_p(x,t) := \Phi(x)^* \boldsymbol{\gamma} + \sum_{i=1}^{p} (t - t_{i-1}) \mathbb{1}_{(t_{i-1},t_i]}(t) \Phi(x)^* \boldsymbol{c}^{(i)} \tag{39}$$

$$(V_p g)(x,t) := \int_\Omega \int_0^T k^{(p)}_{x,t,y,s} g(y,s) \mathrm{d}s \mathrm{d}y \tag{40}$$

$$\boldsymbol{c}^{(i)} := Q^{-1} \left[F^{(i)} - A\boldsymbol{\gamma} \right] \tag{41}$$

$$k^{(p)}_{x,t,y,s} := \sum_{i=p}^{N-1} \lambda_j \phi_j(x) \phi_j(y) k^{(p)}(t,s) \tag{42}$$

$$k^{(p)}(t,s) := \sum_{i=p}^{N-1} (t - t_i)(s - t_i) \mathbb{1}_{(t_i,t_{i+1}]^2}(t,s) \tag{43}$$

This can probably be shown by induction. The above implies that conditioning at all N time points will yield the following degenerate Gaussian distribution:

$$u | \{ \tilde{A}_{t_1} u = F^{(1)}, \ldots, \tilde{A}_{t_N} u = F^{(N)}, f \} \sim \mathcal{N}(m_N, 0) = \delta_{m_N} \tag{44}$$

since $V_N = 0$. This is a Dirac point mass located at the function:

$$m_N(x,t) = \Phi(x)^* \boldsymbol{\gamma} + \sum_{i=1}^{N} (t - t_{i-1}) \mathbb{1}_{(t_{i-1},t_i]}(t) \Phi(x)^* \boldsymbol{c}^{(i)} \tag{45}$$

We now want to perform the marginalisation over the RHS noise term f. In order to do this it will help to rewrite m_N as follows:

$$\begin{aligned}
m_N(x,t) &= \Phi(x)^* \boldsymbol{\gamma} + \sum_{i=1}^{N} (t - t_{i-1}) \mathbb{1}_{(t_{i-1},t_i]}(t) \Phi(x)^* Q^{-1} \left[F^{(i)} - A\boldsymbol{\gamma} \right] \\
&= \Phi(x)^* \boldsymbol{\gamma} - \sum_{i=1}^{N} (t - t_{i-1}) \mathbb{1}_{(t_{i-1},t_i]}(t) \Phi(x)^* Q^{-1} A \boldsymbol{\gamma} + \sum_{i=1}^{N} (t - t_{i-1}) \mathbb{1}_{(t_{i-1},t_i]}(t) \Phi(x)^* Q^{-1} \mathcal{I}_{t_i} f \\
&= \Phi(x)^* Q^{-1} \left[Q - \sum_{i=1}^{N} (t - t_{i-1}) \mathbb{1}_{(t_{i-1},t_i]}(t) A \right] \boldsymbol{\gamma} + \left[\sum_{i=1}^{N} (t - t_{i-1}) \mathbb{1}_{(t_{i-1},t_i]}(t) \Phi(x)^* Q^{-1} \mathcal{I}_i \right] f \\
&= c(x,t) + (Lf)(x,t)
\end{aligned}$$

i.e. $m_N = Lf + c$ where:

$$c(x,t) := \Phi(x)^* Q^{-1} \left[Q - \sum_{i=1}^{N} (t - t_{i-1}) \mathbb{1}_{(t_{i-1},t_i]}(t) A \right] \boldsymbol{\gamma} \tag{46}$$

$$L := \sum_{i=1}^{N} L_{(i)} \tag{47}$$

$$L_{(i)} := \Psi^*_{(i)} Q^{-1} \mathcal{I}_{t_i} \tag{48}$$

where the $\{\Psi^*_{(i)}\}_{i=1}^N$ are operators from $\mathbb{R}^J \to L^2(\Omega \times [0,T])$ given by:

$$(\Psi^*_{(i)} \boldsymbol{v})(x,t) := (t - t_{i-1}) \mathbb{1}_{(t_{i-1},t_i]}(t) \Phi(x)^* \boldsymbol{v} \tag{49}$$

It is easy to show that the adjoint of $\Psi^*_{(i)}$ is $\Psi_{(i)}$ which is defined as:

$$\Psi_{(i)} g := \boldsymbol{\nu}_g^{(i-1)} \tag{50}$$

Note: see equation (23) *for the definition of* $\boldsymbol{\nu}_g^{(i)}$.

We will now marginalise over f in order to obtain the averaged conditional distribution. To do this we will need the following Lemma (which we prove below):

Lemma 0.1. Let $f \sim \mathcal{N}(\bar{f}, K)$ where we assume that this Gaussian measure is on a Hilbert space of functions $\mathcal{H}_1 \subset \mathbb{R}^{\mathcal{X}}$. Suppose that for a fixed realisation of f we have

$$y|f \sim \mathcal{N}(Lf + c, V)$$

where L is a bounded linear operator from $\mathcal{H}_1$ to another Hilbert space $\mathcal{H}_2 \subset \mathbb{R}^{\mathcal{X}}$ (so y lies in $\mathcal{H}_2$), c is a deterministic function and the covariance opertor V does not depend on f. Then marginalizing over f yields:

$$y \sim \mathcal{N}(L\bar{f} + c, LKL^* + V)$$

as the averaged distribution of y.

Proof: The fact that $y|f \sim \mathcal{N}(Lf + c, V)$ is equivalent to saying that:

$$y = Lf + c + \tilde{y}$$

where $\tilde{y} \sim \mathcal{N}(0, V)$ is independent of f. Thus we have:

$$\begin{pmatrix} f \\ \tilde{y} \end{pmatrix} = \mathcal{N}\left(\begin{pmatrix} \bar{f} \\ 0 \end{pmatrix}, \begin{pmatrix} K & 0 \\ 0 & V \end{pmatrix}\right)$$

Since we can write:

$$y = \begin{pmatrix} L & 1 \end{pmatrix} \begin{pmatrix} f \\ \tilde{y} \end{pmatrix} + c$$

we deduce:

$$y \sim \mathcal{N}\left(L\bar{f} + c, \begin{pmatrix} L & 1 \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} L^* \\ 1 \end{pmatrix}\right) = \mathcal{N}(L\bar{f} + c, LKL^* + V)$$

as required. ∎

We can now perform the marginalisation over f noting that $V_N = 0$ does not depend on f to obtain:

$$\int u|\{\tilde{A}_{t_1} u = F^{(1)}, \ldots, \tilde{A}_{t_N} u = F^{(N)}, f\} \mathrm{d}f \sim \mathcal{N}(L\bar{f} + c, LKL^*) \tag{51}$$