We now focus on the following time-dependent PDE:

$$\mathcal{L}u(x,t) := \partial_t u(x,t) - \nabla\cdot(a(x)\nabla u(x,t)) = f(x,t), \quad x\in\Omega,\ t\in[0,T] \tag{1}$$

$$u(x,t) = 0, \qquad x\in\partial\Omega,\ t\in[0,T] \tag{2}$$

$$u(x,0) = u_0(x), \qquad x\in\Omega \tag{3}$$

where the function $f\sim\mathcal{N}(\bar{f},K)$. The solution u is thus also random.

We will set up a prior on the solution u to the above problem. To do so we first let $v_h\in S_h$ be some approximation of the initial condition $u_0(x)$ in the FEM space S_h. To be more specific we will assume that $v_h(x) = \Phi(x)^*\boldsymbol{\gamma} := \sum_{i=1}^J \phi_i(x)\gamma_i$. Note that $\Phi(x) := (\phi_1(x),\dots,\phi_J(x))^T$. We take the prior on u to be:

$$u\sim\mathcal{N}(m_0,V_0) \tag{4}$$

where $m_0(x,t) := v_h(x) = \Phi(x)^*\boldsymbol{\gamma}$ (m_0 is constant in time). The prior covariance operator V_0 is defined as:

$$(V_0 g)(x,t) = \int_\Omega\int_0^T k^{(0)}_{x,t,y,s}\, g(y,s)\mathrm{d}s\mathrm{d}y \tag{5}$$

where $k^{(0)}_{x,t,y,s}$ is defined as follows:

$$k^{(0)}_{x,t,y,s} := \sum_{i=1}^J \lambda_j\phi_j(x)\phi_j(y)k^{(0)}(t,s) \tag{6}$$

We now introduce a uniform time grid:

$$t_n = n\delta,\ \ n=0,1,\dots,N$$

where δ is the spacing between consecutive times and $N=\frac{T}{\delta}$ (*assume that N is an integer*). The time kernel $k^{(0)}(t,s)$ will be taken to be:

$$k^{(0)}(t,s) := \sum_{i=0}^{N-1} l^{(i)}(t)l^{(i)}(s) \tag{7}$$

Where the functions $\{l^{(i)}\}_{i=0}^{N-1}$ are defined as follows:

$$l^{(i)}(t) = \begin{cases} (t-t_i)\mathbb{1}_{(t_i,t_{i+1}]}(t) + \delta\mathbb{1}_{(t_{i+1},t_N]}(t), & i=0,\dots,N-2 \\ (t-t_{N-1})\mathbb{1}_{(t_{N-1},t_N]}, & i=N-1 \end{cases} \tag{8}$$

i.e. for $i=0,\dots,N-2$ we have:

$$l^{(i)}(t) = \begin{cases} 0, & t\le t_i \\ t-t_i, & t_i<t\le t_{i+1} \\ \delta, & t>t_{i+1} \end{cases} \tag{9}$$

while for for $i=N-1$ we have:

$$l^{(i)}(t) = l^{(N-1)}(t) = \begin{cases} 0, & t\le t_{N-1} \\ t-t_i, & t>t_{N-1} \end{cases} \tag{10}$$

note: we are working only with times in the interval $[0,T]$ here.

We now introduce the following information operators $\mathcal{I}_s := (I_1(s),\dots,I_J(s))^T$ where:

$$I_i(s)g := \int_\Omega \phi_i(x)g(x,s)\mathrm{d}x \tag{11}$$

To update our belief in the distribution of u we will condition on the following events: $\mathcal{I}_{t_i}\mathcal{L}u = \mathcal{I}_{t_i}f =: F^{(i)}$ sequentially for $i=1,\dots,N$. Letting $\tilde{A}_t := \mathcal{I}_t\mathcal{L}$ we seek, for a fixed realisation of f (and hence of the $\{F^{(i)}\}$), the following conditional distributions:

$$u|\{\tilde{A}_{t_1}u = F^{(1)},\dots,\tilde{A}_{t_p}u = F^{(p)}, f\}\sim\mathcal{N}(m_p,V_p) \tag{12}$$

for $p\in\{1,\dots,N\}$. We make the following claim:

Proposition 0.1. With the prior specified as above we have that m_p and V_p are given as follows:

$$m_p(x,t) := \Phi(x)^*\boldsymbol{\gamma} + \sum_{i=1}^{p} l^{(i-1)}(t)\Phi(x)^*\boldsymbol{c}^{(i)} \tag{13}$$

$$(V_p g)(x,t) := \int_\Omega \int_0^T k^{(p)}_{x,t,y,s} g(y,s)\mathrm{d}s\mathrm{d}y \tag{14}$$

$$\boldsymbol{c}^{(i)} := Q^{-1}\left[F^{(i)} - A\boldsymbol{\gamma}_{i-1}\right] \text{ for } i = 1,\ldots,p \tag{15}$$

$$k^{(p)}_{x,t,y,s} := \sum_{j=1}^{J} \lambda_j \phi_j(x)\phi_j(y) k^{(p)}(t,s) \tag{16}$$

$$k^{(p)}(t,s) := \sum_{i=p}^{N-1} l^{(i)}(t) l^{(i)}(s) \tag{17}$$

and where the $\{\boldsymbol{\gamma}_i\}_{i=0}^p$ are defined recursively by:

$$\boldsymbol{\gamma}_0 := \boldsymbol{\gamma}, \tag{18}$$

$$\boldsymbol{\gamma}_i := Q^{-1}\left[M\boldsymbol{\gamma}_{i-1} + \delta F^{(i)}\right] \text{ for } i \geq 1 \tag{19}$$

The matrices M and A are the Galerkin Mass and Stiffness matrices respectively, i.e. $M_{ij} := \int_\Omega \phi_i(x)\phi_j(x)\mathrm{d}x$ and $A_{ij} := \int_\Omega a(x)\nabla\phi_i(x)\cdot\nabla\phi_j(x)\mathrm{d}x$. The matrix $Q := (M + \delta A)$. Further, we have that evaluating the conditional mean m_p at time t_p yields the following:

$$m_p(x,t_p) = \Phi(x)^*\boldsymbol{\gamma}_p \tag{20}$$

Thus, we can see that this choice of prior yields (for a fixed realisation of f) what the classical Backward-Euler Galerkin method yields.

Proof: We proceed via proof by induction.

For $p = 1$ it follows that the distribution of $u|\{\tilde{A}_\delta u = F^{(1)}, f\}$ is Gaussian $\mathcal{N}(m_1, V_1)$ by considering the following joint distribution:

$$\begin{pmatrix} u \\ \tilde{A}_\delta u \end{pmatrix} = \begin{pmatrix} I \\ \tilde{A}_\delta \end{pmatrix} u \sim \mathcal{N}\left(\begin{pmatrix} m_o \\ \tilde{A}_\delta m_0 \end{pmatrix}, \begin{pmatrix} V_0 & V_0\tilde{A}_\delta^* \\ \tilde{A}_\delta V_0 & \tilde{A}_\delta V_0 \tilde{A}_\delta^* \end{pmatrix}\right)$$

It follows that the conditional distribution is Gaussian and the mean and covariance are given by:

$$m_1 = m_0 + V_0\tilde{A}_\delta^*(\tilde{A}_\delta V_0 \tilde{A}_\delta^*)^{-1}(F^{(1)} - \tilde{A}_\delta m_0) \tag{21}$$

$$V_1 = V_0 - V_0\tilde{A}_\delta^*(\tilde{A}_\delta V_0\tilde{A}_\delta^*)^{-1}\tilde{A}_\delta V_0 \tag{22}$$

To make progress we must now start computing various terms needed for our mean and covariance update rules. We start with $V_0\tilde{A}_\delta^*$. We have:

$$V_0\tilde{A}_\delta^* = V_0\mathcal{L}^*(I_1(\delta)^*,\ldots,I_J(\delta)^*)$$

We can thus see that we need to be able to compute terms of the form $V_0\mathcal{L}^* I_i(\delta)^* = V_0(I_i(\delta)\mathcal{L})^*$. Now since the operator $I_i(\delta)\mathcal{L}$ takes in a function on $\Omega\times[0,T]$ and outputs a real number its adjoint should take in a real number and output a function on $\Omega\times[0,T]$. This adjoint should satisfy the following relation:

$$\alpha(I_i(\delta)\mathcal{L}g) = \int_\Omega\int_0^T ((I_i(\delta)\mathcal{L})^*\alpha)(x,t)g(x,t)\mathrm{d}t\mathrm{d}x \ \forall g,\ \forall\alpha\in\mathbb{R} \tag{23}$$

Using this we can now compute:

$$\begin{aligned}(V_0(I_i(\delta)\mathcal{L})^*\alpha)(x,s) &= \int_\Omega\int_0^T k^{(0)}_{x,s,y,w}((I_i(\delta)\mathcal{L})^*\alpha)(y,w)\mathrm{d}w\mathrm{d}y \\ &= \alpha(I_i(\delta)\mathcal{L}k^{(0)}_{x,s,\cdot,\cdot}) \\ &= \alpha\int_\Omega \phi_i(y)(\mathcal{L}k^{(0)}_{x,s,\cdot,\cdot})(y,\delta)\mathrm{d}y\end{aligned}$$

We now work out $(\mathcal{L}k^{(0)}_{x,s,\cdot,\cdot})(y,\delta)$ taking care to remember that x, s are fixed and so $\mathcal{L}$ acts on the variables y, δ:

$$\begin{aligned}(\mathcal{L}k^{(0)}_{x,s,\cdot,\cdot})(y,\delta) &= \partial_2 k^{(0)}(s,\delta)\sum_{j=1}^{J}\lambda_j\phi_j(x)\phi_j(y) - k^{(0)}(s,\delta)\nabla_y\cdot\left(a(y)\nabla_y\sum_{j=1}^{J}(\lambda_j\phi_j(x)\phi_j(y)\right) \\ &= \partial_2 k^{(0)}(s,\delta)\sum_{j=1}^{J}\lambda_j\phi_j(x)\phi_j(y) - k^{(0)}(s,\delta)\sum_{j=1}^{J}\lambda_j\phi_j(x)\nabla_y\cdot(a(y)\nabla_y\phi_j(y))\end{aligned}$$

So we can now compute:

$$\begin{aligned}(V_0(I_i(\delta)\mathcal{L})^*\alpha)(x,s) &= \alpha\int_\Omega \phi_i(y)\partial_2 k^{(0)}(s,\delta)\sum_{j=1}^{J}\lambda_j\phi_j(x)\phi_j(y)\mathrm{d}y - \alpha\int_\Omega \phi_i(y)k^{(0)}(s,\delta)\sum_{j=1}^{J}\lambda_j\phi_j(x)\nabla_y\cdot(a(y)\nabla_y\phi_j(y))\,\mathrm{d}y \\ &= \alpha\partial_2 k^{(0)}(s,\delta)\sum_{j=1}^{J}\lambda_j\phi_j(x)M_{ji} + \alpha k^{(0)}(s,\delta)\sum_{j=1}^{J}\lambda_j\phi_j(x)A_{ji}\end{aligned}$$

Using this result we can deduce that:

$$(V_0\tilde{A}^*_\delta\boldsymbol{v})(x,s) = \partial_2 k^{(0)}(s,\delta)\Phi(x)^*\Lambda M\boldsymbol{v} + k^{(0)}(s,\delta)\Phi(x)^*\Lambda A\boldsymbol{v} \tag{24}$$

for any $\boldsymbol{v}\in\mathbb{R}^J$, where $\Lambda = \mathrm{diag}\{\lambda_i\}_{i=1}^J$.

For our time kernel we can deduce:

$$\partial_1 k^{(0)}(t,s) = \sum_{i=0}^{N-1} l^{(i)\prime}(t)l^{(i)}(s) \tag{25}$$

$$\partial_2 k^{(0)}(t,s) = \sum_{i=0}^{N-1} l^{(i)}(t)l^{(i)\prime}(s) \tag{26}$$

$$\partial_1\partial_2 k^{(0)}(t,s) = \sum_{i=0}^{N-1} l^{(i)\prime}(t)l^{(i)\prime}(s) \tag{27}$$

$$\tag{28}$$

We also have:

$$l^{(i)\prime}(t) = \mathbb{1}_{(t_i,t_{i+1}]}(t) \tag{29}$$

for all $i = 0,\dots,N-1$. (*Note: this includes even the case of* $i = N-1$.) Noting that the kernel $k^{(0)}$ is symmetric we have:

$$\begin{aligned}k^{(0)}(s,\delta) &= k^{(0)}(\delta,s) \\ &= \sum_{i=0}^{N-1} l^{(i)}(\delta)l^{(i)}(s) \\ &= \sum_{i=0}^{N-1}\delta\delta_{i,0}l^{(i)}(s) \\ &= \delta l^{(0)}(s)\end{aligned}$$

and

$$\begin{aligned}\partial_2 k^{(0)}(s,\delta) &= \partial_1 k^{(0)}(\delta,s) \\ &= \sum_{i=0}^{N-1} l^{(i)\prime}(\delta)l^{(i)}(s) \\ &= \sum_{i=0}^{N-1}\delta_{i,0}l^{(i)}(s) \\ &= l^{(0)}(s)\end{aligned}$$

where we have used the following properties of the functions $\{l^{(i)}\}$ which can easily be shown:

$$l^{(i)}(t_j) = \delta \cdot \delta_{i,j-1} \tag{30}$$

$$l^{(i)\prime}(t_j) = \delta_{i,j-1} \tag{31}$$

for $i = 0, \dots, N-1$ and $j = 1, \dots, N$.

We can now simplify (24) to:

$$\begin{aligned}(V_0 \tilde{A}_\delta^* \boldsymbol{v})(x,s) &= l^{(0)}(s)\Phi(x)^*\Lambda M\boldsymbol{v} + \delta l^{(0)}(s)\Phi(x)^*\Lambda A\boldsymbol{v} \\ &= l^{(0)}(s)\Phi(x)^*\Lambda(M+\delta A)\boldsymbol{v} \\ &= l^{(0)}(s)\Phi(x)^*\Lambda Q\boldsymbol{v}\end{aligned} \tag{32}$$

We now move onto computing:

$$\begin{aligned}\tilde{A}_\delta V_0 \tilde{A}_\delta^* &= \mathcal{I}_\delta \mathcal{L} V_0 \mathcal{L}^* \mathcal{I}_\delta^* \\ &= \begin{pmatrix} I_1(\delta) \\ \vdots \\ I_J(\delta)\end{pmatrix} \mathcal{L} V_0 \mathcal{L}^* \begin{pmatrix} I_1(\delta)^* & \dots & I_J(\delta)^*\end{pmatrix}\end{aligned}$$

This operator has ij-th entry which is given by:

$$\begin{aligned}(\tilde{A}_\delta V_0 \tilde{A}_\delta^*)_{ij}\alpha &= I_i(\delta)\mathcal{L}V_0\mathcal{L}^* I_j(\delta)^*\alpha \\ &= \int_\Omega \phi_i(x)\left[(\mathcal{L}V_0(I_j(\delta)\mathcal{L})^*\alpha)(x,\delta)\right]\mathrm{d}x \\ &= \int_\Omega \phi_i(x)\Bigg[\alpha\partial_1\partial_2 k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l\phi_l(x)M_{lj} + \alpha\partial_1 k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l\phi_l(x)A_{lj} \\ &- \alpha\partial_2 k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l M_{lj}\nabla\cdot(a(x)\nabla\phi_l(x)) - \alpha k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l A_{lj}\nabla\cdot(a(x)\nabla\phi_l(x))\Bigg]\mathrm{d}x \\ &= \alpha\partial_1\partial_2 k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l M_{il}M_{lj} + \alpha\partial_1 k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l M_{il}A_{lj} \\ &+ \alpha\partial_2 k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l M_{lj}A_{il} + \alpha k^{(0)}(\delta,\delta)\sum_{l=1}^J \lambda_l A_{il}A_{lj} \\ &= \alpha\partial_1\partial_2 k^{(0)}(\delta,\delta)(M\Lambda M)_{ij} + \alpha\partial_1 k^{(0)}(\delta,\delta)(M\Lambda A)_{ij} + \alpha\partial_2 k^{(0)}(\delta,\delta)(A\Lambda M)_{ij} + \alpha k^{(0)}(\delta,\delta)(A\Lambda A)_{ij}\end{aligned}$$

We can thus conclude that $\tilde{A}_\delta V_0 \tilde{A}_\delta^*$ is the $J \times J$ matrix given by:

$$\tilde{A}_\delta V_0 \tilde{A}_\delta^* = \partial_1\partial_2 k^{(0)}(\delta,\delta)M\Lambda M + \partial_1 k^{(0)}(\delta,\delta)M\Lambda A + \partial_2 k^{(0)}(\delta,\delta)A\Lambda M + k^{(0)}(\delta,\delta)A\Lambda A \tag{33}$$

Using our time kernel and its derivatives we can easily conclude:

$$\begin{aligned}k^{(0)}(\delta,\delta) &= \delta^2 \\ \partial_1 k^{(0)}(\delta,\delta) &= \delta \\ \partial_2 k^{(0)}(\delta,\delta) &= \delta \\ \partial_1\partial_2 k^{(0)}(\delta,\delta) &= 1\end{aligned}$$

And thus:

$$\begin{aligned}\tilde{A}_\delta V_0 \tilde{A}_\delta^* &= M\Lambda M + \delta M\Lambda A + \delta A\Lambda M + \delta^2 A\Lambda A \\ &= M\Lambda(M+\delta A) + \delta A\Lambda(M+\delta A) \\ &= (M\Lambda + \delta A\Lambda)(M+\delta A) \\ &= (M+\delta A)\Lambda(M+\delta A) = Q\Lambda Q\end{aligned}$$

We can now make progress with the mean update equation. We first work out the following term using (32):

$$\begin{aligned}(V_0\tilde{A}_\delta^*(\tilde{A}_\delta V_0\tilde{A}_\delta^*)^{-1}F^{(1)})(x,s) &= l^{(0)}(s)\Phi(x)^*\Lambda Q(Q\Lambda Q)^{-1}F^{(1)} \\ &= l^{(0)}(s)\Phi(x)^*\Lambda QQ^{-1}\Lambda^{-1}Q^{-1}F^{(1)} \\ &= l^{(0)}(s)\Phi(x)^*Q^{-1}F^{(1)}\end{aligned}$$

For the other term involving m_0 in the mean update equation we must first work out $\tilde{A}_\delta m_0 = \mathcal{I}_\delta \mathcal{L} m_0$. To do this we compute:

$$\begin{aligned}(\mathcal{L}m_0)(x,t) &= \partial_t m_0 - \nabla\cdot(a(x)\nabla m_0(x,t)) \\ &= -\nabla(a(x)\nabla\Phi(x)^*\boldsymbol{\gamma}) \\ &= -\sum_{j=1}^J \gamma_j \nabla\cdot(a(x)\nabla\phi_j(x))\end{aligned}$$

Thus, the i-th entry of $\tilde{A}_\delta m_0$ can be computed as:

$$\begin{aligned}(\tilde{A}_\delta m_0)_i &= I_i(\delta)\mathcal{L}m_0 \\ &= \int_\Omega \phi_i(x)\left(-\sum_{j=1}^J \gamma_j\nabla\cdot(a(x)\nabla\phi_j(x))\right)\mathrm{d}x \\ &= \sum_{j=1}^J \gamma_j A_{ij} = (A\boldsymbol{\gamma})_i\end{aligned}$$

So $\tilde{A}_\delta m_0 = A\boldsymbol{\gamma}$. We can now compute the second term involving m_0 as follows (again using (32)):

$$\begin{aligned}((V_0\tilde{A}_\delta^*(\tilde{A}_\delta V_0\tilde{A}_\delta^*)^{-1}\tilde{A}_\delta)m_0)(x,s) &= (V_0\tilde{A}_\delta^*(Q\Lambda Q)^{-1}A\boldsymbol{\gamma})(x,s) \\ &= l^{(0)}(s)\Phi(x)^*\Lambda QQ^{-1}\Lambda^{-1}Q^{-1}A\boldsymbol{\gamma} \\ &= l^{(0)}(s)\Phi(x)^*Q^{-1}A\boldsymbol{\gamma}\end{aligned}$$

Using (21) we can now compute:

$$\begin{aligned}m_1(x,s) &= \Phi(x)^*\boldsymbol{\gamma} + l^{(0)}(s)\Phi(x)^*Q^{-1}F^{(1)} - l^{(0)}(s)\Phi(x)^*Q^{-1}A\boldsymbol{\gamma} \\ &= \Phi(x)^*\boldsymbol{\gamma} + l^{(0)}(s)\Phi(x)^*Q^{-1}\left[F^{(1)} - A\boldsymbol{\gamma}\right] \\ &= \Phi(x)^*\boldsymbol{\gamma} + l^{(0)}(s)\Phi(x)^*\boldsymbol{c}^{(1)}\end{aligned}$$

From this we can note:

$$\begin{aligned}m_1(x,t_1) &= \Phi(x)^*\boldsymbol{\gamma} + \delta\Phi(x)^*\boldsymbol{c}^{(1)} \\ &= \Phi(x)^*\boldsymbol{\gamma} + \delta\Phi(x)^*Q^{-1}\left[F^{(1)} - A\boldsymbol{\gamma}\right] \\ &= \Phi(x)^*Q^{-1}\left[Q\boldsymbol{\gamma} + \delta F^{(1)} - \delta A\boldsymbol{\gamma}\right] \\ &= \Phi(x)^*Q^{-1}\left[M\boldsymbol{\gamma} + \delta F^{(1)}\right] \\ &= \Phi(x)^*\boldsymbol{\gamma}_1\end{aligned}$$

We now move on to computing the covariance V_1. To do this we must first work out $\tilde{A}_\delta V_0$. Computing this involves determining how $I_j(\delta)\mathcal{L}V_0$ acts on functions g for $j = 1, \dots, J$. We have:

$$I_j(\delta)\mathcal{L}V_0 g = \int_\Omega \phi_j(x)(\mathcal{L}V_0 g)(x,\delta)\mathrm{d}x$$

Now recalling that $V_0 g(x,\delta) = \int_\Omega \int_0^T k^{(0)}_{x,\delta,y,s} g(y,s)\mathrm{d}s\mathrm{d}y$ we deduce:

$$
\begin{aligned}
(\mathcal{L}V_0 g)(x,\delta) &= \int_\Omega \int_0^T (\mathcal{L}k^{(0)}_{\cdot,\cdot,y,s})(x,\delta) g(y,s)\mathrm{d}s\mathrm{d}y \\
&= \int_\Omega \int_0^T \left(\partial_1 k^{(0)}(\delta,s) \sum_{i=1}^J \lambda_i \phi_i(x)\phi_i(y) - k^{(0)}(\delta,s) \sum_{i=1}^J \lambda_i \nabla_x \cdot (a(x)\nabla_x \phi_i(x))\phi_i(y) \right) g(y,s)\mathrm{d}s\mathrm{d}y
\end{aligned}
$$

We can now perform the integration to obtain:

$$
\begin{aligned}
I_j(\delta)\mathcal{L}V_0 g &= \int_\Omega \phi_j(x) \left(\int_\Omega \int_0^T \left(\partial_1 k^{(0)}(\delta,s) \sum_{i=1}^J \lambda_i \phi_i(x)\phi_i(y) - k^{(0)}(\delta,s) \sum_{i=1}^J \lambda_i \nabla_x \cdot (a(x)\nabla_x \phi_i(x))\phi_i(y) \right) g(y,s)\mathrm{d}s\mathrm{d}y \right) \mathrm{d}x \\
&= \int_\Omega \int_0^T \left(\partial_1 k^{(0)}(\delta,s) \sum_{i=1}^J \lambda_i M_{ij} \phi_i(y) g(y,s) + k^{(0)}(\delta,s) \sum_{i=1}^J \lambda_i A_{ij} \phi_i(y) g(y,s) \right) \mathrm{d}s\mathrm{d}y \\
&= \sum_{i=1}^J \lambda_i \int_\Omega \int_0^T (\partial_1 k^{(0)}(\delta,s) M_{ij} + k^{(0)}(\delta,s) A_{ij})\phi_i(y) g(y,s)\mathrm{d}s\mathrm{d}y \\
&= \sum_{i=1}^J \lambda_i \left[\int_0^T M_{ij} \partial_1 k^{(0)}(\delta,s)(I_i(s)g)\mathrm{d}s + \int_0^T A_{ij} k^{(0)}(\delta,s)(I_i(s)g)\mathrm{d}s \right] \\
&= \sum_{i=1}^J \lambda_i \left[M_{ij} \left(\int_0^T \partial_1 k^{(0)}(\delta,s)(\mathcal{I}_s g)\mathrm{d}s \right)_i + A_{ij} \left(\int_0^T k^{(0)}(\delta,s)(\mathcal{I}_s g)\mathrm{d}s \right)_i \right] \\
&= \left(M\Lambda \int_0^T \partial_1 k^{(0)}(\delta,s)(\mathcal{I}_s g)\mathrm{d}s + A\Lambda \int_0^T k^{(0)}(\delta,s)(\mathcal{I}_s g)\mathrm{d}s \right)_j
\end{aligned}
$$

Thus we can deduce:

$$
\tilde{A}_\delta V_0 g = M\Lambda \int_0^T \partial_1 k^{(0)}(\delta,s)(\mathcal{I}_s g)\mathrm{d}s + A\Lambda \int_0^T k^{(0)}(\delta,s)(\mathcal{I}_s g)\mathrm{d}s \tag{34}
$$

We now utilise the specific form of the time kernel to simplify this to:

$$
\begin{aligned}
\tilde{A}_\delta V_0 g &= M\Lambda \int_0^T l^{(0)}(s)(\mathcal{I}_s g)\mathrm{d}s + \delta A\Lambda \int_0^T l^{(0)}(s)(\mathcal{I}_s g)\mathrm{d}s \qquad (35) \\
&= (M + \delta A)\Lambda \int_0^T l^{(0)}(s)(\mathcal{I}_s g)\mathrm{d}s \qquad (36) \\
&= Q\Lambda \boldsymbol{\nu}_g^{(0)} \qquad (37)
\end{aligned}
$$

where $\boldsymbol{\nu}_g^{(i)} := \int_0^T l^{(i)}(s)(\mathcal{I}_s g)\mathrm{d}s$ for $i = 0, \ldots, N-1$.

We can use this to now compute:

$$
\begin{aligned}
(V_0 \tilde{A}_\delta^* (\tilde{A}_\delta V_0 \tilde{A}_\delta^*)^{-1} \tilde{A}_\delta V_0 g)(x,t) &= (V_0 \tilde{A}_\delta^* (Q\Lambda Q)^{-1} Q\Lambda \boldsymbol{\nu}_g^{(0)})(x,t) \\
&= l^{(0)}(t)\Phi(x)^* \Lambda Q Q^{-1} \Lambda^{-1} Q^{-1} Q\Lambda \boldsymbol{\nu}_g^{(0)} \\
&= l^{(0)}(t)\Phi(x)^* \Lambda \boldsymbol{\nu}_g^{(0)}
\end{aligned}
$$

One can also easily show that the action of V_0 on functions can be rewritten as follows:

$$
(V_0 g)(x,t) = \sum_{i=0}^{N-1} l^{(i)}(t)\Phi(x)^* \Lambda \boldsymbol{\nu}_g^{(i)} \tag{38}
$$

Thus, we can conclude using (22) that:

$$
\begin{aligned}
(V_1 g)(x,t) &= \sum_{i=1}^{N-1} l^{(i)}(t)\Phi(x)^* \Lambda \boldsymbol{\nu}_g^{(i)} \\
&= \int_\Omega \int_0^T k_{x,t,y,s}^{(1)} g(y,s) \mathrm{d}s \mathrm{d}y
\end{aligned} \tag{39}
$$

where:

$$
k_{x,t,y,s}^{(1)} := \sum_{j=1}^{J} \lambda_j \phi_j(x) \phi_j(y) k^{(1)}(t,s) \tag{40}
$$

$$
k^{(1)}(t,s) := \sum_{i=1}^{N-1} l^{(i)}(t) l^{(i)}(s) \tag{41}
$$

We thus see that the result holds for $p = 1$.