Defn:

A random function $\underline{u} : \mathbb{R}^k \to \mathbb{R}^d$ is a *multivariate GP* on $\mathbb{R}^k$,

$$\underline{u} \sim MGP(\underline{\mu}, \kappa, \Omega)$$

with *mean function* $\underline{\mu} : \mathbb{R}^k \to \mathbb{R}^d$,

kernel function $\kappa : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ &

positive semi-definite *parameter covariance matrix*

$\Omega \in \mathbb{R}^{d \times d}$ if the vectorization of any finite collection of evaluations of $\underline{u}$ at given locations have a joint multivariate

Gaussian distribution. I.e., if $\forall n \in \mathbb{N}$ & any $\{\underline{s}_i\}_{i=1}^n \subset \mathbb{R}^k$ we have:

$$\mathrm{vec}\Big(\underbrace{[\underline{u}(\underline{s}_1), \dots, \underline{u}(\underline{s}_n)]}_{\in \mathbb{R}^{d \times n}}\Big) \sim N_{dn}(\mathrm{vec}(M), K \otimes \Omega)$$

where $M = [\underline{\mu}(\underline{s}_1), \dots, \underline{\mu}(\underline{s}_n)] \in \mathbb{R}^{d \times n}$,

$K \in \mathbb{R}^{n \times n}$ has ijth entry $K_{ij} = \kappa(\underline{s}_i, \underline{s}_j)$

Notes :

- denoting $U_s = [\underline{u}(\underline{s}_1), \ldots, \underline{u}(\underline{s}_n)]$

we have $U_s \sim N_{d,n}(M, \Omega, K)$

i.e. U_s has a Matrix Normal distribution.

- since $K \otimes \Omega = \begin{pmatrix} K_{11}\Omega & \cdots & K_{1n}\Omega \\ \vdots & \ddots & \vdots \\ K_{n1}\Omega & \cdots & K_{nn}\Omega \end{pmatrix}$

&

$$vec(U_s) = \begin{pmatrix} \underline{u}(\underline{s}_1) \\ \vdots \\ \underline{u}(\underline{s}_n) \end{pmatrix}$$

we have

$$cov(\underline{u}(\underline{s}_i)) = K_{ii}\Omega = k(\underline{s}_i, \underline{s}_i)\Omega \qquad \forall i \in \{1, \ldots, n\}$$

&

$$cov(\underline{u}(\underline{s}_i), \underline{u}(\underline{s}_j)) = K_{ij}\Omega \qquad \forall i \neq j \in \{1, \ldots, n\}$$

- $\mathbb{E}[\underline{u}(\underline{s}_i)] = \mu(\underline{s}_i) \qquad \forall i \in \{1, \ldots, n\}$