Defn: A random Grassmann-valued function

$\mathcal{P} : \mathbb{R}^k \to Gr(d,n)$ is a Grassmann-valued

Gaussian process,

$$\mathcal{P} \sim GrGP(\mu, k, \Omega, \mathcal{U})$$

with parameters $\mu, k, \Omega, \mathcal{U}$ where:

- $\mu : \mathbb{R}^k \to \mathbb{R}^{d \times n}$ (mean func) ??
- $k : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ is a covariance function
- $\Omega \in \mathbb{R}^{dn \times dn}$ is a positive semi-definite parameter covariance matrix ??
- $\mathcal{U} \in St(d,n)$ is an representative of $[\mathcal{U}] \in Gr(d,n)$. $[\mathcal{U}]$ is called the anchor point.

if for all $\underline{s} \in \mathbb{R}^k$ we have

$$\mathcal{P}(\underline{s}) = \mathrm{Exp}_{[\mathcal{U}]}(\Delta(\underline{s}))$$

where $\Delta(\underline{s}) := (I_d - \mathcal{U}\mathcal{U}^T) V(\underline{s})$

where $V : \mathbb{R}^k \to \mathbb{R}^{d\times n}$ follows the

Matrix-valued GP distribution:

$$V \sim \mathrm{MatGP}(\mu, \kappa, \Omega).$$

$\{\underline{s}_i\}_{i=1}^{N} \subset \mathbb{R}^k$

Let $V_s = [V(\underline{s}_1), \ldots, V(\underline{s}_N)] \in \mathbb{R}^{d\times nN}$

& $M = [\mu(\underline{s}_1), \ldots, \mu(\underline{s}_N)] \in \mathbb{R}^{d\times nN}$

$K \in \mathbb{R}^{N\times N}$, $K_{ij} := \kappa(\underline{s}_i, \underline{s}_j)$

$$\mathrm{vec}(V_s) \sim N_{dnN}(\mathrm{vec}(M), K \otimes \Omega)$$

Let $\underline{v}_s = \mathrm{vec}(V_s)$.

Let $\Delta_s = [\Delta(\underline{s}_1), \ldots, \Delta(\underline{s}_N)] \overset{?}{=} (I_d - UU^T)V_s$

where $\Delta(\underline{s}) = (I_d - UU^T)V(\underline{s})$

Let $P_s = [\overbrace{\mathrm{Exp}_{[u]}(\Delta(s_1))}^{\in \mathbb{R}^{d\times n}}, \dots, \mathrm{Exp}_{[u]}(\Delta(s_N))]$

$\overset{?}{=} \underline{\mathrm{Exp}}_{[u]}(\Delta_s)$

$P_s \in \mathbb{R}^{d\times nN}$

Assume our data is

$\mathcal{D} = \{(s_i, [W_i])\}_{i=1}^{N}$

where $\{W_i\}_{i=1}^{N} \subset St(d, n)$.

Let $V_i := V(s_i)$, $\Delta_i := \Delta(s_i)$

$P_i := \mathrm{Exp}_{[u]}(\Delta_i)$

Possible likelihoods:

- $\underbrace{p(W_j \mid V_j)}_{p(W_j \mid \Delta_j)} \propto \exp\left(-\frac{\|\overbrace{(I_d - UU^T)V_j}^{\Delta_j} - \mathrm{Log}_{[u]}(W_j)\|_F^2}{2\sigma^2}\right)$

$j = 1, \dots, N$

$$\mathrm{Log}_{[u]} : Gr(d,n) \longrightarrow T_{[u]}\, Gr(d,n).$$

$$\Rightarrow \; p(\mathcal{D} \mid V_s, \sigma) \propto \prod_{j=1}^{N} p(W_j \mid V_j)$$

(tangent form)

- $$p(W_j \mid P_j) \propto \exp\left(- \frac{\| P_j - W_j \|_F^2}{2\sigma^2} \right)$$

$$\Rightarrow \; p(\mathcal{D} \mid P_s, \sigma) \propto \prod_{j=1}^{N} p(W_j \mid P_j)$$

(skinny grass form)

- $$p(W_j \mid P_j) \propto \exp\left(- \frac{\| P_j P_j^T - W_j W_j^T \|_F^2}{2\sigma^2} \right)$$

$$\Rightarrow \; p(\mathcal{D} \mid P_s, \sigma) \propto \prod_{j=1}^{N} p(W_j \mid P_j).$$

(projector grass form)