Defn:

$X \in \mathbb{R}^{n \times p}$ random

$X \sim N_{n,p}(M, U, V)$

$\iff$ X has pdf

$$p(X \mid M, U, V) = \frac{\exp\left(-\frac{1}{2} tr\left[V^{-1}(X-M)^T U^{-1}(X-M)\right]\right)}{(2\pi)^{np/2} |V|^{n/2} |U|^{p/2}}$$

$\iff$ $vec(X) \sim N_{np}(vec(M), V \otimes U)$

Here: $\cdot\ M \in \mathbb{R}^{n \times p}$

$\cdot\ U \in \mathbb{R}^{n \times n}, \quad U^T = U$

$\cdot\ V \in \mathbb{R}^{p \times p}, \quad V^T = V$

Pf:

$tr\left[V^{-1}(X-M)^T U^{-1}(X-M)\right] =$ $\quad tr(AB) = tr(BA)$

$= tr\left[(X-M)^T U^{-1}(X-M)V^{-1}\right]$ $\quad tr(A^T B) = vec(A)^T vec(B)$

$= vec(X-M)^T vec(U^{-1}(X-M)V^{-1})$ $\quad vec(ABC) = (C^T \otimes A)\, vec B$

$$= \mathrm{vec}(X-M)^T (V^{-1} \otimes U^{-1})\, \mathrm{vec}(X-M)$$

& $V^{-T} = V^{-1}$

$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

$$= \left(\mathrm{vec}(X) - \mathrm{vec}(M)\right)^T (V \otimes U)^{-1} \left(\mathrm{vec}(X) - \mathrm{vec}(M)\right)$$

also $|V \otimes U| = |V|^n |U|^p$

Properties:

- $\mathbb{E} X = M$
- $\mathbb{E}[(X-M)(X-M)^T] = U \operatorname{tr}(V)$
- $\mathbb{E}[(X-M)^T (X-M)] = V \operatorname{tr}(U)$
- $\mathbb{E}[XAX^T] = U \operatorname{tr}(A^T V) + MAM^T$
- $\mathbb{E}[X^T B X] = V \operatorname{tr}(U B^T) + M^T B M$
- $\mathbb{E}[XCX] = V C^T U + MCM$

- Transformations:

  - Transpose:

$$X^T \sim N_{p,n}(M^T, V, U)$$

  - Linear transform:

If $D \in \mathbb{R}^{r \times n}$, $C \in \mathbb{R}^{p \times s}$ &

rank$(D) = r \leq n$, rank$(C) = s \leq p$

then $DXC \sim N_{r,s}(DMC, DUD^T, C^TVC)$

Sampling: To sample $X \sim N_{n,p}(M, U, V)$:

1. Compute A, B s.t. $U = AA^T$, $V = B^TB$
2. Sample $Z \sim N_{n,p}(0, I_n, I_p)$ via $Z_{ij} \overset{iid}{\sim} N(0,1)$.
3. Set $X = M + AZB$