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SUPERVISOR'S USE ONLY

OUTSTANDING SCHOLARSHIP EXEMPLAR

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Tick this box if there is no writing in this booklet

Scholarship 2020
Calculus

9.30 a.m. Monday 16 November 2020
Time allowed: Three hours
Total score: 40

ANSWER BOOKLET

There are five questions in this examination. Answer ALL FIVE questions.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Question	Score
ONE	
TWO	
THREE	
FOUR	
FIVE	
TOTAL	/40

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QUESTION NUMBER

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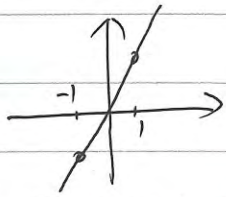
Q1 a) $x \neq \pm 1$ $\quad f(x) = \dfrac{(x-1)^2}{(x-1)} + \dfrac{(x+1)^2}{(x+1)}$

~~$f'(x)$~~

$$\begin{array}{r} x-1 \\ x-1 \,\overline{)\, x^2-2x+1} \\ \underline{x^2-x} \\ -x+1 \\ \underline{-x+1} \\ 0 \end{array} \qquad \begin{array}{r} x+1 \\ x+1 \,\overline{)\, x^2+2x+1} \\ \underline{x^2+x} \\ x+1 \end{array}$$

$f(x) = (x-1) + (x+1)$
$= 2x$

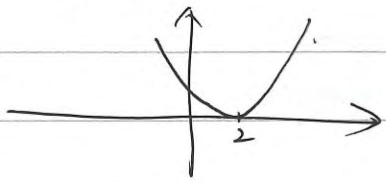


$x < -1 \cap -1 < x < 1 \cap x > 1$

~~b)~~

b) $-(x-2)^2 > 0$ $\quad$ ~~$-(x-2)^2 \pm x$~~

$(x-2)^2 < 0$ $\quad x^2 + 4 - 4x < 0$



$\because x$ is real
$\therefore$ there is no value for $f(x)$ is real

c) i) A B C

1 2 3

${}^6C_1 \times {}^5C_2 \times {}^3C_3$

$= 60$

QUESTION NUMBER

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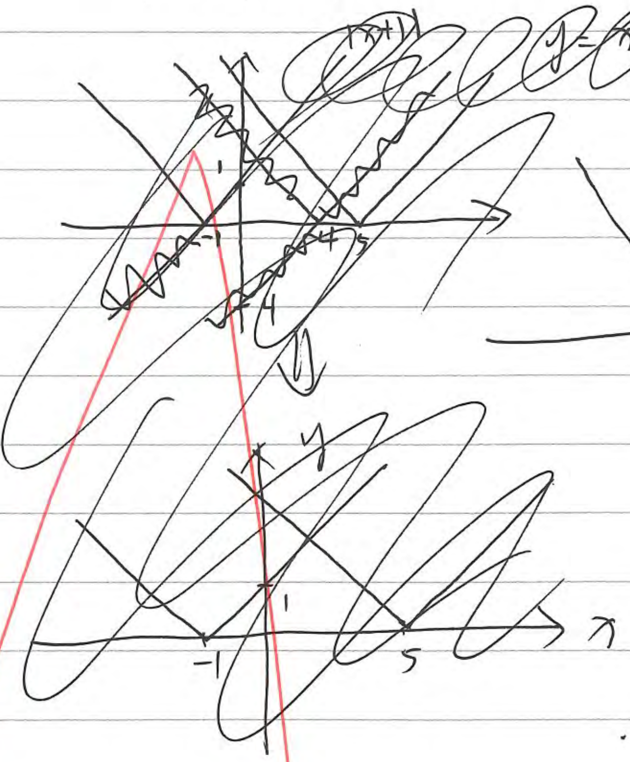
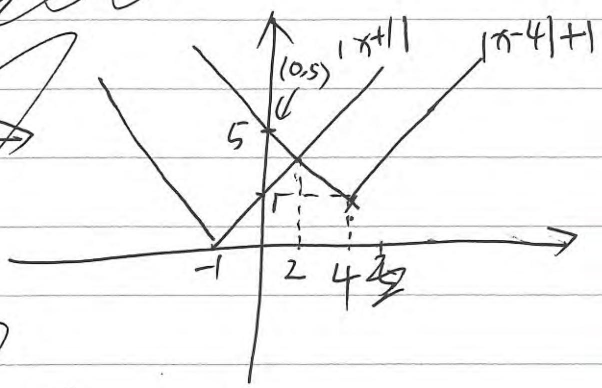
ii) A B C

$^{6}C_2 \times {}^{4}C_2 \times {}^{2}C_2 = 90$

d) $|x+1| - |x-4| \geq 1$

$|x+1| \geq 1 + |x-4|$





$m = \frac{5-1}{0-4} = \frac{4}{-4}$

$= -1$

$\therefore \cancel{y - 0 = -1(x-5)}$

$y - 5 = (-1)x$

$y - 5 = -x$

$y = -x + 5$

$y = -x + 5 = y = x + 1$

$-x + 5 = x + 1$

$4 = 2x$

$x = 2.$

when $x = 2$. $y = 3$

$\therefore x \geq 2$

QUESTION NUMBER

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e) $\sin^4 A + \cos^4 A = \frac{2}{3}$

$\sin^4 A + (\cos^2 A)^2 = \frac{2}{3}$

$(1-\sin^2 A)^2$

$\sin^4 A + 1 + \sin^4 A - 2\sin^2 A = \frac{2}{3}$

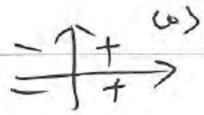
$2\sin^4 A \cancel{\ } - 2\sin^2 A + 1 = \frac{2}{3}$

$2\sin^4 A - 2\sin^2 A + \frac{1}{3} = 0$

$\sin^2 A = x$

$2x^2 - 2x + \frac{1}{3} = 0$

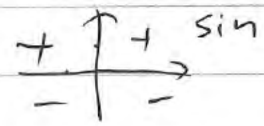
$x = \frac{3+\sqrt{3}}{6}$ ✓ $\quad x = \frac{3-\sqrt{3}}{6}$ ✓



$\because \sin^2 A > 0 \Rightarrow x > 0$

$\cos A < 0$

$90° < A < 180°$



$\therefore \sin A > 0$

$\because$ ~~$\sin 2A = 2\sin A\cos A$~~ $\quad \cos^2 A = 1 - \sin^2 A$

~~$= 2\sin A (1$~~ $\quad = \frac{4-\sqrt{3}}{6}$

~~$1 - \cos 2A$~~

or $= \frac{4+\sqrt{3}}{6}$

1° when $\sin^2 A = \frac{3+\sqrt{3}}{6} \quad \cos^2 A = \frac{4-\sqrt{3}}{6}$

$\sin A = \sqrt{\frac{3+\sqrt{3}}{6}} \quad \cos A = -\sqrt{\frac{4-\sqrt{3}}{6}}$

$\therefore \sin 2A = 2\sin A\cos A$

$= 2\sqrt{\frac{3+\sqrt{3}}{6}} \cdot \sqrt{\frac{4-\sqrt{3}}{6}}$

$= -1.09$ (3sf)

2° when $\sin^2 A = \frac{3-\sqrt{3}}{6} \quad \cos^2 = \frac{4+\sqrt{3}}{6}$

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$$\sin 2A = 2\sin A\cos A$$
$$= -2\sqrt{\frac{3-\sqrt{3}}{6}} \times \sqrt{\frac{4+\sqrt{3}}{6}}$$ NS
$$= -0.899 \ (3s.f.)$$

Q2 $$\left(\sqrt{\frac{x-2}{x}} - \sqrt{\frac{x}{x-2}}\right)^2 = \left(\frac{k}{4}\right)^2$$

$$\frac{x-2}{x} + \frac{x}{x-2} - 2\sqrt{\frac{\cancel{x(x-2)}}{\cancel{x(x-2)}}} = \frac{k^2}{16}$$

$$\frac{(x-2)^2}{x(x-2)} + \frac{x^2}{x(x-2)} - 2 = \frac{k^2}{16}$$

$$\frac{x^2+x^2+4-4x}{x(x-2)} - 2 = \frac{k^2}{16}$$

$$\frac{2x^2-4x+4}{x(x-2)} - \frac{2x(x-2)}{x(x-2)} = \frac{k^2}{16}$$

$$\frac{\cancel{2x^2}\cancel{-4x}+4\cancel{-2x^2}\cancel{+4x}}{x(x-2)} = \frac{k^2}{16}$$

$$\frac{-4}{x^2-2x} = \frac{k^2}{16}$$

$$-64 = k^2x^2 - 2k^2x$$

$$k^2x^2 - 2k^2x + 64 = 0$$

$\Delta < 0 \quad b^2 - 4ac < 0$

$$4k^4 - (4)(k^2)(+64) < 0$$

$$4k^4 - 256k^2 < 0 \implies k^2 < 64$$

~~let $k^2 = y$~~ $4y^2 + \cancel{256y} < 0$ $\because k^2 > 0$

~~$-64 < y < 0$~~

Next page $\Rightarrow$

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$\cancel{-64 \cdot k^2 < 0}$

$\because k$ is real $\therefore k^2 > 0$ and $k \neq 0$

$4k^2 + 256 < 0$

$k^2 < -64$

$\cancel{-64 < k^2 < 0}$

$64 > k^2 > 0$

$8 > k > 0$

b) $\log_{10}(x^2+y^2) - \log_{10} 13 = 1$

$\log_{10}\left(\frac{x^2+y^2}{13}\right) = \log_{10} 10$

$\frac{x^2+y^2}{13} = 10$

$x^2+y^2 = 130$

$x^2+y^2-130 = 0$

$\log_{10}\left(\frac{x+y}{x-y}\right) = \cancel{3}\log_{10} 2^3$

$\frac{x+y}{x-y} = 8$

$x+y = 8x - 8y$

$0 = 7x - 9y$

QUESTION NUMBER

ASSESSOR'S USE ONLY

$$\begin{cases} x^2+y^2=130 \\ 7x-9y=0 \end{cases}$$

$7x=9y$

$x=\frac{9}{7}y$

$\frac{81}{49}y^2+y^2=130$

$\frac{\cancel{130}}{49}y^2=\cancel{130}$

$y^2=49 \quad y=\pm 7$

$y=7 \qquad y=-7$

$x=9 \qquad x=-9$

impossible

$\because x+y>0 \quad (x-y)>0$

when ~~x=7~~ $y=-7$ $x=-9$

$x+y=-16<0$ (X)

$\therefore x=9 \quad y=7$

QUESTION NUMBER

ASSESSOR USE ONLY

c) $y = r\sin\theta \quad x = r\cos\theta$

$r^2\sin^2\theta = r^2\cos^2\theta - r^4\cos^4\theta$

$r^2\cos^4\theta = \cos^2\theta - \sin^2\theta$

$r^2 = \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^4\theta}$

$= \sec^2\theta - \frac{1-\cos^2\theta}{\cos^4\theta}$

$= \sec^2\theta - \frac{1}{\cos^4\theta} + \frac{1}{\cos^2\theta}$

$= 2\sec^2\theta - \sec^4\theta$

$4 \times \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \, d\theta$

$= 2\int_0^{\frac{\pi}{2}} 2\sec^2\theta - \sec^4\theta \, d\theta$

$= 2\left[2\int_0^{\frac{\pi}{2}} \sec^2\theta \, d\theta - \int_0^{\frac{\pi}{2}} \sec^4\theta \, d\theta\right]$

$2\left\{2[\tan\theta]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (1+\tan^2\theta) \, d\tan\theta\right\}$

$[\tan\theta]_0^{\frac{\pi}{2}} + \left[\frac{1}{3}\tan^3\theta\right]_0^{\frac{\pi}{2}}\}$

$= 2\left\{2[\tan\theta]_0^{\frac{\pi}{2}} - [\tan\theta]_0^{\frac{\pi}{2}} - \left[\frac{1}{3}\tan^3\theta\right]_0^{\frac{\pi}{2}}\right\}$

$= 2\left\{[\tan\theta]_0^{\frac{\pi}{2}} - \left[\frac{1}{3}\tan^3\theta\right]_0^{\frac{\pi}{2}}\right\}$

$=$ [illegible]

QUESTION NUMBER

ASSESSOR'S USE ONLY

c) $y^2 = x^2 - x^4$

$y = \sqrt{x^2 - x^4}$

$= x\sqrt{1-x^2}$

$A = 4\int_0^1 x\sqrt{1-x^2}\,dx$

$= 4\times\frac{1}{2}\int_0^1 \sqrt{1-x^2}\,d(x^2)$

$= -2\int_0^1 \sqrt{1-x^2}\cdot(-x^2+1)$

$= -2\left[\frac{2}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1$

$= -\frac{4}{3}\left[(1-x^2)^{\frac{3}{2}}\right]_0^1$

$= -\frac{4}{3}\left(0 - 1^{\frac{3}{2}}\right)$

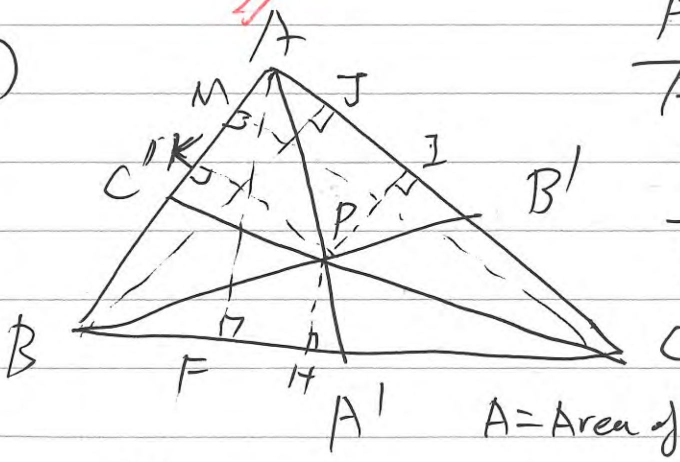
$= \frac{4}{3}$

$\frac{PC'}{CC'} = \frac{PK}{CM}$

$\frac{PA'}{AA'} = \frac{PH}{AF}$

$\frac{PB'}{BB'} = \frac{PI}{BJ}$

d)



A = Area of triangle

$CM = \frac{2A}{AB}$ $AF = \frac{2A}{BC}$ $PI = \frac{2A}{AC}$

QUESTION NUMBER

ASSESSOR USE ONLY

β, γ

let α ~~become~~ a fraction of Area of triangle

$PK = \frac{2\alpha A}{AB}$ same for PH and PJ

define: $\alpha = \frac{A_{\Delta APB}}{A_{\Delta ABC}}$

$\beta = \frac{A_{BCP}}{A_{\Delta ABC}}$

$\gamma = \frac{A_{\Delta ACP}}{A_{\Delta ABC}}$

$\therefore \frac{PC'}{CC'} \quad \frac{\frac{2\alpha A}{\cancel{AB}}}{\frac{2A}{\cancel{AB}}} = \alpha.$

$PH = \frac{2\beta A}{BC}$ ~~B~~ ~~is~~ ~~area~~ ~~fraction~~ ~~of~~ ~~area~~ ~~of~~ ~~triangle~~

$\therefore \frac{PA'}{AA'} = \frac{\frac{2\beta A}{\cancel{BC}}}{\frac{2A}{\cancel{BC}}} = \beta$

$\frac{PB'}{BB'} = \cancel{2} \frac{\frac{2\gamma A}{\cancel{AC}}}{\frac{2A}{\cancel{AC}}} = \gamma$

$\therefore$ ~~$\alpha + \beta + \gamma$~~.

$\frac{PC'}{CC'} + \frac{PA'}{AA'} + \frac{PB'}{BB'} = \alpha + \beta + \gamma = 1$

QUESTION NUMBER

ASSESSOR'S USE ONLY

Q3 a) $f'(4) = \lim_{h \to 0} \left[\dfrac{f(4+h) - f(4)}{h} \right]$

$f(4) = 9$

$\lim_{h \to 0} \left[\dfrac{f(4+h) - 9}{h} \right]$

$\cancel{= \lim_{h \to 0} \dfrac{f(4+h)}{h} - \lim_{h \to 0} \dfrac{9}{h}}$

$f(4+h) = (16 + h^2 + 8h - 4(4+h) + 3)^2$

$= (\cancel{16} + h^2 + 8h - \cancel{16} - 4h + 3)^2$

$(h^2 + 4h + 3)^2$

$\lim_{h \to 0} \left(\dfrac{(h^2+4h+3)^2 - 9}{h} \right)$

$\cancel{\dfrac{h^4(1 + \frac{4}{h} + \frac{3}{h^2})^2 - \frac{9}{h^2}}{\frac{1}{h}}}$

"$\dfrac{0}{0}$" type $\Rightarrow \dfrac{((h^2+4h+3)^2 - 9)'}{(h)'}$

differentiate upper and low separately

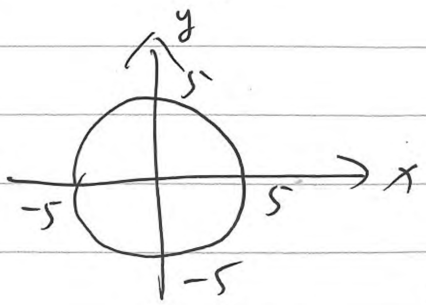
$= \lim_{h \to 0} \dfrac{2(h^2+4h+3)(2h+4)}{1}$

$= 2(3)(4) = 24$

QUESTION NUMBER

ASSESSOR USE ONLY

b) $x^2+y^2=5^2$



when $P(3,4)$

$\frac{dy}{dt} = -2.$

$\frac{dx}{dt} = ?$

~~$(x^2+y^2=5^2)'$~~

$(x^2+y^2)' = (5^2)'$

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y}$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-x}{y}$

$= \frac{-2}{\frac{dx}{dt}} = \frac{-x}{y}$

$\Rightarrow \frac{-2}{\frac{dx}{dt}} = \frac{-3}{4}$

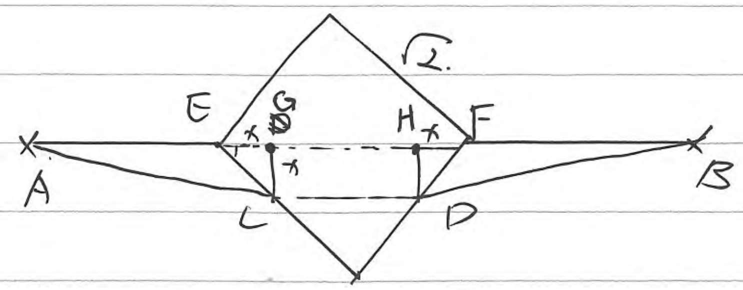
$\frac{dx}{dt} = \cancel{+}2 \times (\cancel{4}\ \frac{4}{\cancel{+}3}$

$= \cancel{8}\ \frac{8}{3}$

QUESTION NUMBER

ASSESSOR'S USE ONLY

c)



$EG = x \qquad HF = x$

$EF = \sqrt{2}\cdot\sqrt{2} = 2.$

$\therefore GH = 2-2x$

$CD = GH = 2-2x$

Time taken for boat: $\frac{2-2x}{2.5}$

$AE + FB = 4-2 = 2 \quad 4-2+2x$

$= 2+2x$

Time taken for tramping $\frac{2+2x}{3}$

$\therefore$ Time tot $= \frac{2+2x}{3} + \frac{2-2x}{2.5}$

Let $y = \frac{2+2x}{3} + \frac{2-2x}{2.5}$

$y' = \frac{1}{3}(2)$

$\therefore FB = AE = \frac{4-2}{2} = 1$

$\therefore AC = \sqrt{(1+x)^2 + x^2} = DB$

$= \sqrt{1+x^2+2x+x^2}$

$= \sqrt{2x^2+2x+1}$

Tot distant travelled by tramping

$= 2\sqrt{2x^2+2x+1}$

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Time taken by travelling

$$= \frac{2\sqrt{2x^2+2x+1}}{3}$$

$$\therefore t_{tot} = \frac{2\sqrt{2x^2+2x+1}}{3} + \frac{2-2x}{2.5}$$

$$t'_{tot} = \cancel{\frac{2}{3} \times \frac{1}{2} \frac{1}{\sqrt{(2x^2+2x+1)}}}$$

$$\frac{2}{3} \times \frac{1}{2} \frac{1}{\sqrt{2x^2+2x+1}} (4x+2) \cancel{+ \left(\frac{-2}{2.5}\right)} - \frac{4}{5} = 0$$

$$\frac{4x+2}{\sqrt{2x^2+2x+1}} = \frac{2}{5} \times 6$$

$$\frac{4x+2}{\sqrt{2x^2+2x+1}} = \frac{12}{5}$$

$$\left(12\sqrt{2x^2+2x+1}\right)^2 = (20x+10)^2$$

$$144(2x^2+2x+1) = 100 + 400x^2 + 400x$$

$$288x^2 + 288x + 144 = 100 + 400x^2 + 400x$$

$$112x^2 + 112x - 44 = 0$$

$$x = \frac{-7+3\sqrt{14}}{14} \qquad x = \frac{-7-3\sqrt{14}}{14} \ (\times)$$

$$\because x > 0$$

boat positioned at $x = \frac{-7+3\sqrt{14}}{14}$ (≈ 0.302)

QUESTION NUMBER

ASSESSOR'S USE ONLY

Q4

$$P(m) = a - \frac{a-b}{t_p} t$$

$$P(f) = 1 - a + \frac{a-b}{t_p} t$$

$$T = \frac{1}{t_p}\left\{ \int_0^{t_p} \left(a - \frac{a-b}{t_p}t\right)^2 dt + \int_0^{t_p} \left(1 - a + \frac{a-b}{t_p}t\right) dt \right\}$$

$$-\frac{t_p}{a-b}\int_0^{t_p} \left(a - \frac{a-b}{t_p}t\right)^2 d\left(-\frac{a-b}{t_p}t + a\right)$$

$$-\frac{t_p}{3(a-b)}\left[\left(a - \frac{a-b}{t_p}t\right)^3\right]_0^{t_p}$$

$$= \frac{t_p}{3(a-b)}(b^3 - a^3)$$

$$= \frac{t_p}{3\cancel{(a-b)}}\cancel{(b-a)}(b^2+ab+a^2)$$

$$= \frac{t_p}{3}(b^2+ab+a^2)$$

$$\int_0^{t_p} \left(1 - a + \frac{a-b}{t_p}t\right)^2 dt$$

$$\cancel{= \left[(1-a)t\right]_0^{t_p} + \int_0^{t_p} \frac{a-b}{t_p} t \, dt}$$

$$\cancel{\frac{a-b}{2t_p}\left[t^2\right]_0^{t_p}}$$

$$\cancel{= (1-a)t_p + \frac{(a-b)}{2}t_p}$$

$$\cancel{\therefore T = \frac{1}{t_p}\left(\frac{t_p}{3} + (1-a)t_p + \frac{(a-b)t_p}{2}\right)}$$

c. on next page

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$$\frac{t_p}{a-b}\int_0^{t_p}\left(1-a+\frac{a-b}{t_p}t\right)^2 d\left(\frac{a-b}{t_p}t+a-1\right)$$

$$=\frac{t_p}{a-b}\,\frac{1}{3}\left[\left(1-a+\frac{a-b}{t_p}t\right)^3\right]_0^{t_p}$$

$$=\frac{t_p}{3(a-b)}\left((1-b)^3-(1-a)^3\right)$$

$$=\frac{t_p}{\cancel{3(a-b)}}\cancel{(1-b-1+a)}\left((1-b)^2+\underbrace{(1-b)(1-a)}_{1-a-b+ab}+(1-a)^2\right)$$

$$=\frac{t_p}{3}\left(1+b^2-2b+1-a-b+ab+1+a^2-2a\right)$$

$$=\frac{t_p}{3}\left(b^2+a^2-3b+3-3a+ab\right)$$

$$\therefore T=\frac{1}{3}\left(b^2+ab+a^2+b^2+a^2-3b-3a+ab+3\right)$$

$$=\frac{1}{3}\left(2b^2+2a^2+2ab-3a-3b+3\right)$$

$-6ab+6ab$

$-4ab$

$$\frac{1}{3}\left(2b^2+2a^2-4ab+6ab-3a-3b+3\right)$$

$$=\frac{2}{3}(a-b)^2+1+2ab-b-a$$

$$=\frac{2}{3}(a-b)^2+1+b(2a-1)-a$$

$$=1-a+b(2a-1)+\frac{2}{3}(a-b)^2$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

b) $\dfrac{dy}{dx} = \dfrac{y^2}{4x^2} - \dfrac{2xy}{4x^2}.$

$\dfrac{dy}{dx} = \dfrac{y^2 - 2xy}{\cancel{4x^2 - 4x^2}\; 4x^2}$

$4x^2 dy = (y^2 - 2xy)dx$

~~$(y^2 - 2xy)dx + (-4x^2)dy = 0$~~

~~let $y = ux$~~

let $y = ux$ $\qquad \dfrac{dy}{dx} = x\dfrac{du}{dx} + u$

$dy = x\,du + u\,dx$

$4x^2(x\,du + u\,dx) = (u^2x^2 - 2x^2u)dx$

$4x^3du + 6x^2u\,dx = u^2x^2dx \cancel{- 2x^2udx}$

$4x^3du = -5x^2u^2dx$

$4x\,du = -5u^2dx$

$\dfrac{1}{-5u^2}du = \dfrac{1}{4x}dx$

$-\dfrac{1}{5}\int u^{-2}du = \dfrac{1}{4}\int \dfrac{1}{4x}d(4x)$

$\dfrac{1}{5}u^{-1} = \dfrac{1}{4}\ln 4x + C.$

$\dfrac{1}{5u} = \dfrac{1}{4}\ln 4x + C.$

QUESTION NUMBER

ASSESSOR'S USE ONLY

$$\frac{1}{5u} = \frac{1}{4}\ln 4x + C$$

$$y = ux$$

$$u = \frac{y}{x}$$

$\cancel{\tfrac{1}{4}}\ln C$

$$\frac{1}{5}\cdot\frac{x}{y} = \frac{1}{4}\ln 4x + C \quad " \frac{1}{4}\ln(4Cx) "$$

$$\cancel{\frac{x}{5y}}\ \frac{5y}{x} = \frac{1}{\frac{1}{4}\ln(4Cx)}$$

$$\cancel{5}y = \frac{4x}{5\cancel{4}\ln(4Cx)}$$

$$y = \frac{4x}{5\ln(4Cx)}$$

when $x=1 \quad y=-6$

$$\frac{-\cancel{6}3}{\cancel{4}2} = \frac{\cancel{4}}{5\ln(4C)}$$

$$-\frac{3}{2} = \frac{1}{5\ln(4C)}$$

$$\cancel{5}\ln(4C) = -\frac{2}{\cancel{3}15}$$

$$4C = e^{-\frac{2}{15}}$$

$$C = \frac{1}{4}e^{-\frac{2}{15}}$$

$$\therefore y = \frac{4x}{5\ln(\cancel{4}\,\cancel{\tfrac{1}{4}}e^{-\frac{2}{15}}x)}$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

$$y = \frac{4x}{5\ln(e^{-\frac{2}{15}}x)}$$

$$= \frac{4x}{5\ln e^{-\frac{2}{15}} + 5\ln x}$$

$$= \frac{4x}{-\frac{2}{3} + 5\ln x}$$

$$= \frac{12x}{-2 + 15\ln x}$$

when $x = 4$.

$$y = \frac{12\times4}{-2+15\ln4} = \frac{12\times4}{-2+30\ln2}$$

$$= 2.55 \ (3sf)$$

$$~~= \frac{48}{-2+30\ln2}~~$$

$$~~= 2.55 \ (3sf)~~$$

$$~~\frac{48}{2(15\ln2-1)}~~$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

Q5 a) $\tan\frac{\theta}{2} = \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$

$=$

$\cancel{w = \cos\theta + i\sin\theta}$

$\dfrac{w-1}{w+1} = \dfrac{(w-1)^2}{w^2-1}$

$\cancel{\cos\theta + i\sin\theta}$

$\dfrac{(\cos\theta + i\sin\theta - 1)^2}{\quad}$

$\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta - 1$

$\cos^2\theta - \sin^2\theta - \sin^2\theta - \cos^2\theta + \cancel{2\sin 2\theta}$ $i\sin 2\theta$

$= -2\sin^2\theta + i\,\cancel{\sin 2\theta}$ $2i\sin\theta\cos\theta$

$\cancel{\cos^2\theta +}$

$\dfrac{(w-1)^2}{w^2-1} =$

$\dfrac{w-1}{w+1} = \dfrac{w+1-2}{w+1} = 1 - \dfrac{2}{w+1}$

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$$\cancel{1-\frac{zw-2}{w^2-1}}$$

$$\sin^2 A = \frac{1-\cos 2A}{2}$$

$$\sin^2\frac{A}{2} = \frac{1-\cos A}{2}$$

$$\cos^2\frac{A}{2} = \frac{1+\cos A}{2}$$

$$\therefore \tan\frac{\theta}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$\cancel{\frac{w-1}{w+1} = \frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1}\quad \frac{(w-1)^2}{w^2-1}}$$

if $z = \cos\theta + i\sin\theta$

$$2\cos A = z + \frac{1}{z}$$

$$\cos A = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$= \frac{1}{2}\left(\frac{z^2+1}{z}\right)$$

$$\therefore \tan\frac{\theta}{2} = \sqrt{\frac{\frac{2z}{2z} - \frac{z^2+1}{2z}}{\frac{2z}{2z} + \frac{z^2+1}{2z}}}$$

$$= \sqrt{\frac{\frac{2z-z^2-1}{\cancel{2z}}}{\frac{2z+z^2+1}{\cancel{2z}}}}$$

$$= \sqrt{\frac{2z-z^2-1}{2z+z^2+1}} = \sqrt{\frac{-(1-z)^2}{(z+1)^2}}$$

$$= i\cdot\frac{1-z}{z+1}$$

$\cancel{z=w}$

$w = z = \cos\theta + i\sin\theta$

$$\therefore i\tan\frac{\theta}{2} = i^2\,\frac{1-w}{w+1} \Rightarrow$$

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$\therefore i\tan\frac{\theta}{2} = (-1)\frac{1-w}{w+1}$

$= \frac{w-1}{w+1}$

b) i) $\tan\phi = \frac{b\sin\theta}{a\cos\theta} = \frac{b}{a}\tan\theta$

ii) $\theta - \phi \rightarrow$ greatest

$\tan\phi = \frac{b}{a}\tan\theta \Rightarrow \tan\theta = \frac{a}{b}\tan\phi$

$\tan(\theta-\phi)$ needs to be greatest

$= \frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}$

$\frac{\frac{a}{b}\tan\phi - \tan\phi}{1+\frac{a}{b}\tan^2\phi}$

$\tan(\theta-\phi) = \frac{a\tan\phi - b\tan\phi}{b + a\tan^2\phi} = \frac{\tan\phi(a-b)}{b+a\tan^2\phi}$

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$\cancel{\tan'(\theta-\phi)}$

$$\cancel{= \frac{}{(b+a\tan^2\phi)^2}}$$

$$= \frac{(a-b)}{\frac{b}{\tan\phi} + a\tan\phi}$$

$$\cancel{\phi} \leq \frac{a-b}{2\sqrt{\frac{b}{\cancel{\tan\phi}} \cdot a\cancel{\tan\phi}}}$$

$$\text{max for } \tan(\theta-\phi) = \frac{a-b}{2\sqrt{ab}}$$

$$\text{when } \frac{b}{\tan\phi} = a\tan\phi.$$

$$b = a\tan^2\phi$$

$$\frac{b}{a} = \tan^2\phi$$

$$\cancel{\tan^2\phi}\ \tan\phi = \sqrt{\frac{b}{a}}.$$

$$\phi = \tan^{-1}\left(\sqrt{\frac{b}{a}}\right)$$

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Question	Mark	Annotation
1	**8**	The candidate showed competence in manipulating complicated algebraic expressions in solving system of equations in **1cii**. The setting out of their solution was concise and the reasoning was clear.
2	**8**	Although the candidate made a sign error in finding the second derivative of the function, their points of reflection were correct. They further argued consistently the concavity of the curve in **2bii**.
3	**8**	In **3a**, the candidate acknowledged the necessity in using left/right limit at x=0. They also demonstrated mathematical rigour in **3c** by showing the angle ABC is a right angle before using right angled trigonometry.
4	**7**	The candidate showed elegance in manipulating abstract expressions while applying 'First Principle of Differentiation'. They also identified the integrand varies in signs over the interval, therefore need to be integrated separately in finding the areas between the curve and the x-axis.
5	**8**	The candidate exhibited aptitude in applying **5ai** into the proof of **5aii**. They yet again displayed strong algebra skills in **5d,** one of the most difficult question of the paper. They showed insight in identifying patterns and could systematically discuss each case in solving the problem.