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SUPERVISOR'S USE ONLY

OUTSTANDING SCHOLARSHIP EXEMPLAR

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2016
Calculus

9.30 a.m. Friday 25 November 2016
Time allowed: Three hours
Total marks: 40

ANSWER BOOKLET

There are five questions in this examination. Answer ALL FIVE questions, choosing ONE option from part (b) of Question Five.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

The graph for Question Five (b) is repeated on pages 26 and 27 of this booklet.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

This examination consists of five questions.
Answer all FIVE questions, choosing ONE option from part (b) of Question Five.

ASSESSOR'S USE ONLY

QUESTION NUMBER

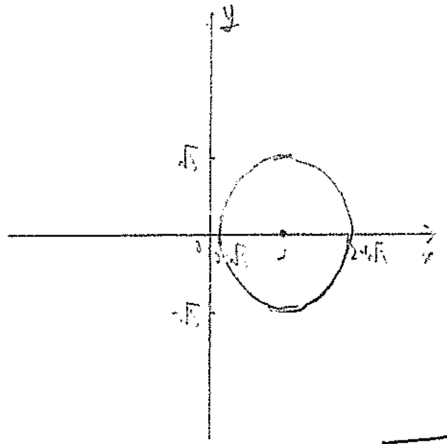
ONE(a) $\because (x, y)$ is on $(x-2)^2+y^2=3$

$\therefore \quad y^2 = 3-(x-2)^2$

$y = \sqrt{3-(x-2)^2}$ or $-\sqrt{3-(x-2)^2}$

$\therefore \quad \frac{y}{x} = \frac{\sqrt{3-(x-2)^2}}{x}$ or $-\frac{\sqrt{3-(x-2)^2}}{x}$

From the graph:



$x > 0 \Rightarrow$ maximum is not at $-\frac{\sqrt{3-(x-2)^2}}{x}$

$\therefore \quad \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{d}{dx}\left(x^{-1}\left(3-(x-2)^2\right)^{\frac{1}{2}}\right)$

$= -x^{-2}\left(3-(x-2)^2\right)^{\frac{1}{2}} - x^{-1}(x-2)\left(3-(x-2)^2\right)^{-\frac{1}{2}} = 0$

$\therefore \quad \frac{\sqrt{3-(x-2)^2}}{x^2} + \frac{x-2}{x\sqrt{3-(x-2)^2}} = 0$

$3-(x-2)^2 + x(x-2) = 0$

$\cancel{(3-x+2+x)(x-2)}$

$3-(x^2-4x+4)+x^2-2x=0$

$3-x^2+4x-4+x^2-2x=0$

$2x = 1$

$x = \frac{1}{2}$

$\therefore \quad y = \frac{\sqrt{3}}{2}$

$\therefore \quad \frac{y}{x} = \sqrt{3}$ when maximised

QUESTION NUMBER | ASSESSOR'S USE ONLY

ONE (b)

(i) $\frac{d f(x)}{dx} = 2x \ln(x+1) + x^2 \times \frac{1}{x+1}$

$= 2x \ln(x+1) + x^2(x+1)^{-1}$

$\frac{d^2 f(x)}{dx} = 2\ln(x+1) + 2x \times \frac{1}{x+1} + 2x(x+1)^{-1} + x^2 \times -(x+1)^{-2}$

$= 2\ln(x+1) + \frac{2x}{x+1} + \frac{2x}{x+1} - \frac{x^2}{(x+1)^2}$

$= 2\ln(x+1) + \frac{4x}{x+1} - \frac{x^2}{(x+1)^2}$

when $x=0$, $\frac{d^2(f(x))}{dx^2} = 2\ln 1 + 0 - 0 = 0$ //

(ii) $\frac{d^3 f(x)}{dx} = 2(x+1)^{-1} + 4(x+1)^{-1} - 4x(x+1)^{-2} - 2x(x+1)^{-2} + 2x^2(x+1)^{-3}$

$= 6(x+1)^{-1} - 6x(x+1)^{-2} + 2x^2(x+1)^{-3}$

$\frac{d^4 f(x)}{dx} = -6(x+1)^{-2} - 6(x+1)^{-2} + 12x(x+1)^{-3} + 4x(x+1)^{-3} - 6x^2(x+1)^{-4}$

$= -12(x+1)^{-2} + 16x(x+1)^{-3} - 6x^2(x+1)^{-4}$

$\frac{d^5 f(x)}{dx} = 24(x+1)^{-3} + 16(x+1)^{-3} - 48x(x+1)^{-4} - 12x(x+1)^{-4} + 24x^2(x+1)^{-5}$

$= 40(x+1)^{-3} - 60x(x+1)^{-4} + 24x^2(x+1)^{-5}$

when $x=0$, the term that determines the value of the derivative is the first term

For $f^{(2016)}(0) = -A(x+1)^{-2014}$ (A is a constant)

$= -A$

~~$A \in \frac{2016!}{2014}$~~ From above, $f^{(n)}(0) = (-1)^{n+1} \times \frac{n!}{n-2}$

$\therefore f^{(2016)}(0) = -\frac{2016!}{2014}$ //

QUESTION NUMBER

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TWO(a)

$$\sin^5 x + \cos^5 x$$
$$= \sin^2 x \sin^3 x + \cos^3 x \cos^2 x$$
$$= (1-\cos^2 x)\sin^3 x + (1-\sin^2 x)\cos^3 x$$
$$= \sin^3 x - \cos^2 x \sin^3 x + \cos^3 x - \sin^2 x \cos^3 x$$
$$= \sin^2 x \sin x - \cos^2 x \sin^2 x \sin x + \cos^2 x \cos x - \sin^2 x \cos^2 x \cos x$$
$$= (1-\cos^2 x)\sin x - \cos^2 x(1-\cos^2 x)\sin x + (1-\sin^2 x)\cos x - \sin^2 x(1-\sin^2 x)\cos x$$
$$= \sin x - \sin x\cos^2 x - \cos^2 x \sin x + \cos^4 x \sin x + \cos x - \cos x \sin^2 x - \sin^2 x \cos x + \sin^4 x \cos x$$
$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5 x + \cos^5 x \, dx$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x - 2\sin x \cos^2 x + \sin x \cos^4 x + \cos x - 2\cos x \sin^2 x + \cos x \sin^4 x \, dx$$
$$= \left[-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= 1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5} = \frac{16}{15} \text{ //}$$

(b)

$$I_n = \int_0^{\frac{\pi}{2}} \frac{2\sin(nx)\cos(nx)}{\sin x} \, dx$$
$$I_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin 2(n-1)x}{\sin x} \, dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{2\sin((n-1)x)\cos((n-1)x)}{\sin x} \, dx$$
$$I_n - I_{n-1} = \int_0^{\frac{\pi}{2}} \frac{2\sin(nx)\cos(nx) - 2\sin((n-1)x)\cos((n-1)x)}{\sin x} \, dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin(2nx) - \sin(2nx-2x)}{\sin x} \, dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin(2nx) - \sin 2nx \cos 2x + \cos 2nx \sin 2x}{\sin x} \, dx \text{ //}$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

TWO(c)

$$\log_{(\tan\theta+\cot\theta)}(\cos\theta)=k$$

$$\frac{\log(\cos\theta)}{\log(\tan\theta+\cot\theta)}=k$$

$$\log(\cos\theta)=k\log(\tan\theta+\cot\theta)$$

$$\log(\cos\theta)=k\log\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

$$\log(\cos\theta)=k\log\left(\frac{1}{\sin\theta\cos\theta}\right)$$

$$\log(\cos\theta)=-k\log(\sin\theta\cos\theta)$$

$$\log(\cos\theta)=-k\log(\sin\theta)-k\log(\cos\theta)$$

$$(k+1)\log(\cos\theta)=-k\log(\sin\theta)$$

$$\log(\cos\theta)=-\frac{k}{k+1}\log(\sin\theta)$$

$$\log_{\tan\theta}(\sin\theta)=\frac{\log(\sin\theta)}{\log(\tan\theta)}$$

$$=\frac{\log(\sin\theta)}{\log(\sin\theta)-\log(\cos\theta)}$$

$$=\frac{\log(\sin\theta)}{\log(\sin\theta)+\frac{k}{k+1}\log(\sin\theta)}$$

$$=\frac{1}{1+\frac{k}{k+1}}$$

$$=\frac{1}{\frac{2k+1}{k+1}}=\frac{k+1}{2k+1}$$ //

QUESTION NUMBER

ASSESSOR'S USE ONLY

THREE

(a) For $\int_0^{\frac{\pi}{2}} f(x)\sin x\,dx$

$\Rightarrow$ let $u = f(x)$ $\quad v' = \sin x$

then $u' = f'(x)$ $\quad v = -\cos x$ ✓

$$\int_0^{\frac{\pi}{2}} f(x)\sin x\,dx = \left[-\cos(x) f(x)\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos(x) f'(x)\,dx$$

$$= f(0) + \int_0^{\frac{\pi}{2}} \cos(x) f'(x)\,dx$$

$\cancel{= f(0) +}$ $\quad u = \cos(x) \quad v = f(x)$

$u' = -\sin(x) \quad v' = f'(x)$

$$= f(0) + \left[\cos(x) f(x)\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \sin(x) f(x)\,dx$$ //

$\cancel{2\int_0^{\frac{\pi}{2}} f(x)\sin x\,dx = f(0) +}$

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THREE (b) (i)

$\cancel{\frac{dy}{dx} = x - 2y}$

$$\frac{d}{dx} e^{2x} y = 2e^{2x} y + e^{2x} \frac{dy}{dx}$$

$$= 2e^{2x} y + e^{2x}(x - 2y)$$

$$= 2e^{2x} y + xe^{2x} - 2e^{2x} y$$

$$= xe^{2x}$$

(ii)

$$d(e^{2x} y) = xe^{2x}\, dx$$

$$\therefore \int e^{2x} y \, dy = \int xe^{2x}\, dx$$

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FOUR

(a) For $\frac{x^2}{2^2} - \frac{y^2}{6^2} = 1$:

$a = 2,\ b = 6$

$\therefore$ asymptotes $y = 3x$ or $y = -3x$

$\therefore$ the acute angle α between the asymptote and the x axis $= \tan^{-1} 3$

$\therefore \sin\alpha = \frac{3}{\sqrt{10}}$

$\frac{1}{2}\theta = 90^\circ - \alpha$

$\therefore \sin(\frac{1}{2}\theta) = \frac{1}{\sqrt{10}}$ $\quad \cos(\frac{1}{2}\theta) = \frac{3}{\sqrt{10}}$

$\sin\theta = 2\sin(\frac{1}{2}\theta)\cos(\frac{1}{2}\theta)$

$= 2 \times \frac{3}{10}$

$= 0.6$

(b) Suppose $P(m, n)$ $(m > 0)$

$\frac{m^2}{4} - \frac{n^2}{36} = 1$

$9m^2 - n^2 = 36$

$n^2 = 9m^2 - 36$

$n = \sqrt{9m^2 - 36}$ or $-\sqrt{9m^2-36}$

$= 3\sqrt{m^2-4}$ or $-3\sqrt{m^2-4}$

$\therefore P(m, 3\sqrt{m^2-4})$ or $P(m, -3\sqrt{m^2-4})$

$l_1: y = 3x \quad l_2: y = -3x$

~~PA =~~ For the distance d from a point (x_1, y_1) to a line $ax + by = 0$:

$$d = \frac{|ax_1 + by_1|}{\sqrt{a^2+b^2}}$$

For $P(m, 3\sqrt{m^2-4})$

$\therefore PA = \frac{|3\sqrt{m^2-4} - 3m|}{\sqrt{1+9}}$ $\quad PB = \frac{|3\sqrt{m^2-4} + 3m|}{\sqrt{1+9}}$

$= \frac{|3\sqrt{m^2-4} - 3m|}{\sqrt{10}}$ $\quad = \frac{|3\sqrt{m^2-4} + 3m|}{\sqrt{10}}$

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$\therefore$ ~~$PA \times PB = \frac{(3\sqrt{m^2-4}-3m)(3\sqrt{m^2-4}+3m)}{10}$~~

~~$= \frac{9(m^2-4)-9m^2}{10}$~~

From the graph: $3\sqrt{m^2-4} < 3m$, $3\sqrt{m^2-4} > 0$, $3m > 0$

$\therefore \quad PA \times PB = \frac{(3m - 3\sqrt{m^2-4})(3m + 3\sqrt{m^2-4})}{10}$

$= \frac{9m^2 - 9(m^2-4)}{10}$

$= \frac{36}{10} = 3.6$

When $P(m, -3\sqrt{m^2-4})$

similar to above: $PA = \frac{|-3\sqrt{m^2-4}-3m|}{\sqrt{10}} = \frac{3\sqrt{m^2-4}+3m}{\sqrt{10}}$ $\quad PB = \frac{|-3\sqrt{m^2-4}+3m|}{\sqrt{10}} = \frac{3m-3\sqrt{m^2-4}}{\sqrt{10}}$

This gives the same answer 3.6 //

(c) P is in the 1st quadrant $\Rightarrow x > 0$, $y > 0$

$CP : PD = 1 : \lambda$

$PD = \lambda CP$

Let $CP = a$, $PD = \lambda a$

$\therefore CD = (\lambda + 1)a$ //

QUESTION NUMBER

ASSESSOR'S USE ONLY

FIVE $z^{11} = 1$

(a) Let $z = \text{cis}\,\theta$

$(\text{cis}\,\theta)^{11} = 1$

$\therefore \text{cis}\,11\theta = 1$

$\therefore 11\theta = 0\pi, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi,$

$12\pi, 14\pi, 16\pi, 18\pi, 20\pi$

$\therefore \theta = 0, \frac{2}{11}\pi, \frac{4}{11}\pi, \frac{6}{11}\pi, \frac{8}{11}\pi, \frac{10}{11}\pi,$

$\frac{12}{11}\pi, \frac{14}{11}\pi, \frac{16}{11}\pi, \frac{18}{11}\pi, \frac{20}{11}\pi$

$\therefore z = 1, \text{cis}\frac{2}{11}\pi, \text{cis}\frac{4}{11}\pi, \text{cis}\frac{6}{11}\pi, \text{cis}\frac{8}{11}\pi, \text{cis}\frac{10}{11}\pi,$

$\text{cis}\frac{12}{11}\pi, \text{cis}\frac{14}{11}\pi, \text{cis}\frac{16}{11}\pi, \text{cis}\frac{18}{11}\pi, \text{cis}\frac{20}{11}\pi$ //

$\cancel{\text{cis}\frac{2}{11}\pi + \text{cis}\frac{12}{11} + \text{cis}\frac{4}{11}\pi +}$

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FIVE (b) EITHER

$3\frac{1}{2}$ hours = 210 min

$2\frac{1}{2}$ hours = 150 min

$2x_1 + 3x_2 + 4x_3 \leq 210$

$2x_1 + 3x_2 \leq 150$

$\cancel{4x_3 \geq 60}$ $4x_3 \geq 60 \Rightarrow x_3 \geq 15$

The total marks $M = 4x_1 + 5x_2 + 6x_3$

we want M to be maximised

as shown on page 26, ~~the~~ M is maximised

~~when $x_1 = 35$, $x_2 = 43.75$, $x_3 = 52.5$~~

Annotated Exemplar for 93202 Calculus Outstanding Scholarship **Candidate 37 – E125**		**Total Score**	**32**
Question	**Mark**	**Annotation**	
1	**8**	The candidate gained full marks for this question but only about 50% of candidates at outstanding scholarship level did this. **1a** correct y/x and dy/dx solved = 0, correct answer (sqr root)3 **1bi** correct first and second derivative and solution = 0 **1bii** correct third derivative and recognises pattern	
2	**8**	The candidate has provided evidence in **2a** of competently substituting trig identities and simplifying. In **2b** the candidate attempted to use trig identities but was unsuccessful. Only very top candidates gained full marks on this part question – hence it was a good question to discriminate for scholarship and outstanding candidates. **2c** is a typical correct response.	
3	**4**	In **3a** the candidate gives the most common response at all levels – a correct first step but then fails to proceed meaningfully. **3bi** is a typical response at outstanding level with succinct working. Question **3bii** was not well done by even the top candidates. This is a typical incorrect response to try to integrate both sides.	
4	**8**	The candidate has used an exact method with double angle formulae and gradient of asymptote in **4a** - a frequently seen successful approach. Many candidates at outstanding level did well on question **4b** - this one is typical in finding expressions for point P and PA and PB. No progress made on **4c**. Only very top candidates made any progress on this question and if they did so it was most commonly using a special case when CD is tangent to the hyperbola.	
5	**4**	The candidate has made progress in **5a** to obtain the exact particular solutions only. This was typical of many candidates who did not continue to prove the desired result. In **5b** this candidate is typical of most answers in recognising the constraints and the objective function in the Linear Programming option. Better outstanding candidates solved the equations algebraically or used points per minute to determine the optimal solution.	