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93202A

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SUPERVISOR'S USE ONLY

OUTSTANDING SCHOLARSHIP EXEMPLAR

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2015
Calculus

2.00 p.m. Tuesday 17 November 2015
Time allowed: Three hours
Total marks: 40

ANSWER BOOKLET

There are five questions in this examination. Answer ALL FIVE questions, choosing ONE option from part (b) of Question Four.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

The diagram for Question Four (b) Option 2 is on page 27 of this booklet.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

This examination consists of five questions.
Answer all FIVE questions, choosing ONE option from part (b) of Question Four.

ASSESSOR'S USE ONLY

QUESTION NUMBER

1a $\int_1^3 \left(2\pi\left(x^3+\frac{1}{12x}\right)\sqrt{1+\left(3x^2-\frac{1}{12x^2}\right)^2}\right)dx$

$=2\pi\int_1^3 \left(x^3+\frac{1}{12x}\right)\sqrt{1+\left(9x^4-\frac{1}{2}+\frac{1}{144x^4}\right)}\,dx$

$=2\pi\int_1^3 \left(x^3+\frac{1}{12x}\right)\sqrt{9x^4+0.5+\frac{1}{144x^4}}\,dx$

$=2\pi\int_1^3 \left(x^3+\frac{1}{12x}\right)\sqrt{\left(3x^2+\frac{1}{12x^2}\right)^2}\,dx$

$=2\pi\int_1^3 \left(x^3+\frac{1}{12x}\right)\left(3x^2+\frac{1}{12x^2}\right)dx$

$=2\pi\int_1^3 \left(3x^5+\frac{x}{4}+\frac{x}{12}+\frac{1}{144x^3}\right)dx$

$=2\pi\int_1^3 \left(3x^5+\frac{x}{3}+\frac{1}{144x^3}\right)dx$

~~$=2\pi\int_1^3$~~

~~$2\pi\left[(3)x^4+\frac{1}{3}+\right.$~~

$=2\pi\left[\frac{x^6}{2}+\frac{x^2}{6}-\frac{1}{288x^2}\right]_1^3$

$=2\pi\left(\left(\frac{3^6}{2}+\frac{3^2}{6}-\frac{1}{288\times3^2}\right)-\left(\frac{1^6}{2}+\frac{1^2}{6}-\frac{1}{288\times1^2}\right)\right)$

$=2\pi\left(\left(\frac{729}{2}+\frac{3}{2}-\frac{1}{2592}\right)-\left(\frac{1}{2}+\frac{1}{6}-\frac{1}{288}\right)\right)$

≈ 2295.48

3

1.b. $f'(x)=(f(x))^3$

$y=f(x)$

$\frac{dy}{dx}=y^3$

$\frac{1}{y^3}\frac{dy}{dx}=1$

$\int\left(\frac{1}{y^3}\frac{dy}{dx}\right)dx=\int 1\,dx$

$\int\frac{1}{y^3}\,dy=\int 1\,dx$

$-\frac{1}{2y^2}=x+C$

$-\frac{1}{x+C}=2y^2$

$-\frac{1}{2x+C'}=y^2$

$y=\pm\sqrt{\frac{1}{2x+C'}}$ MEI

As $f(0)=2$

$2=\pm\sqrt{\frac{1}{2\times0+C'}}$

QUESTION NUMBER

ASSESSOR'S USE ONLY

1. b. (cont) $2 = \pm\sqrt{\frac{1}{C}}$

Must be + as $\sqrt{x}$ is never negative.

$2 = \sqrt{\frac{1}{C}}$

$4 = \frac{1}{C}$

$C = \frac{1}{4}$

$y = \sqrt{\frac{1}{(2x+\frac{1}{4})}}$ CON

c. Initial amount of salt is $200 \times 0.5 = 100$ kg

m = mass of salt in tank

$\frac{dm}{dt} = (0.8 \times 6) - (\frac{m}{200} \times 6)$

$\frac{dm}{dt} = 4.8 - 0.03m$

$\frac{1}{4.8 - 0.03m}\frac{dm}{dt} = 1$

$-\frac{100}{3}\left(\frac{1}{m-160}\right)\frac{dm}{dt} = 1$

$-\frac{100}{3}\int\left(\frac{1}{m-160}\frac{dm}{dt}\right)dt = \int 1\, dt$

$-\frac{100}{3}\int\frac{1}{m-160}\,dm = \int 1\, dt$

$-\frac{100}{3}\ln|m-160| = t + C$

$m - 160 = e^{-\frac{3t}{100}+C}$

$m = Ae^{-\frac{3t}{100}} + 160$

$m = 160 + Ae^{-0.03t}$

$100 = 160 + Ae^{-0.03 \times 0}$

$100 = 160 + A \times 1$

$A = -60$

$m = 160 - 60e^{-0.03t}$

3

3

QUESTION NUMBER

ASSESSOR'S USE ONLY

1.e. (cont.)

$130 = 160 - 60e^{-0.03t}$

$60e^{-0.03t} = 30$

$e^{-0.03t} = \frac{1}{2}$

$\ln(e^{-0.03t}) = \ln(\frac{1}{2})$

$-0.03t = \ln(\frac{1}{2})$

$t = \frac{\ln(\frac{1}{2})}{-0.03}$

$t \approx 23.1$ minutes.

3

QUESTION NUMBER | ASSESSOR'S USE ONLY

2(a)

$9^{2x+y} - 9^{x} \times 3^{y} = 6$

$9^{2x+y} - 3^{2x} \times 3^{y} = 6$

$9^{2x+y} - 3^{2x+y} = 6$

$3^{4x+2y} - 3^{2x+y} = 6$

$3^{2x+y}(3^{2x+y} - 1) = 6$

$u = 3^{2x+y}$

$u(u-1) = 6$

$u^2 - u - 6 = 0$

$(u+2)(u-3) = 0$

$u = -2$ or 3

$3^{2x+y} = -2$ or 3

3^n is always positive

$3^{2x+y} = 3$

$2x + y = 1$

$y = 1 - 2x$

$\log_{x+1}(1-2x+3) + \log_{x+1}(1-2x+x+4) = 3$

$\log_{x+1}(4-2x) + \log_{x+1}(5-x) = 3$

$\log_{x+1}((4-2x)(5-x)) = 3$

$\log_{x+1}(20-10x-4x+2x^2) = 3$

$\log_{x+1}(2x^2-14x+20) = 3$

$x+1^{\log_{x+1}(2x^2-14x+20)} = (x+1)^3$

$2x^2 - 14x + 20 = (x+1)(x^2+2x+1)$

$2x^2 - 14x + 20 = x^3 + 3x^2 + 3x + 1$

$0 = x^3 + x^2 + 17x - 19$

$x = 1$ (only real root, by calculator)

$y = 1 - 2x$

$y = 1 - 2$

$y = -1$

4

4

QUESTION NUMBER

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Assuming cars headlights illuminate straight line directly ahead.

2b. $y = ax^2$

$y = 100,\ x = -100$

$100 = a(-100)^2$

$a = \frac{1}{100}$

$y = \frac{x^2}{100}$

$y_{stat} = 50,\ x_{stat} = 100$

$\frac{dy}{dx} = \frac{x}{50}$

$\frac{x}{50} = \frac{50-x}{100-y}$

$y = \frac{x^2}{100}$

$\frac{x}{50} = \frac{50-x}{100-\frac{x^2}{100}}$

$\frac{1}{50}\left(x\left(100-\frac{x^2}{100}\right)\right) = 50-x$

$\frac{1}{50}\left(100x - \frac{x^3}{100}\right) = 50-x$

$2x - \frac{x^3}{5000} = 50-x$

$0 = \frac{x^3}{5000} - 3x + 50$

$x \approx 113.09,\ 16.99$ or -130.08 (by calculator)

$x \approx 130.08$ is before the car starts

$x \approx 113.09$ is when the rear lights illuminate the statue

$\therefore x \approx 16.99\ 394433$

$y = \frac{x^2}{100}$

$y \approx 2.888$ //

~~$x = ay^2$~~

~~$y = 100,\ x = 100$~~

~~$-100 = a(100)^2$~~

~~$a = -\frac{1}{100}$~~

~~$x = -\frac{y^2}{100}$~~

~~$y_{stat} = 50,\ x_{stat} = 100$~~

~~$\frac{dx}{dy} = -\frac{y}{50}$~~

2c. $\frac{dS}{dt} = kS(N-S)$

$\frac{dS}{dt} = k(SN - S^2)$

$\frac{1}{SN-S^2}\frac{dS}{dt} = k$

$\int \frac{1}{SN-S^2}\frac{dS}{dt}\,dt = \int k\,dt$

$\int \frac{1}{SN-S^2}\,dS = kt + c$ //

QUESTION NUMBER | ASSESSOR'S USE ONLY

2.c. (cont.)

$\frac{1}{S(N-S)} = \frac{A}{S} + \frac{B}{N-S}$

$1 = A(N-S) + BS$

$1 = AN - AS + BS$

$A = B$

$1 = AN$

$A = \frac{1}{N} = B$

$\frac{1}{S(N-S)} = \frac{1}{NS} + \frac{1}{N^2 - SN}$

$\int \frac{1}{NS} + \frac{1}{N^2 - SN} \, dS = kt + C$

$\frac{1}{N} \int \frac{1}{S} + \frac{1}{N-S} \, dS = kt + C$

$\frac{1}{N}(\ln|S| - \ln|N-S|) = kt + C$

$\ln\left(\frac{S}{N-S}\right) = Nkt + C$

$\frac{S}{N-S} = e^{Nkt+C}$

$S = (N-S)e^{Nkt+C}$

$S = (N-S)Ae^{Nkt}$

$2 = (N-2)Ae^{Nk\times 0}$

$2 = (N-2)A \times 1$

$A = \frac{2}{N-2}$

$S = (N-S)\frac{2}{N-2}e^{Nkt}$

$S\left(1 + \frac{2}{N-2}e^{Nkt}\right) = N\frac{2}{N-2}e^{Nkt}$

$S = \frac{N\frac{2}{N-2}e^{Nkt}}{1 + \frac{2}{N-2}e^{Nkt}}$

$S = \frac{N}{1 + \frac{1}{\left(\frac{2}{N-2}e^{Nkt}\right)}}$

$S = \frac{N}{1 + \frac{1}{\left(\frac{2}{N-2}\right)}e^{-Nkt}}$

$S = \frac{N}{1 + \frac{N-2}{2}e^{-Nkt}}$

$S = \frac{N}{1 + \frac{1}{2}e^{-kNt}(N-2)}$

3

3

QUESTION NUMBER

ASSESSOR'S USE ONLY

3a $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$z^n + \frac{1}{z^n} = (\cos\theta + i\sin\theta)^n + \frac{1}{(\cos\theta + i\sin\theta)^n}$

QUESTION NUMBER

ASSESSOR'S USE ONLY

~~3b.~~

3a. (cont)

$z = \cos\theta$

$\cos^6\theta + \frac{1}{\cos^6\theta} = 2\cos 6\theta$

$\cos^6\theta = 2\cos 6\theta - \frac{1}{\cos^6\theta}$

QUESTION NUMBER

ASSESSOR'S USE ONLY

3.b. $\cancel{\log}\ \log(2\sin x-1)-\log(2)=\log(1-y)-\log(2\sin x-1)$.

$\log\left(\frac{2\sin x-1}{2}\right)=\log\left(\frac{1-y}{2\sin x-1}\right)$

$\frac{2\sin x-1}{2}=\frac{1-y}{2\sin x-1}$

$(2\sin x-1)^2=2-2y$

$\cancel{4\sin^2 x-4\sin x+1=2-2y}$

$-1\leq \sin x\leq 1$

$-2\leq 2\sin x\leq 2$

$-3\leq 2\sin x-1\leq 1$

$0\leq(2\sin x-1)^2\leq 9$

$0\leq 2-2y\leq 9$

$-2\leq -2y\leq 7$

$-7\leq 2y\leq 2$

$-\frac{7}{2}\leq y\leq 1$

2 2

3.c. $\frac{4\cos^2 2x-4\cos^2 x+3\sin^2 x}{4\cos^2(\frac{5\pi}{2}-x)-\sin^2(2(x-\pi))}$

$=\frac{4(\overset{2\cos^2 x-1}{\cancel{\cos^2 x-\sin^2 x}})^2-4\cos^2 x+3\sin^2 x}{4\cos^2(x-\frac{5\pi}{2})-\sin^2(2x-2\pi)}$

$=\frac{4(\overset{4\cos^4 x-4\cos^2 x+1}{\cancel{\cos^4 x-2\cos^2 x\sin^2 x+\sin^4 x}})-4\cos^2 x+3\sin^2 x}{4\cos^2(x-\frac{\pi}{2})-\sin^2 2x}$

$=\frac{16\cos^4 x-16\cos^2 x+4-4\cos^2 x+3\sin^2 x}{4\sin^2(x)-(2\sin x\cos x)^2}$.

$=\frac{16\cos^4 x-20\cos^2 x+3\sin^2 x+4}{4\sin^2 x-4\sin^2 x\cos^2 x}$

$=\frac{16\cos^4 x-20\cos^2 x+3(1-\cos^2 x)+4}{4\sin^2 x(1-\cos^2 x)}$

$=\frac{16\cos^4 x-20\cos^3 x+3-3\cos^2 x+4}{4\sin^2 x(\sin^2 x)}$

QUESTION NUMBER

3.c. (cont)

$$= \frac{16\cos^4 x - 23\cos^2 x + 7}{4\sin^4 x}$$

$$= \frac{16(\cos^2 x - 1)(\cos^2 x - \frac{7}{16})}{4\sin^4 x}$$

$$= \frac{16(1-\cos^2 x)(\frac{7}{16} - \cos^2 x)}{4\sin^4 x}$$

$$= \frac{16\sin^2 x(\frac{7}{16} - \cos^2 x)}{4\sin^4 x}$$

$$= \frac{16(\frac{7}{16} - \cos^2 x)}{4\sin^2 x}$$

$$\cancel{= \frac{(1-\cos^2 x) - \frac{9}{16}}{4\sin^2 x}}$$

$$\cancel{= \frac{\sin^2 x - \frac{9}{16}}{4\sin^2 x}}$$

$$\cancel{= \frac{1}{4} - \frac{\frac{9}{16}}{4\sin^2 x}}$$

MEI

$$= \frac{7 - 16\cos^2 x}{4\sin^2 x}$$

$$= \frac{16\cos^2 x - 7}{-4\sin^2 x}$$

$$= \frac{8(2\cos^2 x - 1) + 1}{-4\sin^2 x}$$

$$= \frac{8\cos 2x + 1}{-4\sin^2 x}$$

$$= \frac{8\cos 2x + 1}{(-4\sin^2 x)}$$

$$= \frac{8\cos 2x + 1}{(2((1 - 2\sin^2 x) - 1)}$$

$$= \frac{8\cos 2x + 1}{(2(\cos 2x - 1))}$$

$$= \frac{8\cos 2x + 1}{2(\cos 2x - 1)}$$

3

3

QUESTION NUMBER

ASSESSOR'S USE ONLY

4a. $3z^3+(2-3ai)z^2+(6+2bi)z+4=0$

0

QUESTION NUMBER

ASSESSOR'S USE ONLY

4.b.i $11x + 10y + 9z \geq 1000$ ①

$3x + 4y + 5z > 6x + 2y + 2z$

$2y + 3z > 3x$ ②

$6x + 2y + 2z \leq 500$ ③

$3x + 4y + 5z \leq 400$ ④

$2x + 4y + 2z \leq 300$ ⑤

4.b.ii $x + y + z = 100$

$z = 100 - x - y$

$11x + 10y + 900 - 9x - 9y \geq 1000$

$2x + y \geq 100$ ①

$2y + 300 - 3x - 3y > 3x$

$300 - y > 6x, \quad y \lneq 300 - 6x$ ②

$6x + 2y + 200 - 2x - 2y \leq 500$

$4x \leq 300, \quad x \leq 75$ ③

$3x + 4y + 500 - 5x - 5y \leq 400$

$-2x - y \leq -100$

$2x + y \geq 100$ ④

$2x + 4y + 200 - 2x - 2y \leq 300$

$2y \leq 100$

$y \leq 50$ ⑤

Feasible region shaded pg. 27

→ See page 27

QUESTION NUMBER | ASSESSOR'S USE ONLY

5a. $y = kx^2$

$\frac{dy}{dx} = 2kx$

at x_0, tangent $m = 2kx_0$

normal $m = -\frac{1}{\text{tangent}}$

$= -\frac{1}{2kx_0}$

$y = -\frac{1}{2kx_0}x + c$

$kx_0^2 = -\frac{1}{2kx_0}x_0 + c$

$c = kx_0^2 + \frac{1}{2kx_0}x_0$

$c = kx_0^2 + \frac{1}{2k}$

$y = \frac{-1}{2kx_0}x + kx_0^2 + \frac{1}{2k}$

2

5.b. Intersection with other arm a (x_1, y_1)

$y_1 = \frac{-1}{2kx_0}x_1 + kx_0^2 + \frac{1}{2k}$

$y_1 = kx_1^2$

$kx_1^2 = \frac{-1}{2kx_0}x_1 + kx_0^2 + \frac{1}{2k}$

$kx_1^2 + \frac{1}{2kx_0}x_1 - \left(kx_0^2 + \frac{1}{2k}\right) = 0$

$x_1 = \frac{-\frac{1}{2kx_0} \pm \sqrt{\frac{1}{4k^2x_0^2} + 4k\left(kx_0^2 + \frac{1}{2k}\right)}}{2k}$

(plus gives greater value of x_1, so original intersection, so minus needed)

$x_1 = -\frac{1}{4k^2x_0} - \frac{\sqrt{\frac{1}{4k^2x_0^2} + 4k^2x_0^2 + 2}}{2k}$

~~$\frac{dx_1}{dx_0} = \frac{1}{4k^2x_0^2} - \frac{1}{2k}\left(-\frac{1}{2k^2x_0^3} + 8k^2x_0 + \frac{1}{2}\left(\frac{1}{4k^2x_0^2} + 4k^2x_0^2 + 2\right)^{-\frac{1}{2}}\right)$~~

~~$0 =$~~

QUESTION NUMBER

ASSESSOR'S USE ONLY

$y_1 = kx_1^2$

$\frac{y_1}{k} = x_1^2$

$-\sqrt{\frac{y_1}{k}} = x_1$ (negative 'cause looking for left arm)

$y_1 = \frac{a-1}{2kx_0} \times -\sqrt{\frac{y_1}{k}} + kx_0^2 + \frac{1}{2k}$

$y_1 = \frac{1}{2kx_0}\sqrt{\frac{y_1}{k}} + kx_0^2 + \frac{1}{2k}$

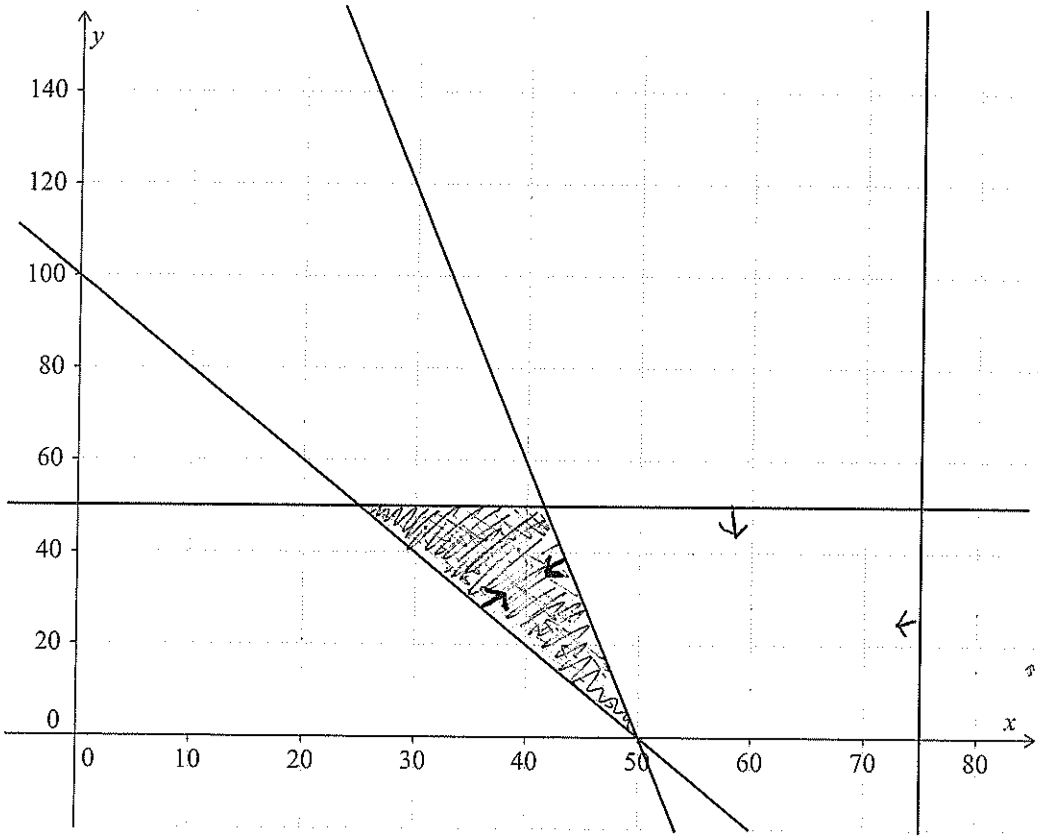
$y_1 = \frac{1}{2k\sqrt{k}x_0}(y_1)^{\frac{1}{2}} + kx_0^2 + \frac{1}{2k}$

$\frac{dy_1}{dx_0} = \left(\frac{1}{2k\sqrt{k}x_0} \times \frac{dy_1}{dx_0}\frac{1}{2}(y_1)^{-\frac{1}{2}}\right) + \left(-\frac{1}{2k\sqrt{k}x_0^2}(y_1)^{\frac{1}{2}}\right) + kx_0^2 + \frac{1}{2k}$

Minimum when $\frac{dy_1}{dx_0} = 0$

~~$0 = \left(\frac{1}{2k\sqrt{k}x_0} \times 0\frac{1}{2}(y_1)^{-\frac{1}{2}}\right) +$~~

Diagram for Question Four (b).



ASSESSOR'S USE ONLY

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Annotated Exemplar for 93202 Calculus Outstanding Scholarship			**Total Score**	**32**
Question	**Mark**	**Annotation**		
1	**8**	The candidate has shown algebraic competence in **1a** in recognising the perfect square and used efficient integration techniques in **1a** and **1b**. The candidate also set up the correct differential equation in **1c** and found its solution.		
2	**8**	The candidate has provided evidence in **2a** of competently solving simultaneous equations with logarithms and exponents. The candidate recognised and correctly used partial fractions in **2c**. The candidate correctly understood the relationship between the tangent and the curve in **2b**.		
3	**6**	The candidate did not give an adequate proof or expansion in **3a**. The candidate made progress in **3b** to obtain an equation in one variable but did not recognise the restricted range of the expression. The candidate has shown competence in **3c** in simplifying and expanding trigonometric expressions and sound use of trigonometric identities.		
4	**6**	The candidate was unable to explain conditions for a solution in **4a**. In **4b Option 2**, the candidate has communicated clearly the equations which describe the linear programming context. The equations have been correctly simplified to be in two variables and the feasible region shaded.		
5	**4**	The candidate has made progress to obtain the equation of the normal to the curve at a general point in **5a**. Inadequate attempts were made at **5b** and **5c**.		