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SUPERVISOR'S USE ONLY

93202A

932021

TOP SCHOLAR

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2017
Calculus

9.30 a.m. Friday 10 November 2017
Time allowed: Three hours
Total marks: 40

ANSWER BOOKLET

There are five questions in this examination. Answer ALL FIVE questions.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Question	Mark
ONE	
TWO	
THREE	
FOUR	
FIVE	
TOTAL	/40

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QUESTION NUMBER

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1a) find x, y integers in $x^4 - y^2 = 71$

$$(x^2+y)(x^2-y) = 71$$

71 is prime so either $\begin{cases} x^2+y = 1, \\ x^2-y = 71 \end{cases}$

[as both x^2 and y are integer, hence $x^2 \pm y$ is integer] or $\begin{cases} x^2+y = 71 \\ x^2-y = 1 \end{cases}$

either : $2x^2 = 72$, $x = \pm 6$, $y = -35$

or : $2x^2 = 72$, $x = \pm 6$, $y = +35$

$x = \pm 6$, $y = \pm 35$

in any combination

b) $(x^2 - bx)(p+1) = (ax+c)(p-1)$, $p \neq \pm 1$

$x^2 p - bpx + x^2 - bx = apx + cp - ax - c$

$x^2 \underline{(p+1)} + x \underline{(-bp - b - ap + a)} - \underline{cp + c} = 0.$

quadratic formula $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

hence ['b' = 0,]

[$b^2 - 4ac > 0$.]

~~$-bp - b - ap + a = 0$~~,

~~[$-4ac > 0$. , $ac < 0$.]~~

~~$c(p+1)(p-1) > 0$.~~ , ~~$-1 < 1$~~, ~~$p > 1$~~ ~~$p < 1$~~.

~~$\Rightarrow$~~ ~~$p < -1, 1 < p$.~~

~~$-p(b+a) + a - b = 0$.~~

~~$p = \frac{a-b}{a+b}$~~

~~$(-(p+1)b - a(p-1))^2$~~

~~$+4c(p+1)(p-1) > 0$~~,

continue pg 13.

~~continue pg 13~~

QUESTION NUMBER

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$y = \pm 2\sqrt{ax}$

c) $y^2 = 4ax$ $\qquad \frac{dy}{dx} = \frac{4a}{2y} = \pm\sqrt{\frac{a}{x}}$

$A(x_0, y_0) \qquad F(a, 0)$

tangent at A: $\quad y = \pm\sqrt{\frac{a}{x_0}}\,(x - x_0) + y_0$

assume $y_0 > 0$, $\quad y = \pm\sqrt{\frac{a}{x_0}}\,x \mp \sqrt{ax_0} + y_0$

when $y = 0$, $\quad \sqrt{\frac{a}{x_0}}\,x = \sqrt{ax_0} - y_0$

$$= \sqrt{ax_0} - 2\sqrt{ax_0}$$

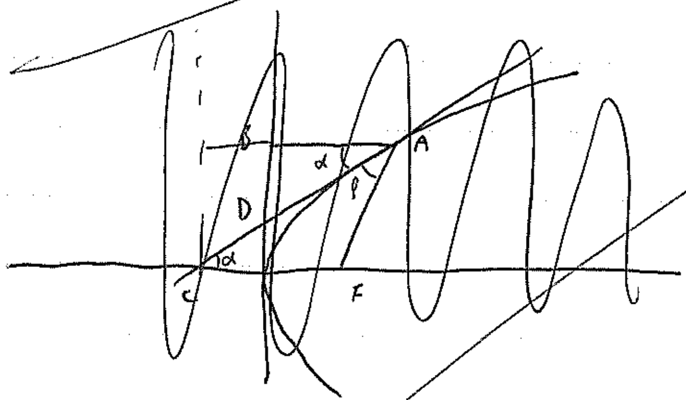
$$= -\sqrt{ax_0}$$

$$x = -x_0$$

$$C(-x_0, 0)$$

~~gradient of $\overline{AC}$: $\sqrt{\frac{a}{x_0}}$~~

~~gradient of $\overline{AF}$: $\frac{y_0}{x_0 - a}$~~



~~one property of the parabola is that the distance to the directrix = distance to the focal point.~~

~~notice that $\overline{OC}$ and $\overline{BA}$ are the same length, x_0, and same angles means that $\triangle OCD$ and $\triangle DAB$ are the same triangle of same length and angles.~~

~~$\angle AFC = 180 - \alpha - \beta$ cointerior~~

~~$\angle CDO = 90 - \alpha$ right angled triangle~~

~~$\angle ADO = \alpha + 90$ same line~~

continued pg 13

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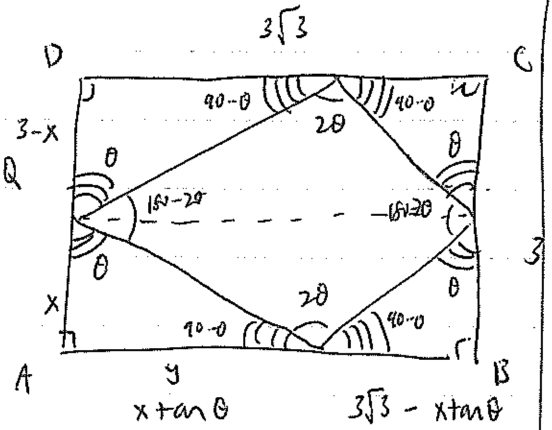
2a) due to right angles,

$\angle BRQ = \angle CRS = \cancel{\angle SDT}\ \angle DPS = \angle APQ$



$\tan\theta = \frac{y}{x}, \quad y = x\tan\theta$

~~the The inner shape is a kite as it has two identical triangles, same angles and lengths.~~

~~$\overline{PQ} = \sec\theta\, x$~~

~~$\overline{QR} = \overline{QR} = \frac{1}{\sin\theta}\ 3\sqrt{3} - x\tan\theta$~~

~~$= \frac{3\sqrt{3}}{\sin\theta} - \frac{x}{\sec\theta}$~~

~~$\sin\theta = \frac{3\sqrt{3} - x\tan\theta}{QR}$~~

~~$QR = \frac{3\sqrt{3}}{\sin\theta} - \frac{x}{\sec\theta}$~~

~~perimeter:~~

~~due to the symmetry of the kite and its the kite must lay symmetrically within the rectangle for all the angles to be equal to one another.~~ (eg θ, ~~$90-\theta$~~).

~~hence x =~~

i) $\overline{PQ} = (\sec\theta)x$

$\overline{QR} = \cancel{\frac{3\sqrt{3}}{\sin\theta} - \frac{x}{\sec\theta}} \quad \frac{3\sqrt{3} - x\tan\theta}{\sin\theta} = \frac{3\sqrt{3}}{\sin\theta} - \cancel{\sec x}\ x\sec\theta$

~~perimeter $= 2\left(\frac{3\sqrt{3}}{\sin\theta} + 2x\left(\frac{1}{\cos\theta} - \cos\theta\right)\right)$~~

$= \cancel{2\left(\frac{3\sqrt{3}}{\sin\theta} + x\tan\theta\right)}$

perimeter $= 2\left(\frac{3\sqrt{3}}{\sin\theta}\right) = \frac{6\sqrt{3}}{\sin\theta}$.

2aii) on pg 12

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2b)

$x+y = z+1 \quad \cdots ①$

$x^2+y^2+2xy = z^2+5 \quad \cdots ②$

$\cancel{x+y = \sqrt{z^2+5}}$ $① \times ① = \quad x^2+2xy+y^2 = z^2+2z+1$

$2z+1 = 5 \quad , \quad z = 2.$

$x+y = 3$

$x^3+y^3+3xy = 43+8 = 51$

$x^3+3x^2y+3xy^2+y^3 = 27$

$3x^2y+3xy^2-3xy = -24$

$x^2y+xy^2-xy = -8$

$x+y-1 = -\frac{8}{xy}$

sub $x+y=3$,

$2 = -\frac{8}{xy}$

$xy = -4$

$y = -\frac{4}{x}$

$x - \frac{4}{x} = 3 \quad$ (sub in $x+y=3$)

$x^2-3x-4=0, \quad x = 4, -1$

$y = -1, 4$

$z = 2.$

$$\begin{cases} x = 4 \\ y = -1 \\ z = 2 \end{cases} \quad \begin{cases} x = -1 \\ y = 4 \\ z = 2 \end{cases}$$

this or this

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3a) $\ln y = x^x \cdot \ln x$

$m = x^x$ let $m = x^x$ to find $\frac{d}{dx}(x^x)$

$\ln m = x \ln x$

$\frac{1}{m}\frac{dm}{dx} = \ln x + 1$

$\frac{dm}{dx} = x^x(\ln x + 1)$

hence

$\frac{1}{y}\frac{dy}{dx} = x^x(\ln x + 1)\cdot \ln x + \frac{1}{x}\cdot x^x$

$= x^x(\ln x(\ln x + 1) + \frac{1}{x})$

$\frac{dy}{dx} = x^{2x}(\ln x)(\ln x(\ln x+1) + \frac{1}{x})$

$\left.\frac{dy}{dx}\right|_{x=2} = 16(\ln 2)(\ln 2(\ln 2 + 1) + \frac{1}{2})$

≈ 18.5608

b) i) $\frac{dy}{dx} = e^x \cdot \cos x + e^x \sin x$

$= e^x(\sin x + \cos x)$

$= e^x(\sin x + \sin(x + \frac{\pi}{2}))$

$= e^x(2\sin(x + \frac{\pi}{4})\cos(\frac{\pi}{4}))$

$= \frac{2}{\sqrt{2}} e^x \sin(x + \frac{\pi}{4})$

$= 2^{\frac{1}{2}} e^x \sin(x + \frac{\pi}{4})$ as required.

ii) $\frac{d^2y}{dx^2} = \cancel{\left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right)y} =$

$-e^x \sin x + e^x \cos x + e^x \sin x + e^x \cos x$

$= 2e^x \cos x$

$= 2e^x \sin(x + \frac{\pi}{2})$

iii) notice the x 2 and sin shift of $\frac{\pi}{2}$.

$\frac{d^ny}{dx^n} = 2^{\frac{n}{2}} e^x \sin(x + \frac{n\pi}{4})$.

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when $x=0$, $\frac{d^ny}{dx^n} = 2^{\frac{n}{2}} \cdot 1 \cdot \sin\left(\frac{n\pi}{4}\right)$

$= 2^{\frac{n}{2}} \cdot \sin\left(\frac{n\pi}{4}\right)$.

where ~~as~~ $\sin\left(\frac{n\pi}{4}\right)$ can only be fixed values:

$0, \pm\frac{1}{\sqrt{2}}, \pm 1,$ dependent on n.

c) $\cancel{\int \frac{1}{\sqrt{x^2+1}}\,dx = dy}$ $\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$

integration via by substitution, let $x = \sinh u$

$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{1}{4}(e^{2u}-2+e^{-2u})+1}}$

$= \frac{1}{\frac{1}{2}e^{u}+e^{-u}}$

$= \cancel{\frac{1}{\cosh u}} \cdot \frac{1}{\cosh u}$

$y = \int \frac{1}{\cosh u}\,dx$.

$\frac{dx}{du} = \frac{d}{dx}\left(\frac{1}{2}(e^{x}-e^{-x})\right)$

$= \frac{1}{2}(e^{x}+e^{-x})$

$= \cosh u$

$y = \int 1\,du$

$= u + C$

$y = \cancel{\sin} \sinh^{-1}x + C$

hence $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{x^2+1}}$.

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4a) differentiate the RHS.

$$\frac{-\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x} + \frac{\sin 3x}{\cos 3x}$$

$$= -\tan x - \tan 2x + \tan 3x$$

$$\cancel{\tan 2x = \frac{2\tan x}{1-\tan^2 x}} \qquad \cancel{\tan 3x = \frac{\tan x + \tan 2x}{1-\tan x \tan 2x}}$$

$$\cancel{\text{LHS} : \tan x \tan 2x \tan 3x} =$$

$$= \cancel{\frac{\sin x \sin 2x}{\cos x \cos 2x} \tan 3x} \qquad \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$= \cancel{\frac{\cos x - \cos 3x}{\cos x + \cos 3x} \tan 3x}$$

$$\tan 3x = \frac{\tan x + \frac{2\tan x}{1-\tan^2 x}}{1 - \frac{2\tan^2 x}{1-\tan^2 x}} = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

let $\tan x = A$.

$$\text{RHS (differentiated)} = -A - \frac{2A}{1-A^2} + \frac{3A - A^3}{1-3A^2}$$

$$= A\left(-1 - \frac{2}{1-A^2} + \frac{3-A^2}{1-3A^2}\right)$$

$$= A\left(-\frac{2}{1-A^2} + \frac{2+2A^2}{1-3A^2}\right)$$

$$= A\left(\frac{-2+6A^2+2-2A^2+2A^2-2A^4}{(1-A^2)(1-3A^2)}\right)$$

$$= A\left(\frac{6A^2-2A^4}{(1-A^2)(1-3A^2)}\right) = A\left(\frac{2A}{1-A^2}\right)\left(\frac{3A-A^3}{1-3A^2}\right)$$

$$= \tan x \tan 2x \tan 3x = \text{LHS}$$. (differentiated)

as required

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4b) $r = a(1-\cos\theta)$ $\qquad \frac{dr}{d\theta} = a\sin\theta$

$= a - a\cos\theta$

$$S = 2\int_0^{\pi} \sqrt{a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}\ d\theta$$

$$= 2a\int_0^{\pi} \sqrt{1 - 2\cos\theta + 1}\ d\theta$$

$$= 2a\int_0^{\pi} \sqrt{2 - 2\cos\theta}\ d\theta$$

let $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

$$= 2a\int_0^{\pi} \sqrt{4\sin^2\frac{\theta}{2}}\ d\theta = 2a\int_0^{\pi} \cdot 2\sin\frac{\theta}{2}\ d\theta$$

$$= 4a\left[-2\cos\frac{\theta}{2}\right]_0^{\pi} = 4a(0+2)$$

$$= 8a$$

4c) $a = -\frac{gR^2}{(x+R)^2}$ $\qquad \frac{d}{dt}x = v$

$\int a\ dt = v.$

let initial velocity $= v_0$, acceleration : $-\frac{gR^2}{(x+R)^2}$.

$\frac{dv}{dt} = -\frac{gR^2}{(x+R)^2}$ $\qquad dv = -\frac{gR^2}{(x+R)^2}\ dt$

$= -\frac{gR^2}{(x+R)^2}\ dx \cdot \frac{dt}{dx}$

$v\ dv = -\frac{gR^2}{(x+R)^2}\ dx$

$\frac{v^2}{2} = +\frac{gR^2}{(x+R)^1} + C$

when ~~t=0~~, $v=0$, $x=h$.

continue pg 12

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5a) i) $\cos 2\theta = 2\cos^2\theta - 1$

$\cos 3\theta = \cos\theta \cdot (2\cos^2\theta - 1) - \sin\theta\,(2\sin\theta\cos\theta)$

$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$

$= 4\cos^3\theta - 3\cos\theta$

$\sin 3\theta = \sin\theta\,(1 - 2\sin^2\theta) + \cos\theta\,(2\sin\theta\cos\theta)$

$= \sin\theta - 2\sin^3\theta + 2\sin\theta\,(1 - \sin^2\theta)$

$= 3\sin\theta - 4\sin^3\theta$

$\cos 5\theta = (2\cos^2\theta - 1)(4\cos^3\theta - 3\cos\theta) - (3\sin\theta - 4\sin^3\theta)(2\sin\theta\cos\theta)$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 6(1-\cos^2\theta)\cos\theta + 8\sin^4\theta\cos\theta$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 6\cos\theta + 6\cos^3\theta + 8(1 - 2\cos^2\theta + \cos^4\theta)\cos\theta$

$= 16\cos^5\theta - 26\cos^3\theta + 11\cos\theta - 6\cos\theta + 6\cos^3\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ as required

ii) ~~$\left(x - \cos\frac{2\pi}{9}\right)\left(x - \cos\frac{4\pi}{9}\right)\left(x - \cos\frac{8\pi}{9}\right)$~~

when $\cos 5\theta = \cos 4\theta$,

$5\theta = 4\theta \pm 2\pi n \Rightarrow \theta = \pm 2\pi n$

or $-4\theta \pm 2\pi n \Rightarrow 9\theta = \pm 2\pi n$

$\theta = \pm \frac{2\pi n}{9}$

hence $\cos 5\theta - \cos 4\theta = 0$ will have these roots.

$\cos 4\theta = 2\cos^2 2\theta - 1 = 2(2\cos^2\theta - 1)^2 - 1$

$= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

$\cos 5\theta - \cos 4\theta = 16\cos^5\theta - 8\cos^4\theta - 20\cos^3\theta + 8\cos^2\theta + 5\cos\theta - 1 = 0$

will have these roots (and more).

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b)

1	2	3	4	5	6	7
2	7	$\frac{9}{2}$	$\frac{23}{4}$	$\frac{41}{8}$		
a	b	$\frac{a+b}{2}$	$\frac{a+3b}{4}$	$\frac{3a+5b}{8}$	$\frac{5a+11b}{16}$	$\frac{11a+21b}{32}$

3	4	5	6	7	8	9
$\frac{(1,1)}{2}$	$\frac{(1,3)}{4}$	$\frac{(3,5)}{8}$	$\frac{(5,11)}{16}$	$\frac{(11,21)}{32}$	$\frac{(21,43)}{64}$	$\frac{(43,85)}{128}$

for odd $n \geq 3$,

a lags b by one

$$\underline{a_{n+1} + a_n = 2^{n-2}}, \quad \cancel{a_{n+2} = 2^{n-1}}$$

$$\cancel{a_{n+2} + a_{n+1} = 2^{n-1}}$$

$$\cancel{a_{n+3} + a_{n+2} = 2^n}$$

$$\cancel{a_{n+2} = \frac{1}{2}a_{n+1} + \frac{1}{2}a_n}$$

$$\cancel{\frac{3}{2}a_{n+1} + \frac{1}{2}a_n}$$

$$\cancel{a_{n+3} - a_n =}$$

n	a_n	2^{n-3}	
3	1	1	0
4	1	2	-1
5	3	4	-1
6	5	8	-3
7	11	16	-5
8	21	32	-11
9	43	64	-21

	a_n	2^{n-2}	difference
3	1	2	1
4	1	4	3 → a_{n+1}
5	3	8	5
6	5	16	11
7	11	32	21
8	21	64	43
9	43		

notice that the sums of a and ~~the~~ b are always 2^{n-2},

and that it converges ~~to~~ towards $\frac{1}{3}$ and $\frac{2}{3}$.

ii as ~~a_n app~~ n approaches infinity a_n ~~a_{n+1}~~ ~~it~~ will be $\frac{1}{3}a_1 + \frac{2}{3}b\,a_2$

$= \frac{16}{3}$.

QUESTION NUMBER

2aii)

$PQ = 2 \cdot 2 = 4$

$PR = 3\sqrt{3} \cdot \frac{2}{\sqrt{3}} - 4 = 2.$ $\qquad \angle PSR = 2\theta = \frac{2\pi}{3}$

$$\overline{PR}^2 = a^2 + b^2 - 2ab\cos\theta$$
$$= 16 + 4 - 16\cos\frac{2\pi}{3}$$
$$= 20 + 8$$
$$= 28$$
$$\overline{PR} = \sqrt{28} \text{ units.}$$

4c) continued

$$\frac{v^2}{2} = +\frac{gR^2}{(x+R)} + C$$

when $v=0$, $x = h$. $\qquad C = -\frac{gR^2}{(\cancel{x+h})\ R+h}$

$$\frac{v^2}{2} = +\frac{gR^2}{x+R} - \frac{gR^2}{\cancel{x+h}\ R+h}$$

when $x=0$, $v = v_0$.

$$\frac{v^2}{2} = gR^2 \cancel{\left(\frac{1}{R} - \frac{1}{R+h}\right)} \left(\frac{1}{\cancel{x+}R} - \frac{1}{R+h}\right)$$
$$= \cancel{gR^2(} \ \cancel{gRh}\ gR^2\left(\frac{h}{R(R+h)}\right)$$
$$= \frac{gRh}{(R+h)}$$
$$v = \sqrt{\frac{2gRh}{R+h}} \quad \text{as required}$$

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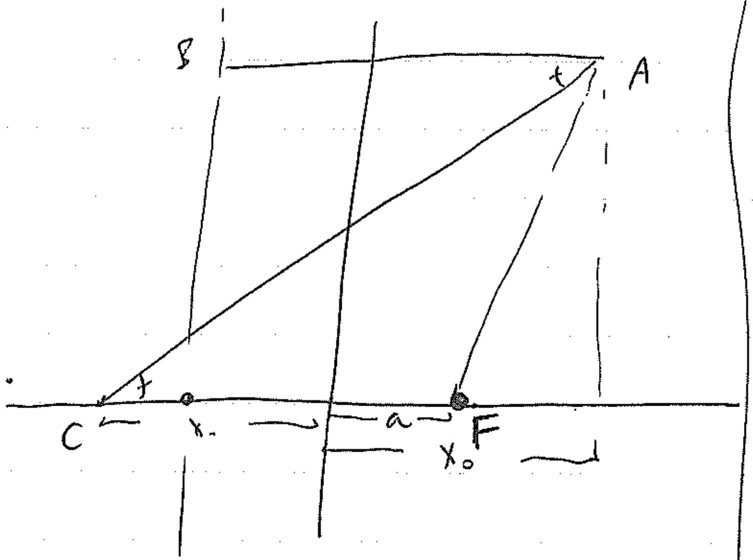
1c) ~~F(f, 0)~~

$F(a, 0)$

$A(x_0, y_0)$



$\overline{AF}^2 = (x_0 - a)^2 + y_0^2$

$= x_0^2 - 2ax_0 + a^2 + 4ax_0$

$= x_0^2 + 2ax_0 + a^2$

$\overline{AF} = x_0 + a$.

As $\overline{AF} = \overline{FC}$, the triangle is isoceles

and $\angle FAC = \angle FCA$,

$\angle FAC = \angle BAC$,

hence $\overline{AC}$ ~~to ang~~ bisects $\angle BAF$.

1b) $\underbrace{\cancel{(p+1)^2 b^2}}_{\text{positive}} + 2ab(p+1)(p-1) + \underbrace{a^2(p-1)^2}_{\text{positive}}$

$+ 4\cancel{c(p+1)(p-1)} > 0.$

~~$2ab(p+1)(p-1) > -4c(p+1)(p-1)$~~

$\frac{ab}{c} > -2.$

since ~~multiplying and dividing maintains~~

~~the~~ signs,

$\frac{bc}{a}$

$D = (-b(p+1) - a(p-1))^2 + 4(p-1)(p+1)c > 0$

$\underbrace{b^2(p+1)^2}_{\text{always}} + \underbrace{a^2(p-1)^2}_{\text{positive}} + ab(p+1)(p-1) + 4c(p-1)(p+1) > 0.$

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$(p+1)(p-1)(ab+4c) > 0.$

as the roots have equal magnitudes

$$b(p+1) + a(p-1) = 0.$$

Top Script for 93202 Calculus Outstanding Scholarship		Total Score	40
Question	**Mark**	**Annotation**	
		This paper was remarkable because it was completed in only 14 pages. It was awarded the top script because of the use of succinct and exact answers, showing flair and clear communication throughout. One of the very few students who gave a correct answer to **4c** Impressive succinct answer to **1c** on page 13	