S

93202A

932021

SUPERVISOR'S USE ONLY

SCHOLARSHIP EXEMPLAR

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2016
Calculus

9.30 a.m. Friday 25 November 2016
Time allowed: Three hours
Total marks: 40

ANSWER BOOKLET

There are five questions in this examination. Answer ALL FIVE questions, choosing ONE option from part (b) of Question Five.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

The graph for Question Five (b) is repeated on pages 26 and 27 of this booklet.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

This examination consists of five questions.
Answer all FIVE questions, choosing ONE option from part (b) of Question Five.

ASSESSOR'S USE ONLY

QUESTION NUMBER

1(a) $(x-2)^2 + y^2 = 3$

$2(x-2) + 2y \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2(x-2)}{2y}$

$= \frac{-(x-2)}{y}$

$\therefore \frac{-(x-2)}{y} = 0$ (stationary point)

$-x+2 = 0$

$x = 2$

when $x=2$, $y \Rightarrow \sqrt{3}$ or $-\sqrt{3}$

$\therefore$ max value of $\frac{y}{x} = \frac{\sqrt{3}}{2} = 0.866$ //

1(b i) $f(x) = x^2 \ln(x+1)$ (product rule)

$\frac{d f(x)}{dx} = 2x \cdot \ln(x+1) + x^2 \times \frac{1}{x+1}$

$\frac{d f(x)}{dx} = 2x \ln(x+1) + \frac{x^2}{x+1}$

$\frac{d^2 f(x)}{dx^2} = 2\ln(x+1) + \frac{2x}{x+1} + \frac{(x+1) \times 2x - x^2}{(x+1)^2}$

$\therefore$ second derivative $= 2\ln(x+1) + \frac{2x}{x+1} + \frac{2x(x+1) - x^2}{(x+1)^2}$ //

QUESTION NUMBER

ASSESSOR'S USE ONLY

Sub $x=0$ into second derivative

$$\frac{d^2 f(x)}{dx^2} = 2\ln(0+1) + \frac{2(0)}{0+1} + \frac{2(0)(0+1) - 0^2}{(0+1)^2}$$

$$= 2\ln(1)$$

$$= 0 \;//$$

1bii) second derivative $= 2\ln(x+1) + \frac{2x}{x+1} + \frac{2x(x+1) - x^2}{(x+1)^2}$

$$= 2\ln(x+1) + \frac{2x(x+1) + 2x(x+1) - x^2}{(x+1)^2}$$

~~third derivative~~

~~third derivative $= \ln(x+1) + \frac{2}{x+1} + \frac{\quad}{(x+1)^4}$~~

$$= 2\ln(x+1) + \frac{4x(x+1)}{(x+1)^2} - \frac{x^2}{(x+1)^2}$$

$$= 2\ln(x+1) + \frac{4x^2}{(x+1)^2} + \frac{4x}{(x+1)^2} - \frac{x^2}{(x+1)^2}$$

third derivative $= \ln(x+1) + \frac{2}{x+1} + \frac{8x(x+1)^2 - 2(x+1) \times 4x^2}{(x+1)^4}$

$$+ \frac{4(x+1)^2 - 2(x+1) \cdot 4x}{(x+1)^4} - \frac{2x(x+1)^2 - 2(x+1) \times x^2}{(x+1)^4}$$

$\therefore$ 2016 derivative $= 0! = 1 \;//$

QUESTION NUMBER

ASSESSOR'S USE ONLY

2a) $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^5 x + \cos^5 x)\, dx$

~~let $\sin x = u$~~

~~$\frac{du}{dx} = \cos x$~~

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \times \sin^2 x \times \sin x + \cos^5 x \quad dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos^2 x)(1-\cos^2 x)\sin x + \cos^5 x \quad dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - 2\cos^2 x + \cos^4 x)\sin x + \cos^5 x \quad dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x' \; \cancel{dx} + \cos^5 x - 2\cos^2 x \cdot \sin x + \cos^4 x \sin x \, dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x + \cos^4 x \sin x - 2\cos^2 x \sin x \, dx$

$= \left[-\cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x (\cos^4 x - 2\cos^2 x) dx$

$= 0 + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x (\cos^4 x - 2\cos^2 x) dx$

x	$\pi/2$	$-\pi/2$
u	0	0

let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$\therefore dx = \frac{du}{-\sin x}$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, dx + \int_0^0 -(u^4 - 2u^2)\, du$

QUESTION NUMBER

ASSESSOR'S USE ONLY

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, dx + \int_0^0 -u^4 + 2u^2 \, du$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, dx + 0$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^2 x)(1 - \sin^2 x) \cdot \cos x \, dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \left(1 - 2\sin^2 x + \sin^4 x\right) dx$$

~~let $u = \cos x$~~ ~~let sin~~

~~$\frac{du}{dx} = -\sin x$~~

let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

x	$\pi/2$	$-\pi/2$
u	1	-1

$$\therefore dx = \frac{du}{\cos x}$$

$$= \int_{-1}^{1} 1 - 2u^2 + u^4 \, du$$

$$= \left[u - \frac{2u^3}{3} + \frac{u^5}{5}\right]_{-1}^{1}$$

$$= \left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(-1 - \frac{2(-1)^3}{3} + \frac{(-1)^5}{5}\right)$$

$$= \frac{16}{15} \text{ units}^2$$

ASSESSOR'S USE ONLY

QUESTION NUMBER

2b) $= \int_0^{\frac{\pi}{2}} \frac{\sin(2nx)}{\sin x}\,dx - \int_0^{\frac{\pi}{2}} \frac{\sin(2(n-1)x)}{\sin x}\,dx$

$2nx$ and $2(n-1)x$ have a difference of 2

Looking at $\int \frac{\sin(2nx)}{\sin x}$ only

if $n=1$

$$\int \frac{\sin 2x}{\sin x} = \int \frac{2\sin x \cdot \cos x}{\sin x}$$

$$= \int 2\cos x$$

if $n=2$

$$\int \frac{\sin 4x}{\sin x} = \int \frac{2\sin 2x \cdot \cos 2x}{\sin x}$$

$$= \int \frac{2(2\cancel{\sin x}\cos x(\cos 2x))}{\cancel{\sin x}}$$

$$= \int 4\cos x(\cos 2x)$$

$$= \int 4\cos x(2\cos^2 x - 1)$$

$$= \int 8\cos^3 x - 4\cos x$$

$\therefore$ Each integral is multiplied by $2\cos 2x$

QUESTION NUMBER

ASSESSOR'S USE ONLY

if $n=1$

$= \int_0^{\pi/2} 2\cos x \, dx$

$= \left[2\sin x\right]_0^{\frac{\pi}{2}}$

$= 2$

if $n=2$

$= \int_0^{\pi/2} 8\cos^3 x - 4\cos x \, dx$

$= \int_0^{\frac{\pi}{2}} 8\cos^3 x - \int_0^{\frac{\pi}{2}} 4\cos x \, dx$

$= \int_0^{\frac{\pi}{2}} (1-\sin^2 x)\cdot\cos x - \left[4\sin x\right]_0^{\frac{\pi}{2}}$

let $u = \sin x$

x	$\pi/2$	0
u	1	0

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$= \int_0^1 1 - u^2 \, du - 4$

$= \left[u - \frac{u^3}{3}\right]_0^1 - 4$

$= \frac{-10}{3}$

$\therefore \frac{-10}{3} - 2 = \frac{-2}{3}$

similarily in $I_n - I_{n-1} = \frac{2(-1)^{n-1}}{2n-1}$,

if $n=2$

then $I_2 - I_{(2-1)} = \frac{2(-1)^{(2-1)}}{2(2)-1} = \frac{-2}{3}$

$\therefore$ since same answer – RAWW

$I_n - I_{n-1} = \frac{2(-1)^{n-1}}{2n-1}$ //

QUESTION NUMBER

ASSESSOR'S USE ONLY

2c) $\log_x y = \dfrac{\log y}{\log x}$

$\therefore \dfrac{\log(\cos\theta)}{\log(\tan\theta + \cot\theta)} = K$

$\log(\cos\theta) = K \cdot \log(\tan\theta + \cot\theta)$

$10^{\log(\cos\theta)} = 10^K \times 10^{\log(\tan\theta + \cot\theta)}$

$\cos\theta = 10^K \times (\tan(\theta) + \cot(\theta))$

$\therefore \dfrac{\cos\theta}{\tan\theta + \cot\theta} = 10^K$

$\dfrac{\cos\theta}{\tan\theta + \dfrac{1}{\tan\theta}} = 10^K$

$\dfrac{\cos\theta}{\dfrac{\tan^2\theta + 1}{\tan\theta}} = 10^K$

$\dfrac{\tan\theta \times \cos\theta}{\sec^2\theta} = 10^K$

$\dfrac{\sin\theta}{\sec^2\theta} = 10^K$

$\sin\theta = 10^K(1 + \tan^2\theta)$

$\sin\theta = 10^K + 10^K \cdot \tan^2\theta$

$\log\sin\theta = K + \log(10^K \cdot \tan^2\theta)$

$\log\sin\theta = K + K + \log\tan^2\theta$

$\log\sin\theta = 2K + 2\log\tan\theta$ //

QUESTION NUMBER

ASSESSOR'S USE ONLY

$\log \sin\theta - 2\log \tan\theta = 2K$

$\log\left(\frac{\sin\theta}{\tan^2\theta}\right) = 2K$

$\frac{1}{2}\log\left(\frac{\sin\theta}{\tan\theta}\right) = 2K$

$\therefore \log_{\tan\theta}(\sin\theta) = 4K$

3a)

ASSESSOR USE ONLY

QUESTION NUMBER

3a) $f(x) = x - \int_0^{\frac{\pi}{2}} f(x) \sin x \, dx$

$y = f(x) \qquad g' = \sin x$
$y' = f'(x) \qquad g = -\cos x$

$\therefore f(x) = x - \left(\left[-f(x)\cos x \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f'(x) \cos x \, dx \right)$

$f(x) = x - \left(\left[-f(x)\cos x \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f'(x) \cos x \, dx \right)$

$f(x) = x - \left((0 + f(x)) + \int_0^{\frac{\pi}{2}} f'(x) \cos x \, dx \right)$

$f(x) = x - \left(f(x) + \int_0^{\frac{\pi}{2}} f'(x) \cos x \, dx \right)$

NB: $\int_0^{\frac{\pi}{2}} f(x) \sin x = f(x) + \int_0^{\frac{\pi}{2}} f'(x) \cos x \, dx$

$\begin{bmatrix} y = \cos x & g' = f'(x) \\ y' = -\sin x & g = f(x) \end{bmatrix}$

$\therefore \int_0^{\frac{\pi}{2}} f(x) \sin x = f(x) + \left[f(x) \cos x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} f(x) \sin x$

$2 \int_0^{\frac{\pi}{2}} f(x) \sin x = f(x) + (0 - f(x))$
$= 0$

$\therefore f(x) = x - 0$
$f(x) = x$

QUESTION NUMBER

ASSESSOR'S USE ONLY

3bi)

$$= e^{2x} \cdot y$$

$$= 2e^{2x} \cdot y + e^{2x} \cdot \frac{dy}{dx}$$

$$-2e^{2x} y = e^{2x} \cdot \frac{dy}{dx}$$

$$-2y = \frac{dy}{dx}$$

$$\frac{dy}{dx} = -2(e^{2x} y)$$

$$-2y = x$$

$$\therefore \frac{dy}{dx} = xe^{2x} \ //$$

3bii)

$$\frac{dy}{dx} = x - 2y$$

$$\frac{1}{x-2y} dy = \cancel{xy} \, dx$$

$$\frac{1}{x-2y} \times \frac{(x+2y)}{(x+2y)} dy = dx$$

$$\frac{x+2y}{x^2 - 4y^2} dy = dx$$

$$x \, dy + 2y \, dy = x^2 dx - 4y^2 dx \ //$$

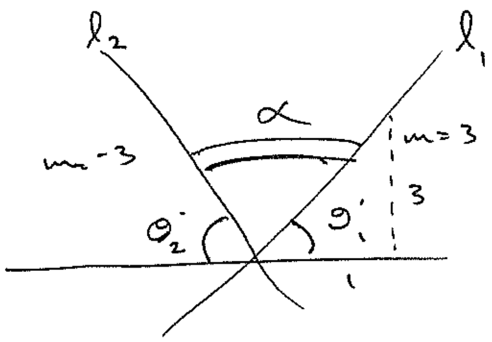
ASSESSOR'S USE ONLY

QUESTION NUMBER

4a) $\frac{x^2}{4} - \frac{y^2}{36} = 1$

$m(l_1) = \frac{6}{2} = 3$

$m(l_2) = -\frac{6}{2} = -3$



$\theta_1' = \theta_2'$

$\tan\theta = \frac{3}{1}$

$\theta_1' = 71.56\ldots$

$\therefore \theta_2' = 71.56\ldots$

$\therefore \alpha = 180° - 2 \times 71.56\ldots$

$= 36.86\ldots$

$\therefore \sin(36.86\ldots) = 0.6$

$\therefore \sin\alpha$ or θ (as per diagram) $= 0.6$

(3b) Equation (l_1) = $y = 3x$

Equation (l_2) = $y = -3x$

let P be point $(\sqrt{8}, 6)$

$\frac{x^2}{4} - \frac{y^2}{36} = 1$

$\frac{2x}{4} - \frac{2y}{36} \cdot \frac{dy}{dx} = 0$

$\frac{x}{2} - \frac{y}{18} \cdot \frac{dy}{dx} = 0$

$9x = \frac{dy}{dx}$

$\therefore m(P) = 9\sqrt{8}$

$\therefore m(\text{normal}) = \frac{-1}{9\sqrt{8}}$

$\therefore y - 6 = \frac{-1}{9\sqrt{8}}(x - \sqrt{8})$

$y = \frac{-1}{9\sqrt{8}}x + \frac{1}{9} + 6$

$\therefore y = \frac{-1}{9\sqrt{8}}x + \frac{55}{9}$

QUESTION NUMBER

ASSESSOR'S USE ONLY

~~Equating normal with l_1:~~

$$\therefore 3x = \frac{-1}{9\sqrt{8}}x + \frac{55}{9}$$

$$x = 2.0107\ldots$$

$$\therefore y = 3x = 6.0321\ldots$$

$$\therefore x_1 = \sqrt{8} \qquad x_2 = 2.0107\ldots \quad (A)$$

$$y_1 = 6 \qquad y_2 = 6.0321\ldots \quad (B)$$

$$D\ (PA) = \sqrt{(\sqrt{8} - 2.0107\ldots)^2 + (6 - 6.0321\ldots)^2}$$

$$= 0.818$$

~~Equating~~

3b) $m(l_1) = 3$ $\qquad\qquad$ $m(l_2) = -3$

$\therefore m(\text{normal}) = \frac{-1}{3}$ $\qquad\qquad$ $\therefore m(\text{normal}) = \frac{1}{3}$

$$\frac{x^2}{4} - \frac{y^2}{36} = 1$$

$$\therefore \frac{dy}{dx} = 9x$$

let P be point $(\sqrt{8}, 6)$

$$\therefore m(\text{normal}) = \frac{-1}{9\sqrt{8}}$$

with l_1:

$$3x = \frac{-1}{9\sqrt{8}}x + \frac{55}{9}$$

$$x = 2.0107\ldots$$

$$y = 3x = 6.0321\ldots$$

with l_2:

$$-3x = \frac{-1}{9\sqrt{8}}x + \frac{55}{9}$$

$$x = -2.064\ldots$$

$$y = -3x = 6.1921\ldots$$

QUESTION NUMBER

for l_1:

$x_1 = \sqrt{8}$ $\qquad$ $x_2 = 2.0107...$

$y_1 = 6$ $\qquad$ $y_2 = 6.0321$

$\therefore D(AP) = \sqrt{(\sqrt{8} - 2.0107..)^2 + (6 - 6.0321..)^2}$

$= 0.818$

for l_2:

$x_1 = \sqrt{8}$ $\qquad$ $x_2 = -2.064....$

$y_1 = 6$ $\qquad$ $y_2 = 6.1921...$

$\therefore D(BP) = \sqrt{(\sqrt{8} - {-2.064})^2 + (6 - 6.1921..)^2}$

$= 4.896....$

QUESTION NUMBER

ASSESSOR'S USE ONLY

5a) See page 16

b) OR – critical path.

Start time: | Time Duration | End time
0 | 3 → A | 3

(start when A is 1/3 complete)

1 | 4 → B | 5

(when B is half complete $= \frac{1}{2} \times 4 + 1$)

3 | 5 → C | ($\frac{1}{2}$ one hour after B is complete) 8.5

(when A is 2/3 complete)

2 | 2 → D | 6

$\therefore$ critical path time is 8.5 hours.

A → B → D → C

3 → 5 → 6 → 8.5. (time completed at end time)

QUESTION NUMBER

5a) $z^{11} = 1$

$(a+bi)^{11} = 1$

$r = 1$

$\theta = 0$

$\therefore \; = r^{\frac{1}{n}} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right)$

~~for~~ $k = 0$

$= 1 \times (\cos 0 + i\sin 0)$

$= 1$

~~for~~ $k = 1$

$= \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$

$= 0.841 + 0.54i$

~~for~~ $k = 2$

$= 0.415 + 0.910i$

~~for~~ $k = 3$

$= -0.142 + 0.99i$

~~for~~ $k = 4$

$= 0.99 + 0.0399i$

~~for~~ $k = 5$

$= 0.99 + 0.049i$

$\therefore$ roots:

1) 1

2) $0.99 + 0.009i$

3) $0.99 + 0.019i$

4) $0.99 + 0.029i$

5) $0.99 + 0.039i$

6) $0.99 + 0.049i$

7) $0.99 + 0.059i$

QUESTION NUMBER

ASSESSOR'S USE ONLY

~~8) 0.99 + 0.069i~~
~~9) 0.99 + 0.079i~~
~~10) 0.99 + 0.089i~~
~~11) 0.99 + 0.099i~~

$$\cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right) = \frac{-1}{2}$$

$$\cos\left(\pi - \frac{9\pi}{11}\right)$$

Annotated Exemplar for 93202 Calculus Scholarship Candidate 54-M129		Total Score	20
Question	**Mark**	**Annotation**	
1	**4**	The candidate gave typical answers to the three parts of this question **1a** dy/dx is correct but it is then equated to zero (which is incorrect). Value of x = 2 is found and substituted into y/x to get (sqr root 3)/2 (incorrect answer) **1bi** correct first and second derivatives and f"(0) = 0 **1bii** incorrect third derivative and no progress made	
2	**6**	The candidate has provided evidence in **2a** of competently substituting trig identities and simplifying. In **2b** the candidate attempts to use trig identities but the answer is fudged. In **2c** there is correct change of base of logarithm and an attempt to use trig identities further but fails to proceed with meaningful steps	
3	**2**	The candidate gives the most common answer in **3a** – correct first step but then fails to proceed meaningfully. The candidate has shown competence in **3bi** in implicit differentiation but then fails to simplify correctly. (most candidates gaining scholarship gained 2 or 3 marks to this part question). **3bii** is a typical answer also – tries to integrate x-2y by separating the variables instead of integration by parts as suggested	
4	**4**	In **4a** the candidate has used tan inverse and the property of angles on a line adding to 180 degrees. This was a typical approach which gave a non-exact answer. **4b and 4c** Hardly any candidates made a promising start on these two questions.	
5	**4**	The candidate has made sufficient progress in **5a** to find the general solution. But has not been able to proceed further. In **5b** the candidate has given a typical answer to the Networks option. It shows A and B overlapping and an overall time of 8.5 hours. But has not explained the handling of Tasks C and C sufficiently.	