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93202A

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SUPERVISOR'S USE ONLY

OUTSTANDING SCHOLARSHIP EXEMPLAR

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2017
Calculus

9.30 a.m. Friday 10 November 2017
Time allowed: Three hours
Total marks: 40

ANSWER BOOKLET

There are five questions in this examination. Answer ALL FIVE questions.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Question	Mark
ONE	
TWO	
THREE	
FOUR	
FIVE	
TOTAL	/40

ASSESSOR'S USE ONLY

$\cos 5\theta \pm 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

QUESTION NUMBER

5a) $z = \text{cis}\,\theta$

i. $z = \cos\theta + i\sin\theta$

$\frac{1}{z} = \bar{z} = \cos\theta - i\sin\theta$ $\qquad z^5 + \frac{1}{z^5} = 2\cos 5\theta$

$z + \frac{1}{z} = 2\cos\theta$

$z^n = \cos(n\theta) + i\sin(n\theta)$

$\frac{1}{z^n} = \cos(n\theta) - i\sin(n\theta)$

$z^n + \frac{1}{z^n} = 2\cos(n\theta)$ $\qquad z^3 + \frac{1}{z^3} = \left(z+\frac{1}{z}\right)^3 - 3z\cdot\frac{1}{z}\left(z+\frac{1}{z}\right)$

$= \left(z+\frac{1}{z}\right)^3 - 3\left(z+\frac{1}{z}\right)$

$\cancel{z^5 = \text{cis}\,5\theta = \cos 5\theta + i\sin 5\theta}$

$\cancel{\left(z+\frac{1}{z}\right)^5 = (2\cos\theta)^5 = 2^5\cos^5 5\theta}$ ~~by D.M. theorem~~

$\left(z+\frac{1}{z}\right)^5 = z^5 + 5z^4\cdot\frac{1}{z} + 10z^3\cdot\frac{1}{z^2} + 10z^2\cdot\frac{1}{z^3} + 5z\cdot\frac{1}{z^4} + \frac{1}{z^5}$

$= z^5 + \frac{1}{z^5} + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z+\frac{1}{z}\right)$

$= z^5 + \frac{1}{z^5} + 5\left[\left(z+\frac{1}{z}\right)^3 - 3\left(z+\frac{1}{z}\right)\right] + 10\left(z+\frac{1}{z}\right)$

$= z^5 + \frac{1}{z^5} + 5\left(z+\frac{1}{z}\right)^3 - 15\left(z+\frac{1}{z}\right) + 10\left(z+\frac{1}{z}\right)$

$= z^5 + \frac{1}{z^5} + 5\left(z+\frac{1}{z}\right)^3 - 5\left(z+\frac{1}{z}\right)$

$z^5 + \frac{1}{z^5} = \left(z+\frac{1}{z}\right)^5 - 5\left(z+\frac{1}{z}\right)^3 + 5\left(z+\frac{1}{z}\right)$

$2\cos 5\theta = (2\cos\theta)^5 - 5(2\cos\theta)^3 + 5(2\cos\theta)$

$2\cos 5\theta = 2^5\cos^5\theta - 5\cdot 2^3\cos^3\theta + 5\cdot 2\cos\theta$

$\cos 5\theta = 2^4\cos^5\theta - 5\cdot 2^2\cos^3\theta + 5\cos\theta$

$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ (proven)

QUESTION NUMBER | ASSESSOR'S USE ONLY

3b) i.

$$y = e^x \sin x$$

$$\frac{dy}{dx} = e^x \sin x + e^x \cos x$$

$$= e^x (\sin x + \cos x)$$

$$= \sqrt{2}\, e^x \left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x\right)$$

$$= 2^{\frac{1}{2}} e^x \left(\cos \frac{\pi}{4} \sin x + \sin \frac{\pi}{4} \cos x\right)$$

$$\frac{dy}{dx} = 2^{\frac{1}{2}} e^x \sin\left(x + \frac{\pi}{4}\right) \quad // \text{(proven).}$$

3b) ii. From $\frac{dy}{dx} = e^x (\sin x + \cos x)$

$$\frac{d^2y}{dx^2} = e^x (\sin x + \cos x) + e^x (\cos x - \sin x)$$

$$= e^x (2\cos x)$$

$$= 2e^x \cos x$$

$$= 2e^x \left(1 - 2\sin^2 \frac{x}{2}\right)$$

$$= 2e^x - 4e^x \sin^2 \frac{x}{2} \quad //$$

3a) $y = x^{(x^x)}$ so $\ln y = \ln x^{(x^x)} = x^x \ln x$

$\ln y = x^x \ln x$ Let x^x be a, $\qquad a = x^x$

$\ln y = a \ln x$ $\qquad \ln a = x \ln x$

$\frac{1}{y}\frac{dy}{dx} = \frac{da}{dx} \ln x + \frac{a}{x}$ $\qquad \frac{1}{a}\frac{da}{dx} = \ln x + 1$

$\frac{1}{y}\frac{dy}{dx} = x^x \ln x (\ln x + 1) + x^{x-1}$ $\qquad \frac{da}{dx} = a(\ln x + 1)$

$\frac{1}{y}\frac{dy}{dx} = x^x \left(\ln x (\ln x + 1) + \frac{1}{x}\right)$ $\qquad = x^x (\ln x + 1)$

$$\frac{dy}{dx} = x^{x^x} \cdot x^x \left(\ln^2 x + \ln x + \frac{1}{x}\right)$$

$$\frac{dy}{dx} = x^{x^x + x} \left(\ln^2 x + \ln x + \frac{1}{x}\right)$$

~~3b)~~ when $x = 2$,

$$\frac{dy}{dx} = 2^{4+2} \left(\ln^2 2 + \ln 2 + \frac{1}{2}\right)$$

$$= 2^6 \left(\ln^2 2 + \ln 2 + \frac{1}{2}\right) = 64 \left(\ln^2 2 + \ln 2 + \frac{1}{2}\right)$$

$$= 107.1104 \text{ (correct to 4dp)}$$

QUESTION NUMBER

~~Let $y = \sinh x$ so $y^{-1} = \sinh^{-1} x$~~

~~$y = \frac{1}{2}(e^{x} - e^{-x})$~~

~~$2y = e^{x} - e^{-x}$~~

~~$\ln 2y = x - (-x)$~~

~~$\ln 2y = 2x$~~

~~$x = \frac{1}{2}\ln 2y$~~

~~$\therefore y^{-1} = \frac{1}{2}\ln 2x$~~

~~$\sinh^{-1}x = \frac{1}{2}\ln 2x$~~

~~For $\sinh^{-1}x$, when $x=2$~~

~~$y = 1.4436$~~

~~For $\sinh x$, when $x = 1.4436$~~

~~$y = 2$.~~

~~when $\sinh^{-1}x = \frac{1}{\sqrt{x^2+1}}$ and $\sinh x = \frac{1}{2}(e^{x} - e^{-x})$~~

~~$y = g(x) = 5x-6$, $x = \frac{y+6}{5}$~~

~~so $g^{-1}(x) = \frac{x+6}{5}$~~

~~$x \xrightarrow{g(x)} y$, $y \xrightarrow{g^{-1}(x)} x$~~

3c

$\int \frac{1}{\sqrt{x^2+1}}\,dx$ using the substitution $x = \tan\theta$ gives

then,

$= \int \frac{\sec^2\theta}{\sec\theta}\,d\theta$

$= \int \sec\theta\,d\theta$

$= \ln(\sec\theta + \tan\theta)$

$= \ln(\sqrt{1+x^2} + x) + c$

$= \sinh^{-1}x + c$

$$\boxed{\therefore \frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{x^2+1}}}$$

thirdly (proven) since $\int \frac{dy}{dx}\,dx = y$

Firstly

Let $y = \sinh x = \frac{1}{2}(e^{x} - e^{-x})$

so, $\sinh(x) = \frac{1}{2}(e^{x} - e^{-x})$

and its inverse function:

$\sinh^{-1}x = \ln\left(\sqrt{1+x^2} + x\right)$

To find inverse function $\sinh^{-1}x$, reflection in the line $y = x$

swap the variables y and x

$x = \frac{1}{2}(e^{y} - e^{-y})$

$2x = e^{y} - e^{-y}$

$2xe^{y} = e^{2y} - 1$

$e^{2y} - 2xe^{y} + x^2 = 1 + x^2$

$(e^{y} - x)^2 = (\sqrt{1+x^2})^2$

$e^{y} - x = \sqrt{1+x^2}$ so $e^{y} = \sqrt{1+x^2} + x$

$y = \ln(\sqrt{1+x^2} + x)$

$\frac{\cdots}{(p+1)} = -\alpha^2$

QUESTION NUMBER | ASSESSOR'S USE ONLY

(b) $\frac{x^2-bx}{p-1} = \frac{ax+c}{p+1}$

$(x^2-bx)(p+1) = ax+c\,(p-1)$

$(p+1)x^2 + [-b(p+1) - a(p-1)]x - c(p-1) = 0$ quadratic in x.

~~since has 2 real roots: $\Delta = b^2-4ac>0$~~

~~roots are~~ roots are equal magnitude but opp. sign ie. $\alpha, -\alpha$

so sum of roots $= \alpha - \alpha = 0 = \frac{-b(p+1)-a(p-1)}{p+1}$

$0 = -b - \frac{a(p-1)}{p+1}$

$b = \frac{a(p-1)}{p+1}$

$b(p+1) = a(p-1)$

$\frac{1}{b(p-1)} = \frac{a}{p+1}$

since has 2 real roots: $\Delta = b^2-4ac>0$

$[-b(p+1) - a(p-1)]^2 - 4(p+1)\cdot(-c)(p-1) > 0$

$[-a(p-1) - a(p-1)]^2 - 4c(p+1)(p-1) > 0$

$[-2a(p-1)]^2 - 4c(p+1)(p-1) > 0$

$4a^2(p-1)^2 - 4c(p+1)(p-1) > 0$

$4(p-1)\ [a^2(p-1) - c(p+1)] > 0$

$4(p-1)\ [ab(p+1) - c(p+1)] > 0$

$4(p-1)(p+1)[ab-c] > 0$ $\quad ab > c$

$4\frac{a}{b}(p-1)(p-1)\ [ab-c] > 0$

$\frac{a}{b}(p-1)^2\ [ab-c] > 0$

$a^2(p-1)^2 - \frac{ac}{b}(p-1) > 0$

$a^2(p-1)^2 > \frac{ac}{b}(p-1)$

$0 > \frac{c}{ab(p-1)}$

$0 > \frac{bc}{a}$

$\therefore \frac{bc}{a} < 0$ (proven)

QUESTION NUMBER

ASSESSOR'S USE ONLY

1c

gradient of tangent to parabola $\frac{dy}{dx} = \frac{2a}{y}$

" at A $\frac{dy}{dx} = \frac{2a}{y_0}$

eqn of tangent to parabola at A $y - y_0 = \frac{2a}{y_0}(x - x_0)$

$yy_0 - y_0^2 = 2ax - 2ax_0$

at C, $y = 0$ so $-y_0^2 = 2ax - 2ax_0$

$2ax = 2ax_0 - y_0^2$

$\because (x_0, y_0)$ lies on parabola, $y_0^2 = 4ax_0$ so substitute this value

$2ax = 2ax_0 - 4ax_0$

$2ax = -2ax_0$

$x = -x_0$

C has coordinates $(-x_0, 0)$

$BC = \sqrt{x_0^2 + y_0^2}$

$CF = a + x_0$

$AB = x_0$

$AF = \sqrt{(a-x_0)^2 + y_0^2}$ ~~$= a + x_0$~~

$= \sqrt{a^2 - 2ax_0 + x_0^2 + 4ax_0}$

$= \sqrt{a^2 + 2ax_0 + x_0^2} = \sqrt{(a+x_0)^2}$

$= a + x_0$

AB = AF

QUESTION NUMBER | ASSESSOR USE ONLY

2a) Let $AP = x$, $CR = z$. Given $AD = BC = 3$, $PD = 3-x$, $BR = 3-z$

i. in $\triangle PQA$ $\quad PQ = x\sec\theta$

in $\triangle PDS$ $\quad PS = (3-x)\sec\theta$

in $\triangle CRS$ $\quad SR = z\sec\theta$

in $\triangle BQR$ $\quad QR = (3-z)\sec\theta$

perimeter of PQRS $= PQ + PS + SR + QR$

$= (3-x+x+z+3-z)\sec\theta$

$\therefore$ Perimeter $= 6\sec\theta$

ie. independent of x. //

2b) $x+y-z=1$ — ① From ①: $z = x+y-1$ — ④

Sub ④ into ② $\quad x^2+y^2-(x+y-1)^2 = 5-2xy$

$x^2+y^2-[x^2+y^2+2xy-2(x+y)+1] = 5-2xy$

$-2xy+2(x+y)-1 = 5-2xy$

$2(x+y) = 6$

$x+y = 3$ $\quad (x+y)$

sub $x+y=3$ into ④ $\quad z = 3-1 = 2$

Substitute the value of z into ③

$x^3+y^3-2^3 = 43-3xy$

$x^3+y^3-8 = 43-3xy$

$x^3+y^3+3xy = 51$

$\because x^3+y^3 = (x+y)^3-3xy(x+y)$ and $x+y=3$

$= 27-3xy(3)$

$= 27-9xy$

$27-9xy+3xy = 51$

$6xy = 27-51 = -24 \Rightarrow xy = -4$ //

QUESTION NUMBER

From $xy=-4 \rightarrow x=-\frac{4}{y}$ — ⑤

sub ⑤ into $x+y=3$

$$-\frac{4}{y}+y=3$$
$$-4+y^2=3y$$
$$y^2-3y-4=0$$
$$(y-4)(y+1)=0$$
$$y=4 \text{ or } -1$$
$$x=-1 \text{ or } 4$$

$$\begin{cases} x=-1 \\ y=4 \\ z=2 \end{cases} \text{ or } \begin{cases} x=4 \\ y=-1 \\ z=2 \end{cases}$$

4a) $\tan x\tan(2x)\tan(3x)$ where $\tan 3x=\tan(2x+x)$

$$=\frac{\tan 2x+\tan x}{1-\tan 2x\tan x}$$

$$=\tan x\cdot\tan 2x\cdot\frac{\tan 2x+\tan x}{1-\tan 2x\tan x}$$

$$=\frac{\tan x\tan^2 2x+\tan^2 x\tan 2x}{1-\tan 2x\tan x}$$

$$=\frac{-(\tan x+\tan 2x)(1-\tan 2x\tan x)+\tan 2x+\tan x}{1-\tan 2x\tan x}$$

$$=-(\tan x+\tan 2x)+\frac{\tan 2x+\tan x}{1-\tan 2x\tan x}$$

$$=-\tan x-\tan 2x+\tan 3x$$

$$\int \tan x\tan 2x\tan 3x\,dx=\int(-\tan x-\tan 2x+\tan 3x)\,dx$$
$$=\int \frac{-\sin x}{\cos x}+\frac{2(-\sin 2x)}{2\cos 2x}+\frac{3\sin 3x}{3\cos 3x}\,dx$$
$$=\ln|\cos x|+\frac{1}{2}\ln|\cos 2x|-\frac{1}{3}\ln|\cos 3x|+C$$

(proven)

QUESTION NUMBER

ASSESSOR'S USE ONLY

4b) $r = a(1-\cos\theta)$ $\qquad \frac{dr}{d\theta} = a\sin\theta$

~~= a ac~~ $r^2 = a^2(1-\cos\theta)^2$ $\qquad \left(\frac{dr}{d\theta}\right)^2 = a^2\sin^2\theta$

$= a^2(1-2\cos\theta+\cos^2\theta)$ $\qquad = a^2(1-\cos^2\theta)$

$$r^2+\left(\frac{dr}{d\theta}\right)^2 = a^2(1-2\cos\theta+\cos^2\theta) + a^2(1-\cos^2\theta)$$

$$= a^2(1-2\cos\theta+\cos^2\theta+1-\cos^2\theta)$$

$$= a^2(2-2\cos\theta)$$

$\cos\theta = 1-2\sin^2\theta$

$1-\cos\theta = 2\sin^2\theta$

$$= 2a^2(1-\cos\theta)$$

$$= 2a^2(2\sin^2\theta)$$

$$= 4a^2\sin^2\theta$$

$$S = \int_{\theta_1}^{\theta_2}\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta = \int_{\theta_1}^{\theta_2}\sqrt{4a^2\sin^2\theta}\,d\theta$$

$$= \int_{\theta_1}^{\theta_2} 2a\sin\theta\,d\theta$$

$$= 2a\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta$$

$$= -2a\left[\cos\theta\right]_{\theta_1}^{\theta_2}$$

$$= -2a(\cos\theta_2-\cos\theta_1)$$

$$\therefore S = 2a(\cos\theta_1-\cos\theta_2)$$

4c) $a = \frac{dv}{dt}$ and $a = -\frac{gR^2}{(x+R)^2}$ acceleration from $V=V_0$ to $V=0$

$a = \frac{v-u}{t}$

5a ii) $\cos 5\theta = \cos 4\theta$

$\cos 5\theta - \cos 4\theta = 0$

~~cos~~ $-2\sin\frac{9\theta}{2}\sin\frac{\theta}{2} = 0$

$\sin\frac{9\theta}{2} = 0$ ~~= kπ~~ or $\sin\frac{\theta}{2} = 0$ ~~= kπ~~ where k is ~~a positive~~ any integer

$\frac{9\theta}{2} = k\pi$ $\qquad \frac{\theta}{2} = k\pi$

$\theta = \frac{2}{9}k\pi$ $\qquad \theta = 2k\pi$

Annotated Exemplar for 93202 Calculus Outstanding Scholarship		Total Score	34
Question	**Mark**	**Annotation**	
1	**8**	The candidate did not attempt **1a.** In **1b** candidate has correctly formed a quadratic and recognised that the coefficient of x is zero and used discriminant for required result – a typical approach. In **1c** candidate has made a good start with equation of tangent and establishing FA = FC. Final statement (AB =AF) is incorrect and candidate has not carried on to prove bisection by use of isosceles triangle as required.	
2	**8**	The candidate has provided evidence in **2a (i)** of competently and succinctly establishing lengths of sides of parallelogram PQRS and hence perimeter. **2a(ii)** was not attempted. This was not typical of outstanding candidates as most used cosine rule for exact answer. In **2b)** was a good question to discriminate for scholarship and outstanding candidates because it involved sound algebra skills with complicated simultaneous equations.	
3	**8**	In **3a** the candidate has correctly taken logs and differentiated implicitly. **3bi** and **3bii** are also well done with correct differentiation and substitution. This was typical of scholarship and outstanding scholarship candidates. No attempt at **3b(iii).** One of few candidates who correctly proved result in **3c** - a challenging question which requires understanding of inverse functions – Cambridge and IB students had an advantage here.	
4	**5**	The candidate was unusual in proving **4a** by integration – most successful candidates used differentiation. Good use of trig identities. This candidate gave typical response to **4b** by differentiating and substituting correctly but then did not use the correct trig substitution in order to integrate. No progress made on **4c –** a challenging question for all but the very top candidates.	
5	**5**	The candidate has completed proof in **5a(i)** This was typical of many candidates who then could not link it to **5a(ii)** . Candidate has made a start on **5a(ii)** converting sum to product. No attempt at **5b** - this was a typical response as few candidates could come up with the nth term.	