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TOP SCHOLAR

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2016
Calculus

9.30 a.m. Friday 25 November 2016
Time allowed: Three hours
Total marks: 40

ANSWER BOOKLET

There are five questions in this examination. Answer ALL FIVE questions, choosing ONE option from part (b) of Question Five.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

The graph for Question Five (b) is repeated on pages 26 and 27 of this booklet.

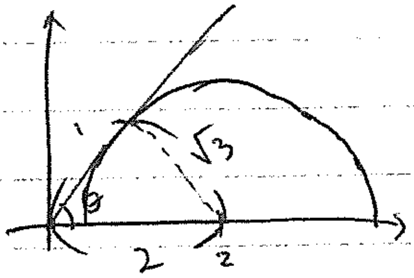
YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

This examination consists of five questions.
Answer all FIVE questions, choosing ONE option from part (b) of Question Five.

ASSESSOR'S USE ONLY

QUESTION NUMBER

1 a) $\frac{y}{x}=k$. $y=kx$. Maximum gradient k.



$k=\tan\theta=\sqrt{3}$.

$\sqrt{3}$

b) i) $\frac{df}{dx}=2x\ln(x+1)+\frac{x^2}{x+1}$.

$\frac{d^2f}{dx^2}=\frac{2x}{x+1}+2\ln(x+1)+\frac{2x^2+2x-x^2}{(x+1)^2}$

$=\frac{2x}{x+1}+2\ln(x+1)+1-\frac{1}{(x+1)^2}$.

sub. $x=0$, $f^{(2)}(0)=0$.

ii) $\frac{d^3f}{dx^3}=\frac{2}{x+1}+\frac{2}{(x+1)^2}+\frac{2}{(x+1)^3}$. $f^{(3)}(0)=2+2+2$

$\frac{d^4f}{dx^4}=-2(x+1)^{-2}-4(x+1)^{-3}-6(x+1)^{-4}$ $f^{(4)}(0)=-2-4-6$

$\frac{d^5f}{dx^5}=4(x+1)^{-3}+12(x+1)^{-4}+24(x+1)^{-5}$ $f^{(5)}(0)=2\times2!+2\times3!+4!$

$\frac{d^6f}{dx^6}=-12(x+1)^{-4}-48(x+1)^{-5}-120(x+1)^{-6}$ $f^{(6)}(0)=-2\times3!-2\times4!-5!$

⋮

$f^{(2016)}(0)=-2\times2013!-2\times2014!-2015!$

QUESTION NUMBER

ASSESSOR'S USE ONLY

2 a) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5 x\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos^2 x)^2 \sin x\, dx$ $\quad \cos x = u$, $-\sin x = \frac{du}{dx}$

$= \int_0^0 -(1-u^2)^2 du = 0$.

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin^2 x)^2 \cos x\, dx$ $\quad \sin x = u$, $\cos x\, dx = du$

$= \int_{-1}^{1} (1-u^2)^2 du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_{-1}^{1}$

$= 1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5} = \frac{16}{15}$ //

3

3

b) $I_n - I_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin 2nx - \sin 2(n-1)x}{\sin x} dx$

$= \int_0^{\frac{\pi}{2}} \frac{2 \times \cos(2n-1)x \times \cancel{\sin x}}{\cancel{\sin x}} dx = \int_0^{\frac{\pi}{2}} 2\cos(2n-1)x$

$= \frac{2}{2n-1}\left[\sin(2n-1)x\right]_0^{\frac{\pi}{2}}$

$= \frac{2}{2n-1} \sin \frac{2n-1}{2}\pi$

when $n = \text{odd}$, $\frac{2}{2n-1}$

when $n = \text{even}$, $-\frac{2}{2n-1}$

$\therefore I_n - I_{n-1} = \frac{2(-1)^{n-1}}{2n-1}$ //

QUESTION NUMBER

ASSESSOR'S USE ONLY

c) $(\tan\theta + \cot\theta)^k = \cos\theta$. $\left(\tan\theta + \frac{1}{\tan\theta}\right)^k = \cos\theta$.

$\tan\theta + \frac{1}{\tan\theta} = \frac{\tan^2\theta + 1}{\tan\theta} = \frac{\sec^2\theta}{\tan\theta} = \frac{1}{\tan\theta} \times \frac{1}{\cos^2\theta}$.

$\left(\frac{1}{\tan\theta} \times \frac{1}{\cos^2\theta}\right)^k = \cos\theta$.

$\frac{1}{\tan^k\theta} \times \frac{1}{\cos^{2k}\theta} = \cos\theta$. $\frac{1}{\tan^k\theta} = \cos^{2k+1}\theta$.

multiply $\tan^{2k+1}\theta$ both side.

$\tan^{k+1}\theta = \sin^{2k+1}\theta$. $(k+1)\log\tan\theta = (2k+1)\log\sin\theta$.

$\frac{\log\sin\theta}{\log\tan\theta} = \frac{k+1}{2k+1} = \log_{(\tan\theta)}(\sin\theta)$

$\therefore \frac{k+1}{2k+1}$

3 a) $\int_0^{\frac{\pi}{2}} f(x)\sin x\,dx$ is a constant.

$\int_0^{\frac{\pi}{2}} f(x)\sin x\,dx = a$, $f(x) = x - a$, therefore

$\int_0^{\frac{\pi}{2}} (x-a)\sin x\,dx = a$.

$\left[-(x-a)\cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x\,dx = \left[-(x-a)\cos x\right]_0^{\frac{\pi}{2}}$

$+ \left[\sin x\right]_0^{\frac{\pi}{2}} = 1 - a = a$. $a = \frac{1}{2}$.

$\therefore f(x) = x - \frac{1}{2}$.

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b) i) $\dfrac{d(e^{2x}y)}{dx} = 2e^{2x}y + e^{2x} \times \dfrac{dy}{dx}$

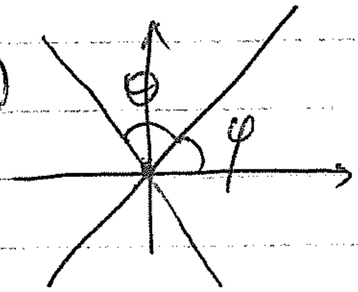
$= 2e^{2x}y + e^{2x}(x-2y)$

$= 2e^{2x}y + e^{2x}x - 2e^{2x}y$

$= xe^{2x}$

ii)

4. a)



$\tan\varphi = 3$

$\tan(\frac{\pi}{2} - \varphi) = \frac{1}{3}$

$\theta = 2 \times (\frac{\pi}{2} - \varphi)$

$\tan 2\times(\frac{\pi}{2} - \varphi) = \dfrac{\frac{1}{3} + \frac{1}{3}}{1 - \frac{1}{3} \times \frac{1}{3}} = \frac{3}{4} = \tan\theta$

$\cot^2\theta + 1 = \csc^2\theta = \frac{25}{9}$ $\therefore \sin\theta = \frac{3}{5}$

b) $P(a, b)$. $\overline{AP} = \dfrac{|3a-b|}{\sqrt{3^2+1^2}} = \dfrac{|3a-b|}{\sqrt{10}}$

$\overline{BP} = \dfrac{|3a+b|}{\sqrt{3^2+1^2}} = \dfrac{|3a-b|}{\sqrt{10}}$ MEI

$\overline{AP} \times \overline{BP} = \dfrac{|9a^2 - b^2|}{10}$ If $9a^2 - b^2$ is a constant, $\overline{AP} \times \overline{BP}$ is constant.

As P is on $\frac{x^2}{4} - \frac{y^2}{36} = 1$, it satisfies

$\frac{a^2}{4} - \frac{b^2}{36} = 1$ $\therefore 9a^2 - b^2 = 36$.

$\overline{AP} \times \overline{BP} = \frac{36}{10} = \frac{18}{5}$

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Area is minimum when CD is tangent to the curve.

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c) $P(a, 3\sqrt{a^2-4})$

$$\frac{2x}{4} - \frac{2y}{36} \times \frac{dy}{dx} = 0. \quad \frac{dy}{dx} = \frac{x}{y} \times 9.$$

$$\frac{dy}{dx} = \frac{a \times 9}{3\sqrt{a^2-4}} = \frac{3a}{\sqrt{a^2-4}}.$$

$$CD: y = \frac{3a}{\sqrt{a^2-4}}(x-a) + 3\sqrt{a^2-4}$$

$$\frac{\cancel{3}a}{\sqrt{a^2-4}}(x-a) + \cancel{3}\sqrt{a^2-4} = \pm \cancel{3}x.$$

$$\frac{ax}{\sqrt{a^2-4}} - \frac{a^2}{\sqrt{a^2-4}} + \sqrt{a^2-4} = \pm x.$$

$$\cancel{\frac{a \pm \sqrt{a^2-4}}{\sqrt{a^2-4}} x = \frac{a^2 - a^2 + 4}{\sqrt{a^2-4}} = \frac{4}{\sqrt{a^2-4}}}$$

$$\cancel{x = \frac{4}{a \pm \sqrt{a^2-4}} = \frac{4}{a \pm \sqrt{a^2-4}} \times \frac{a \mp \sqrt{a^2-4}}{a \mp \sqrt{a^2-4}}}$$

$$\cancel{= a \mp \sqrt{a^2-4}}$$

$$\frac{a \mp \sqrt{a^2-4}}{\sqrt{a^2-4}} x = \frac{a^2 - a^2 + 4}{\sqrt{a^2-4}} = \frac{4}{\sqrt{a^2-4}}$$

$$x = \frac{4}{a \mp \sqrt{a^2-4}} = \frac{4}{a \mp \sqrt{a^2-4}} \times \frac{a \pm \sqrt{a^2-4}}{a \pm \sqrt{a^2-4}} = a \pm \sqrt{a^2-4}.$$

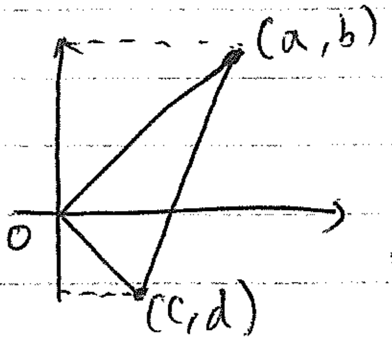
$C(a + \sqrt{a^2-4}, 3a + 3\sqrt{a^2-4})$

$P(a, 3\sqrt{a^2-4})$

$D(a - \sqrt{a^2-4}, -3a + 3\sqrt{a^2-4})$

$\overline{CP} = \sqrt{a^2-4+9a^2} = \sqrt{10a^2-4}$

$\overline{PD} = \sqrt{a^2-4+9a^2} = \sqrt{10a^2-4}.$ $\quad \lambda = 1$



Area = □ Trapezium − △ triangle − △ triangle

$$= \frac{(a+c)(-d+b) - ab - c \times (-d)}{2}$$

$$= \frac{-ad + ab - cd + bc - ab + cd}{2}$$

$$= \frac{bc - ad}{2}$$

$$= \frac{(3a+3\sqrt{a^2-4})(a-\sqrt{a^2-4}) - (a+\sqrt{a^2-4})(-3a+3\sqrt{a^2-4})}{2}$$

$$= \frac{1}{2}(3a^2 - 3a^2 + 12 + 3a^2 - 3a^2 + 12) = 12$$

5. a) $1,\ \cos\frac{2\pi}{11} + i\sin\frac{2\pi}{11},\ \cos\frac{4\pi}{11} + i\sin\frac{4\pi}{11},\ \cos\frac{6\pi}{11} + i\sin\frac{6\pi}{11},$
$\cos\frac{8\pi}{11} + i\sin\frac{8\pi}{11},\ \cos\frac{10\pi}{11} + i\sin\frac{10\pi}{11},\ \cos\frac{12}{11}\pi + i\sin\frac{12}{11}\pi,$
$\cos\frac{14}{11}\pi + i\sin\frac{14}{11}\pi,\ \cos\frac{16}{11}\pi + i\sin\frac{16}{11}\pi,\ \cos\frac{18}{11}\pi + i\sin\frac{18}{11}\pi,$
$\cos\frac{20}{11}\pi + i\sin\frac{20}{11}\pi$, because $\text{cis}\, 2n\pi = z^{11}$
$= \text{cis}\, 11\theta$.

When all these roots are added, the result should be zero because the coefficient of $z^{10} = 0$.

$1 + \cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \cos\frac{6\pi}{11} + \cos\frac{8\pi}{11} + \cos\frac{10}{11}\pi$
$\ldots \cos\frac{20}{11}\pi = 0.$

$\cos\frac{10}{11}\pi = \cos\frac{12}{11}\pi$ because $\frac{12}{10}\pi = 2\pi - \frac{10}{10}\pi$.

$\cos\frac{8}{11}\pi = \cos\frac{14}{11}\pi \ \ldots \ \cos\frac{2\pi}{11} = \cos\frac{20}{11}\pi$

Therefore, $1 + 2 \times \left(\cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \cos\frac{6\pi}{11} + \cos\frac{8\pi}{11} + \cos\frac{10\pi}{11}\right) = 0$.

$\therefore \cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \cos\frac{6\pi}{11} + \cos\frac{8\pi}{11} + \cos\frac{10\pi}{11} = 0$.

QUESTION NUMBER

ASSESSOR USE ONLY

b) $x_1 + x_2 + x_3 \leq 100$

$2x_1 + 3x_2 \leq 150$

$2x_1 + 3x_2 + 4x_3 \leq 210$.

$f(x_1, x_2, x_3) = 4x_1 + 5x_2 + 6x_3$

Logically, the group 1 Questions give $\frac{4}{2} = 2$ marks/minute

group 2 : $\frac{5}{3} = 1.67$ marks/minute

group 3 : $\frac{6}{4} = 1.5$ marks/minute.

Therefore, group 1 questions are done before losing motivation and for the rest of the time group 3 questions are done.

In 150 minutes, 75 group 1 questions can be done, which is 300 marks.

In 60 minutes, 15 group 3 questions can be done, which is 90 marks.

$\therefore$ 390 marks

3 b) ii)

Top Script for 93202 Calculus Outstanding Scholarship **Candidate 37 – A111**		**Total Score**	**40**

Question	Mark	Annotation
		This paper was remarkable because it was completed in only 8 pages. It was awarded the top script because of the use of succinct and exact answers, showing flair and clear communication throughout.