LaTeX template for scripts

by Yannick Kees University Bonn

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1 Examples for using this class

Definition 1.1:

We define the **Laplace-operator** as

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

Theorem 1.2 (Fundamentallemma of calcus of variations):

Let Ω be an open subset of \mathbb{R}^2 and $\Phi: \Omega \to \mathbb{R}$ be lokal integrable. If for any function $v: \Omega \to \mathbb{R}$ with compact support the integral

$$\int\limits_{\Omega} \Phi(x)v(x)\,\mathrm{d}x$$

vanishes, then $\Phi(x) = 0$ nearly everywhere.

Theorem 1.3 (FIRST GREEN IDENTITY):

Let $\omega \subset \mathbb{R}^n$ be compact and ϕ and ψ are two functions on Ω , where ϕ is once and ψ is twice differentable. Then

$$\int_{\Omega} \phi \Delta(\psi) + \nabla \phi \cdot \nabla \psi \, \mathrm{d} m^d = \int_{\partial \Omega} \phi \frac{\partial \psi}{\partial n} \, \mathrm{d} m^{d-1}$$

Proof: We use the gaußsche Integral formula to see that

$$\int_{\partial\Omega} \phi \frac{\partial \psi}{\partial n} dm^{d-1} = \int_{\partial\Omega} (\phi \nabla \psi) \cdot \vec{n} dm^{d-1}$$

$$= \int_{\Omega} \nabla \text{div}(\phi \nabla \psi)$$

$$= \int_{\Omega} \phi \Delta(\psi) + \nabla \phi \cdot \nabla \psi dm^{d}$$

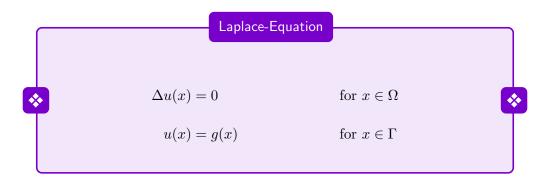
Korollar 1.4:

If $\mathcal{L}u \geq 0$ in Ω , then u obtains its minimum in the boundary of Ω .

Definition 1.5:

A problem is called **well posed** if there exists a solution, that is unique and depends continuously on its data.





Lemma 1.6:

Stability implies $||A_h^{-1}||_{\infty} \leq C_s$ and this bound is independent of h.

Proof: Let \vec{v}_h be the coefficient vector of v_h and $\vec{w} := A_h \vec{v}_h$ is the coefficient vector of $\mathcal{L}_h v_h$

$$||A_h^{-1}\vec{w}_h||_{\infty} = ||\vec{v}_h||_{\infty} = ||v_h||_{\Omega} \le C_s ||\mathcal{L}_h v_h||_{\overline{\Omega_h}} = C_s ||A_h \vec{v}_h||_{\infty} = C_s ||\vec{w}_h||_{\infty}$$

Then $||A_h^{-1}||_{\infty}$ is the smallest number for which this inequality holds.

Remark

In contrast to Dirichlet boundary conditions, Neumann boundary conditions are not directly built into the search space. Therefore they are also calles **natural boundary conditions**.

Lemma 1.7 (COMPARISON PRINCIPLE):

If $\mathcal{L}u \leq \mathcal{L}v$ in Ω and $u \leq v$ in the boundary, then $u \leq v$ in $\overline{\Omega}$.

Proof: We use the maximum principle for

$$\mathcal{L}(v-u) \le 0$$

Then $(v-u) \leq 0$ in $\overline{\Omega}$.

Lemma 1.8:

Let \mathcal{L} be uniformly elliptic. Then there exists a constant $c := c(\Omega, \alpha)$, such that

$$|u(\vec{x})| \le \max_{\vec{z} \in \Gamma} |u(\vec{z})| + c \cdot \sup_{\vec{z} \in \Omega} |(\mathcal{L}u)(\vec{z})| \qquad \forall u \in C^2(\Omega) \cap C(\overline{\Omega}), \ \vec{x} \in \Omega$$



${\bf Theorem~1.9:}$

The space C^{∞} is dense in $H^m(\Omega)$.

Proof: This theorem was proofen in 1964 by Meyers and Serrin. (H=W).

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