LaTeX template for scripts

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1 Examples for using this class

Definition 1.1:
We define the **Laplace-operator** as

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

Theorem 1.2 (FUNDAMENTALLEMMA OF CALCUS OF VARIATIONS):
Let Ω be an open subset of $\mathbb{R}^2$ and $\Phi : \Omega \to \mathbb{R}$ be lokal integrable. If for any function $v : \Omega \to \mathbb{R}$ with compact support the integral

$$\int_{\Omega} \Phi(x) v(x) \, \mathrm{d}x$$

vanishes, then $\Phi(x) = 0$ nearly everywhere.

Theorem 1.3 (FIRST GREEN IDENTITY):
Let $\omega \subset \mathbb{R}^n$ be compact and ϕ and ψ are two functions on Ω, where ϕ is once and ψ is twice differentable. Then

$$\int_{\Omega} \phi \Delta(\psi) + \nabla \phi \cdot \nabla \psi \, \mathrm{d}m^d = \int_{\partial\Omega} \phi \frac{\partial \psi}{\partial n} \, \mathrm{d}m^{d-1}$$

Proof: We use the gaußsche Integral formula to see that

$$\begin{aligned}
\int_{\partial\Omega} \phi \frac{\partial \psi}{\partial n} \, \mathrm{d}m^{d-1} &= \int_{\partial\Omega} (\phi \nabla \psi) \cdot \vec{n} \, \mathrm{d}m^{d-1} \\
&= \int_{\Omega} \nabla \mathrm{div}(\phi \nabla \psi) \\
&= \int_{\Omega} \phi \Delta(\psi) + \nabla \phi \cdot \nabla \psi \, \mathrm{d}m^d
\end{aligned}$$

□

Korollar 1.4:
If $\mathcal{L}u \geq 0$ in Ω, then u obtains its minimum in the boundary of Ω.

Definition 1.5:
A problem is called **well posed** if there exists a solution, that is unique and depends continuously on its data.

Laplace-Equation

$$\Delta u(x) = 0 \qquad \text{for } x \in \Omega$$
$$u(x) = g(x) \qquad \text{for } x \in \Gamma$$

Lemma 1.6:
Stability implies $\|A_h^{-1}\|_\infty \leq C_s$ and this bound is independent of h.

Proof: Let $\vec{v}_h$ be the coefficient vector of v_h and $\vec{w} := A_h \vec{v}_h$ is the coefficient vector of $\mathcal{L}_h v_h$

$$\|A_h^{-1}\vec{w}_h\|_\infty = \|\vec{v}_h\|_\infty = \|v_h\|_\Omega \leq C_s \|\mathcal{L}_h v_h\|_{\overline{\Omega_h}} = C_s \|A_h \vec{v}_h\|_\infty = C_s \|\vec{w}_h\|_\infty$$

Then $\|A_h^{-1}\|_\infty$ is the smallest number for which this inequality holds.

□

Remark

In contrast to Dirichlet boundary conditions, Neumann boundary conditions are not directly built into the search space. Therefore they are also calles **natural boundary conditions**.

Lemma 1.7 (COMPARISON PRINCIPLE):
If $\mathcal{L}u \leq \mathcal{L}v$ in Ω and $u \leq v$ in the boundary, then $u \leq v$ in $\overline{\Omega}$.

Proof: We use the maximum principle for

$$\mathcal{L}(v - u) \leq 0$$

Then $(v - u) \leq 0$ in $\overline{\Omega}$.

□

Lemma 1.8:
Let $\mathcal{L}$ be uniformly elliptic. Then there exists a constant $c := c(\Omega, \alpha)$, such that

$$|u(\vec{x})| \leq \max_{\vec{z}\in\Gamma} |u(\vec{z})| + c \cdot \sup_{\vec{z}\in\Omega} |(\mathcal{L}u)(\vec{z})| \qquad \forall u \in C^2(\Omega) \cap C(\overline{\Omega}),\ \vec{x} \in \Omega$$

Theorem 1.9:
The space C^∞ is dense in $H^m(\Omega)$.

Proof: This theorem was proofen in 1964 by Meyers and Serrin. $(H = W)$.

$\square$