Sheet 1 - Scientific Computing 1

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Exercise 3

Show that the two dimensional Laplace operator in polar coordinates has the form

$$\Delta u(r,\phi) = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r^2}u_{\phi\phi}$$

First we see that

$$D\begin{pmatrix} x(r,\phi) \\ y(r,\phi) \end{pmatrix} = D\begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \end{pmatrix} = \begin{pmatrix} \cos(\phi) & -r\sin(\phi) \\ \sin(\phi) & r\cos(\phi) \end{pmatrix}$$

Now we can compute the second derivatives with the chain rule

$$\begin{aligned} u_{rr} &= \frac{\partial}{\partial r}(u_x x_r + u_y y_r) \\ &= \frac{\partial}{\partial r}(u_x\cos(\phi) + u_y\sin(\phi)) \\ &= \cos(\phi)(u_{xx}x_r + u_{uy}y_r) + \sin(\phi)(u_{yx}x_r + u_{yy}y_r) \\ &= \cos^2(\phi)u_{xx} + 2\cos(\phi)\sin(\phi)u_{xy} + \sin^2(\phi)u_{yy} \end{aligned}$$

and

$$\begin{aligned} u_{\phi\phi} &= \frac{\partial}{\partial\phi}(u_x x_\phi + u_y y_\phi) \\ &= \frac{\partial}{\partial\phi}(-r\sin(\phi)u_x + r\cos(\phi)u_y) \\ &= -r\cos(\phi)u_x - r\sin(\phi)u_{x\phi} - r\sin(\phi)u_y + r\cos(\phi)u_{y\phi} \\ &= -r\cos(\phi)u_x - r\sin(\phi)(u_{xx}x_\phi u_{xy}y_\phi) - r\sin(\phi)u_y + r\cos(\phi)(u_{xy}x_\phi + u_{yy}y_\phi) \\ &= -r\underbrace{(\cos(\phi)u_x + \sin(\phi)u_y)}_{=u_r} + r^2\left(\sin^2(\phi)u_{xx} - 2\cos(\phi)\sin(\phi)u_{xy} + \cos^2(\phi)u_{yy}\right) \\ \Leftrightarrow \quad \frac{1}{r^2}u_{\phi\phi} &= -\frac{1}{r}u_r + \sin^2(\phi)u_{xx} - 2\cos(\phi)\sin(\phi)u_{xy} + \cos^2(\phi)u_{yy} \end{aligned}$$

By adding both we obtain

$$\begin{aligned} u_{rr} + \frac{1}{r^2}u_{\phi\phi} &= -\frac{1}{r}u_r + u_{xx} + u_{yy} \\ \Leftrightarrow \quad \Delta u(r,\phi) &= \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r^2}u_{\phi\phi} \end{aligned}$$

□

Exercise 4

Solve the Laplace equation on the unit disk with boundary conditions $u_r = g$ where g is given as Fourier expansion.

$$g(\cos(\phi), \sin(\phi)) := \alpha_0 + \sum_k r^k(\alpha_k\cos(k\phi) + \beta_k\sin(k\phi))$$

We need to show that

$$u(r\cos(\phi), r\sin(\phi)) = a_0 + \sum_k r^k(a_k\cos(k\phi) + b_k\sin(k\phi))$$

fulfills the Laplace equation. Because of exercise three we can confirm that

$$\begin{aligned}
\Delta u(r,\phi) &= \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r^2}u_{\phi\phi} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\sum_k kr^k(a_k\cos(k\phi) + b_k\sin(k\phi)) - \frac{1}{r^2}\sum_k k^2r^k(a_k\cos(k\phi) + b_k\sin(k\phi)) \\
&= \sum_k k^2r^{k-2}(a_k\cos(k\phi) + b_k\sin(k\phi)) - \sum_k k^2r^{k-2}(a_k\cos(k\phi) + b_k\sin(k\phi)) \\
&= 0
\end{aligned}$$

Now we need to make sure, that the boundary condition fits. We need that

$$u_r = \sum_k kr^{k-1}(a_k\cos(k\phi) + b_k\sin(k\phi)) \stackrel{!}{=} \alpha_0 + \sum_k r^k(\alpha_k\cos(k\phi) + \beta_k\sin(k\phi))$$

for $r = 1$, so we choose

$$\alpha_k = \frac{a_k}{k} \qquad \beta_k = \frac{b_k}{k}$$

$\square$