Exercise Sheet 1

1. Verify that the columns of the following matrix are linearly independent. Then apply the Gram-Schmidt process.
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

2. Let $w \in \mathbb{K}^n$ be a nonzero vector and consider the reflection matrix
$$H = I - \frac{2}{w^* w} w w^* \in \mathbb{K}^{n \times n}.$$
Multiplication with H reflects every vector $x \in \mathbb{K}^n$ across the $(n-1)$-dimensional subspace orthogonal to w.
   (a) Show that H is self-adjoint, unitary/orthogonal and involutory ($HH = I$).
   (b) Determine the eigenvalues, determinant and singular values of H.

3. Show that for every $x \in \mathbb{K}^n$, we have $\|x\|_\infty \leq \|x\|_2$ and $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$. For each inequality find a nonzero vector that attains equality.

4. Show that for $x \in \mathbb{K}^n$
$$\lim_{p \to +\infty} \|x\|_p = \|x\|_\infty.$$

5. The *trace* of a matrix, written $\operatorname{tr} A$, is the sum of its diagonal entries. Show that $\|A\|_F = (\operatorname{tr}(A^*A))^{\frac{1}{2}} = (\operatorname{tr}(AA^*))^{\frac{1}{2}}$ for every matrix A.

6. Compute the condition numbers $\kappa_1(A)$, $\kappa_2(A)$ and $\kappa_\infty(A)$ of the following matrix
$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

7. Consider the linear system $Ax = b$ with
$$A = \begin{bmatrix} 1 & 4 \\ 1 & 4.001 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$
Is it well- or ill-conditioned? Perturb the right-hand side by $\Delta b = [0,\, 0.001]^\top$ and determine the relative error of the solution.