Exercise Sheet 6

1. (a) Determine the condition numbers $\kappa_1(F_n)$, $\kappa_2(F_n)$ and $\kappa_\infty(F_n)$.
   (b) Exploit the symmetry of the Fourier matrix ($F_n = F_n^\top$) to obtain a factorization of F_n that is different from the one given in equation (3.27) of the lecture notes.
   (c) Determine two different factorizations for F_n^{-1}.

2. Determine the real trigonometric interpolant over $x_j = 2\pi j/n$, $j = 0, \ldots, n-1$, for
   (a) $n = 4$ and $y = (1, 0, 1, 0)^\top$,
   (b) $n = 4$ and $y = (0, 1, 0, 1)^\top$,
   (c) $n = 8$ and $y = (1, 0, 1, 0, 1, 0, 1, 0)^\top$,
   (d) $n = 8$ and $y = (0, 1, 0, 1, 0, 1, 0, 1)^\top$.

3. Write a program that implements trigonometric interpolation over $x_j = 2\pi j/n$, $j = 0, \ldots, n-1$. It should
   (a) take as input a vector $y \in \mathbb{R}^n$,
   (b) return the coefficients a_k, b_k of the real trigonometric interpolant,
   (c) produce a plot showing data points and interpolant.

   You can use the FFT function (of Matlab, Python, etc.). Upload your program via Moodle.

4. A matrix $A \in \mathbb{C}^{n \times n}$ is *circulant*, if there is a vector $a = (a_0, \ldots, a_{n-1})^\top \in \mathbb{C}^n$ such that

$$A = \begin{bmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & \cdots & a_2 \\ a_2 & a_1 & a_0 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix}.$$

   Circulant matrices are diagonalized by the Fourier matrix (You do not have to show this):

$$F_n A F_n^{-1} = \operatorname{diag}(F_n a).$$

   For the case $n = 2^p, p \in \mathbb{N}$, describe an algorithm based on this fact that solves $Ax = b$ in essentially $Cn \log_2 n$ operations, where $C > 0$ is a moderate constant.

5. Consider the quadrature rule

$$Q[f] = w_0 f(-1) + w_1 f(0) + w_2 f(1)$$

for estimating the integral

$$I[f] = \int_{-1}^{1} f(x)\,dx.$$

Determine the weights w_0, w_1 and w_2 such that $Q[f]$ is exact for polynomials of degree 2. What happens if you apply it to a cubic polynomial?

6. Write a program that implements the composite trapezoidal rule. It should

   (a) take as input a vector of nodes $x \in \mathbb{R}^n$, where $x_i < x_{i+1}$, and a vector of function values $y \in \mathbb{R}^n$,

   (b) apply the trapezoidal rule to each subinterval $[x_i, x_{i+1}]$,

   (c) return the final estimate for the integral.

   Upload your program via Moodle.