Exercise Sheet 7

1. Proof that (i) a hermitian matrix has real eigenvalues, and (ii) a positive definite matrix has positive eigenvalues. (iii) Show that similar matrices have the same characteristic polynomial (*Hint: Use that for square matrices A and B we have* $\det(AB) = \det(A)\det(B)$).

2. Let A be hermitian with eigendecomposition $A = Q\Lambda Q^*$. Show that the eigendecomposition of the rank-1 perturbation $A + yy^*$ (y an appropriate sized column vector) can be obtained from the eigendecomposition of the perturbed diagonal matrix $\Lambda + zz^*$ with $z = Q^*y$.

3. Write a program/script where you investigate convergence of the power iteration and the Rayleigh quotient estimate to the eigenvalue. For that purpose, generate a symmetric matrix. Investigate the convergence of the Rayleigh quotient estimate to the largest eigenvalue and the error in approximation of the eigenvector. Plot the errors against iteration number in a semi-logarithmic plot. Also illustrate the "speed of convergence" given by q^k, where $q = |\lambda_2/\lambda_1|$, the ratio of the largest and second largest eigenvalue in absolute terms.
   Hints: Generate a random matrix and take its symmetric part; Use built-in/package functions to compute the reference solutions used to calculate the errors; Note that the reference and the approximated eigenvector should be normalized and point in the same direction. The latter can be guaranteed by investigating the sign of the inner product of the reference and the approximated eigenvector.

4. Apply the power iteration on the matrix
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
with starting vector $\frac{1}{\sqrt{3}}(1,1,1)^T$. Does it converge? Why? *Hint: You do not need a computer program for this, but it is allowed.*

5. Consider the webpage linking graph corresponding to the links
$$A \to B, D, E$$
$$B \to A, C$$
$$C \to A, B$$
$$D \to B$$
$$E \to A, C$$
Calculate the PageRank in a little program/script with power iteration where you use (i) $p = 0.15$ and (ii) $p = $ 1e-03.

6. Consider the test matrix
$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$
Implement the inverse iteration and test it with the matrix and shifts $\mu = 5$ and 7. Now, do a slight modification in your code: Start with the same initialization and the shifts as before but replace the shifts inside the loop by the Rayleigh quotients. Do the same test as before.

7. Implement the basic QR algorithm and test it for a symmetric matrix (e.g. take the symmetric part of a randomly generated matrix, i.e., $A \leftarrow (A + A^T)/2$). Compare the results with a package function (e.g. `numpy.linalg.eig` or Matlab's/Octave's `eig` function).

8. Consider a randomly generated matrix $A \in \mathbb{R}^{15\times 30}$ and compute its SVD (take a package/built-in function). Compute the rank-10 (low rank) matrix A_{10} from the SVD of A by discarding the smallest five singular values $\sigma_{11}, \ldots, \sigma_{15}$. Calculate the error in the Frobenius norm $\|A - A_{10}\|_F$ and $\sqrt{\sum_{j=11}^{15} \sigma_j^2}$.

9. Read in a photo/image and convert to a matrix. Perform a SVD of the matrix. Reconstruct the photo using only $25\%, 50\%, 75\%$ of the singular values.
(i) Print/Show the reconstructed photo. How good is the quality of the reconstructed photo?
(ii) What is the respective error in the Frobenius norm for the reconstructed images.

10. Consider the overdetermined system $Ax = b$ with
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \\ 1 & 5 & 6 \end{bmatrix}.$$
and $b = (1, 1, 1, 1)^T$. Compute a solution x via SVD and the norm of the residual $\|Ax - b\|_2$.