Exercise Sheet 1

1. Consider the problem $f : x \mapsto \cos x$ for $x \approx 0$. Is it well- or ill-conditioned? Suppose you have a perfect implementation of the cosine that gives you $\tilde{f}(x) = \mathrm{fl}(\cos x)$. Is evaluation of $\tilde{f}$ (backward) stable?

2. Let $L \in \mathbb{K}^{n\times n}$ be regular and lower triangular. Assume that L is also a *band matrix* with *bandwidth* $2p+1$, that is, $\ell_{ij} = 0$ for $|i-j| > p$. Here, p is a nonnegative integer, typically much smaller than n.

   (a) What does such a matrix look like?

   (b) Show that, if unnecessary zero operations are avoided, the number of flops required for solving $Lx = b$ with forward substitution is linear in n.

3. Let $U \in \mathbb{K}^{n\times n}$ be a regular upper triangular matrix and $b \in \mathbb{K}^n$.

   (a) Derive the analogues of formulas (2.1) and (2.2) (in the lecture notes) for *back substitution*.

   (b) Prove that back substitution for $U \in \mathbb{F}^{2\times 2}$ and $b \in \mathbb{F}^2$ is backward stable.

4. Write a program that, for given upper triangular matrix U and vector b, solves $Ux = b$ using back substitution. Upload it via Moodle.

5. Check if the following matrices admit a factorization $A = LU$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 0 & 6 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 0 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 1 & 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 6 & 2 & 1 & 0 \\ 2 & 5 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}.$$

6. Solve the following linear systems using Gaussian elimination.

$$\text{(a)} \quad \begin{aligned} 2x_1 - 2x_2 + x_3 &= 6 \\ x_2 + 2x_3 &= 3 \\ 5x_1 + 3x_2 + x_3 &= 4 \end{aligned} \qquad \text{(b)} \quad \begin{aligned} x_1 - 2x_2 - 3x_3 &= 10 \\ 5x_1 + 6x_2 - x_3 &= 2 \\ x_1 - x_2 - x_3 &= 6 \end{aligned}$$

7. Use Gaussian elimination with partial pivoting (GEPP) to find an LU factorization of PA, where

$$A = \begin{bmatrix} 3 & 17 & 10 \\ 2 & 4 & -2 \\ 6 & 8 & -12 \end{bmatrix},$$

and P is a suitable permutation matrix. Use this factorization to solve $Ax = b$ where $b = (14, 2, 2)^\top$.

8. Write a program that, given a regular $n \times n$ matrix A, computes its LU factorization using Gaussian elimination with partial pivoting. Upload your program via Moodle.