# Determination of complete experiments using new 'directional' graphs

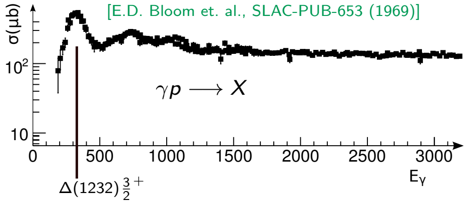

Yannick Wunderlich

HISKP, University of Bonn


November 16, 2021

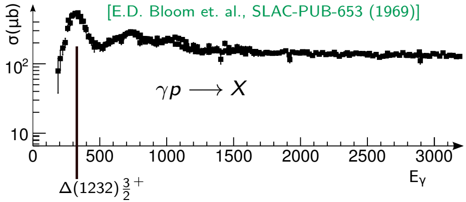
# Introduction: why spin-amplitudes?

$\ast$) Photoproduction is a generic reaction used to study baryon resonances:



# Introduction: why spin-amplitudes?

$\ast$) Photoproduction is a generic reaction used to study baryon resonances:




- $\ast$) Baryon resonances $\left(\Delta(1232)\frac{3}{2}^+, N(1440)\frac{1}{2}^+, \ldots\right)$ are Fermions
  - $\hookrightarrow$ Scatter particles with spin to excite systems with half-integer $J$
- $\ast$) '$T$-matrix' $\mathcal{T}_{fi}$ parameterized by $N$ spin-amplitudes $\{b_i, i = 1, \ldots, N\}$
- $\ast$) The usual reactions under study are:
  - Pion-Nucleon ($\pi N$-) scattering: $\pi N \longrightarrow \pi N$ (2 spin-amplitudes)
  - Pion photoproduction: $\gamma N \longrightarrow \pi N$ (4 spin-amplitudes)
  - Pion electroproduction: $eN \longrightarrow e'\pi N$ (6 spin-amplitudes)
  - 2-Pion photoproduction: $\gamma N \longrightarrow \pi\pi N$ (8 spin-amplitudes)
  - ...

# Algebraic starting point I

$*)$ Generic problem with $N$ amplitudes $\{b_i, i = 1, \ldots, N\}$: the $N^2$ (polarization-) observables are bilinear hermitean forms (def. via orthogonal matrices $\tilde{\Gamma}^{\alpha}$):

$$\mathcal{O}^{\alpha} = \boldsymbol{c}^{\alpha} \sum_{i,j=1}^{N} b_i^{*} \tilde{\Gamma}_{ij}^{\alpha} b_j, \text{ for } \alpha = 1, \ldots, N^2.$$

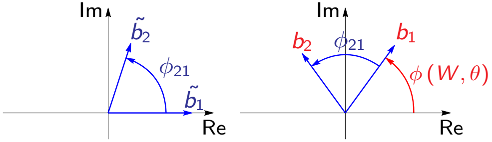
# Algebraic starting point I

$*)$ Generic problem with $N$ amplitudes $\{b_i, i = 1, \ldots, N\}$: the $N^2$ (polarization-) observables are bilinear hermitean forms (def. via orthogonal matrices $\tilde{\Gamma}^{\alpha}$):

$$\mathcal{O}^{\alpha} = \boldsymbol{c}^{\alpha} \sum_{i,j=1}^{N} b_i^{*} \tilde{\Gamma}^{\alpha}_{ij} b_j, \text{ for } \alpha = 1, \ldots, N^2.$$

$\hookrightarrow$ Complete-experiment problem:
What are the minimal subsets of the observables $\mathcal{O}^{\alpha}$, which allow for the unique extraction of the amplitudes $b_i$ up to one unknown overall phase $\phi(W, \theta)$?

- $*)$ Analysis operates on each bin in $(W, \theta)$ individually.
- $*)$ Consider idealized (academic) case without measurement uncertainty!



# Algebraic starting point II

$*)$ Expression $\mathcal{O}^{\alpha} = \boldsymbol{c}^{\alpha} \sum_{i,j=1}^{N} b_i^{*} \tilde{\Gamma}_{ij}^{\alpha} b_j$ can be 'inverted' (using the *completeness* of the $\tilde{\Gamma}$-matrices):

$$b_i^{*} b_j = \frac{1}{\tilde{N}} \sum_{\alpha=1}^{N^2} \left( \tilde{\Gamma}_{ij}^{\alpha} \right)^{*} \left( \frac{\mathcal{O}^{\alpha}}{\boldsymbol{c}^{\alpha}} \right) .$$

$\Rightarrow$ Determine the real- and imaginary parts of a 'minimal' set of $b_i^{*} b_j$

$\Rightarrow$ Obtain (quite large) over-complete set $\{\mathcal{O}^{\alpha}\}$ determined via the RHS

# Algebraic starting point II

$*)$ Expression $\mathcal{O}^{\alpha} = \boldsymbol{c}^{\alpha} \sum_{i,j=1}^{N} b_i^* \tilde{\Gamma}_{ij}^{\alpha} b_j$ can be 'inverted' (using the *completeness* of the $\tilde{\Gamma}$-matrices):

$$b_i^* b_j = \frac{1}{\tilde{N}} \sum_{\alpha=1}^{N^2} \left( \tilde{\Gamma}_{ij}^{\alpha} \right)^* \left( \frac{\mathcal{O}^{\alpha}}{\boldsymbol{c}^{\alpha}} \right) .$$

$\Rightarrow$ Determine the real- and imaginary parts of a 'minimal' set of $b_i^* b_j$
$\Rightarrow$ Obtain (quite large) over-complete set $\{\mathcal{O}^{\alpha}\}$ determined via the RHS

$*)$ Consider alternative 'basis': <u>bilinear products</u>

$b_j^* b_i$, for $i, j = 1, \ldots, N$. [Moravcsik, J. Math. Phys. **26**, 211 (1985).]

# Algebraic starting point II

*) Expression $\mathcal{O}^{\alpha} = \boldsymbol{c}^{\alpha} \sum_{i,j=1}^{N} b_i^{*} \tilde{\Gamma}_{ij}^{\alpha} b_j$ can be 'inverted' (using the *completeness* of the $\tilde{\Gamma}$-matrices):

$$b_i^{*} b_j = \frac{1}{\tilde{N}} \sum_{\alpha=1}^{N^2} \left( \tilde{\Gamma}_{ij}^{\alpha} \right)^{*} \left( \frac{\mathcal{O}^{\alpha}}{\boldsymbol{c}^{\alpha}} \right) .$$

$\Rightarrow$ Determine the real- and imaginary parts of a 'minimal' set of $b_i^{*} b_j$
$\Rightarrow$ Obtain (quite large) over-complete set $\{\mathcal{O}^{\alpha}\}$ determined via the RHS

*) Consider alternative 'basis': bilinear products

$b_j^{*} b_i$, for $i, j = 1, \ldots, N$. [Moravcsik, J. Math. Phys. **26**, 211 (1985).]

*) Standard initial assumption: the moduli $|b_1|, |b_2|, \ldots, |b_N|$ are already known from a certain subset of 'diagonal' observables.
$\Rightarrow$ Have to determine a minimal set of relative phases $\phi_{ij} := \phi_i - \phi_j$ ($b_j = |b_j| e^{i\phi_j}$)

# Algebraic starting point II

$*)$ Expression $\mathcal{O}^{\alpha} = \boldsymbol{c}^{\alpha} \sum_{i,j=1}^{N} b_i^{*} \tilde{\Gamma}_{ij}^{\alpha} b_j$ can be 'inverted' (using the *completeness* of the $\tilde{\Gamma}$-matrices):

$$b_i^{*} b_j = \frac{1}{\tilde{N}} \sum_{\alpha=1}^{N^2} \left( \tilde{\Gamma}_{ij}^{\alpha} \right)^{*} \left( \frac{\mathcal{O}^{\alpha}}{\boldsymbol{c}^{\alpha}} \right) .$$

$\Rightarrow$ Determine the real- and imaginary parts of a 'minimal' set of $b_i^{*} b_j$
$\Rightarrow$ Obtain (quite large) over-complete set $\{\mathcal{O}^{\alpha}\}$ determined via the RHS

$*)$ Consider alternative 'basis': bilinear products

$b_j^{*} b_i$, for $i, j = 1, \ldots, N$. [Moravcsik, J. Math. Phys. **26**, 211 (1985).]

$*)$ Standard initial assumption: the moduli $|b_1|, |b_2|, \ldots, |b_N|$ are already known from a certain subset of 'diagonal' observables.
$\Rightarrow$ Have to determine a minimal set of relative phases $\phi_{ij} := \phi_i - \phi_j$ ($b_j = |b_j| e^{i\phi_j}$)

$*)$ Finding a generic solution for such problems, for arbitrary $N$, can be quite tough in the $\mathcal{O}^{\alpha}$-basis.
However: In the $b_i^{*} b_j$-basis, a general solution exists:

Moravcsik's Theorem!

# Discrete ambiguities

'Cosine-type' ambiguities:

The real part

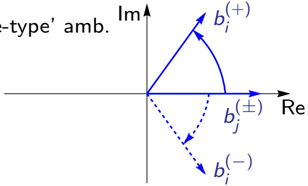
$$\mathrm{Re}\left[b_j^* b_i\right] = |b_i|\,|b_j|\,\mathrm{Re}\left[e^{i\phi_{ij}}\right] = |b_i|\,|b_j|\cos\phi_{ij},$$

fixes the relative phase $\phi_{ij}$ up to the discrete ambiguity:

$$\phi_{ij} \longrightarrow \phi_{ij}^{\pm} = \begin{cases} +\alpha_{ij}, \\ -\alpha_{ij}, \end{cases}$$

with a unique $\alpha_{ij} \in [0, \pi]$.

'Cosine-type' amb.



# Discrete ambiguities

<u>'Cosine-type' ambiguities:</u>

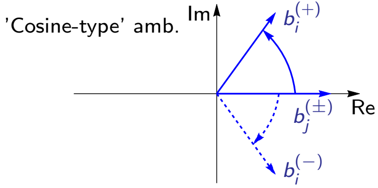
The real part

$$\mathrm{Re}\left[b_j^* b_i\right] = |b_i|\,|b_j|\,\mathrm{Re}\left[e^{i\phi_{ij}}\right] = |b_i|\,|b_j|\cos\phi_{ij},$$

fixes the relative phase $\phi_{ij}$ up to the discrete ambiguity:

$$\phi_{ij} \longrightarrow \phi_{ij}^{\pm} = \begin{cases} +\alpha_{ij}, \\ -\alpha_{ij}, \end{cases}$$

with a unique $\alpha_{ij} \in [0, \pi]$.




<u>'Sine-type' ambiguities:</u>

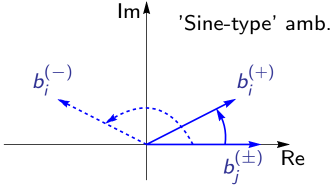
The imaginary part

$$\mathrm{Im}\left[b_j^* b_i\right] = |b_i|\,|b_j|\,\mathrm{Im}\left[e^{i\phi_{ij}}\right] = |b_i|\,|b_j|\sin\phi_{ij},$$

fixes the relative phase $\phi_{ij}$ up to the discrete ambiguity:

$$\phi_{ij} \longrightarrow \phi_{ij}^{\pm} = \begin{cases} +\alpha_{ij}, \\ \pi - \alpha_{ij}, \end{cases}$$

with a unique $\alpha_{ij} \in [-\pi/2, \pi/2]$.



# Discrete ambiguities

'Cosine-type' ambiguities:

The real part

$$\mathrm{Re}\left[b_j^* b_i\right] = |b_i|\,|b_j|\,\mathrm{Re}\left[e^{i\phi_{ij}}\right] \\ = |b_i|\,|b_j|\cos\phi_{ij},$$

fixes the relative phase $\phi_{ij}$ up to the discrete ambiguity:

$$\phi_{ij} \longrightarrow \phi_{ij}^{\pm} = \begin{cases} +\alpha_{ij}, \\ -\alpha_{ij}, \end{cases}$$

with a unique $\alpha_{ij} \in [0, \pi]$.

'Sine-type' ambiguities:

The imaginary part

$$\mathrm{Im}\left[b_j^* b_i\right] = |b_i|\,|b_j|\,\mathrm{Im}\left[e^{i\phi_{ij}}\right] \\ = |b_i|\,|b_j|\sin\phi_{ij},$$

fixes the relative phase $\phi_{ij}$ up to the discrete ambiguity:

$$\phi_{ij} \longrightarrow \phi_{ij}^{\pm} = \begin{cases} +\alpha_{ij}, \\ \pi - \alpha_{ij}, \end{cases}$$

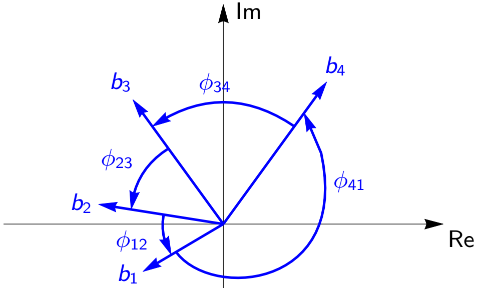
with a unique $\alpha_{ij} \in [-\pi/2, \pi/2]$.

$\hookrightarrow$ Discrete ambiguities for a subset of real-and imaginary parts of bilinear products $b_j^* b_i$, defined by $N$ amplitudes $\{b_i, i = 1, \ldots, N\}$, are 'direct (or Kronecker-) products' of these fundamental discrete ambiguities.

$\Rightarrow$ Such ambiguities turn up time and again in the discussion of complete experiments! Is there help? Yes! $\rightarrow$ Consistency Relations

# Consistency relations

$\ast$) Consider amplitude-arrangement in the complex plane (e.g.: $N = 4$):

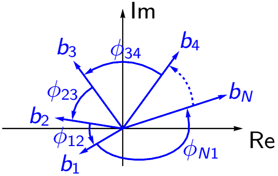



$\ast$) Natural constraint satisfied by this constellation: <u>consistency relation</u>

$$\phi_{12} + \phi_{23} + \phi_{34} + \phi_{41} = 0 \text{ (up to add. of multiples of } 2\pi\text{)}.$$

# Consistency relations

$*$) Consider amplitude-arrangement in the complex plane (arbitrary $N$):

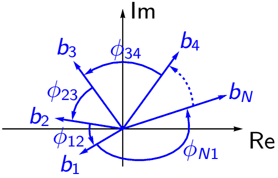



$*$) Fundamental consistency relation for a problem with $N$ amplitudes:

$$\phi_{12} + \phi_{23} + \ldots + \phi_{N-1,N} + \phi_{N1} = 0 \text{ (modulo add. of } 2\pi\text{).}$$

# Consistency relations

*) Consider amplitude-arrangement in the complex plane (arbitrary $N$):




*) Fundamental consistency relation for a problem with $N$ amplitudes:

$$\phi_{12} + \phi_{23} + \ldots + \phi_{N-1,N} + \phi_{N1} = 0 \text{ (modulo add. of } 2\pi).$$

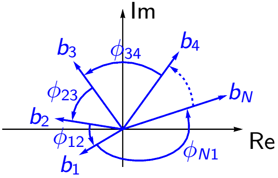
*) Consistency relations may look trivial, but they are very important for the resolution of discrete ambiguities: in case all the possible cases

$$\phi_{12}^{\pm} + \phi_{23}^{\pm} + \ldots + \phi_{N-1,N}^{\pm} + \phi_{N1}^{\pm} = 0,$$

are a fully non-degenerate set of equations, i.e. $\nexists$ any equivalent pairs of equations, the corresponding set of observables is complete!

# Consistency relations

*) Consider amplitude-arrangement in the complex plane (arbitrary $N$):




*) Fundamental consistency relation for a problem with $N$ amplitudes:

$$\phi_{12} + \phi_{23} + \ldots + \phi_{N-1,N} + \phi_{N1} = 0 \text{ (modulo add. of } 2\pi\text{)}.$$

*) Consistency relations may look trivial, but they are very important for the resolution of discrete ambiguities: in case all the possible cases

$$\phi_{12}^{\pm} + \phi_{23}^{\pm} + \ldots + \phi_{N-1,N}^{\pm} + \phi_{N1}^{\pm} = 0,$$

are a fully non-degenerate set of equations, i.e. $\nexists$ any equivalent pairs of equations, the corresponding set of observables is complete!

↪ Moravcsik's Theorem is a systematic study of all cases where such non-degeneracies are obtained, in the $b_j^* b_i$-basis.

# Moravcsik's Theorem (modified form)

From [YW, P. Kroenert, F. Afzal, A. Thiel, Phys. Rev. C **102**, no.3, 034605 (2020)], based on [Moravcsik, J. Math. Phys. **26**, 211 (1985).]:

# Moravcsik's Theorem (modified form)

From [YW, P. Kroenert, F. Afzal, A. Thiel, Phys. Rev. C **102**, no.3, 034605 (2020)], based on [Moravcsik, J. Math. Phys. **26**, 211 (1985).]:

'Geometrical (graphical) analog': Represent every amplitude $b_1, \ldots, b_N$ by a *point* and every product $b_j^* b_i$, or rel.-phase $\phi_{ij}$, by a *line connecting points 'i' and 'j'*.

Furthermore: $\hookrightarrow$ Represent every $\mathrm{Re}\left[b_i^* b_j\right] \propto \cos\phi_{ij}$ by a *solid line*,

$\hookrightarrow$ Represent every $\mathrm{Im}\left[b_i^* b_j\right] \propto \sin\phi_{ij}$ by a *dashed line*.

# Moravcsik's Theorem (modified form)

From [YW, P. Kroenert, F. Afzal, A. Thiel, Phys. Rev. C **102**, no.3, 034605 (2020)], based on [Moravcsik, J. Math. Phys. **26**, 211 (1985).]:

'Geometrical (graphical) analog': Represent every amplitude $b_1, \ldots, b_N$ by a *point* and every product $b_j^* b_i$, or rel.-phase $\phi_{ij}$, by a *line connecting points 'i' and 'j'*.

Furthermore: $\hookrightarrow$ Represent every $\mathrm{Re}\left[b_i^* b_j\right] \propto \cos\phi_{ij}$ by a *solid line*,

$\hookrightarrow$ Represent every $\mathrm{Im}\left[b_i^* b_j\right] \propto \sin\phi_{ij}$ by a *dashed line*.

Moravcsik's Theorem (modified): The thus constructed graph is *fully complete*, i.e. it allows for neither any continuous nor any discrete ambiguities, if it satisfies:

# Moravcsik's Theorem (modified form)

From [YW, P. Kroenert, F. Afzal, A. Thiel, Phys. Rev. C **102**, no.3, 034605 (2020)], based on [Moravcsik, J. Math. Phys. **26**, 211 (1985).]:

'Geometrical (graphical) analog': Represent every amplitude $b_1, \ldots, b_N$ by a *point* and every product $b_j^* b_i$, or rel.-phase $\phi_{ij}$, by a *line connecting points 'i' and 'j'*.

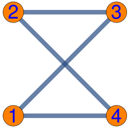
Furthermore: $\hookrightarrow$ Represent every $\mathrm{Re}\left[b_i^* b_j\right] \propto \cos \phi_{ij}$ by a *solid line*,

$\hookrightarrow$ Represent every $\mathrm{Im}\left[b_i^* b_j\right] \propto \sin \phi_{ij}$ by a *dashed line*.

Moravcsik's Theorem (modified): The thus constructed graph is *fully complete*, i.e. it allows for neither any continuous nor any discrete ambiguities, if it satisfies:

(i) the graph is fully *connected* and all points have to have *order two* (i.e. are attached to two lines):
  - all continuous ambiguities are resolved,
  - existence of *consistency relation* is ensured.
    $\hookrightarrow$ crucial for resolving discrete ambiguities

Example:




$\Leftrightarrow \phi_{13} + \phi_{32} + \phi_{24} + \phi_{41} = 0$

# Moravcsik's Theorem (modified form)

From [YW, P. Kroenert, F. Afzal, A. Thiel, Phys. Rev. C **102**, no.3, 034605 (2020)], based on [Moravcsik, J. Math. Phys. **26**, 211 (1985).]:

'Geometrical (graphical) analog': Represent every amplitude $b_1, \ldots, b_N$ by a *point* and every product $b_j^* b_i$, or rel.-phase $\phi_{ij}$, by a *line connecting points* '$i$' and '$j$'.

Furthermore: $\hookrightarrow$ Represent every $\mathrm{Re}\left[b_i^* b_j\right] \propto \cos\phi_{ij}$ by a *solid line*,

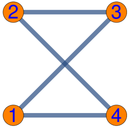
$\hookrightarrow$ Represent every $\mathrm{Im}\left[b_i^* b_j\right] \propto \sin\phi_{ij}$ by a *dashed line*.

Moravcsik's Theorem (modified): The thus constructed graph is *fully complete*, i.e. it allows for neither any continuous nor any discrete ambiguities, if it satisfies:

(i) the graph is fully *connected* and all points have to have *order two* (i.e. are attached to two lines):

- all continuous ambiguities are resolved,
- existence of *consistency relation* is ensured.
  $\hookrightarrow$ crucial for resolving discrete ambiguities

Example:

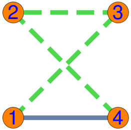



$\Leftrightarrow \phi_{13} + \phi_{32} + \phi_{24} + \phi_{41} = 0$

(ii) the graph has to have an *odd* number of dashed lines, as well as *any* number of solid lines:

- all discrete ambiguities are resolved.

E.g.:




is complete ✓

# Example: photoproduction (observables)

| Observable | Bilinear form | Shape-class |
|---|---|---|
| $\sigma_0 = \frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^1\,\lvert b\rangle$ | |
| $-\check{\Sigma} = \frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 - \lvert b_4\rvert^2\right)$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^4\,\lvert b\rangle$ | $\mathcal{S} = \mathrm{D}$ |
| $-\check{T} = \frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 - \lvert b_4\rvert^2\right)$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^{10}\,\lvert b\rangle$ | |
| $\check{P} = \frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^{12}\,\lvert b\rangle$ | |
| $\mathcal{O}^a_{1+} = \lvert b_1\rvert\lvert b_3\rvert\sin\phi_{13} + \lvert b_2\rvert\lvert b_4\rvert\sin\phi_{24} = \mathrm{Im}\left[b_3^* b_1 + b_4^* b_2\right] = -\check{G}$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^3\,\lvert b\rangle$ | |
| $\mathcal{O}^a_{1-} = \lvert b_1\rvert\lvert b_3\rvert\sin\phi_{13} - \lvert b_2\rvert\lvert b_4\rvert\sin\phi_{24} = \mathrm{Im}\left[b_3^* b_1 - b_4^* b_2\right] = \check{F}$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^{11}\,\lvert b\rangle$ | $a = \mathcal{BT} = \mathrm{PR}$ |
| $\mathcal{O}^a_{2+} = \lvert b_1\rvert\lvert b_3\rvert\cos\phi_{13} + \lvert b_2\rvert\lvert b_4\rvert\cos\phi_{24} = \mathrm{Re}\left[b_3^* b_1 + b_4^* b_2\right] = -\check{E}$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^9\,\lvert b\rangle$ | |
| $\mathcal{O}^a_{2-} = \lvert b_1\rvert\lvert b_3\rvert\cos\phi_{13} - \lvert b_2\rvert\lvert b_4\rvert\cos\phi_{24} = \mathrm{Re}\left[b_3^* b_1 - b_4^* b_2\right] = \check{H}$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^5\,\lvert b\rangle$ | |
| $\mathcal{O}^b_{1+} = \lvert b_1\rvert\lvert b_4\rvert\sin\phi_{14} + \lvert b_2\rvert\lvert b_3\rvert\sin\phi_{23} = \mathrm{Im}\left[b_4^* b_1 + b_3^* b_2\right] = \check{O}_{z'}$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^7\,\lvert b\rangle$ | |
| $\mathcal{O}^b_{1-} = \lvert b_1\rvert\lvert b_4\rvert\sin\phi_{14} - \lvert b_2\rvert\lvert b_3\rvert\sin\phi_{23} = \mathrm{Im}\left[b_4^* b_1 - b_3^* b_2\right] = -\check{C}_{x'}$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^{16}\,\lvert b\rangle$ | $b = \mathcal{BR} = \mathrm{AD}$ |
| $\mathcal{O}^b_{2+} = \lvert b_1\rvert\lvert b_4\rvert\cos\phi_{14} + \lvert b_2\rvert\lvert b_3\rvert\cos\phi_{23} = \mathrm{Re}\left[b_4^* b_1 + b_3^* b_2\right] = -\check{C}_{z'}$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^2\,\lvert b\rangle$ | |
| $\mathcal{O}^b_{2-} = \lvert b_1\rvert\lvert b_4\rvert\cos\phi_{14} - \lvert b_2\rvert\lvert b_3\rvert\cos\phi_{23} = \mathrm{Re}\left[b_4^* b_1 - b_3^* b_2\right] = -\check{O}_{x'}$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^{14}\,\lvert b\rangle$ | |
| $\mathcal{O}^c_{1+} = \lvert b_1\rvert\lvert b_2\rvert\sin\phi_{12} + \lvert b_3\rvert\lvert b_4\rvert\sin\phi_{34} = \mathrm{Im}\left[b_2^* b_1 + b_4^* b_3\right] = -\check{L}_{x'}$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^8\,\lvert b\rangle$ | |
| $\mathcal{O}^c_{1-} = \lvert b_1\rvert\lvert b_2\rvert\sin\phi_{12} - \lvert b_3\rvert\lvert b_4\rvert\sin\phi_{34} = \mathrm{Im}\left[b_2^* b_1 - b_4^* b_3\right] = -\check{T}_{z'}$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^{13}\,\lvert b\rangle$ | $c = \mathcal{TR} = \mathrm{PL}$ |
| $\mathcal{O}^c_{2+} = \lvert b_1\rvert\lvert b_2\rvert\cos\phi_{12} + \lvert b_3\rvert\lvert b_4\rvert\cos\phi_{34} = \mathrm{Re}\left[b_2^* b_1 + b_4^* b_3\right] = -\check{L}_{z'}$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^{15}\,\lvert b\rangle$ | |
| $\mathcal{O}^c_{2-} = \lvert b_1\rvert\lvert b_2\rvert\cos\phi_{12} - \lvert b_3\rvert\lvert b_4\rvert\cos\phi_{34} = \mathrm{Re}\left[b_2^* b_1 - b_4^* b_3\right] = \check{T}_{x'}$ | $\frac{1}{2}\langle b\rvert\,\tilde{\Gamma}^6\,\lvert b\rangle$ | |

# Example: photoproduction (further preliminaries)

$\ast$) Standard assumption: moduli are known from group $\mathcal{S}$ observables:

$$|b_1| = \sqrt{\frac{1}{2}\left(\sigma_0 - \check{\Sigma} + \check{T} - \check{P}\right)}, |b_2| = \sqrt{\frac{1}{2}\left(\sigma_0 - \check{\Sigma} - \check{T} + \check{P}\right)},$$
$$|b_3| = \sqrt{\frac{1}{2}\left(\sigma_0 + \check{\Sigma} - \check{T} - \check{P}\right)}, |b_4| = \sqrt{\frac{1}{2}\left(\sigma_0 + \check{\Sigma} + \check{T} + \check{P}\right)}.$$

# Example: photoproduction (further preliminaries)

$*)$ Standard assumption: moduli are known from group $\mathcal{S}$ observables:

$$|b_1| = \sqrt{\frac{1}{2}\left(\sigma_0 - \check{\Sigma} + \check{T} - \check{P}\right)}, |b_2| = \sqrt{\frac{1}{2}\left(\sigma_0 - \check{\Sigma} - \check{T} + \check{P}\right)},$$
$$|b_3| = \sqrt{\frac{1}{2}\left(\sigma_0 + \check{\Sigma} - \check{T} - \check{P}\right)}, |b_4| = \sqrt{\frac{1}{2}\left(\sigma_0 + \check{\Sigma} + \check{T} + \check{P}\right)}.$$

$*)$ Define a basis of 'decoupled' observables $\tilde{\mathcal{O}}^n_{\nu\pm}$, which isolate the real- and imaginary parts of the bilinear products $b_j^* b_i$:

$$\tilde{\mathcal{O}}^n_{1\pm} := \frac{1}{2}\left(\mathcal{O}^n_{1+} \pm \mathcal{O}^n_{1-}\right), \; n = a, b, c,$$
$$\tilde{\mathcal{O}}^n_{2\pm} := \frac{1}{2}\left(\mathcal{O}^n_{2+} \pm \mathcal{O}^n_{2-}\right), \; n = a, b, c.$$

# Example: photoproduction (further preliminaries)

$*)$ Standard assumption: moduli are known from group $\mathcal{S}$ observables:

$$|b_1| = \sqrt{\frac{1}{2}\left(\sigma_0 - \check{\Sigma} + \check{T} - \check{P}\right)}, |b_2| = \sqrt{\frac{1}{2}\left(\sigma_0 - \check{\Sigma} - \check{T} + \check{P}\right)},$$
$$|b_3| = \sqrt{\frac{1}{2}\left(\sigma_0 + \check{\Sigma} - \check{T} - \check{P}\right)}, |b_4| = \sqrt{\frac{1}{2}\left(\sigma_0 + \check{\Sigma} + \check{T} + \check{P}\right)}.$$

$*)$ Define a basis of 'decoupled' observables $\tilde{\mathcal{O}}^n_{\nu\pm}$, which isolate the real- and imaginary parts of the bilinear products $b_j^* b_i$:

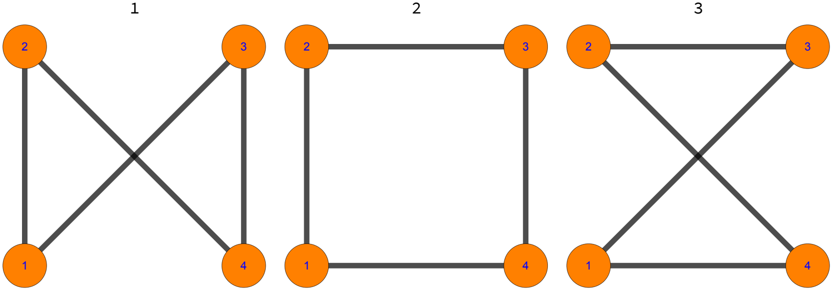
$$\tilde{\mathcal{O}}^n_{1\pm} := \frac{1}{2}\left(\mathcal{O}^n_{1+} \pm \mathcal{O}^n_{1-}\right),\ n = a, b, c,$$
$$\tilde{\mathcal{O}}^n_{2\pm} := \frac{1}{2}\left(\mathcal{O}^n_{2+} \pm \mathcal{O}^n_{2-}\right),\ n = a, b, c.$$

- Example:

$$\mathrm{Im}\left[b_4^* b_2\right] = |b_2|\,|b_4| \sin\phi_{24} = \tilde{\mathcal{O}}^a_{1-} = \tfrac{1}{2}\left(\mathcal{O}^a_{1+} - \mathcal{O}^a_{1-}\right) = \tfrac{1}{2}\left(-\check{G} - \check{F}\right).$$

# Example: photoproduction (à la Moravcsik) I

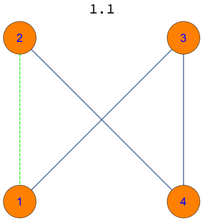
$*)$ For $N = 4$ amplitudes, one gets $\frac{(N-1)!}{2} = \frac{3!}{2} = 3$ possible graph-topologies :




$\hookrightarrow$ *Each* of these topologies can be used as a *starting point* to derive complete sets of observables, by inserting *odd* numbers of dashed lines . . .

# Example: photoproduction (à la Moravcsik) II

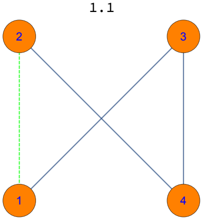
$*$) Example (1.1) (fully complete):




$\longrightarrow \{\sin\phi_{12}, \cos\phi_{24}, \cos\phi_{34}, \cos\phi_{13}\}$

# Example: photoproduction (à la Moravcsik) II

$*$) Example (1.1) (fully complete):



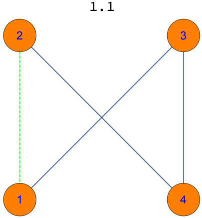

$\longrightarrow \{\sin\phi_{12}, \cos\phi_{24}, \cos\phi_{34}, \cos\phi_{13}\}$

$\hookrightarrow$ Map this result to observables (in $\tilde{\mathcal{O}}$- and $\mathcal{O}$-basis):

$$|b_1|\,|b_2| \sin\phi_{12} = \tilde{\mathcal{O}}^c_{1+} = (1/2)\left[\mathcal{O}^c_{1+} + \mathcal{O}^c_{1-}\right] = (1/2)\left[-\check{L}_{x'} - \check{T}_{z'}\right],$$
$$|b_2|\,|b_4| \cos\phi_{24} = \tilde{\mathcal{O}}^a_{2-} = (1/2)\left[\mathcal{O}^a_{2+} - \mathcal{O}^a_{2-}\right] = (1/2)\left[-\check{E} - \check{H}\right],$$
$$|b_3|\,|b_4| \cos\phi_{34} = \tilde{\mathcal{O}}^c_{2-} = (1/2)\left[\mathcal{O}^c_{2+} - \mathcal{O}^c_{2-}\right] = (1/2)\left[-\check{L}_{z'} - \check{T}_{x'}\right],$$
$$|b_1|\,|b_3| \cos\phi_{13} = \tilde{\mathcal{O}}^a_{2+} = (1/2)\left[\mathcal{O}^a_{2+} + \mathcal{O}^a_{2-}\right] = (1/2)\left[-\check{E} + \check{H}\right].$$

# Example: photoproduction (à la Moravcsik) II

$*)$ Example (1.1) (fully complete):




$\longrightarrow \{\sin\phi_{12}, \cos\phi_{24}, \cos\phi_{34}, \cos\phi_{13}\}$

$\hookrightarrow$ Map this result to observables (in $\tilde{\mathcal{O}}$- and $\mathcal{O}$-basis):

$$|b_1|\,|b_2|\sin\phi_{12} = \tilde{\mathcal{O}}^{c}_{1+} = (1/2)\left[\mathcal{O}^{c}_{1+} + \mathcal{O}^{c}_{1-}\right] = (1/2)\left[-\check{L}_{x'} - \check{T}_{z'}\right],$$
$$|b_2|\,|b_4|\cos\phi_{24} = \tilde{\mathcal{O}}^{a}_{2-} = (1/2)\left[\mathcal{O}^{a}_{2+} - \mathcal{O}^{a}_{2-}\right] = (1/2)\left[-\check{E} - \check{H}\right],$$
$$|b_3|\,|b_4|\cos\phi_{34} = \tilde{\mathcal{O}}^{c}_{2-} = (1/2)\left[\mathcal{O}^{c}_{2+} - \mathcal{O}^{c}_{2-}\right] = (1/2)\left[-\check{L}_{z'} - \check{T}_{x'}\right],$$
$$|b_1|\,|b_3|\cos\phi_{13} = \tilde{\mathcal{O}}^{a}_{2+} = (1/2)\left[\mathcal{O}^{a}_{2+} + \mathcal{O}^{a}_{2-}\right] = (1/2)\left[-\check{E} + \check{H}\right].$$

$\Rightarrow$ Extract the 'Moravcsik-complete' set (combined with $\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\}$):

$$\left\{\mathcal{O}^{a}_{2+}, \mathcal{O}^{a}_{2-}, \mathcal{O}^{c}_{1+}, \mathcal{O}^{c}_{1-}, \mathcal{O}^{c}_{2+}, \mathcal{O}^{c}_{2-}\right\} \equiv \left\{\check{E}, \check{H}, \check{L}_{x'}, \check{T}_{z'}, \check{L}_{z'}, \check{T}_{x'}\right\}.$$

# Example: photoproduction (à la Moravcsik) III

∗) Similar procedure, applied to all the remaining relevant graphs, leads to 12 non-redundant 'Moravcsik-complete' sets for photoproduction (always in combination with $\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\}$):

| Set-Nr. | Observables | | | Set-Nr. | Observables | | |
|---|---|---|---|---|---|---|---|
| 1 | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^c_{1\pm}$ | $\mathcal{O}^c_{2\pm}$ | 7 | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^c_{1\pm}$ | $\mathcal{O}^c_{2\pm}$ |
| 2 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^c_{2\pm}$ | 8 | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^b_{2\pm}$ | $\mathcal{O}^c_{1\pm}$ |
| 3 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^c_{1\pm}$ | $\mathcal{O}^c_{2\pm}$ | 9 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^b_{2\pm}$ |
| 4 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^c_{1\pm}$ | 10 | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^b_{2\pm}$ |
| 5 | $\mathcal{O}^b_{2\pm}$ | $\mathcal{O}^c_{1\pm}$ | $\mathcal{O}^c_{2\pm}$ | 11 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^b_{1\pm}$ |
| 6 | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^b_{2\pm}$ | $\mathcal{O}^c_{2\pm}$ | 12 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^b_{2\pm}$ |

# Example: photoproduction (à la Moravcsik) III

$*$) Similar procedure, applied to all the remaining relevant graphs, leads to 12 non-redundant 'Moravcsik-complete' sets for photoproduction (always in combination with $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}$):

| Set-Nr. | Observables | | | Set-Nr. | Observables | | |
|---|---|---|---|---|---|---|---|
| 1 | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^c_{1\pm}$ | $\mathcal{O}^c_{2\pm}$ | 7 | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^c_{1\pm}$ | $\mathcal{O}^c_{2\pm}$ |
| 2 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^c_{2\pm}$ | 8 | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^b_{2\pm}$ | $\mathcal{O}^c_{1\pm}$ |
| 3 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^c_{1\pm}$ | $\mathcal{O}^c_{2\pm}$ | 9 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^b_{2\pm}$ |
| 4 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^c_{1\pm}$ | 10 | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^b_{2\pm}$ |
| 5 | $\mathcal{O}^b_{2\pm}$ | $\mathcal{O}^c_{1\pm}$ | $\mathcal{O}^c_{2\pm}$ | 11 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^b_{1\pm}$ |
| 6 | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^b_{2\pm}$ | $\mathcal{O}^c_{2\pm}$ | 12 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^b_{2\pm}$ |

Observation: Moravcsik-complete sets contain 2 observables more than complete sets with an absolutely minimal amount of observables, i.e. with $2N = 8$ observables [Chiang & Tabakin (1997), Nakayama (2018)].

# Example: photoproduction (à la Moravcsik) III

$*$) Similar procedure, applied to all the remaining relevant graphs, leads to 12 non-redundant 'Moravcsik-complete' sets for photoproduction (always in combination with $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}$):

| Set-Nr. | Observables | | | Set-Nr. | Observables | | |
|---|---|---|---|---|---|---|---|
| 1 | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^c_{1\pm}$ | $\mathcal{O}^c_{2\pm}$ | 7 | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^c_{1\pm}$ | $\mathcal{O}^c_{2\pm}$ |
| 2 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^c_{2\pm}$ | 8 | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^b_{2\pm}$ | $\mathcal{O}^c_{1\pm}$ |
| 3 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^c_{1\pm}$ | $\mathcal{O}^c_{2\pm}$ | 9 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^b_{2\pm}$ |
| 4 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^c_{1\pm}$ | 10 | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^b_{2\pm}$ |
| 5 | $\mathcal{O}^b_{2\pm}$ | $\mathcal{O}^c_{1\pm}$ | $\mathcal{O}^c_{2\pm}$ | 11 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^a_{2\pm}$ | $\mathcal{O}^b_{1\pm}$ |
| 6 | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^b_{2\pm}$ | $\mathcal{O}^c_{2\pm}$ | 12 | $\mathcal{O}^a_{1\pm}$ | $\mathcal{O}^b_{1\pm}$ | $\mathcal{O}^b_{2\pm}$ |

Observation: Moravcsik-complete sets contain 2 observables more than complete sets with an absolutely minimal amount of observables, i.e. with $2N = 8$ observables [Chiang & Tabakin (1997), Nakayama (2018)].

$\leftrightarrow$ What happened?!

- Not fully clear yet. Possible method to reduce this mismatch $\rightarrow$ new graphs

# Motivation for new 'directional' graphs

<u>Observation:</u> Moravcsik-complete sets tend to be slightly over-complete, i.e. to contain *more than* $2N$ observables, for problems with $N \geq 4$ amplitudes

# Motivation for new 'directional' graphs

Observation: Moravcsik-complete sets tend to be slightly over-complete, i.e. to contain *more than* $2N$ observables, for problems with $N \geq 4$ amplitudes

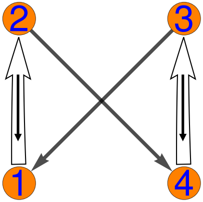
$\hookrightarrow$ One can improve the situation using new kind of graphs, containing additional *directional information*. [YW, Phys. Rev. C **104**, no. 4, 045203 (2021)]

# Motivation for new 'directional' graphs

Observation: Moravcsik-complete sets tend to be slightly over-complete, i.e. to contain *more than* $2N$ observables, for problems with $N \geq 4$ amplitudes

$\hookrightarrow$ One can improve the situation using new kind of graphs, containing additional *directional information*. [YW, Phys. Rev. C **104**, no. 4, 045203 (2021)]

$*$) Example:



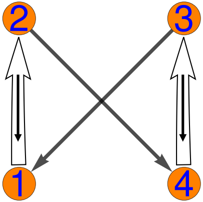

$\Leftrightarrow$ complete photoproduction-set ($2N = 8$ obs.'s in combination with 4 'diagonal' obs.'s $\left\{ \sigma_0, \check{\Sigma}, \check{T}, \check{P} \right\}$):
$\left\{ \mathcal{O}^a_{2+}, \mathcal{O}^a_{2-}, \mathcal{O}^c_{1+}, \mathcal{O}^c_{2-} \right\} = \left\{ \check{E}, \check{H}, \check{L}_{x'}, \check{T}_{x'} \right\}$.

# Motivation for new 'directional' graphs

Observation: Moravcsik-complete sets tend to be slightly over-complete, i.e. to contain *more than* $2N$ observables, for problems with $N \geq 4$ amplitudes

$\hookrightarrow$ One can improve the situation using new kind of graphs, containing additional *directional information*. [YW, Phys. Rev. C **104**, no. 4, 045203 (2021)]

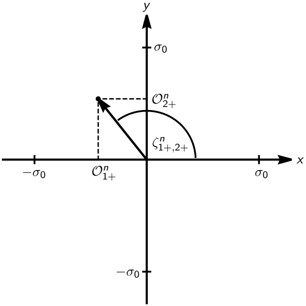
∗) Example:




$\Leftrightarrow$ complete photoproduction-set ($2N = 8$ obs.'s in combination with 4 'diagonal' obs.'s $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}$):

$$\left\{\mathcal{O}^a_{2+}, \mathcal{O}^a_{2-}, \mathcal{O}^c_{1+}, \mathcal{O}^c_{2-}\right\} = \left\{\check{E}, \check{H}, \check{L}_{x'}, \check{T}_{x'}\right\}.$$

- Single-lined arrows: same as in Moravcsik's Theorem
- Double-lined arrows: 'crossed' selection $\mathcal{O}^c_{1\pm} \oplus \mathcal{O}^c_{2\pm}$
- 'Outer' direction $\Leftrightarrow$ 'directional convention' for consistency rel.: $\phi_{12} + \phi_{24} + \phi_{43} + \phi_{31} = 0$.
- Direction of 'inner' arrows: sign of '$\zeta$-angle' (cf. Figure on the right) in discrete-ambiguity formulas



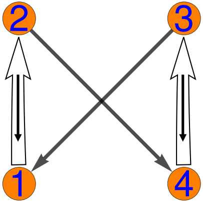

$\hookrightarrow$ Confirm photoprod.; new sets for $e^-$-production

# Motivation for new 'directional' graphs

<u>Observation:</u> Moravcsik-complete sets tend to be slightly over-complete, i.e. to contain *more than* $2N$ observables, for problems with $N \geq 4$ amplitudes

$\hookrightarrow$ One can improve the situation using <u>new kind of graphs</u>, containing additional *directional information*. [YW, Phys. Rev. C **104**, no. 4, 045203 (2021)]

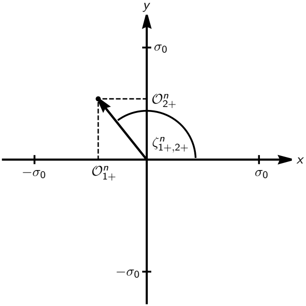
$*$) <u>Example:</u>




$\Leftrightarrow$ complete photoproduction-set ($2N = 8$ obs.'s in combination with 4 'diagonal' obs.'s $\left\{ \sigma_0, \check{\Sigma}, \check{T}, \check{P} \right\}$):

$\left\{ \mathcal{O}^a_{2+}, \mathcal{O}^a_{2-}, \mathcal{O}^c_{1+}, \mathcal{O}^c_{2-} \right\} = \left\{ \check{E}, \check{H}, \check{L}_{x'}, \check{T}_{x'} \right\}$.

- Single-lined arrows: same as in Moravcsik's Theorem
- Double-lined arrows: 'crossed' selection $\mathcal{O}^c_{1\pm} \oplus \mathcal{O}^c_{2\pm}$
- 'Outer' direction $\Leftrightarrow$ 'directional convention' for consistency rel.: $\phi_{12} + \phi_{24} + \phi_{43} + \phi_{31} = 0$.
- Direction of 'inner' arrows: sign of <u>'$\zeta$-angle'</u> (cf. Figure on the right) in discrete-ambiguity formulas




Now: explain new graphs in more detail ...

# Reminder: 'shape-classes' of photoproduction observables

| Observable | Relative-phases | Shape-class |
|---|---|---|
| $\sigma_0 = \frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$ | | |
| $-\check{\Sigma} = \frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 - \lvert b_4\rvert^2\right)$ | | $\mathcal{S} = \mathrm{D}$ |
| $-\check{T} = \frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 - \lvert b_4\rvert^2\right)$ | | |
| $\check{P} = \frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$ | | |
| $\mathcal{O}^a_{1+} = \lvert b_1\rvert \lvert b_3\rvert \sin\phi_{13} + \lvert b_2\rvert \lvert b_4\rvert \sin\phi_{24} = -\check{G}$ | | |
| $\mathcal{O}^a_{1-} = \lvert b_1\rvert \lvert b_3\rvert \sin\phi_{13} - \lvert b_2\rvert \lvert b_4\rvert \sin\phi_{24} = \check{F}$ | $\{\phi_{13}, \phi_{24}\}$ | $a = \mathcal{BT} = \mathrm{PR}$ |
| $\mathcal{O}^a_{2+} = \lvert b_1\rvert \lvert b_3\rvert \cos\phi_{13} + \lvert b_2\rvert \lvert b_4\rvert \cos\phi_{24} = -\check{E}$ | | |
| $\mathcal{O}^a_{2-} = \lvert b_1\rvert \lvert b_3\rvert \cos\phi_{13} - \lvert b_2\rvert \lvert b_4\rvert \cos\phi_{24} = \check{H}$ | | |
| $\mathcal{O}^b_{1+} = \lvert b_1\rvert \lvert b_4\rvert \sin\phi_{14} + \lvert b_2\rvert \lvert b_3\rvert \sin\phi_{23} = \check{O}_{z'}$ | | |
| $\mathcal{O}^b_{1-} = \lvert b_1\rvert \lvert b_4\rvert \sin\phi_{14} - \lvert b_2\rvert \lvert b_3\rvert \sin\phi_{23} = -\check{C}_{x'}$ | $\{\phi_{14}, \phi_{23}\}$ | $b = \mathcal{BR} = \mathrm{AD}$ |
| $\mathcal{O}^b_{2+} = \lvert b_1\rvert \lvert b_4\rvert \cos\phi_{14} + \lvert b_2\rvert \lvert b_3\rvert \cos\phi_{23} = -\check{C}_{z'}$ | | |
| $\mathcal{O}^b_{2-} = \lvert b_1\rvert \lvert b_4\rvert \cos\phi_{14} - \lvert b_2\rvert \lvert b_3\rvert \cos\phi_{23} = -\check{O}_{x'}$ | | |
| $\mathcal{O}^c_{1+} = \lvert b_1\rvert \lvert b_2\rvert \sin\phi_{12} + \lvert b_3\rvert \lvert b_4\rvert \sin\phi_{34} = -\check{L}_{x'}$ | | |
| $\mathcal{O}^c_{1-} = \lvert b_1\rvert \lvert b_2\rvert \sin\phi_{12} - \lvert b_3\rvert \lvert b_4\rvert \sin\phi_{34} = -\check{T}_{z'}$ | $\{\phi_{12}, \phi_{34}\}$ | $c = \mathcal{TR} = \mathrm{PL}$ |
| $\mathcal{O}^c_{2+} = \lvert b_1\rvert \lvert b_2\rvert \cos\phi_{12} + \lvert b_3\rvert \lvert b_4\rvert \cos\phi_{34} = -\check{L}_{z'}$ | | |
| $\mathcal{O}^c_{2-} = \lvert b_1\rvert \lvert b_2\rvert \cos\phi_{12} - \lvert b_3\rvert \lvert b_4\rvert \cos\phi_{34} = \check{T}_{x'}$ | | |

# Selection of observable-pairs from a given shape-class

Again: assume $|b_1|, \ldots, |b_N|$ to be fixed via set if $N$ 'diagonal' observables.

Generic 'shape-class' ('class-index' $n$ [e.g.: $n \in \{a, b, c\}$]; $i, j, k, l$ pairwise distinct):

$$\mathcal{O}^n_{1+} = |b_i| \, |b_j| \sin \phi_{ij} + |b_k| \, |b_l| \sin \phi_{kl}, \quad \mathcal{O}^n_{1-} = |b_i| \, |b_j| \sin \phi_{ij} - |b_k| \, |b_l| \sin \phi_{kl},$$
$$\mathcal{O}^n_{2+} = |b_i| \, |b_j| \cos \phi_{ij} + |b_k| \, |b_l| \cos \phi_{kl}, \quad \mathcal{O}^n_{2-} = |b_i| \, |b_j| \cos \phi_{ij} - |b_k| \, |b_l| \cos \phi_{kl},$$

# Selection of observable-pairs from a given shape-class

Again: assume $|b_1|, \ldots, |b_N|$ to be fixed via set if $N$ 'diagonal' observables.

Generic 'shape-class' ('class-index' $n$ [e.g.: $n \in \{a, b, c\}$]; $i, j, k, l$ pairwise distinct):

$$\mathcal{O}^n_{1+} = |b_i|\,|b_j| \sin\phi_{ij} + |b_k|\,|b_l| \sin\phi_{kl}, \quad \mathcal{O}^n_{1-} = |b_i|\,|b_j| \sin\phi_{ij} - |b_k|\,|b_l| \sin\phi_{kl},$$
$$\mathcal{O}^n_{2+} = |b_i|\,|b_j| \cos\phi_{ij} + |b_k|\,|b_l| \cos\phi_{kl}, \quad \mathcal{O}^n_{2-} = |b_i|\,|b_j| \cos\phi_{ij} - |b_k|\,|b_l| \cos\phi_{kl},$$

$\hookrightarrow$ Consider first selections leading to 'decoupled' relative-phases:

A.1) $\left(\mathcal{O}^n_{1+}, \mathcal{O}^n_{1-}\right)$, isolate both sines: $\sin\phi_{ij} = \frac{\mathcal{O}^n_{1+} + \mathcal{O}^n_{1-}}{2|b_i||b_j|}$, $\sin\phi_{kl} = \frac{\mathcal{O}^n_{1+} - \mathcal{O}^n_{1-}}{2|b_k||b_l|}$

$$\Rightarrow \phi^{\lambda}_{ij} = \phi^{\pm}_{ij} = \begin{cases} +\alpha_{ij}, \\ \pi - \alpha_{ij}, \end{cases} \quad \phi^{\lambda'}_{kl} = \phi^{\pm}_{kl} = \begin{cases} +\alpha_{kl}, \\ \pi - \alpha_{kl}. \end{cases}$$

A.2) $\left(\mathcal{O}^n_{2+}, \mathcal{O}^n_{2-}\right)$, isolate both cosines: $\cos\phi_{ij} = \frac{\mathcal{O}^n_{2+} + \mathcal{O}^n_{2-}}{2|b_i||b_j|}$, $\cos\phi_{kl} = \frac{\mathcal{O}^n_{2+} - \mathcal{O}^n_{2-}}{2|b_k||b_l|}$

$$\Rightarrow \phi^{\lambda}_{ij} = \phi^{\pm}_{ij} = \begin{cases} +\alpha_{ij}, \\ -\alpha_{ij}, \end{cases} \quad \phi^{\lambda'}_{kl} = \phi^{\pm}_{kl} = \begin{cases} +\alpha_{kl}, \\ -\alpha_{kl}. \end{cases}$$

# Selection of observable-pairs from a given shape-class

Again: assume $|b_1|, \ldots, |b_N|$ to be fixed via set if $N$ 'diagonal' observables.

Generic 'shape-class' ('class-index' $n$ [e.g.: $n \in \{a, b, c\}$]; $i, j, k, l$ pairwise distinct):

$$\mathcal{O}^n_{1+} = |b_i|\,|b_j| \sin\phi_{ij} + |b_k|\,|b_l| \sin\phi_{kl}, \quad \mathcal{O}^n_{1-} = |b_i|\,|b_j| \sin\phi_{ij} - |b_k|\,|b_l| \sin\phi_{kl},$$
$$\mathcal{O}^n_{2+} = |b_i|\,|b_j| \cos\phi_{ij} + |b_k|\,|b_l| \cos\phi_{kl}, \quad \mathcal{O}^n_{2-} = |b_i|\,|b_j| \cos\phi_{ij} - |b_k|\,|b_l| \cos\phi_{kl},$$

$\hookrightarrow$ Consider first selections leading to 'decoupled' relative-phases:

A.1) $\left(\mathcal{O}^n_{1+}, \mathcal{O}^n_{1-}\right)$, isolate both sines: $\sin\phi_{ij} = \frac{\mathcal{O}^n_{1+} + \mathcal{O}^n_{1-}}{2|b_i||b_j|}$, $\sin\phi_{kl} = \frac{\mathcal{O}^n_{1+} - \mathcal{O}^n_{1-}}{2|b_k||b_l|}$

$$\Rightarrow \phi^{\lambda}_{ij} = \phi^{\pm}_{ij} = \begin{cases} +\alpha_{ij}, \\ \pi - \alpha_{ij}, \end{cases} \quad \phi^{\lambda'}_{kl} = \phi^{\pm}_{kl} = \begin{cases} +\alpha_{kl}, \\ \pi - \alpha_{kl}. \end{cases}$$

A.2) $\left(\mathcal{O}^n_{2+}, \mathcal{O}^n_{2-}\right)$, isolate both cosines: $\cos\phi_{ij} = \frac{\mathcal{O}^n_{2+} + \mathcal{O}^n_{2-}}{2|b_i||b_j|}$, $\cos\phi_{kl} = \frac{\mathcal{O}^n_{2+} - \mathcal{O}^n_{2-}}{2|b_k||b_l|}$

$$\Rightarrow \phi^{\lambda}_{ij} = \phi^{\pm}_{ij} = \begin{cases} +\alpha_{ij}, \\ -\alpha_{ij}, \end{cases} \quad \phi^{\lambda'}_{kl} = \phi^{\pm}_{kl} = \begin{cases} +\alpha_{kl}, \\ -\alpha_{kl}. \end{cases}$$

$\hookrightarrow$ However: These selections are already used in the case of Moravcsik's Theorem

$\Rightarrow$ Not really useful for the selection of minimal complete sets

# Selection of observable-pairs from a given shape-class

Again: assume $|b_1|, \ldots, |b_N|$ to be fixed via set if $N$ 'diagonal' observables.

Generic 'shape-class' ('class-index' $n$ [e.g.: $n \in \{a, b, c\}$]; $i, j, k, l$ pairwise distinct):

$$\mathcal{O}^n_{1+} = |b_i|\,|b_j| \sin\phi_{ij} + |b_k|\,|b_l| \sin\phi_{kl}, \quad \mathcal{O}^n_{1-} = |b_i|\,|b_j| \sin\phi_{ij} - |b_k|\,|b_l| \sin\phi_{kl},$$
$$\mathcal{O}^n_{2+} = |b_i|\,|b_j| \cos\phi_{ij} + |b_k|\,|b_l| \cos\phi_{kl}, \quad \mathcal{O}^n_{2-} = |b_i|\,|b_j| \cos\phi_{ij} - |b_k|\,|b_l| \cos\phi_{kl},$$

$\hookrightarrow$ Consider 'crossed' selections $\mathcal{O}^n_{1\pm} \oplus \mathcal{O}^n_{2\pm}$, for instance:

B.1) $\left(\mathcal{O}^n_{1+}, \mathcal{O}^n_{2+}\right)$: a lengthy algebraic derivation implies the discrete phase-amb.:
cf.: [Nakayama, PRC **100**, 035208 (2019)] & [YW, PRC **104**, 045203 (2021)]

$$\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = -\zeta - \alpha_{kl} + \pi, \end{cases} \text{or} \begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = -\zeta + \alpha_{kl}, \end{cases}$$

where $\alpha_{ij}, \alpha_{kl} \in [-\pi/2, \pi/2]$ are uniquely fixed by the selected observables.

# Selection of observable-pairs from a given shape-class

Generic 'shape-class' ('class-index' $n$ [e.g.: $n \in \{a, b, c\}$]; $i, j, k, l$ pairwise distinct):

$$\mathcal{O}^n_{1+} = |b_i|\,|b_j| \sin\phi_{ij} + |b_k|\,|b_l| \sin\phi_{kl}, \quad \mathcal{O}^n_{1-} = |b_i|\,|b_j| \sin\phi_{ij} - |b_k|\,|b_l| \sin\phi_{kl},$$

$$\mathcal{O}^n_{2+} = |b_i|\,|b_j| \cos\phi_{ij} + |b_k|\,|b_l| \cos\phi_{kl}, \quad \mathcal{O}^n_{2-} = |b_i|\,|b_j| \cos\phi_{ij} - |b_k|\,|b_l| \cos\phi_{kl},$$

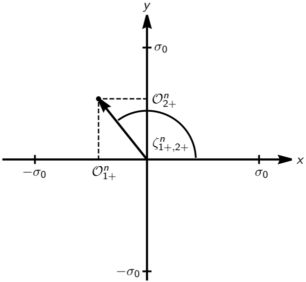
$\hookrightarrow$ Consider 'crossed' selections $\mathcal{O}^n_{1\pm} \oplus \mathcal{O}^n_{2\pm}$, for instance:

B.1) $\left(\mathcal{O}^n_{1+}, \mathcal{O}^n_{2+}\right)$: a lengthy algebraic derivation implies the discrete phase-amb.:

cf.: [Nakayama, PRC **100**, 035208 (2019)] & [YW, PRC **104**, 045203 (2021)]

$$\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = -\zeta - \alpha_{kl} + \pi, \end{cases} \text{or} \begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = -\zeta + \alpha_{kl}, \end{cases}$$

where $\alpha_{ij}, \alpha_{kl} \in [-\pi/2, \pi/2]$ are uniquely fixed by the selected observables.




$\hookrightarrow$ At this stage, the 'transitional angle' $\zeta^n_{1+,2+} \equiv \zeta$, which is uniquely specified by $\left(\mathcal{O}^n_{1+}, \mathcal{O}^n_{2+}\right)$, enters the procedure.

$\Rightarrow$ Very important quantity for the removal of degenerate pairs of consistency relations, i.e. for the determination of complete sets!

# Selection of observable-pairs from a given shape-class

Consider the remaining cases for $\mathcal{O}^n_{1\pm} \oplus \mathcal{O}^n_{2\pm}$ type selections:

B.1) $\left(\mathcal{O}^n_{1+}, \mathcal{O}^n_{2+}\right)$: $\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = -\zeta - \alpha_{kl} + \pi, \end{cases}$ or $\begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = -\zeta + \alpha_{kl}. \end{cases}$

B.2) $\left(\mathcal{O}^n_{1+}, \mathcal{O}^n_{2-}\right)$: $\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = \zeta - \alpha_{kl}, \end{cases}$ or $\begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = \zeta + \alpha_{kl} - \pi. \end{cases}$

B.3) $\left(\mathcal{O}^n_{1-}, \mathcal{O}^n_{2+}\right)$: $\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = \zeta + \alpha_{kl} - \pi, \end{cases}$ or $\begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = \zeta - \alpha_{kl}. \end{cases}$

B.4) $\left(\mathcal{O}^n_{1-}, \mathcal{O}^n_{2-}\right)$: $\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = -\zeta + \alpha_{kl}, \end{cases}$ or $\begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = -\zeta - \alpha_{kl} + \pi. \end{cases}$

# Selection of observable-pairs from a given shape-class

B.1) $\left(\mathcal{O}^{n}_{1+}, \mathcal{O}^{n}_{2+}\right)$: $\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = -\zeta - \alpha_{kl} + \pi, \end{cases}$ or $\begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = -\zeta + \alpha_{kl}. \end{cases}$

B.2) $\left(\mathcal{O}^{n}_{1+}, \mathcal{O}^{n}_{2-}\right)$: $\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = \zeta - \alpha_{kl}, \end{cases}$ or $\begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = \zeta + \alpha_{kl} - \pi. \end{cases}$

B.3) $\left(\mathcal{O}^{n}_{1-}, \mathcal{O}^{n}_{2+}\right)$: $\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = \zeta + \alpha_{kl} - \pi, \end{cases}$ or $\begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = \zeta - \alpha_{kl}. \end{cases}$

B.4) $\left(\mathcal{O}^{n}_{1-}, \mathcal{O}^{n}_{2-}\right)$: $\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = -\zeta + \alpha_{kl}, \end{cases}$ or $\begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = -\zeta - \alpha_{kl} + \pi. \end{cases}$

Important: Signs of $\zeta$'s do *not* change between ambiguous solutions!

# Selection of observable-pairs from a given shape-class

B.1) $(\mathcal{O}^{n}_{1+}, \mathcal{O}^{n}_{2+})$: $\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = -\zeta - \alpha_{kl} + \pi, \end{cases}$ or $\begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = -\zeta + \alpha_{kl}. \end{cases}$

B.2) $(\mathcal{O}^{n}_{1+}, \mathcal{O}^{n}_{2-})$: $\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = \zeta - \alpha_{kl}, \end{cases}$ or $\begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = \zeta + \alpha_{kl} - \pi. \end{cases}$

B.3) $(\mathcal{O}^{n}_{1-}, \mathcal{O}^{n}_{2+})$: $\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = \zeta + \alpha_{kl} - \pi, \end{cases}$ or $\begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = \zeta - \alpha_{kl}. \end{cases}$

B.4) $(\mathcal{O}^{n}_{1-}, \mathcal{O}^{n}_{2-})$: $\begin{cases} \phi_{ij} = -\zeta + \alpha_{ij}, \\ \phi_{kl} = -\zeta + \alpha_{kl}, \end{cases}$ or $\begin{cases} \phi_{ij} = -\zeta - \alpha_{ij} + \pi, \\ \phi_{kl} = -\zeta - \alpha_{kl} + \pi. \end{cases}$

Important: Signs of $\zeta$'s do *not* change between ambiguous solutions!
How can we formulate a criterion for the removal of degenerate consistency rel.'s $\phi^{\lambda}_{1i} + \phi^{\lambda'}_{ij} + \ldots + \phi^{\lambda^{(N-2)}}_{rq} + \phi^{\lambda^{(N-1)}}_{q1} = 0$ using the $\zeta$'s?
$\Rightarrow$ Need new graphs containing directional information!

# Include more information into new graphs

Consider again *connected graph-topologies* and include in addition:

1.) Different (styles of) lines denoting the relative-phases to which different selection-patterns were applied:

   (i) A single line, dashed or solid, for selection A.1 or A.2 ('decoupling')

   (ii) A double-line for any of the selections B.1, B.2, B.3 or B.4 ('crossed')

   $\hookrightarrow$ Distinguish rel.-phases from different shape-classes by *different styles* for double-lines (solid, dashed, dotted, wavy, ...)

# Include more information into new graphs

Consider again *connected graph-topologies* and include in addition:

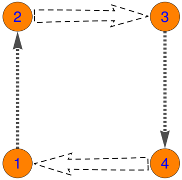
1.) Different (styles of) lines denoting the relative-phases to which different selection-patterns were applied:

   (i) A single line, dashed or solid, for selection A.1 or A.2 ('decoupling')

   (ii) A double-line for any of the selections B.1, B.2, B.3 or B.4 ('crossed')

   $\hookrightarrow$ Distinguish rel.-phases from different shape-classes by *different styles* for double-lines (solid, dashed, dotted, wavy, ...)

2.) An *overall direction* which flows through the graph $\Leftrightarrow$ index placement in consistency-rel., e.g.:



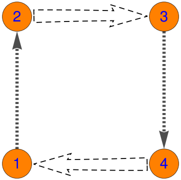

$\Leftrightarrow \phi_{12} + \phi_{23} + \phi_{34} + \phi_{41} = 0$

# Include more information into new graphs

Consider again *connected graph-topologies* and include in addition:

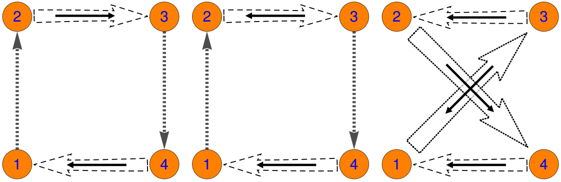
1.) Different (styles of) lines denoting the relative-phases to which different selection-patterns were applied:
   (i) A single line, dashed or solid, for selection A.1 or A.2 ('decoupling')
   (ii) A double-line for any of the selections B.1, B.2, B.3 or B.4 ('crossed')
   $\hookrightarrow$ Distinguish rel.-phases from different shape-classes by *different styles* for double-lines (solid, dashed, dotted, wavy, . . .)

2.) An *overall direction* which flows through the graph $\Leftrightarrow$ index placement in consistency-rel., e.g.:




$\Leftrightarrow \phi_{12} + \phi_{23} + \phi_{34} + \phi_{41} = 0$

3.) A smaller arrow inside of each double-lined arrow which indicates each $\zeta$-sign, e.g.:



# Statement of the new graphical completeness-criterion

The graph thus constructed, when starting from a particular set of observables, is *fully complete* if it contains *at least one pair of double-lines* and furthermore satisfies the following criterion:

(C1) For *at least one of the pairs of double-lines* in the graph, one of the following two conditions has to be fulfilled (both conditions cannot be satisfied at the same time):

- ◇ for *both* double-lines, the directional arrows have to point into the *same* direction as the corresponding $\zeta$-sign arrows,
- ◇ for *both* double-lines, the directional arrows have to point into the direction *opposite* to the direction of the respective $\zeta$-sign arrows.

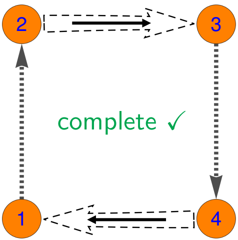
# Statement of the new graphical completeness-criterion

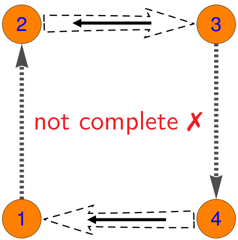
The graph thus constructed, when starting from a particular set of observables, is *fully complete* if it contains *at least one pair of double-lines* and furthermore satisfies the following criterion:

(C1) For *at least one of the pairs of double-lines* in the graph, one of the following two conditions has to be fulfilled (both conditions cannot be satisfied at the same time):

- ◇ for *both* double-lines, the directional arrows have to point into the *same* direction as the corresponding $\zeta$-sign arrows,
- ◇ for *both* double-lines, the directional arrows have to point into the direction *opposite* to the direction of the respective $\zeta$-sign arrows.

Examples:

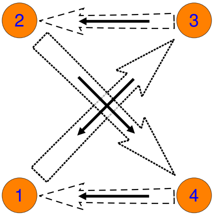


# Example-graph for photoproduction ($N = 4$ amplitudes)

Fully complete example-set (i.e. example-graph):

$$\left\{ \mathcal{O}^{a}_{1+}, \mathcal{O}^{a}_{2-}, \mathcal{O}^{b}_{1+}, \mathcal{O}^{b}_{2+} \right\} \Rightarrow$$



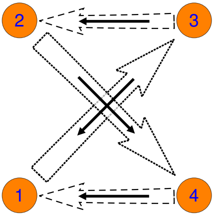

class '$a$': $\{\phi_{13}, \phi_{24}\}$

class '$b$': $\{\phi_{14}, \phi_{23}\}$

# Example-graph for photoproduction ($N = 4$ amplitudes)

Fully complete example-set
(i.e. example-graph):

$$\left\{ \mathcal{O}^a_{1+}, \mathcal{O}^a_{2-}, \mathcal{O}^b_{1+}, \mathcal{O}^b_{2+} \right\} \Rightarrow$$




class '$a$': $\{\phi_{13}, \phi_{24}\}$

class '$b$': $\{\phi_{14}, \phi_{23}\}$

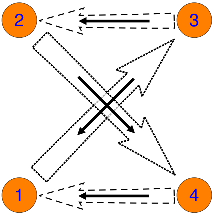
Cases for consistency-rel. $\phi^{\lambda}_{13} + \phi^{\lambda'}_{32} + \phi^{\lambda''}_{24} + \phi^{\lambda'''}_{41} = 0$ ($\zeta \equiv \zeta^a_{1+,2-}$, $\zeta' \equiv \zeta^b_{1+,2+}$):

$$\begin{aligned} & -\zeta + \alpha_{13} + \zeta' + \alpha_{32} - \pi + \zeta - \alpha_{24} + \zeta' - \alpha_{41} \\ & = 2\zeta' + \alpha_{13} + \alpha_{32} - \alpha_{24} - \alpha_{41} - \pi = 0, \end{aligned}$$

# Example-graph for photoproduction ($N = 4$ amplitudes)

Fully complete example-set (i.e. example-graph):

$$\left\{ \mathcal{O}^{a}_{1+}, \mathcal{O}^{a}_{2-}, \mathcal{O}^{b}_{1+}, \mathcal{O}^{b}_{2+} \right\} \Rightarrow$$




class '$a$': $\{\phi_{13}, \phi_{24}\}$

class '$b$': $\{\phi_{14}, \phi_{23}\}$

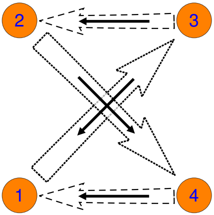
Cases for consistency-rel. $\phi_{13}^{\lambda} + \phi_{32}^{\lambda'} + \phi_{24}^{\lambda''} + \phi_{41}^{\lambda'''} = 0$ ($\zeta \equiv \zeta^{a}_{1+,2-}$, $\zeta' \equiv \zeta^{b}_{1+,2+}$):

$$\begin{aligned}
& -\zeta + \alpha_{13} + \zeta' + \alpha_{32} - \pi + \zeta - \alpha_{24} + \zeta' - \alpha_{41} \\
& = 2\zeta' + \alpha_{13} + \alpha_{32} - \alpha_{24} - \alpha_{41} - \pi = 0, \\
& -\zeta + \alpha_{13} + \zeta' - \alpha_{32} + \zeta - \alpha_{24} + \zeta' + \alpha_{41} - \pi \\
& = 2\zeta' + \alpha_{13} - \alpha_{32} - \alpha_{24} + \alpha_{41} - \pi = 0, \\
& -\zeta - \alpha_{13} + \pi + \zeta' + \alpha_{32} - \pi + \zeta + \alpha_{24} - \pi + \zeta' - \alpha_{41} \\
& = 2\zeta' - \alpha_{13} + \alpha_{32} + \alpha_{24} - \alpha_{41} - \pi = 0, \\
& -\zeta - \alpha_{13} + \pi + \zeta' - \alpha_{32} + \zeta + \alpha_{24} - \pi + \zeta' + \alpha_{41} - \pi \\
& = 2\zeta' - \alpha_{13} - \alpha_{32} + \alpha_{24} + \alpha_{41} - \pi = 0.
\end{aligned}$$

# Example-graph for photoproduction ($N = 4$ amplitudes)

Fully complete example-set
(i.e. example-graph):

$$\left\{ \mathcal{O}^a_{1+}, \mathcal{O}^a_{2-}, \mathcal{O}^b_{1+}, \mathcal{O}^b_{2+} \right\} \Rightarrow$$




class '$a$': $\{\phi_{13}, \phi_{24}\}$

class '$b$': $\{\phi_{14}, \phi_{23}\}$

Cases for consistency-rel. $\phi^{\lambda}_{13} + \phi^{\lambda'}_{32} + \phi^{\lambda''}_{24} + \phi^{\lambda'''}_{41} = 0$ ($\zeta \equiv \zeta^a_{1+,2-}$, $\zeta' \equiv \zeta^b_{1+,2+}$):

$$\begin{aligned}
& -\zeta + \alpha_{13} + \zeta' + \alpha_{32} - \pi + \zeta - \alpha_{24} + \zeta' - \alpha_{41} \\
& = 2\zeta' + \alpha_{13} + \alpha_{32} - \alpha_{24} - \alpha_{41} - \pi = 0, \\
& -\zeta + \alpha_{13} + \zeta' - \alpha_{32} + \zeta - \alpha_{24} + \zeta' + \alpha_{41} - \pi \\
& = 2\zeta' + \alpha_{13} - \alpha_{32} - \alpha_{24} + \alpha_{41} - \pi = 0, \\
& -\zeta - \alpha_{13} + \pi + \zeta' + \alpha_{32} - \pi + \zeta + \alpha_{24} - \pi + \zeta' - \alpha_{41} \\
& = 2\zeta' - \alpha_{13} + \alpha_{32} + \alpha_{24} - \alpha_{41} - \pi = 0, \\
& -\zeta - \alpha_{13} + \pi + \zeta' - \alpha_{32} + \zeta + \alpha_{24} - \pi + \zeta' + \alpha_{41} - \pi \\
& = 2\zeta' - \alpha_{13} - \alpha_{32} + \alpha_{24} + \alpha_{41} - \pi = 0.
\end{aligned}$$

$\Rightarrow$ $\zeta$-variables cancel each other out; $\zeta'$-variables break the degeneracies!

# How to obtain all complete sets using just the graphs?

*Up to now:* select (gr. $\mathcal{S} \oplus 4$) observables $\Rightarrow$ draw graph $\Rightarrow$ check completeness

# How to obtain all complete sets using just the graphs?

*Up to now:* select (gr. $\mathcal{S} \oplus 4$) observables $\Rightarrow$ draw graph $\Rightarrow$ check completeness

$\hookrightarrow$ Can we also determine all complete experiments starting solely from graphs?

# How to obtain all complete sets using just the graphs?

*Up to now:* select (gr. $\mathcal{S} \oplus 4$) observables $\Rightarrow$ draw graph $\Rightarrow$ check completeness

$\hookrightarrow$ Can we also determine all complete experiments starting solely from graphs?

Problem: 'mapping' between graphs and sets of observables is *non-bijective*, i.e.:

observable-set $\rightarrow$ unique graph *vs.* graph $\nrightarrow$ unique observable-set.

# How to obtain all complete sets using just the graphs?

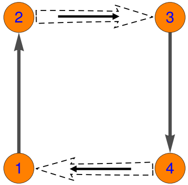
*Up to now:* select (gr. $\mathcal{S} \oplus 4$) observables $\Rightarrow$ draw graph $\Rightarrow$ check completeness

$\hookrightarrow$ Can we also determine all complete experiments starting solely from graphs?

Problem: 'mapping' between graphs and sets of observables is *non-bijective*, i.e.:

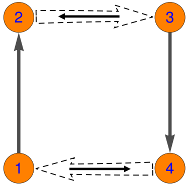
observable-set $\rightarrow$ unique graph *vs.* graph $\nrightarrow$ unique observable-set.

Furthermore, graphs exist from which no selection of obs.'s seems to follow, e.g.:




$\Leftrightarrow \left\{ \mathcal{O}^{b}_{1+}, \mathcal{O}^{b}_{2-}, \mathcal{O}^{c}_{2+}, \mathcal{O}^{c}_{2-} \right\}$

$\& \left\{ \mathcal{O}^{b}_{1-}, \mathcal{O}^{b}_{2+}, \mathcal{O}^{c}_{2+}, \mathcal{O}^{c}_{2-} \right\}$, however:




$\Leftrightarrow$ ?

# How to obtain all complete sets using just the graphs?

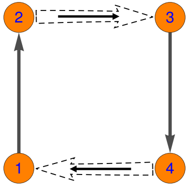
*Up to now:* select (gr. $\mathcal{S} \oplus 4$) observables $\Rightarrow$ draw graph $\Rightarrow$ check completeness

$\hookrightarrow$ Can we also determine all complete experiments starting solely from graphs?

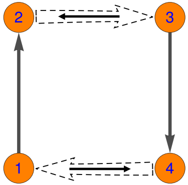
Problem: 'mapping' between graphs and sets of observables is *non-bijective*, i.e.:

observable-set $\rightarrow$ unique graph *vs.* graph $\nrightarrow$ unique observable-set.

Furthermore, graphs exist from which no selection of obs.'s seems to follow, e.g.:

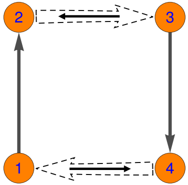



$\Leftrightarrow \left\{ \mathcal{O}^{b}_{1+}, \mathcal{O}^{b}_{2-}, \mathcal{O}^{c}_{2+}, \mathcal{O}^{c}_{2-} \right\}$ & $\left\{ \mathcal{O}^{b}_{1-}, \mathcal{O}^{b}_{2+}, \mathcal{O}^{c}_{2+}, \mathcal{O}^{c}_{2-} \right\}$, however:




$\Leftrightarrow$ ?

Solution: yet unassigned graphs $\leftrightarrow$ observable-combinations with *flipped signs*




$\Leftrightarrow \left\{ \mathcal{O}^{b}_{1+}, -\mathcal{O}^{b}_{2-}, \mathcal{O}^{c}_{2+}, \mathcal{O}^{c}_{2-} \right\}$ & $\left\{ \mathcal{O}^{b}_{1-}, -\mathcal{O}^{b}_{2+}, \mathcal{O}^{c}_{2+}, \mathcal{O}^{c}_{2-} \right\}$, i.e. in principle redundant information!

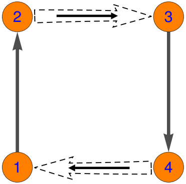
# How to obtain all complete sets using just the graphs?

$\hookrightarrow$ Can we also determine all complete experiments starting solely from graphs?

Problem: 'mapping' between graphs and sets of observables is *non-bijective*, i.e.:

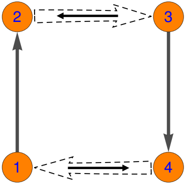
observable-set $\longrightarrow$ unique graph *vs.* graph $\not\longrightarrow$ unique observable-set.

Furthermore, graphs exist from which no selection of obs.'s seems to follow, e.g.:



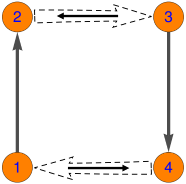

$\Leftrightarrow \left\{ \mathcal{O}^b_{1+}, \mathcal{O}^b_{2-}, \mathcal{O}^c_{2+}, \mathcal{O}^c_{2-} \right\}$

$\& \left\{ \mathcal{O}^b_{1-}, \mathcal{O}^b_{2+}, \mathcal{O}^c_{2+}, \mathcal{O}^c_{2-} \right\}$, however:




$\Leftrightarrow$ ?

Solution: yet unassigned graphs $\leftrightarrow$ observable-combinations with *flipped signs*



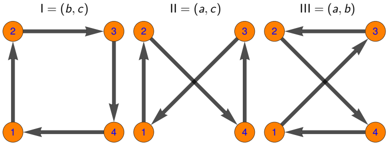

$\Leftrightarrow \left\{ \mathcal{O}^b_{1+}, -\mathcal{O}^b_{2-}, \mathcal{O}^c_{2+}, \mathcal{O}^c_{2-} \right\}$

$\& \left\{ \mathcal{O}^b_{1-}, -\mathcal{O}^b_{2+}, \mathcal{O}^c_{2+}, \mathcal{O}^c_{2-} \right\}$, i.e. in principle redundant information!

$\Rightarrow$ Possible to deduce *graphical standard solution-procedure* ...

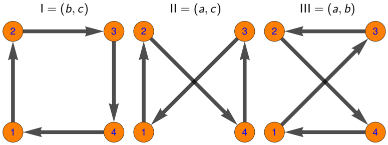
# Example procedure for photoprod. ($N = 4$ amplitudes)

Consider three basic start-topologies (with direction):

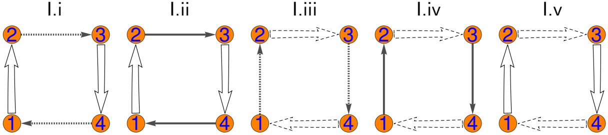


# Example procedure for photoprod. ($N = 4$ amplitudes)

Consider three basic start-topologies (with direction):

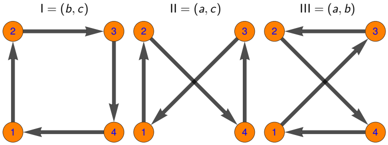



1.) For each topology, draw all combinations of single- and double lines, e.g. for 'I':

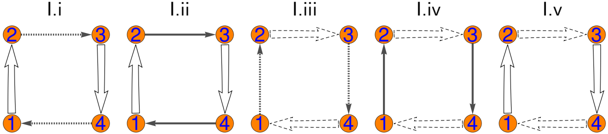


# Example procedure for photoprod. ($N = 4$ amplitudes)

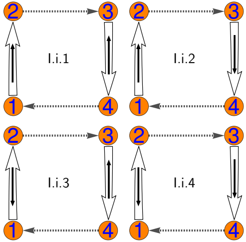
Consider three basic start-topologies (with direction):




1.) For each topology, draw all combinations of single- and double lines, e.g. for 'I':

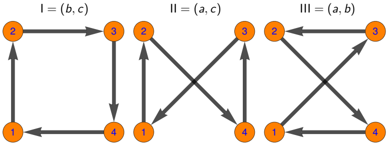



2.) For each graph-type obtained in '1.)', draw all combinations of $\zeta$-sign arrows:

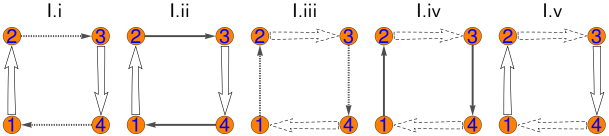


# Example procedure for photoprod. ($N = 4$ amplitudes)

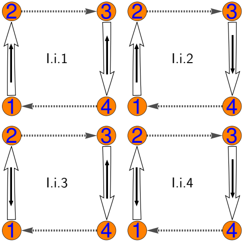
Consider three basic start-topologies (with direction):




1.) For each topology, draw all combinations of single- and double lines, e.g. for 'I':




2.) For each graph-type obtained in '1.)', draw all combinations of $\zeta$-sign arrows:



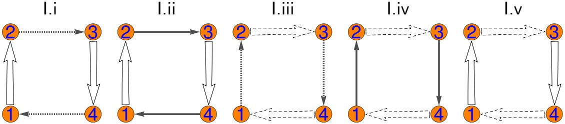

$\Rightarrow$ 3.) Determine all fully complete graphs, as well as the corresponding sets of observables (with possibly flipped signs);
Here: (I.i.2) & (I.i.3) are complete ✓;

(I.i.2) $\Leftrightarrow$ $\left\{ \mathcal{O}^{b}_{1+}, \mathcal{O}^{b}_{1-}, \mathcal{O}^{c}_{1+}, -\mathcal{O}^{c}_{2+} \right\}$ & $\left\{ \mathcal{O}^{b}_{1+}, \mathcal{O}^{b}_{1-}, \mathcal{O}^{c}_{1-}, -\mathcal{O}^{c}_{2-} \right\}$

(I.i.3) $\Leftrightarrow$ $\left\{ \mathcal{O}^{b}_{1+}, \mathcal{O}^{b}_{1-}, \mathcal{O}^{c}_{1+}, \mathcal{O}^{c}_{2+} \right\}$ & $\left\{ \mathcal{O}^{b}_{1+}, \mathcal{O}^{b}_{1-}, \mathcal{O}^{c}_{1-}, \mathcal{O}^{c}_{2-} \right\}$ .

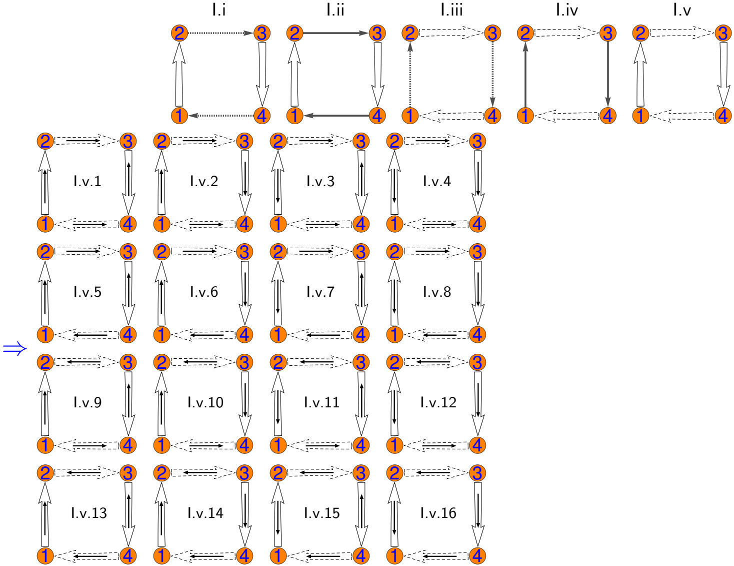
# More involved graph-type for photoproduction

Select now possibility '(I.v)' from the five graph-types



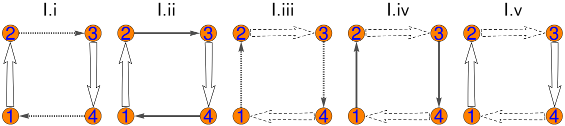
# More involved graph-type for photoproduction

Select now possibility '(I.v)' from the five graph-types

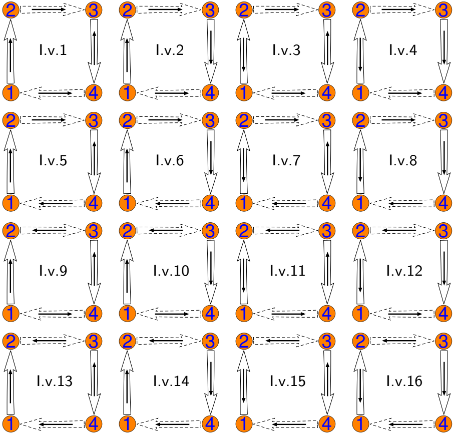


# More involved graph-type for photoproduction

Select now possibility '(I.v)' from the five graph-types




⇒




⇒ Determine all fully complete graphs:

(I.v.2) & (I.v.3) are complete ✓;

(I.v.5), . . ., (I.v.12) are complete ✓;

(I.v.14) & (I.v.15) are complete ✓.

<u>In total:</u> 60 non-redundant minimal complete sets derived using graphs.

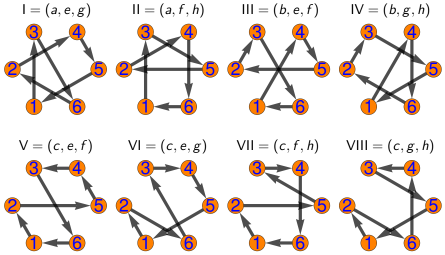
# More involved example: $e^-$-production ($N = 6$ amplitudes)

Electroproduction: $N = 6$ amplitudes vs. $N^2 = 36$ observables
(containing 7 non-diagonal 'shape-classes of 4 observables')

# More involved example: $e^-$-production ($N = 6$ amplitudes)

Electroproduction: $N = 6$ amplitudes vs. $N^2 = 36$ observables
(containing 7 non-diagonal 'shape-classes of 4 observables')

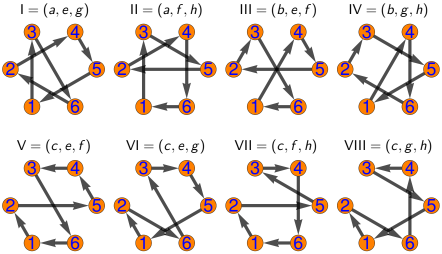
$*$) There exist 8 relevant start-topologies (with direction):



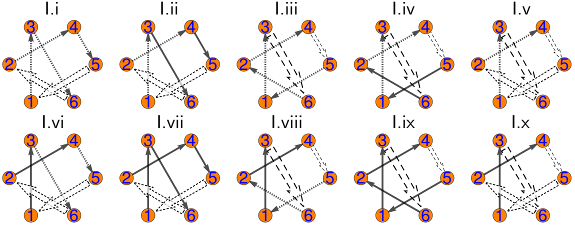
# More involved example: $e^{-}$-production ($N = 6$ amplitudes)

Electroproduction: $N = 6$ amplitudes vs. $N^2 = 36$ observables
(containing 7 non-diagonal 'shape-classes of 4 observables')

$\ast$) There exist 8 relevant start-topologies (with direction):




$\Rightarrow$ Same graphical construction-procedure as in the photoproduction case, e.g.:




. . .

$\Rightarrow$ 1216 non-redundant min. complete sets with $2N = 12$ obs.'s

# Example with $N > 6$ amplitudes: two-meson photoprod.

Two-meson photoproduction: $N = 8$ amplitudes vs. $N^2 = 64$ observables.

# Example with $N > 6$ amplitudes: two-meson photoprod.

Two-meson photoproduction: $N = 8$ amplitudes vs. $N^2 = 64$ observables.
Assume: $|b_1|, \ldots, |b_8|$ known; consider typical 'shape-class' ($n = 1, \ldots, 7$):

$$\mathcal{O}^n_{s1} = |b_i| |b_j| \sin\phi_{ij} + |b_k| |b_l| \sin\phi_{kl} + |b_m| |b_p| \sin\phi_{mp} + |b_q| |b_r| \sin\phi_{qr},$$
$$\mathcal{O}^n_{s2} = |b_i| |b_j| \sin\phi_{ij} + |b_k| |b_l| \sin\phi_{kl} - |b_m| |b_p| \sin\phi_{mp} - |b_q| |b_r| \sin\phi_{qr},$$
$$\mathcal{O}^n_{s3} = |b_i| |b_j| \sin\phi_{ij} - |b_k| |b_l| \sin\phi_{kl} + |b_m| |b_p| \sin\phi_{mp} - |b_q| |b_r| \sin\phi_{qr},$$
$$\mathcal{O}^n_{s4} = |b_i| |b_j| \sin\phi_{ij} - |b_k| |b_l| \sin\phi_{kl} - |b_m| |b_p| \sin\phi_{mp} + |b_q| |b_r| \sin\phi_{qr},$$
$$\mathcal{O}^n_{c1} = |b_i| |b_j| \cos\phi_{ij} + |b_k| |b_l| \cos\phi_{kl} + |b_m| |b_p| \cos\phi_{mp} + |b_q| |b_r| \cos\phi_{qr},$$
$$\mathcal{O}^n_{c2} = |b_i| |b_j| \cos\phi_{ij} + |b_k| |b_l| \cos\phi_{kl} - |b_m| |b_p| \cos\phi_{mp} - |b_q| |b_r| \cos\phi_{qr},$$
$$\mathcal{O}^n_{c3} = |b_i| |b_j| \cos\phi_{ij} - |b_k| |b_l| \cos\phi_{kl} + |b_m| |b_p| \cos\phi_{mp} - |b_q| |b_r| \cos\phi_{qr},$$
$$\mathcal{O}^n_{c4} = |b_i| |b_j| \cos\phi_{ij} - |b_k| |b_l| \cos\phi_{kl} - |b_m| |b_p| \cos\phi_{mp} + |b_q| |b_r| \cos\phi_{qr},$$

with indices $i, j, k, l, m, p, q, r \in \{1, \ldots, 8\}$ all pairwise distinct.

# Example with $N > 6$ amplitudes: two-meson photoprod.

Two-meson photoproduction: $N = 8$ amplitudes vs. $N^2 = 64$ observables.
Assume: $|b_1|, \ldots, |b_8|$ known; consider typical 'shape-class' ($n = 1, \ldots, 7$):

$$\begin{aligned}
\mathcal{O}^n_{s1} &= |b_i||b_j| \sin\phi_{ij} + |b_k||b_l| \sin\phi_{kl} + |b_m||b_p| \sin\phi_{mp} + |b_q||b_r| \sin\phi_{qr}, \\
\mathcal{O}^n_{s2} &= |b_i||b_j| \sin\phi_{ij} + |b_k||b_l| \sin\phi_{kl} - |b_m||b_p| \sin\phi_{mp} - |b_q||b_r| \sin\phi_{qr}, \\
\mathcal{O}^n_{s3} &= |b_i||b_j| \sin\phi_{ij} - |b_k||b_l| \sin\phi_{kl} + |b_m||b_p| \sin\phi_{mp} - |b_q||b_r| \sin\phi_{qr}, \\
\mathcal{O}^n_{s4} &= |b_i||b_j| \sin\phi_{ij} - |b_k||b_l| \sin\phi_{kl} - |b_m||b_p| \sin\phi_{mp} + |b_q||b_r| \sin\phi_{qr}, \\
\mathcal{O}^n_{c1} &= |b_i||b_j| \cos\phi_{ij} + |b_k||b_l| \cos\phi_{kl} + |b_m||b_p| \cos\phi_{mp} + |b_q||b_r| \cos\phi_{qr}, \\
\mathcal{O}^n_{c2} &= |b_i||b_j| \cos\phi_{ij} + |b_k||b_l| \cos\phi_{kl} - |b_m||b_p| \cos\phi_{mp} - |b_q||b_r| \cos\phi_{qr}, \\
\mathcal{O}^n_{c3} &= |b_i||b_j| \cos\phi_{ij} - |b_k||b_l| \cos\phi_{kl} + |b_m||b_p| \cos\phi_{mp} - |b_q||b_r| \cos\phi_{qr}, \\
\mathcal{O}^n_{c4} &= |b_i||b_j| \cos\phi_{ij} - |b_k||b_l| \cos\phi_{kl} - |b_m||b_p| \cos\phi_{mp} + |b_q||b_r| \cos\phi_{qr},
\end{aligned}$$

with indices $i, j, k, l, m, p, q, r \in \{1, \ldots, 8\}$ all pairwise distinct.
Problem: The repeating structure, i.e. typical shape-class, in two-meson photoproduction is larger and more complicated than the basic 'shape-class of 4 observables', which is at the heart of the new graphical procedure!

# Example with $N > 6$ amplitudes: two-meson photoprod.

Two-meson photoproduction: $N = 8$ amplitudes vs. $N^2 = 64$ observables.
<u>Assume:</u> $|b_1|, \ldots, |b_8|$ known; consider typical 'shape-class' $(n = 1, \ldots, 7)$:

$$\mathcal{O}^n_{s1} = |b_i| |b_j| \sin\phi_{ij} + |b_k| |b_l| \sin\phi_{kl} + |b_m| |b_p| \sin\phi_{mp} + |b_q| |b_r| \sin\phi_{qr},$$
$$\mathcal{O}^n_{s2} = |b_i| |b_j| \sin\phi_{ij} + |b_k| |b_l| \sin\phi_{kl} - |b_m| |b_p| \sin\phi_{mp} - |b_q| |b_r| \sin\phi_{qr},$$
$$\mathcal{O}^n_{s3} = |b_i| |b_j| \sin\phi_{ij} - |b_k| |b_l| \sin\phi_{kl} + |b_m| |b_p| \sin\phi_{mp} - |b_q| |b_r| \sin\phi_{qr},$$
$$\mathcal{O}^n_{s4} = |b_i| |b_j| \sin\phi_{ij} - |b_k| |b_l| \sin\phi_{kl} - |b_m| |b_p| \sin\phi_{mp} + |b_q| |b_r| \sin\phi_{qr},$$
$$\mathcal{O}^n_{c1} = |b_i| |b_j| \cos\phi_{ij} + |b_k| |b_l| \cos\phi_{kl} + |b_m| |b_p| \cos\phi_{mp} + |b_q| |b_r| \cos\phi_{qr},$$
$$\mathcal{O}^n_{c2} = |b_i| |b_j| \cos\phi_{ij} + |b_k| |b_l| \cos\phi_{kl} - |b_m| |b_p| \cos\phi_{mp} - |b_q| |b_r| \cos\phi_{qr},$$
$$\mathcal{O}^n_{c3} = |b_i| |b_j| \cos\phi_{ij} - |b_k| |b_l| \cos\phi_{kl} + |b_m| |b_p| \cos\phi_{mp} - |b_q| |b_r| \cos\phi_{qr},$$
$$\mathcal{O}^n_{c4} = |b_i| |b_j| \cos\phi_{ij} - |b_k| |b_l| \cos\phi_{kl} - |b_m| |b_p| \cos\phi_{mp} + |b_q| |b_r| \cos\phi_{qr},$$

with indices $i, j, k, l, m, p, q, r \in \{1, \ldots, 8\}$ all pairwise distinct.
<u>Problem:</u> The repeating structure, i.e. typical shape-class, in two-meson photoproduction is larger and more complicated than the basic 'shape-class of 4 observables', which is at the heart of the new graphical procedure!

$\hookrightarrow$ What can we do?

$\Rightarrow$ Try to *reduce this problem* to the already known cases!

# Reduction-strategies for two-meson photoproduction

Reduce the problem posed by the generic 'shape-class' ($n = 1, \ldots, 7$):

$$\begin{aligned}
\mathcal{O}^n_{s1} &= |b_i|\,|b_j| \sin\phi_{ij} + |b_k|\,|b_l| \sin\phi_{kl} + |b_m|\,|b_p| \sin\phi_{mp} + |b_q|\,|b_r| \sin\phi_{qr}\,, \\
\mathcal{O}^n_{s2} &= |b_i|\,|b_j| \sin\phi_{ij} + |b_k|\,|b_l| \sin\phi_{kl} - |b_m|\,|b_p| \sin\phi_{mp} - |b_q|\,|b_r| \sin\phi_{qr}\,, \\
\mathcal{O}^n_{s3} &= |b_i|\,|b_j| \sin\phi_{ij} - |b_k|\,|b_l| \sin\phi_{kl} + |b_m|\,|b_p| \sin\phi_{mp} - |b_q|\,|b_r| \sin\phi_{qr}\,, \\
\mathcal{O}^n_{s4} &= |b_i|\,|b_j| \sin\phi_{ij} - |b_k|\,|b_l| \sin\phi_{kl} - |b_m|\,|b_p| \sin\phi_{mp} + |b_q|\,|b_r| \sin\phi_{qr}\,, \\
&\vdots
\end{aligned}$$

# Reduction-strategies for two-meson photoproduction

Reduce the problem posed by the generic 'shape-class' ($n = 1, \ldots, 7$):

$$\mathcal{O}^n_{s1} = |b_i|\,|b_j|\sin\phi_{ij} + |b_k|\,|b_l|\sin\phi_{kl} + |b_m|\,|b_p|\sin\phi_{mp} + |b_q|\,|b_r|\sin\phi_{qr},$$
$$\mathcal{O}^n_{s2} = |b_i|\,|b_j|\sin\phi_{ij} + |b_k|\,|b_l|\sin\phi_{kl} - |b_m|\,|b_p|\sin\phi_{mp} - |b_q|\,|b_r|\sin\phi_{qr},$$
$$\mathcal{O}^n_{s3} = |b_i|\,|b_j|\sin\phi_{ij} - |b_k|\,|b_l|\sin\phi_{kl} + |b_m|\,|b_p|\sin\phi_{mp} - |b_q|\,|b_r|\sin\phi_{qr},$$
$$\mathcal{O}^n_{s4} = |b_i|\,|b_j|\sin\phi_{ij} - |b_k|\,|b_l|\sin\phi_{kl} - |b_m|\,|b_p|\sin\phi_{mp} + |b_q|\,|b_r|\sin\phi_{qr},$$
$$\vdots$$

Strategy (i): fully decouple all bilinear products by defining:

$$|b_i|\,|b_j|\sin\phi_{ij} = \frac{1}{4}\left(\mathcal{O}^n_{s1} + \mathcal{O}^n_{s2} + \mathcal{O}^n_{s3} + \mathcal{O}^n_{s4}\right), |b_i|\,|b_j|\cos\phi_{ij} = \frac{1}{4}\left(\mathcal{O}^n_{c1} + \mathcal{O}^n_{c2} + \mathcal{O}^n_{c3} + \mathcal{O}^n_{c4}\right),$$
$$|b_k|\,|b_l|\sin\phi_{kl} = \frac{1}{4}\left(\mathcal{O}^n_{s1} + \mathcal{O}^n_{s2} - \mathcal{O}^n_{s3} - \mathcal{O}^n_{s4}\right), |b_k|\,|b_l|\cos\phi_{kl} = \frac{1}{4}\left(\mathcal{O}^n_{c1} + \mathcal{O}^n_{c2} - \mathcal{O}^n_{c3} - \mathcal{O}^n_{c4}\right),$$
$$|b_m|\,|b_p|\sin\phi_{mp} = \frac{1}{4}\left(\mathcal{O}^n_{s1} - \mathcal{O}^n_{s2} + \mathcal{O}^n_{s3} - \mathcal{O}^n_{s4}\right), |b_m|\,|b_p|\cos\phi_{mp} = \frac{1}{4}\left(\mathcal{O}^n_{c1} - \mathcal{O}^n_{c2} + \mathcal{O}^n_{c3} - \mathcal{O}^n_{c4}\right),$$
$$|b_q|\,|b_r|\sin\phi_{qr} = \frac{1}{4}\left(\mathcal{O}^n_{s1} - \mathcal{O}^n_{s2} - \mathcal{O}^n_{s3} + \mathcal{O}^n_{s4}\right), |b_q|\,|b_r|\cos\phi_{qr} = \frac{1}{4}\left(\mathcal{O}^n_{c1} - \mathcal{O}^n_{c2} - \mathcal{O}^n_{c3} + \mathcal{O}^n_{c4}\right).$$

$\Rightarrow$ Use (modified form of) Moravcsik's Theorem

$\Rightarrow$ Obtain minimal complete sets with 24 observables [P. Kroenert et al. (2021)]

# Reduction-strategies for two-meson photoproduction

Reduce the problem posed by the generic 'shape-class' ($n = 1, \ldots, 7$):

$$\mathcal{O}^n_{s1} = |b_i|\,|b_j| \sin\phi_{ij} + |b_k|\,|b_l| \sin\phi_{kl} + |b_m|\,|b_p| \sin\phi_{mp} + |b_q|\,|b_r| \sin\phi_{qr},$$
$$\mathcal{O}^n_{s2} = |b_i|\,|b_j| \sin\phi_{ij} + |b_k|\,|b_l| \sin\phi_{kl} - |b_m|\,|b_p| \sin\phi_{mp} - |b_q|\,|b_r| \sin\phi_{qr},$$
$$\mathcal{O}^n_{s3} = |b_i|\,|b_j| \sin\phi_{ij} - |b_k|\,|b_l| \sin\phi_{kl} + |b_m|\,|b_p| \sin\phi_{mp} - |b_q|\,|b_r| \sin\phi_{qr},$$
$$\mathcal{O}^n_{s4} = |b_i|\,|b_j| \sin\phi_{ij} - |b_k|\,|b_l| \sin\phi_{kl} - |b_m|\,|b_p| \sin\phi_{mp} + |b_q|\,|b_r| \sin\phi_{qr},$$
$$\vdots$$

Strategy (ii): partially decouple eqs in order to obtain disjoint 'shape-classes of 4':

$$\tilde{\mathcal{O}}^{n,a}_{1+} = \frac{1}{2}\left(\mathcal{O}^n_{s1} + \mathcal{O}^n_{s2}\right) = |b_i|\,|b_j| \sin\phi_{ij} + |b_k|\,|b_l| \sin\phi_{kl},$$
$$\tilde{\mathcal{O}}^{n,a}_{1-} = \frac{1}{2}\left(\mathcal{O}^n_{s3} + \mathcal{O}^n_{s4}\right) = |b_i|\,|b_j| \sin\phi_{ij} - |b_k|\,|b_l| \sin\phi_{kl},$$
$$\tilde{\mathcal{O}}^{n,a}_{2+} = \frac{1}{2}\left(\mathcal{O}^n_{c1} + \mathcal{O}^n_{c2}\right) = |b_i|\,|b_j| \cos\phi_{ij} + |b_k|\,|b_l| \cos\phi_{kl},$$
$$\tilde{\mathcal{O}}^{n,a}_{2-} = \frac{1}{2}\left(\mathcal{O}^n_{c3} + \mathcal{O}^n_{c4}\right) = |b_i|\,|b_j| \cos\phi_{ij} - |b_k|\,|b_l| \cos\phi_{kl},$$

(similarly: define class $\tilde{\mathcal{O}}^{n,b}_{\nu\pm}$ for rel.-phase combination $\{\phi_{mp}, \phi_{qr}\}$.)

$\Rightarrow$ Use new Theorem with 'directional graphs'

$\Rightarrow$ Expectation: Reduce length of minimal complete sets to 20 obs.'s (probably ...)

# Reduction-strategies for two-meson photoproduction

Reduce the problem posed by the generic 'shape-class' ($n = 1, \ldots, 7$):

$$
\begin{aligned}
\mathcal{O}^n_{s1} &= |b_i|\,|b_j| \sin\phi_{ij} + |b_k|\,|b_l| \sin\phi_{kl} + |b_m|\,|b_p| \sin\phi_{mp} + |b_q|\,|b_r| \sin\phi_{qr}, \\
\mathcal{O}^n_{s2} &= |b_i|\,|b_j| \sin\phi_{ij} + |b_k|\,|b_l| \sin\phi_{kl} - |b_m|\,|b_p| \sin\phi_{mp} - |b_q|\,|b_r| \sin\phi_{qr}, \\
\mathcal{O}^n_{s3} &= |b_i|\,|b_j| \sin\phi_{ij} - |b_k|\,|b_l| \sin\phi_{kl} + |b_m|\,|b_p| \sin\phi_{mp} - |b_q|\,|b_r| \sin\phi_{qr}, \\
\mathcal{O}^n_{s4} &= |b_i|\,|b_j| \sin\phi_{ij} - |b_k|\,|b_l| \sin\phi_{kl} - |b_m|\,|b_p| \sin\phi_{mp} + |b_q|\,|b_r| \sin\phi_{qr}, \\
&\;\;\vdots
\end{aligned}
$$

Strategy (iii): new algebraic derivations for selections of pairs of observables from the original 'shape-class of 8' shown above

$\Rightarrow$ Develop graphical criterion for this new case

$\hookrightarrow$ Difficult (!); derivations thus not yet done ...

# Conclusion and Outlook

For a reaction involving particles with spin:
$N$ (transversity) amplitudes $b_i$ vs. $N^2$ pol.-observables $\mathcal{O}^{\alpha} \propto \langle b| \tilde{\Gamma}^{\alpha} |b\rangle$.

# Conclusion and Outlook

For a reaction involving particles with spin:
$N$ (transversity) amplitudes $b_i$ vs. $N^2$ pol.-observables $\mathcal{O}^\alpha \propto \langle b| \tilde{\Gamma}^\alpha |b\rangle$.

We have:

$\ast$) (Re-) derived a modified version of Moravcsik's Theorem

$\hookrightarrow$ Useful solution-tool for *any* number of amplitudes $N$

$\ast$) Treated the example of photoproduction in detail

$\hookrightarrow$ Moravcsik-complete sets with 10 obs.'s <u>vs.</u> minimal sets with $2N = 8$ obs.'s

# Conclusion and Outlook

For a reaction involving particles with spin:
$N$ (transversity) amplitudes $b_i$ vs. $N^2$ pol.-observables $\mathcal{O}^{\alpha} \propto \langle b| \tilde{\Gamma}^{\alpha} |b\rangle$.

We have:

$\ast$) (Re-) derived a modified version of Moravcsik's Theorem

$\hookrightarrow$ Useful solution-tool for *any* number of amplitudes $N$

$\ast$) Treated the example of photoproduction in detail

$\hookrightarrow$ Moravcsik-complete sets with 10 obs.'s <u>vs.</u> minimal sets with $2N = 8$ obs.'s

$\ast$) We considered the mismatch between Moravcsik-complete sets and minimal complete sets of $2N$ observables
$\Rightarrow$ new 'directional' graphs [YW, Phys. Rev. C **104**, 045203 (2021)]

- Photoproduction ($N = 4$) $\rightarrow$ min. complete sets with $2N = 8$ obs.'s ✓
- Electroproduction ($N = 6$) $\rightarrow$ min. compl. sets with $2N = 12$ obs.'s ✓
- Two-meson photoproduction ($N = 8$) $\rightarrow$ algebraic derivation of new phase-ambiguity structures likely required ...

# Conclusion and Outlook

We have:

$*$) (Re-) derived a modified version of Moravcsik's Theorem

$\hookrightarrow$ Useful solution-tool for *any* number of amplitudes $N$

$*$) Treated the example of photoproduction in detail

$\hookrightarrow$ Moravcsik-complete sets with 10 obs.'s vs. minimal sets with $2N = 8$ obs.'s

$*$) We considered the mismatch between Moravcsik-complete sets and minimal complete sets of $2N$ observables
$\Rightarrow$ new 'directional' graphs [YW, Phys. Rev. C **104**, 045203 (2021)]

- Photoproduction ($N = 4$) $\rightarrow$ min. complete sets with $2N = 8$ obs.'s ✓
- Electroproduction ($N = 6$) $\rightarrow$ min. compl. sets with $2N = 12$ obs.'s ✓
- Two-meson photoproduction ($N = 8$) $\rightarrow$ algebraic derivation of new phase-ambiguity structures likely required ...

## Thank You for your attention!

# References


[Moravcsik (1985)]: M. J. Moravcsik, J. Math. Phys. **26**, 211 (1985).

[W. K. A. T. (2020)]: YW, P. Kroenert, F. Afzal and A. Thiel, Phys. Rev. C **102**, no.3, 034605 (2020) [arXiv:2004.14483 [nucl-th]].

[P. Kroenert et al. (2021)]: P. Kroenert, YW, F. Afzal and A. Thiel, Phys. Rev. C **103**, no.1, 014607 (2021) [arXiv:2009.04356 [nucl-th]].

[Chiang & Tabakin (1996)]: W.T. Chiang and F. Tabakin, Phys. Rev., C **55**:2054-2066, 1997.

[Nakayama (2018)]: K. Nakayama, Phys. Rev. C **100**, no. 3, 035208 (2019).

[YW (2021)]: YW, Phys. Rev. C **104**, no. 4, 045203 (2021) [arXiv:2106.00486 [nucl-th]]..