Introduction to partial-wave analysis and amplitude analysis

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July 12, 2021

Basic plan for the lecture

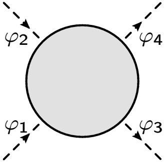
Introduction to partial-wave analysis (PWA) / amplitude analysis (AA) divided into two parts:

I.) Generic concepts and structures of PWA/AA for the (simplest) example of scalar $2 \rightarrow 2$-reactions,

II.) Example: truncated partial-wave analysis with *spin*, for single-meson photoproduction.

Amplitude for scalar $2 \to 2$ scattering

Amplitude for $2 \to 2$ reaction among scalar (i.e. spinless) particles $\varphi_1 \varphi_2 \to \varphi_3 \varphi_4$:



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$\hat{T}$: 'transition (T-) matrix'.

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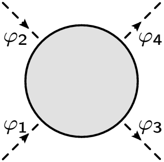

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$*)$ 4-momentum conservation: $p_1 + p_2 = p_3 + p_4$ (or $p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$).

$$\left\{ \text{4-momentum: } p_i^\mu = \begin{bmatrix} E_i \\ \vec{p}_i \end{bmatrix}; \, \mu = 0, 1, 2, 3, \; i = 1, 2, 3, 4. \right\}$$

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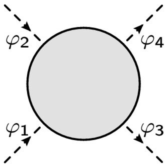

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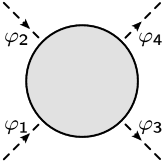

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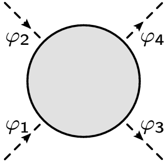

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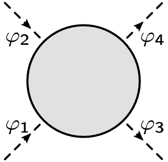

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  $\rightarrow$ 6 scalars, e.g.: $p_1 \cdot p_2$, $p_1 \cdot p_3$, $p_1 \cdot p_4$, $p_2 \cdot p_3$, $p_2 \cdot p_4$, $p_3 \cdot p_4$.

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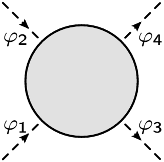

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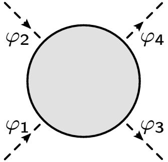

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- Use 4-momentum conservation: $p_4 = p_1 + p_2 - p_3$
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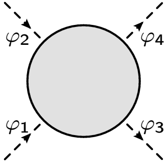

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  $\rightarrow$ 3 independent scalars, e.g.: $p_1 \cdot p_2$, $p_1 \cdot p_3$, $p_2 \cdot p_3$.
- 'Square' the 4-momentum conservation: $p_4 \cdot p_4 = m_4^2 = (p_1 + p_2 - p_3)^2$
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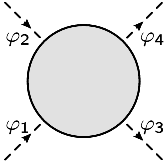

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     $\to$ one finds 2 independent Lorentz-scalars, e.g.: $p_1 \cdot p_2$, $p_1 \cdot p_3$.

*) Standard-convention: use the so-called *Mandelstam variables*:

$$s := (p_1 + p_2)^2 \, , \; t := (p_1 - p_3)^2 , \; u := (p_1 - p_4)^2 .$$

In case all scalars are on-shell: only 2 independent variables
$\to$ Choose the pair (s, t) and write the amplitude as $A = A(s, t)$.

*) 'Center-of-mass' (CMS) coordinates: $s \to W$ 'energy'; $t \to \theta$ 'scatt. angle'
$\to$ Amplitude simply a function of energy and angle: $A = A(W, \theta)$.

The 'observable': differential cross section

Consider basic geometry for 2-body scattering event:

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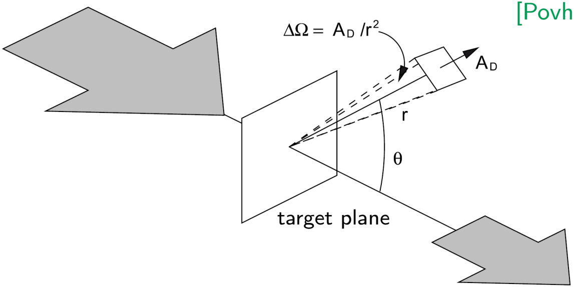

Rate $\dot{N}$ of particles scattered into solid-angle $\Delta\Omega$ at laboratory scattering-angle θ_{LAB}, for beam-energy E_{LAB}, reads:

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with the experiment's 'luminosity' $\mathcal{L}$ and the <u>differential cross section</u> $\left(\frac{d\sigma}{d\Omega}\right)_0$.

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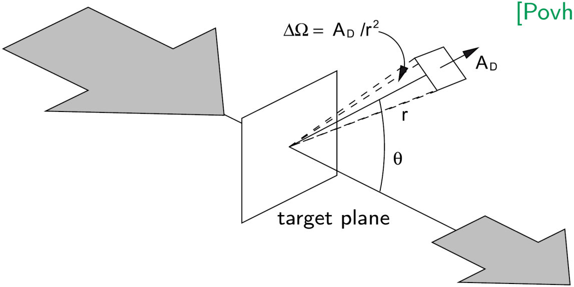

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- $*$) Transform energy and angle from LAB to CMS: $\sigma_0(W, \theta) \equiv \left(\frac{d\sigma}{d\Omega}\right)_0 (W, \theta)$.
- $*$) Relation to the scatt. amplitude (up to kin. factors): $\underline{\sigma_0(W, \theta) = |A(W, \theta)|^2}$.

Why partial waves?

Recall from non-rel. QM [e.g. QM book by Sakurai]:

$$A\left(\vec{k}', \vec{k}\right) \propto \left\langle \vec{k}' \right| \hat{T} \left| \vec{k} \right\rangle \propto \sum_{\ell,m} \sum_{\ell',m'} \left\langle \vec{k}' \middle| \ell', m' \right\rangle \langle \ell', m' | \, \hat{T} \, | \ell, m \rangle \left\langle \ell, m \middle| \vec{k} \right\rangle .$$

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Use the basis-change matrix elements $\left\langle \ell, m \middle| \vec{k} \right\rangle \propto Y^{*}_{\ell,m}(\hat{k})$, to get:

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$$\propto \sum_{\ell} A_{\ell}(W) \underbrace{\sum_{m} Y_{\ell,m}(\hat{k}') Y^{*}_{\ell,m}(\hat{k})}_{= \frac{2\ell+1}{4\pi} P_{\ell}\left(\hat{k}' \cdot \hat{k}\right)} \propto \sum_{\ell} (2\ell + 1) A_{\ell}(W) P_{\ell}\left(\hat{k}' \cdot \hat{k}\right) .$$

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Amazingly, result continues to hold formally for the *relativistic* $(2 \rightarrow 2)$ case:

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The (usually quickly converging) partial-wave expansion is *important* because:

∗) it factorizes the W- and θ dependences,

∗) fixes angular dependence of A without any additional model-dependence (often: truncation at some $\ell_{\mathrm{max}} \rightarrow$ polynomial amplitude),

∗) the p.w.'s $A_{\ell}(W)$ 'filter' intermediate states (resonances) according to their quantum numbers $\left\{ J = \ell, \; P = (-1)^{\ell} \right\}$.

What is partial-wave analysis / amplitude analysis?

$*$) We want to solve 'inverse scattering problems', i.e.:

(1) Use data on the observable effects of a scattering process, e.g. for the case of scalar $2 \rightarrow 2$ scattering ($\pi\pi \rightarrow \pi\pi$), the differential cross section

$$\sigma_0(W, \theta) = \left|A(W, \theta)\right|^2, \; \{W\text{: energy, } \theta\text{: scattering angle}\}$$

in order to ...

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$\hookrightarrow$ Partial waves contain 'J^P-filtered' contributions of *resonances*

$*$) Complications due to:

- Spin, e.g. for meson photoproduction $\gamma p \rightarrow \pi^0 p$
  $\rightarrow$ Larger no. of amplitudes and *polarization observables* [Photoprod.-TPWA]
- More than 2 particles in final state, e.g. $\gamma p \rightarrow \pi^0 \pi^0 p$
  $\rightarrow$ phase-space (cf. (W, θ) above) becomes higher-dimensional

What is partial-wave analysis / amplitude analysis?

$\ast$) We want to solve 'inverse scattering problems', i.e.:

(1) Use data for observable cross section, i.e. $\sigma_0(W,\theta) = |A(W,\theta)|^2$ ...

(2) ... in order to extract maximal information on the amplitude $A(W,\theta)$ (or partial waves $A_\ell(W)$).

$\ast$) Complications due to:

- Spin, e.g. for meson photoproduction $\gamma p \to \pi^0 p$
  $\to$ Larger no. of amplitudes and *polarization observables* [Photoprod.-TPWA]
- More than 2 particles in final state, e.g. $\gamma p \to \pi^0 \pi^0 p$
  $\to$ phase-space (cf. (W,θ) above) becomes higher-dimensional

$\ast$) One can:

Extract the amplitude itself

- Moment analysis
- Truncated partial-wave analysis

Model the amplitude physically

- (Unitarized) Isobar models
- Coupled-channels analyses
- S-Matrix principles (Analyticity, Unitarity, Crossing)

Energy-independent vs. energy-dependent fits

I.) Energy- (or mass-) independent fit

∗) Truncate partial-wave expansion at $\ell_{\mathrm{max}} \geq 0$ and evaluate $\sigma_0 = |A|^2 = A^* A$

$$\Rightarrow \sigma_0(W,\theta) = \frac{q}{k} \sum_{n=0}^{2\ell_{\mathrm{max}}} a_n^{\sigma_0}(W) P_n(\cos\theta),$$

$$a_n^{\sigma_0}(W) = \sum_{\ell,k=0}^{\ell_{\mathrm{max}}} A_\ell^*(W) \mathcal{C}_{\ell k}^n A_k(W).$$

↪ Perform angular (moment-) analysis

↪ Extract $\mathrm{Re}A_\ell$ and $\mathrm{Im}A_\ell$ as fit-parameters (up to 1 overall phase)

∗) Minimal model-dependence

∗) 'Experimental' partial-wave analysis

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$*$) 'Experimental' partial-wave analysis

II.) Energy-dependent fit:

$*$) Write and fit a model for the p.w.'s $A_\ell(W)$ as functions of energy
$\Rightarrow$ Add. dynamical assumptions

$*$) Example: 'isobar-model'

$$A_\ell(W) = \sum_{\text{res.}} a_{\text{res.}}^{f\leftarrow\ell}(W) \mathcal{R}_{\text{res.}}^{\ell}(W) a_{\text{res.}}^{\ell\leftarrow i}(W) + \boldsymbol{B}_\ell(W),$$

with 'couplings' $a_{\text{res.}}$, 'lineshape-function' $\mathcal{R}^\ell$ and 'background' $\boldsymbol{B}_\ell$.

$*$) $\exists$ many (more complicated) models

$*$) 'Resonance' $\equiv$ pole of amplitude (e.g. A_ℓ) in complex energy-plane

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$\hookrightarrow$ Perform angular (moment-) analysis

$\hookrightarrow$ Extract ReA_ℓ and ImA_ℓ as fit-parameters (up to 1 overall phase)

∗) Minimal model-dependence

∗) 'Experimental' partial-wave analysis

$\hookrightarrow$ Problems with ambiguities!

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∗) $\exists$ many (more complicated) models

∗) 'Resonance' $\equiv$ pole of amplitude (e.g. A_ℓ) in complex energy-plane

$\hookrightarrow$ Complications due to multi-channel scattering!

Resonant partial-waves I

Remember basic shape of the simple isobar-Ansatz from before:

$$A_{\ell}(W) = \sum_{\text{res.}} a_{\text{res.}}^{f \leftarrow \ell}(W) \mathcal{R}_{\text{res.}}^{\ell}(W) a_{\text{res.}}^{\ell \leftarrow i}(W) + \boldsymbol{B}_{\ell}(W).$$

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$*$) Often: lineshape-function modelled by <u>(relativistic) Breit-Wigner function</u>:

$$\mathcal{R}_{\mathrm{res.}}^{\ell}(W) = \frac{1}{(m_{\mathbf{res.}}^{\mathbf{0}})^{2} - W^{2} - i\, m_{\mathbf{res.}}^{\mathbf{0}} \Gamma(W)},$$

with 'mass-dependent width' $\Gamma(W)$, subject to modelling.

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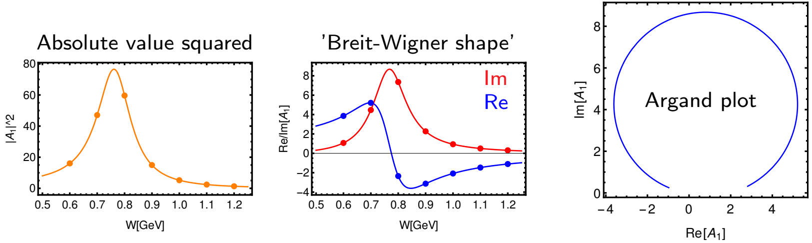
*) Often: lineshape-function modelled by (relativistic) Breit-Wigner function:

$$\mathcal{R}^{\ell}_{\text{res.}}(W) = \frac{1}{(m^{\mathbf{0}}_{\mathbf{res.}})^2 - W^2 - i\, m^{\mathbf{0}}_{\mathbf{res.}}\Gamma(W)},$$

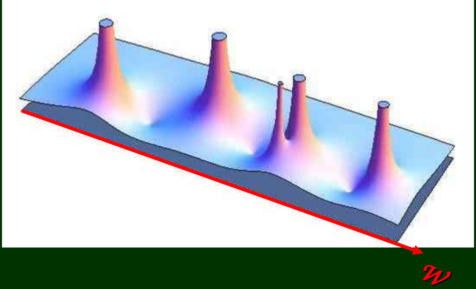
with 'mass-dependent width' $\Gamma(W)$, subject to modelling.

*) Example: Resonance $\rho(770)1^-$ (i.e. $J=1$, $P=(-1)$), coupling to the $(\pi\pi)$-system in a P-wave A_1 (figures are simplifications!):



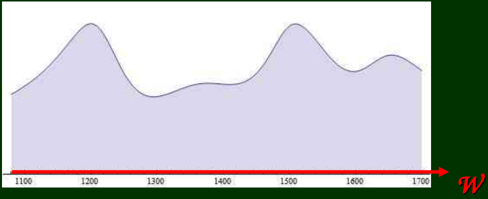
⇒ We want to extract partial waves with such shapes from the data, either for discrete or continuous energies!

Resonant partial-waves II



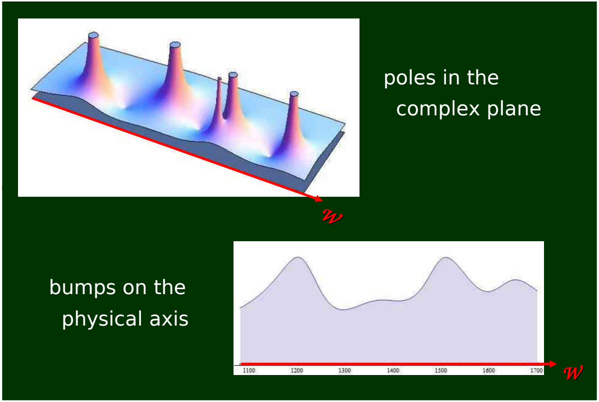
poles in the
complex plane

bumps on the
physical axis



[Figure by L. Tiator]

Resonant partial-waves II



[Figure by L. Tiator]

For *remainder of the lecture*: turn our attention to energy-independent fits for an example-reaction with spin: pseudoscalar meson photoproduction ...

Photoproduction: matrix-element and invariant amplitudes

Consider the reaction of pseudoscalar meson photoproduction: $\underline{\gamma + N \rightarrow \varphi + B}$, with particles:

γ: photon, $J^P = 1^-$
N: target-nucleon, $J^P = \frac{1}{2}^+$
φ: p.s. meson, $(\pi, K, \ldots)$ $J^P = 0^-$
B: recoil-baryon, $(N, \Lambda, \ldots)$ $J^P = \frac{1}{2}^+$

Photoproduction: matrix-element and invariant amplitudes

Consider the reaction of pseudoscalar meson photoproduction: $\underline{\gamma + N \rightarrow \varphi + B}$, with particles, leading to spin-structures in the amplitude:

$$\left.\begin{array}{l} \gamma\text{: photon, } J^P = 1^- \\ N\text{: target-nucleon, } J^P = \frac{1}{2}^+ \\ \varphi\text{: p.s. meson, } (\pi, K, \ldots)\ J^P = 0^- \\ B\text{: recoil-baryon, } (N, \Lambda, \ldots)\ J^P = \frac{1}{2}^+ \end{array}\right\} \Rightarrow \begin{cases} \gamma \text{ pol.-vector: } \epsilon_\mu\text{; Dirac-spinors: } u_N,\ u_B; \\ \text{Dirac-matrices: } \gamma^\mu,\ \gamma_5 \text{ (p.s. meson).} \end{cases}$$

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$\ast$) From p_γ, p_N, p_φ, p_B, def. 4-momenta: $k := p_\gamma$, $q := p_\varphi$, $P := \frac{1}{2}\left(p_N + p_B\right)$.

Photoproduction: matrix-element and invariant amplitudes

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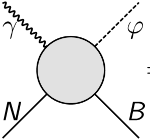
$\Rightarrow$ γ pol.-vector: ϵ_μ; Dirac-spinors: u_N, u_B; Dirac-matrices: γ^μ, γ_5 (p.s. meson).

$*)$ From p_γ, p_N, p_φ, p_B, def. 4-momenta: $k := p_\gamma$, $q := p_\varphi$, $P := \frac{1}{2}(p_N + p_B)$.

$*)$ Write QFT-inspired Ansatz for the photoproduction amplitude:



$$= \mathcal{T}_{fi} = \bar{u}_B \epsilon_\mu J^\mu u_N \equiv \epsilon_\mu \bar{u}_B \left[\sum_{i=1}^{n} A_i(s,t) M_i^\mu (k, q, P; \gamma^\mu, \gamma_5) \right] u_N.$$

$\hookrightarrow$ How many (Lorentz-) vectorial operators M_i^μ can be found that satisfy:

- transformation-properties of a pseudoscalar (φ (!)), i.e. linearity in γ_5,
- gauge-invariance, i.e. $\epsilon_\mu \sum_i A_i M_i^\mu \rightarrow k_\mu \sum_i A_i M_i^\mu = 0$, for $\epsilon_\mu \rightarrow k_\mu$?

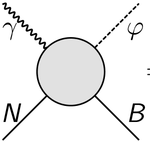
Photoproduction: matrix-element and invariant amplitudes

Consider the reaction of pseudoscalar meson photoproduction: $\underline{\gamma + N \to \varphi + B}$, with particles, leading to spin-structures in the amplitude:

photon pol.-vector: ϵ_μ; Dirac-spinors: u_N, u_B; Dirac-matrices: γ^μ, γ_5 (p.s. meson).

*) From p_γ, p_N, p_φ, p_B, def. 4-momenta: $k := p_\gamma$, $q := p_\varphi$, $P := \frac{1}{2}\left(p_N + p_B\right)$.

*) Write QFT-inspired Ansatz for the photoproduction amplitude:



$$= \mathcal{T}_{fi} = \bar{u}_B \epsilon_\mu J^\mu u_N \equiv \epsilon_\mu \bar{u}_B \left[\sum_{i=1}^{n} A_i(s,t) M_i^\mu \left(k, q, P; \gamma^\mu, \gamma_5\right)\right] u_N .$$

$\hookrightarrow$ How many (Lorentz-) vectorial operators M_i^μ can be found that satisfy:

- transformation-properties of a pseudoscalar (φ (!)), i.e. linearity in γ_5,
- gauge-invariance, i.e. $\epsilon_\mu \sum_i A_i M_i^\mu \to k_\mu \sum_i A_i M_i^\mu = 0$, for $\epsilon_\mu \to k_\mu$?

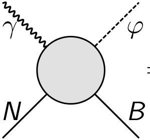
*) [Chew, Goldberger, Low & Nambu, (1957)] found $n = 4$ such structures:

$$M_1^\mu = -\frac{i}{2}\gamma_5 \left(\gamma^\mu \not{k} - \not{k}\gamma^\mu\right), \; M_2^\mu = 2i\gamma_5 \left(P^\mu k \cdot \left(q - \frac{k}{2}\right) - \left(q - \frac{k}{2}\right)^\mu k \cdot P\right),$$

$$M_3^\mu = -i\gamma_5 \left(\gamma^\mu k \cdot q - \not{k} q^\mu\right), \; M_4^\mu = -2i\gamma_5 \left(\gamma^\mu k \cdot P - \not{k} P^\mu\right) - 2 m_N M_1^\mu .$$

$\Rightarrow$ Photoproduction described by $\underline{\text{4 invariant amplitudes: } A_1(s,t), \ldots, A_4(s,t)}$.

$*)$ Evaluate amplitude in CMS (i.e. $k^{\mu} = \left[k, \vec{k}\right]^{T}$, $P_{N}^{\mu} = \left[E_{N}^{\mathrm{CMS}}, -\vec{k}\right]^{T}$, $\ldots$):

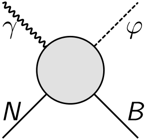


$$= \mathcal{T}_{fi} \propto \chi_{B}^{\dagger}\Big[i\vec{\sigma}\cdot\hat{\epsilon}F_{1} + \vec{\sigma}\cdot\hat{q}\vec{\sigma}\cdot\left(\hat{k}\times\hat{\epsilon}\right)F_{2} + i\vec{\sigma}\cdot\hat{k}\hat{q}\cdot\hat{\epsilon}F_{3} + i\vec{\sigma}\cdot\hat{q}\hat{q}\cdot\hat{\epsilon}F_{4}\Big]\chi_{N},$$

with Pauli-spinors $\chi_{N,B}$ and 4 CGLN-amplitudes: $F_{1}(W,\theta)$, $\ldots$, $F_{4}(W,\theta)$.

Photoproduction: CGLN-amplitudes and multipoles I

$*)$ Evaluate amplitude in CMS (i.e. $k^\mu = \left[k, \vec{k}\right]^T$, $P_N^\mu = \left[E_N^{\text{CMS}}, -\vec{k}\right]^T$, $\ldots$):



$$= \mathcal{T}_{fi} \propto \chi_B^\dagger \Big[i\vec{\sigma} \cdot \hat{\epsilon} F_1 + \vec{\sigma} \cdot \hat{q} \vec{\sigma} \cdot \left(\hat{k} \times \hat{\epsilon} \right) F_2 + i\vec{\sigma} \cdot \hat{k} \hat{q} \cdot \hat{\epsilon} F_3 + i\vec{\sigma} \cdot \hat{q} \hat{q} \cdot \hat{\epsilon} F_4 \Big] \chi_N ,$$

with Pauli-spinors $\chi_{N,B}$ and 4 CGLN-amplitudes: $F_1(W, \theta)$, $\ldots$, $F_4(W, \theta)$.

$*)$ Expand CGLN-amplitudes into electric and magnetic multipoles $E_{\ell\pm}, M_{\ell\pm}$, the partial waves of photoproduction (with $x \equiv \cos\theta$):

$$F_1 (W, \theta) = \sum_{\ell=0}^{\infty} \left[\ell M_{\ell+} (W) + E_{\ell+} (W) \right] P_{\ell+1}^{'} (x) + \left[(\ell + 1) M_{\ell-} (W) + E_{\ell-} (W) \right] P_{\ell-1}^{'} (x) ,$$

$$F_2 (W, \theta) = \sum_{\ell=1}^{\infty} \left[(\ell + 1) M_{\ell+} (W) + \ell M_{\ell-} (W) \right] P_{\ell}^{'} (x) ,$$

$$F_3 (W, \theta) = \sum_{\ell=1}^{\infty} \Big\{ \left[E_{\ell+} (W) - M_{\ell+} (W) \right] P_{\ell+1}^{''} (x) + \left[E_{\ell-} (W) + M_{\ell-} (W) \right] P_{\ell-1}^{''} (x) \Big\} ,$$

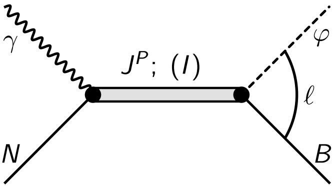
$$F_4 (W, \theta) = \sum_{\ell=2}^{\infty} \left[M_{\ell+} (W) - E_{\ell+} (W) - M_{\ell-} (W) - E_{\ell-} (W) \right] P_{\ell}^{''} (x) .$$

Photoproduction: CGLN-amplitudes and multipoles II

Important concept: expansion of full amplitudes into partial waves:

$$F_1(W,\theta) = \sum_{\ell=0}^{\infty} \left\{ \left[\ell M_{\ell+} + E_{\ell+}\right] P^{'}_{\ell+1}\left(\cos\left(\theta\right)\right) + \left[\left(\ell+1\right) M_{\ell-} + E_{\ell-}\right] P^{'}_{\ell-1}\left(\cos\left(\theta\right)\right) \right\}$$

$$F_2(W,\theta) = \ldots$$



$*)$ $J = |\ell \pm 1/2|,\ P = (-)^{\ell+1}$.

$*)$ s-chn. resonance $J^P;\ (I)$

$\updownarrow$

multipole $E^{(I)}_{\ell\pm},\ M^{(I)}_{\ell\pm}$

Photoproduction: CGLN-amplitudes and multipoles II

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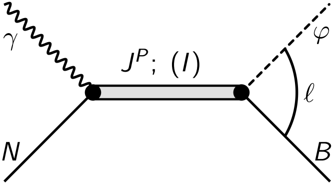
$$F_1(W,\theta) = \sum_{\ell=0}^{\cancel{\infty}\,\ell_{\max}} \Big\{ \left[\ell M_{\ell+} + E_{\ell+}\right] P'_{\ell+1}(\cos(\theta)) + \left[(\ell+1) M_{\ell-} + E_{\ell-}\right] P'_{\ell-1}(\cos(\theta)) \Big\}$$

$$F_2(W,\theta) = \ldots$$



In practice:

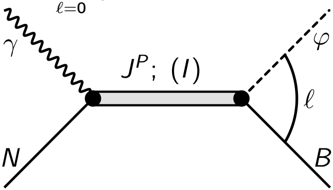
Truncate at some finite $\ell_{\max}$

$\rightarrow$ Try to extract the $4\ell_{\max}$ complex multipoles in a fit to the data.

Photoproduction: CGLN-amplitudes and multipoles II

Important concept: expansion of full amplitudes into partial waves:

$$F_1(W,\theta) = \sum_{\ell=0}^{\infty} \left\{ \left[\ell M_{\ell+} + E_{\ell+}\right] P^{\prime}_{\ell+1}(\cos(\theta)) + \left[(\ell+1) M_{\ell-} + E_{\ell-}\right] P^{\prime}_{\ell-1}(\cos(\theta)) \right\}, \ldots$$



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$*$) s-chn. resonance $J^P;\ (I)$

$\updownarrow$

multipole $E^{(I)}_{\ell\pm},\ M^{(I)}_{\ell\pm}$

$*$) Relate multipoles $\leftrightarrow$ quantum numbers due to well-defined selection-rules:

$J = \left|L_\gamma \pm \frac{1}{2}\right| \equiv \left|\ell \pm \frac{1}{2}\right|$;

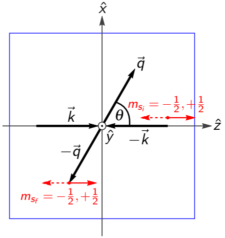
Electric (EL): $\boldsymbol{P} = (-)^{L_\gamma} \equiv (-)^{\ell+1}$,

Magn. (ML): $\boldsymbol{P} = (-)^{L_\gamma+1} \equiv (-)^{\ell+1}$.

γN-system		φB-system				γN-system		φB-system			
L_γ	$\mathcal{M}L$	J	ℓ	$\mathcal{M}_{\ell\pm}$	P	L_γ	$\mathcal{M}L$	J	ℓ	$\mathcal{M}_{\ell\pm}$	P
1	$E1$	$1/2$	0	E_{0+}	$-$	2	$E2$	$3/2$	1	E_{1+}	$+$
			$\cancel{1}$						$\cancel{2}$		
		$3/2$	$\cancel{1}$					$5/2$	$\cancel{2}$		
			2	E_{2-}	$-$				3	E_{3-}	$+$
	$M1$	$1/2$	$\cancel{0}$				$M2$	$3/2$	$\cancel{1}$		
			1	M_{1-}	$+$				2	M_{2-}	$-$
		$3/2$	1	M_{1+}	$+$			$5/2$	2	M_{2+}	$-$
			$\cancel{2}$						$\cancel{3}$		

Photoproduction: helicity- and transversity amplitudes

$*)$ CGLN-amplitudes $F_i \leftrightarrow$ spin-z quantization:



Photoproduction: helicity- and transversity amplitudes

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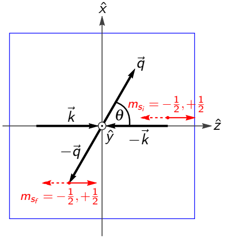
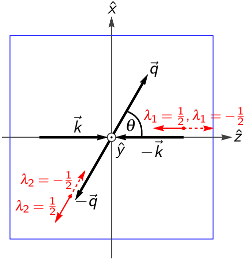


$*)$ Helicity-amplitudes $H_i \leftrightarrow$ quantization along directions of momenta:



Basis-change for the amplitudes:

$$H_1 = \frac{i}{\sqrt{2}} \sin\theta \sin\frac{\theta}{2} F_3 - \frac{i}{\sqrt{2}} \sin\theta \sin\frac{\theta}{2} F_4 ,$$

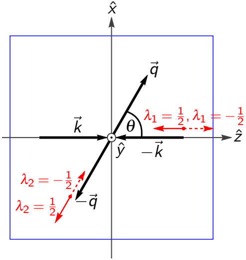
$$H_2 = -i\sqrt{2} \sin\frac{\theta}{2} F_1 - i\sqrt{2} \sin\frac{\theta}{2} F_2 - i\sqrt{2} \sin\frac{\theta}{2} \cos^2\frac{\theta}{2} F_3 - i\sqrt{2} \sin\frac{\theta}{2} \cos^2\frac{\theta}{2} F_4 ,$$

$$H_3 = \frac{i}{\sqrt{2}} \sin\theta \cos\frac{\theta}{2} F_3 + \frac{i}{\sqrt{2}} \sin\theta \cos\frac{\theta}{2} F_4 ,$$

$$H_4 = -i\sqrt{2} \cos\frac{\theta}{2} F_1 + i\sqrt{2} \cos\frac{\theta}{2} F_2 + i\sqrt{2} \cos\frac{\theta}{2} \sin^2\frac{\theta}{2} F_3 - i\sqrt{2} \cos\frac{\theta}{2} \sin^2\frac{\theta}{2} F_4 .$$

Photoproduction: helicity- and transversity amplitudes

$\ast$) Helicity-amplitudes $H_i \leftrightarrow$ quantization along directions of momenta:



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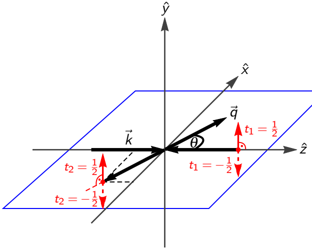
$$H_1 = \frac{i}{\sqrt{2}} \sin\theta \sin\frac{\theta}{2} F_3 - \frac{i}{\sqrt{2}} \sin\theta \sin\frac{\theta}{2} F_4 ,$$

$$H_2 = -i\sqrt{2}\sin\frac{\theta}{2} F_1 - i\sqrt{2}\sin\frac{\theta}{2} F_2 - i\sqrt{2}\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} F_3 - i\sqrt{2}\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} F_4 ,$$

$$H_3 = \frac{i}{\sqrt{2}} \sin\theta \cos\frac{\theta}{2} F_3 + \frac{i}{\sqrt{2}} \sin\theta \cos\frac{\theta}{2} F_4 ,$$

$$H_4 = -i\sqrt{2}\cos\frac{\theta}{2} F_1 + i\sqrt{2}\cos\frac{\theta}{2} F_2 + i\sqrt{2}\cos\frac{\theta}{2}\sin^2\frac{\theta}{2} F_3 - i\sqrt{2}\cos\frac{\theta}{2}\sin^2\frac{\theta}{2} F_4 .$$

$\ast$) Transversity-amp.'s $b_i \leftrightarrow$ quantize $\perp$ to the *reaction-plane* ($\equiv$ Span $\left(\vec{k}, \vec{q}\right)$):



Basis-change for the amplitudes:

$$b_1 = \frac{1}{2}\left[H_1 + H_4 - i\left(H_2 - H_3\right)\right] ,$$

$$b_2 = \frac{1}{2}\left[H_1 + H_4 + i\left(H_2 - H_3\right)\right] ,$$

$$b_3 = \frac{1}{2}\left[H_1 - H_4 + i\left(H_2 + H_3\right)\right] ,$$

$$b_4 = \frac{1}{2}\left[H_1 - H_4 - i\left(H_2 + H_3\right)\right] .$$

Photoproduction: observables I

Generic definition of an observable

$$\check{\Omega} = \left[\left(\frac{d\sigma}{d\Omega} \right)^{(B_1, T_1, R_1)} - \left(\frac{d\sigma}{d\Omega} \right)^{(B_2, T_2, R_2)} \right]$$

Photoproduction: observables I

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$*$) In total, 16 non-redundant observables

$$\check{\Omega}^{\alpha} \left(W, \theta \right) = \frac{1}{2} \frac{q}{k} \sum_{i,j} F_i^{*} \hat{A}_{ij}^{\alpha} F_j, \quad \alpha = 1, \ldots, 16$$

can be defined, involving Beam-, Target- and Recoil Polarization.

Photoproduction: observables I

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$*$) In total, 16 non-redundant observables

$$\check{\Omega}^{\alpha} \left(W, \theta \right) = \frac{1}{2} \frac{q}{k} \sum_{i,j} F_i^* \hat{A}_{ij}^{\alpha} F_j, \quad \alpha = 1, \ldots, 16$$

can be defined, involving Beam-, Target- and Recoil Polarization.

$\hookrightarrow$ Example: Beam-asymmetry $\check{\Sigma}$ is bilinear in the F_i:

$$\begin{aligned} \check{\Sigma} &= \left(\frac{d\sigma}{d\Omega} \right)^{(\perp,0,0)} - \left(\frac{d\sigma}{d\Omega} \right)^{(\parallel,0,0)} \\ &= -\frac{q}{k} \sin^2 \theta \,\mathrm{Re} \Big[\left| F_3 \right|^2 + \left| F_4 \right|^2 + 2 \big\{ F_1^* F_4 + F_2^* F_3 + \cos \theta F_3^* F_4 \big\} \Big]. \end{aligned}$$

Photoproduction: observables II

Observable	Transversity representation	Type
σ_0	$\frac{1}{2}\left(\lvert b_1 \rvert^2 + \lvert b_2 \rvert^2 + \lvert b_3 \rvert^2 + \lvert b_4 \rvert^2 \right)$	
$\check{\Sigma}$	$\frac{1}{2}\left(- \lvert b_1 \rvert^2 - \lvert b_2 \rvert^2 + \lvert b_3 \rvert^2 + \lvert b_4 \rvert^2 \right)$	$\mathcal{S}$
$\check{T}$	$\frac{1}{2}\left(\lvert b_1 \rvert^2 - \lvert b_2 \rvert^2 - \lvert b_3 \rvert^2 + \lvert b_4 \rvert^2 \right)$	
$\check{P}$	$\frac{1}{2}\left(- \lvert b_1 \rvert^2 + \lvert b_2 \rvert^2 - \lvert b_3 \rvert^2 + \lvert b_4 \rvert^2 \right)$	
$\check{G}$	$\mathrm{Im}\left[-b_1 b_3^* - b_2 b_4^* \right]$	
$\check{H}$	$-\mathrm{Re}\left[b_1 b_3^* - b_2 b_4^* \right]$	$\mathcal{BT}$
$\check{E}$	$-\mathrm{Re}\left[b_1 b_3^* + b_2 b_4^* \right]$	
$\check{F}$	$\mathrm{Im}\left[b_1 b_3^* - b_2 b_4^* \right]$	
$\check{O}_{x'}$	$-\mathrm{Re}\left[-b_1 b_4^* + b_2 b_3^* \right]$	
$\check{O}_{z'}$	$\mathrm{Im}\left[-b_1 b_4^* - b_2 b_3^* \right]$	$\mathcal{BR}$
$\check{C}_{x'}$	$\mathrm{Im}\left[b_1 b_4^* - b_2 b_3^* \right]$	
$\check{C}_{z'}$	$\mathrm{Re}\left[b_1 b_4^* + b_2 b_3^* \right]$	
$\check{T}_{x'}$	$-\mathrm{Re}\left[-b_1 b_2^* + b_3 b_4^* \right]$	
$\check{T}_{z'}$	$-\mathrm{Im}\left[b_1 b_2^* - b_3 b_4^* \right]$	$\mathcal{TR}$
$\check{L}_{x'}$	$-\mathrm{Im}\left[-b_1 b_2^* - b_3 b_4^* \right]$	
$\check{L}_{z'}$	$\mathrm{Re}\left[-b_1 b_2^* - b_3 b_4^* \right]$	

$\ast)$ Transversity amplitudes: $b_i = \sum_j M_{ij} F_j$.
(Different scheme of spin-quantization)

$\ast)$ Observables simplify:

$$\check{\Omega}^{\alpha} = \frac{1}{2} \sum_{i,j} b_i^* \tilde{\Gamma}^{\alpha}_{ij} b_j .$$

Photoproduction: general TPWA-formalism

Consider truncation of the multipole-expansion at some angular momentum ℓ_{max}:

$$F_1(W,\theta) = \sum_{\ell=0}^{\cancel{\infty}\,\ell_{\mathbf{max}}} \Big\{ \left[\ell M_{\ell+} + E_{\ell+}\right] P^{'}_{\ell+1}\left(\cos\left(\theta\right)\right) + \left[\left(\ell+1\right) M_{\ell-} + E_{\ell-}\right] P^{'}_{\ell-1}\left(\cos\left(\theta\right)\right) \Big\}$$

$$F_2(W,\theta) = \ldots,$$

and insert this expansion into the (sixteen) observables $\check{\Omega}^{\alpha} = \frac{1}{2}\frac{q}{k}\sum_{i,j} F_i^{*}\hat{A}_{ij}^{\alpha}F_j$.

Photoproduction: general TPWA-formalism

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$$F_2(W,\theta) = \ldots,$$

and insert this expansion into the (sixteen) observables $\check{\Omega}^{\alpha} = \frac{1}{2}\frac{q}{k}\sum_{i,j} F_i^{*}\hat{A}_{ij}^{\alpha}F_j$.

$\Rightarrow$ General form for the TPWA (energy-independent fit) in photoproduction:

(i) Expansion of the angular distributions into Legendre-moments:

$$\check{\Omega}^{\alpha}(W,\theta) = \rho \sum_{k=\beta_{\alpha}}^{2\ell_{\mathrm{max}}+\beta_{\alpha}+\gamma_{\alpha}} (a_L)_k^{\check{\Omega}^{\alpha}}(W)\, P_k^{\beta_{\alpha}}(\cos\theta),$$

(ii) Legendre-moments are bilinear equations in terms of multipoles:

$$(a_L)_k^{\check{\Omega}^{\alpha}}(W) = \left\langle \mathcal{M}_{\ell_{\mathrm{max}}}(W)\right| (\mathcal{C}_L)_k^{\check{\Omega}^{\alpha}} \left|\mathcal{M}_{\ell_{\mathrm{max}}}(W)\right\rangle,$$

Interlude: Legendre coefficients in terms of multipoles

Example: $\check{\Sigma} \propto (a_2)_2^{\check{\Sigma}} \, P_2^2(\cos\theta) + (a_2)_3^{\check{\Sigma}} \, P_3^2(\cos\theta) + (a_2)_4^{\check{\Sigma}} \, P_4^2(\cos\theta)$, i.e. $\ell_{\mathrm{max}} = 2$;

Interlude: Legendre coefficients in terms of multipoles

Example: $\check{\Sigma} \propto (a_2)_2^{\check{\Sigma}} P_2^2(\cos\theta) + (a_2)_3^{\check{\Sigma}} P_3^2(\cos\theta) + (a_2)_4^{\check{\Sigma}} P_4^2(\cos\theta)$, $\ell_{\max} = 2$;

$$
\begin{aligned}
(a_2)_2^{\check{\Sigma}} = \frac{1}{14}\Big[& E_{2-}^{*}\Big(- 7E_{2-} + 7E_{0+} - 2E_{2+} + 7M_{2-} - 7M_{2+}\Big) + 7E_{0+}^{*}\Big(E_{2-} + E_{2+} + M_{2-} - M_{2+}\Big) \\
& + E_{2+}^{*}\Big(- 2E_{2-} + 7E_{0+} - 18(4E_{2+} + M_{2-} - M_{2+})\Big) + M_{2-}^{*}\Big(7E_{2-} + 7E_{0+} - 18E_{2+} \\
& + 21M_{2-} + 9M_{2+}\Big) + M_{2+}^{*}\Big(- 7E_{2-} - 7E_{0+} + 9(2E_{2+} + M_{2-} + 4M_{2+})\Big) \\
& + 7\Big(E_{1+}^{*}\Big(- 3E_{1+} - M_{1-} + M_{1+}\Big) + M_{1-}^{*}\Big(M_{1+} - E_{1+}\Big) + M_{1+}^{*}\Big(E_{1+} + M_{1-} + M_{1+}\Big)\Big)\Big]
\end{aligned}
$$

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\end{aligned}
$$

$$
= \begin{bmatrix} E_{0+}^{*} & E_{1+}^{*} & \ldots & M_{2-}^{*} \end{bmatrix}
\left[\begin{array}{cccccccc}
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
\hline
0 & -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
\hline
\frac{1}{2} & 0 & 0 & 0 & -\frac{36}{7} & -\frac{1}{7} & \frac{9}{7} & -\frac{9}{7} \\
\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{7} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & 0 & 0 & 0 & \frac{9}{7} & -\frac{1}{2} & \frac{18}{7} & \frac{9}{14} \\
\frac{1}{2} & 0 & 0 & 0 & -\frac{9}{7} & \frac{1}{2} & \frac{9}{14} & \frac{3}{2}
\end{array} \right]
\begin{bmatrix} E_{0+} \\ E_{1+} \\ M_{1+} \\ M_{1-} \\ E_{2+} \\ E_{2-} \\ M_{2+} \\ M_{2-} \end{bmatrix}
$$

Interlude: Legendre coefficients in terms of multipoles

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\end{aligned}
$$

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\hline
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0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
\hline
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\frac{1}{2} & 0 & 0 & 0 & -\frac{9}{7} & \frac{1}{2} & \frac{9}{14} & \frac{3}{2}
\end{array} \right]
\begin{bmatrix} E_{0+} \\ E_{1+} \\ M_{1+} \\ M_{1-} \\ E_{2+} \\ E_{2-} \\ M_{2+} \\ M_{2-} \end{bmatrix}
$$

$$
= \left\langle \mathcal{M}_{\ell} \right| \mathcal{C}_2^{\check{\Sigma}} \left| \mathcal{M}_{\ell} \right\rangle \equiv \langle S, D \rangle + \langle P, P \rangle + \langle D, D \rangle
$$

Generally: $(a_{\ell_{\mathrm{max}}})_k^{\check{\Omega}^{\alpha}}$ defined by matrices with $\langle \ell_1, \ell_2 \rangle$-interference blocks

Photoproduction: general TPWA-formalism

Consider <u>truncation</u> of the multipole-expansion at some angular momentum ℓ_{max}:

$$F_1(W,\theta) = \sum_{\ell=0}^{\cancel{\infty}\ell_{\mathbf{max}}} \left\{ [\ell M_{\ell+} + E_{\ell+}] \, P^{'}_{\ell+1}(\cos(\theta)) + [(\ell+1) M_{\ell-} + E_{\ell-}] \, P^{'}_{\ell-1}(\cos(\theta)) \right\}$$

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and insert this expansion into the (sixteen) observables $\check{\Omega}^{\alpha} = \frac{1}{2}\frac{q}{k}\sum_{i,j} F_i^{*} \hat{A}_{ij}^{\alpha} F_j$.

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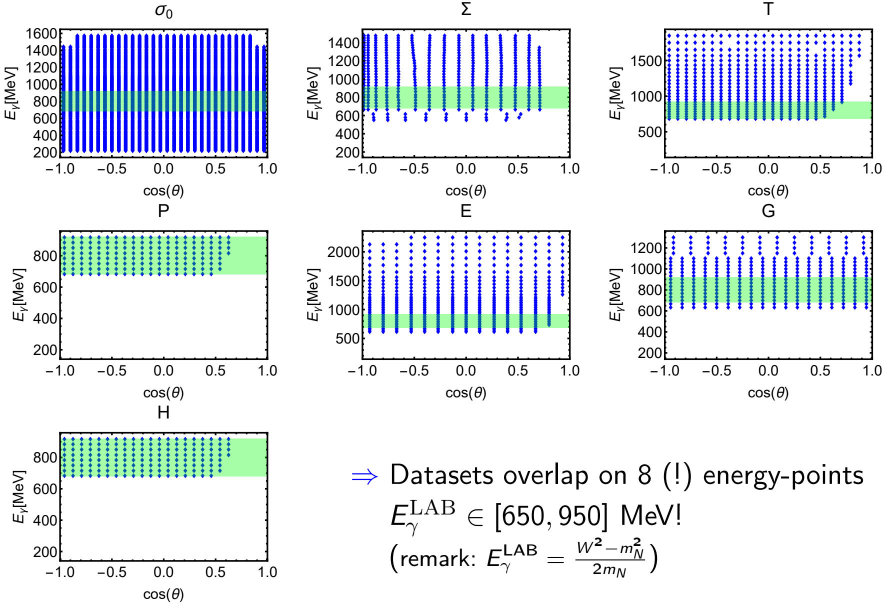
$\rightarrow$ Use this for simple moment-analysis ('ℓ_{max}-fit') of angular distributions.

(ii) Legendre-moments are bilinear equations in terms of multipoles:

$$(a_L)_k^{\check{\Omega}^{\alpha}}(W) = \left\langle \mathcal{M}_{\ell_{\max}}(W) \right| (\mathcal{C}_L)_k^{\check{\Omega}^{\alpha}} \left| \mathcal{M}_{\ell_{\max}}(W) \right\rangle,$$

$\rightarrow$ Solve these equation-systems to obtain multipoles (difficult!).

Example-TPWA in photoproduction: database



$\Rightarrow$ Datasets overlap on 8 (!) energy-points $E_\gamma^{\mathrm{LAB}} \in [650, 950]$ MeV!

$\left(\text{remark: } E_\gamma^{\mathrm{LAB}} = \frac{W^2 - m_N^2}{2m_N}\right)$

Example-TPWA in photoproduction: database

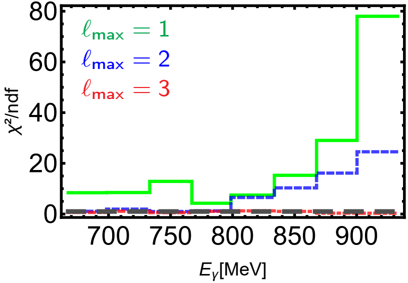
$$\{\sigma_0, \Sigma, T, P, E, G, H\}.$$

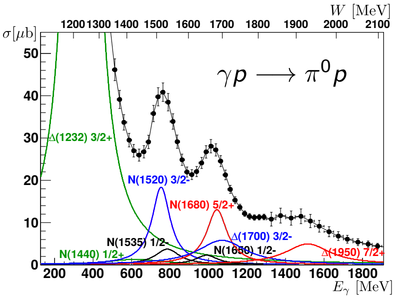
From moment analyses of the angular distributions of the data (and later confirmed by χ^2/ndf in the multipole fit):
truncation at $\underline{\ell_{\mathrm{max}} = 2}$ and/or $\underline{\ell_{\mathrm{max}} = 3}$ can already describe the data.
$\rightarrow$ See now in more detail how this works ...

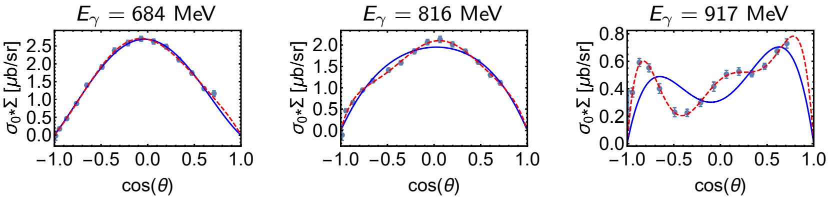
Example-TPWA in photoproduction: moment analysis

E.g.: $\check{\Sigma}(W,\theta) = \sigma^{(\perp)} - \sigma^{(\parallel)} = \frac{q}{k} \sum_{n=2}^{2\ell_{\max}} \left(a_{\ell_{\max}}\right)_n^{\check{\Sigma}}(W)\, P_n^2(\cos\theta)$

[GRAAL-data for beam-asymmetry $\check{\Sigma}$]







Example-TPWA in photoproduction: fit-method I

<u>Ansatz</u>: Use the <u>total</u> cross section $\hat{\sigma}\left(W\right)$. Example: $\ell \leq \ell_{\mathrm{max}} = 1$,
phase constraint $\mathrm{Im}\left[\tilde{E}_{0+}\right] = 0$ & $\mathrm{Re}\left[\tilde{E}_{0+}\right] > 0$:

$$\hat{\sigma}(W) \equiv 4\pi \int_{-1}^{+1} d\cos\theta \sigma_0(W, \cos\theta) \approx 4\pi \frac{q}{k} \Big(\mathrm{Re}\left[\tilde{E}_{0+}\right]^2 + 6\,\mathrm{Re}\left[\tilde{E}_{1+}\right]^2$$
$$+ 6\,\mathrm{Im}\left[\tilde{E}_{1+}\right]^2 + 2\,\mathrm{Re}\left[\tilde{M}_{1+}\right]^2 + 2\,\mathrm{Im}\left[\tilde{M}_{1+}\right]^2 + \mathrm{Re}\left[\tilde{M}_{1-}\right]^2 + \mathrm{Im}\left[\tilde{M}_{1-}\right]^2 \Big)$$

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*) $\hat{\sigma}(W)$ constrains the intervals of the multipoles:

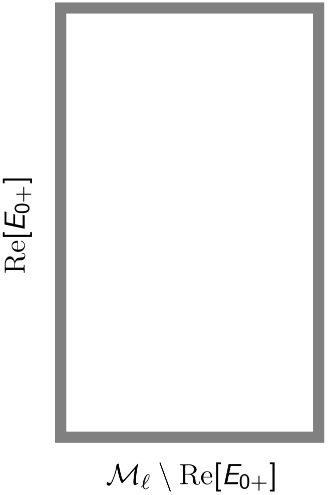
$$\mathrm{Re}\left[\tilde{E}_{0+}\right] \in \left[0, \sqrt{\frac{k}{q}\frac{\sigma(W)}{4\pi}}\right], \ldots, \mathrm{Im}\left[\tilde{M}_{1-}\right] \in \left[-\sqrt{\frac{k}{q}\frac{\sigma(W)}{4\pi}}, \sqrt{\frac{k}{q}\frac{\sigma(W)}{4\pi}}\right]$$

*) The total cross section, being quadratic form in the multipoles, also defines an ellipsoid in the multipole space.

1. The total cross section

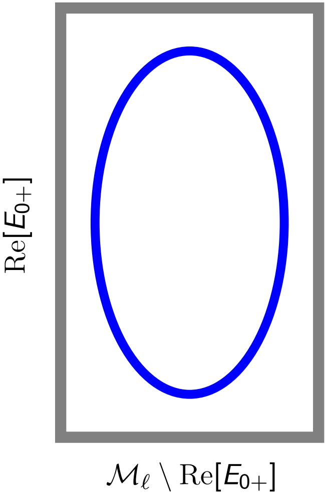
$$\hat{\sigma}\left(W\right) = \sum_{\mathcal{M}_\ell}^{\ell_{\mathrm{max}}} c_{\mathcal{M}_\ell} \left|\mathcal{M}_\ell\right|^2$$

constrains the $(8\ell_{\mathrm{max}} - 1)$-dimensional multipole space $\mathcal{M}_\ell$.



Example-TPWA in photoproduction: fit-method II

1. The total cross section
$$\hat{\sigma}\left(W\right) = \sum_{\mathcal{M}_\ell}^{\ell_{\mathrm{max}}} c_{\mathcal{M}_\ell} \left|\mathcal{M}_\ell\right|^2$$
constrains the $(8\ell_{\mathrm{max}} - 1)$-dimensional multipole space $\mathcal{M}_\ell$.
2. $\hat{\sigma}\left(W\right)$ defines an $(8\ell_{\mathrm{max}} - 2)$-dimensional ellipsoid in $\mathcal{M}_\ell$.



Example-TPWA in photoproduction: fit-method II

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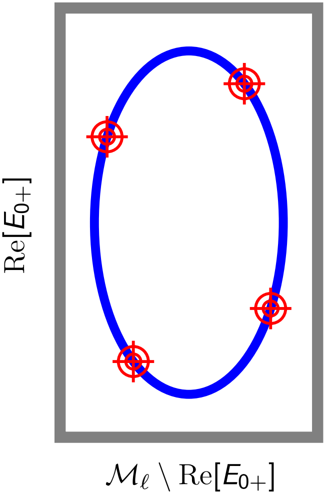
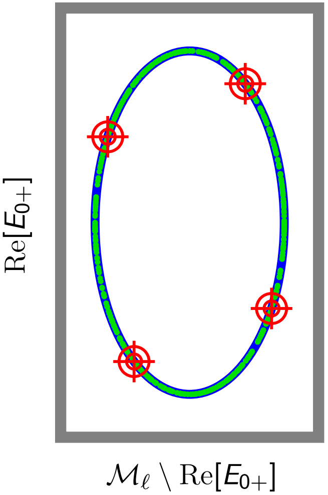
$$\hat{\sigma}(W) = \sum_{\mathcal{M}_\ell}^{\ell_{\mathrm{max}}} c_{\mathcal{M}_\ell} \left|\mathcal{M}_\ell\right|^2$$
constrains the $(8\ell_{\mathrm{max}} - 1)$-dimensional multipole space $\mathcal{M}_\ell$.
2. $\hat{\sigma}(W)$ defines an $(8\ell_{\mathrm{max}} - 2)$-dimensional ellipsoid in $\mathcal{M}_\ell$.
3. Solutions to the TPWA problem lie on the ellipsoid defined by $\hat{\sigma}(W)$.



Example-TPWA in photoproduction: fit-method II

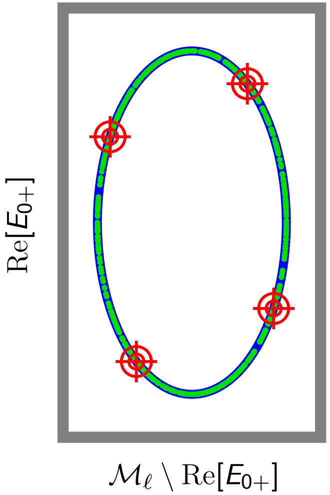
1. The total cross section
$$\hat{\sigma}\left(W\right) = \sum_{\mathcal{M}_\ell}^{\ell_{\max}} c_{\mathcal{M}_\ell} \left|\mathcal{M}_\ell\right|^2$$
constrains the $(8\ell_{\mathrm{max}} - 1)$-dimensional multipole space $\mathcal{M}_\ell$.
2. $\hat{\sigma}\left(W\right)$ defines an $(8\ell_{\mathrm{max}} - 2)$-dimensional ellipsoid in $\mathcal{M}_\ell$.
3. Solutions to the TPWA problem lie on the ellipsoid defined by $\hat{\sigma}\left(W\right)$.
4. The start values for the $\mathrm{FindMinimum}$-Fit are chosen randomly on the $\hat{\sigma}\left(W\right)$-ellipsoid.

   $\Rightarrow$ Monte Carlo sampling of the multipole space.



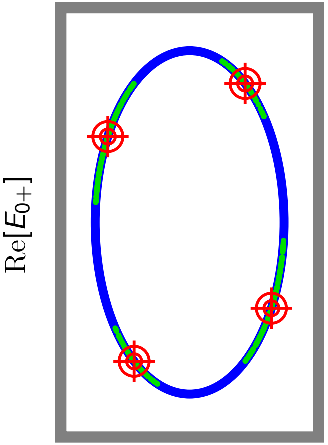
5. A $\mathrm{FindMinimum}$-minimization is performed for each of the randomly generated start configurations.

$$\Rightarrow N_{MC} = \# \text{ of M.C. start configurations} = \# \text{ of (possibly redundant) solutions}$$



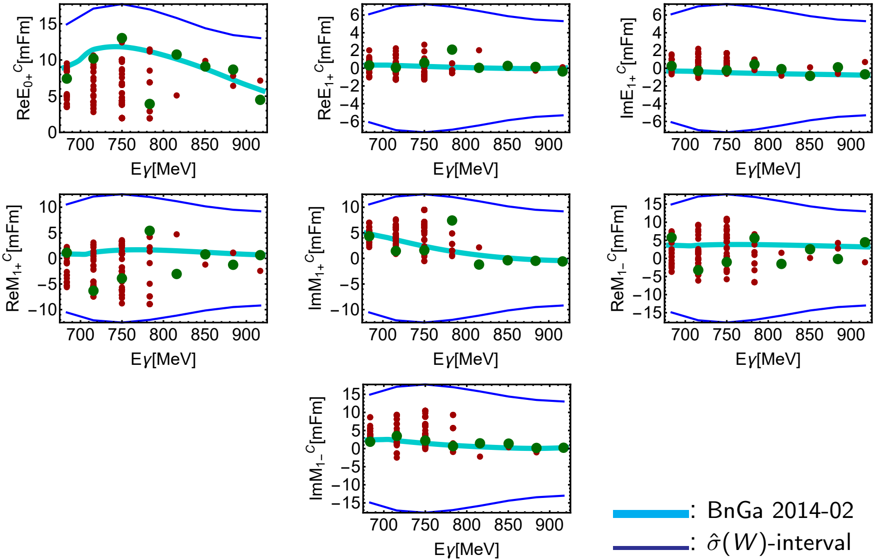
Example-TPWA in photoproduction: fit-method II

6. Analysis described up to now is fully model-independent.
   However: if wished for or needed, individual partial-wave parameters can be fixed to model-constraints quite freely.
7. In this way, map out the global minimum as well as all local minima of the χ^2-function in the TPWA step (ii). (I.e., the function to be minimized to solve for the multipoles:



$$\chi^2 = \sum_{i,j} \left[a_{L,i} - \langle \mathcal{M}_\ell | \mathcal{C}_{L,i} | \mathcal{M}_\ell \rangle \right] \boldsymbol{Cov}_{ij}^{-1} \left[a_{L,j} - \langle \mathcal{M}_\ell | \mathcal{C}_{L,j} | \mathcal{M}_\ell \rangle \right] \;) \mathcal{M}_\ell \setminus \mathrm{Re}[E_{0+}]$$

TPWA in photoproduction: all multipoles unconstrained



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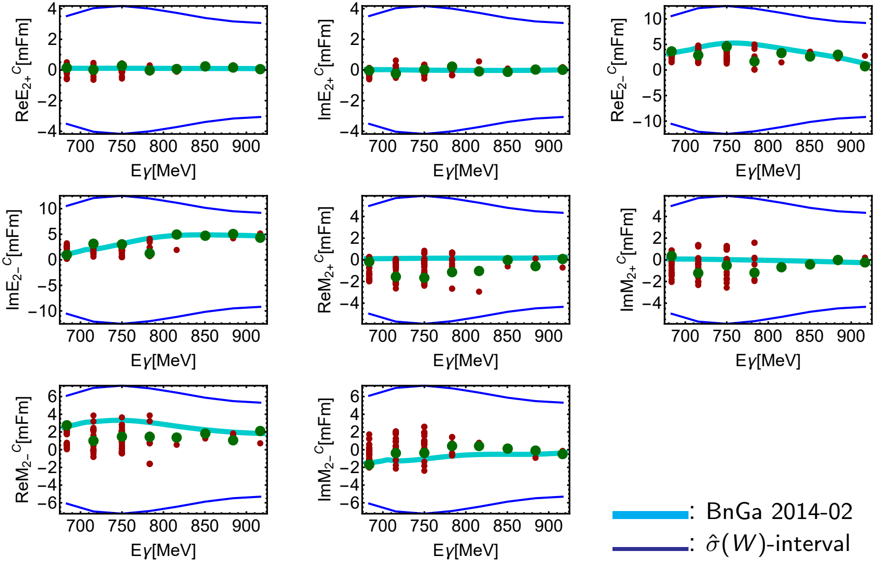

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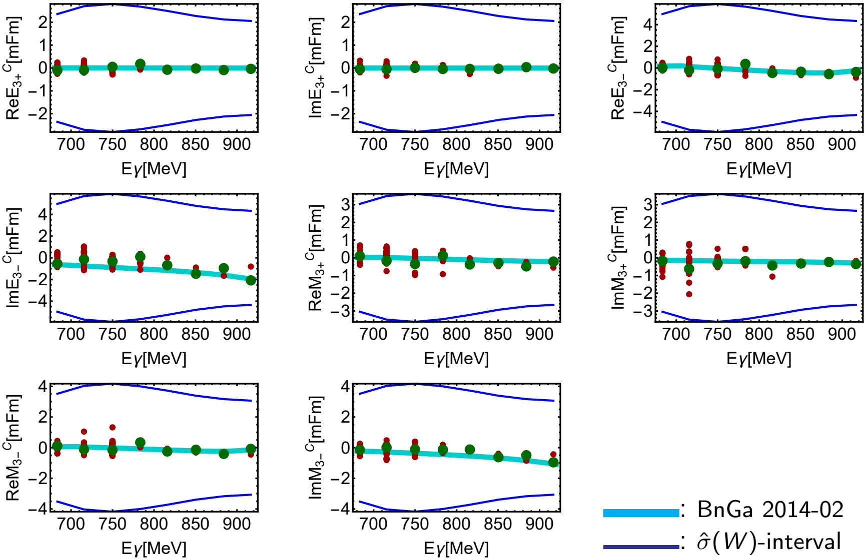
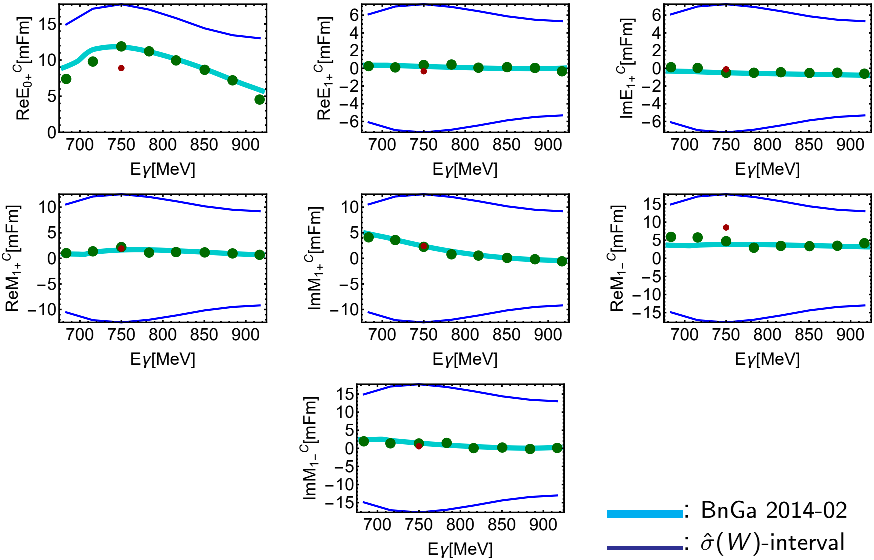


TPWA in photoproduction: constraints on F-waves



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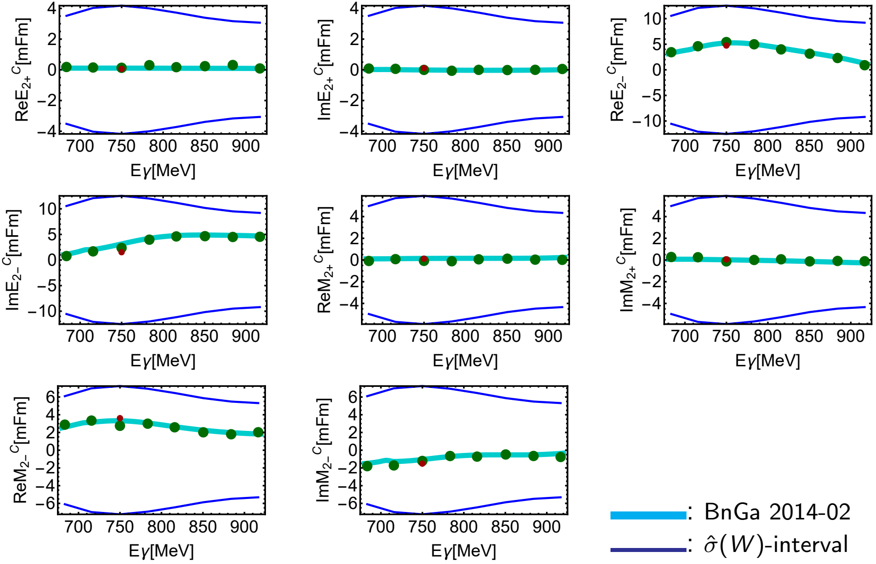
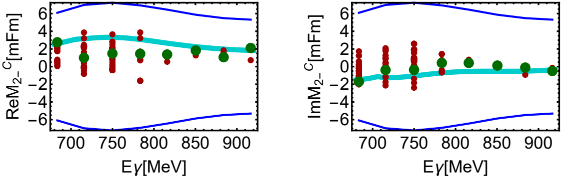
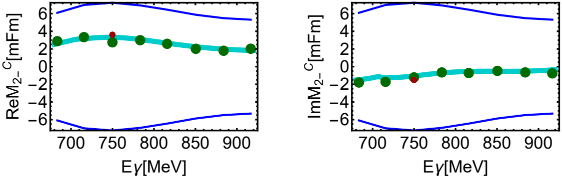


Example-TPWA in photoproduction: resonant multipole

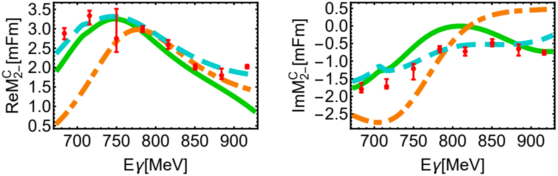
Consider multipole $M_{2-} \leftrightarrow$ resonance $N(1520)\frac{3}{2}^{-}$ (i.e. $J = \frac{3}{2}$, $P = (-)$, $\ell = 2$):



All multipoles unconstrained



F-waves constrained to BnGa



Errors: bootstrapping

: BnGa 2014-02

: MAID-07

: SAID CM12

Thank You!

Appendices

Energy-independent fits: ambiguities

$*)$ A general truncated (i.e. polynomial-) amplitude for arbitrary $L = \ell_{\mathrm{max}}$ has a <u>linear factorization</u>. Example: $L = 2$, $A = \sum_{\ell=0}^{2} (2\ell + 1) A_{\ell} P_{\ell}(\cos\theta)$ leads to:

$$\begin{aligned} A &= A_0 + 3A_1 P_1(\cos\theta) + 5A_2 P_2(\cos\theta) \\ &= \lambda \left(\cos\theta - \alpha_1\right)\left(\cos\theta - \alpha_2\right), \text{ with } \lambda \propto A_2 . \end{aligned}$$

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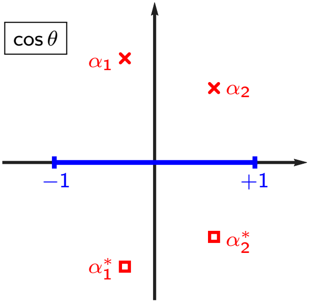
*) roots $(\lambda, \{\alpha_1, \alpha_2\}) \leftrightarrow$ partial waves $\{A_0, A_1, A_2\}$

*) Define 'mappings' $\boldsymbol{\pi}_n$, which comprise all possibilities to complex conjugate subsets of the roots: $\alpha_i \longrightarrow \boldsymbol{\pi}_n(\alpha_i)$, i.e.:

$$\boldsymbol{\pi}_0 \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \boldsymbol{\pi}_1 \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1^* \\ \alpha_2 \end{bmatrix},$$
$$\boldsymbol{\pi}_2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2^* \end{bmatrix}, \boldsymbol{\pi}_3 \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1^* \\ \alpha_2^* \end{bmatrix}.$$

$\hookrightarrow$ There exist in total $2^2 = 4$ such maps $\boldsymbol{\pi}_n$.



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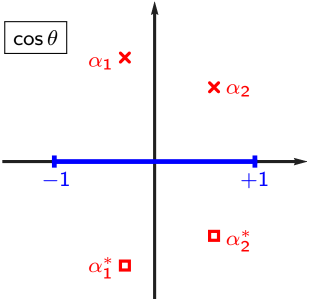

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$\ast$) One can transform to $2^2 = 4$ ambiguous amplitudes:

$$A^{(n)} = \lambda \prod_{i=1}^{L} \left(\cos\theta - \boldsymbol{\pi}_n\left[\alpha_i\right]\right) \equiv \sum_{\ell=0}^{L} (2\ell + 1) A_\ell^{(n)}(W) P_\ell(\cos\theta),$$

which <u>all</u> have the same diff. c.s. $\sigma_0 = \left|\lambda\right|^2 \prod_{i=1}^{L} \left(\cos\theta - \alpha_i^*\right)\left(\cos\theta - \alpha_i\right)$.

Energy-dependent fits: single vs. coupled reactions I

Writing and fitting an energy-dependent model for the amplitude A_{fi} describing a single transition $i \rightarrow f$ is possible but also in some sense *artificial*, because:

In Relativistic QM: conversions mass $\leftrightarrow$ energy, i.e. creation/destruction of particles

$\Rightarrow$ Multiple different scattering-states, or 'channels', open with ascending energy!

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$$\left| I \right\rangle = \left| \varphi_1^I , \varphi_2^I \right\rangle, \ \left| II \right\rangle = \left| \varphi_1^{II} , \varphi_2^{II} \right\rangle, \ \left| III \right\rangle = \left| \varphi_1^{III} , \varphi_2^{III} \right\rangle \text{ (different masses!).}$$

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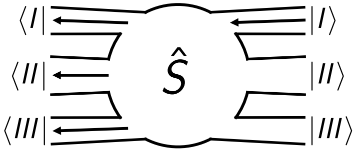
$$\hat{S} = \mathbb{1} + 2i\hat{T} \text{ (remember: } A_{fi} \propto \langle f| \hat{T} |i\rangle\text{).}$$

Conservation of probabilities, i.e. $\sum_b P_{a\rightarrow b} = 1$, is equivalent to the *unitarity of S*: $\underline{\hat{S}^\dagger \hat{S} = \mathbb{1}}$.

'QM irrigation-system'



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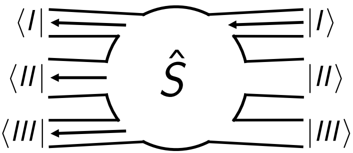

$*$) Derive unitarity-condition for the T-matrix:

$$\hat{S}^{\dagger}\hat{S} = \left(\mathbb{1} - 2i\hat{T}^{\dagger}\right)\left(\mathbb{1} + 2i\hat{T}\right) = \mathbb{1} - 2i\hat{T}^{\dagger} + 2i\hat{T} + 4\hat{T}^{\dagger}\hat{T} \stackrel{!}{=} \mathbb{1},$$

$$\Rightarrow \underline{\hat{T} - \hat{T}^{\dagger} = 2i\hat{T}^{\dagger}\hat{T}} \text{ ('optical theorem in operator-notation').}$$

Energy-dependent fits: single vs. coupled reactions II

$*$) (2-body) unitarity takes simple form in partial-wave basis:
$A_\ell^{b\leftarrow a} - \left(A_\ell^{b\leftarrow a}\right)^* = i \sum_c \left(A_\ell^{b\leftarrow c}\right)^* \rho_c A_\ell^{c\leftarrow a}$ (ρ_c: 'phase-space volume' of $|c\rangle$).

Any possible model-construction that fulfills this constraint is unitary!

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$\ast$) Trick: go back to abstract operators and write

$$\hat{T} - \hat{T}^{\dagger} = 2 i \hat{T}^{\dagger} \hat{T} \Leftrightarrow (\hat{T}^{\dagger})^{-1} - \hat{T}^{-1} = 2 i \mathbb{1} \Leftrightarrow \left(\hat{T}^{-1} + i \mathbb{1} \right)^{\dagger} = \left(\hat{T}^{-1} + i \mathbb{1} \right).$$

Define: 'K-Matrix' $\hat{K}^{-1} := \hat{T}^{-1} + i \mathbb{1} \Leftrightarrow \hat{T} = \hat{K} \left(\mathbb{1} - i \hat{K} \right)^{-1}$.

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∗) N coupled 2-body channels: implement unitarity via K-Matrix models, e.g.:

$$K_{\ell}^{b\leftarrow a}(s) := \sum_{j} \frac{g_j^b g_j^a}{s - m_j^2} + \boldsymbol{B}_{\ell}^{b\leftarrow a}(s) \equiv \sum_{j} \text{'}K\text{-Mat. pole of resonance } j\text{'} + \text{'backgr.'}\ .$$

↪ Then fit the 'observables', i.e. diff. cross-sections $\sigma_0^{b\leftarrow a} \equiv \left|A^{b\leftarrow a}\right|^2$, of all 'reactions' $a \rightarrow b$ at once!

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$*)$ However: nature is not so kind, because:

(i) $\exists$ 3-, 4-, ... body channels,

(ii) particles generally have spin $\neq 0$.

$\Rightarrow$ Takes years to write and fit a coupled-channels model

$\hookrightarrow$ Dedicated groups: [Bonn-Gatchina], [Jülich-Bonn], ...