Stochastic Block Models

COS 424/524, SML 302: Fundamentals of Machine Learning

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COS424/524, SML302

Lecture 19

Mixed membership models

In previous lectures, we learned three ways to perform exploratory data analysis using dimension reduction:

- principal component analysis (PCA);
- factor analysis (FA);
- latent Dirichlet allocation (LDA), a mixed membership model.

Today we will learn a second mixed membership model, this time for the goal of *community detection in networks*: the **mixed membership stochastic block model**.

Community detection: problem definition

Community detection is the problem of labeling communities of nodes in a network, where community membership drives the probability of a direct edge between the two nodes.

- in networks, global properties of a network are a function of local network structure
- our goal is to describe a latent set of communities among nodes based on their global network structure

Networks: two differences from previous approaches

The data we discuss today have two main differences from previous data analytic problems:

- the samples are not independent and identically distributed (IID)
- the structure on the samples is a network, which is often difficult to work with

Network analysis: challenges

- Network representation:
    - missing edges
    - smooth structure
    - network motifs
    - network semantics?
- Population statistics:
    - how does sample size (non-IID) influence analysis?
    - variability among features? edges?
- Diffusion of information in network:
    - what is background? what is network-based interaction?
    - are nodes only activating? or can they repress?

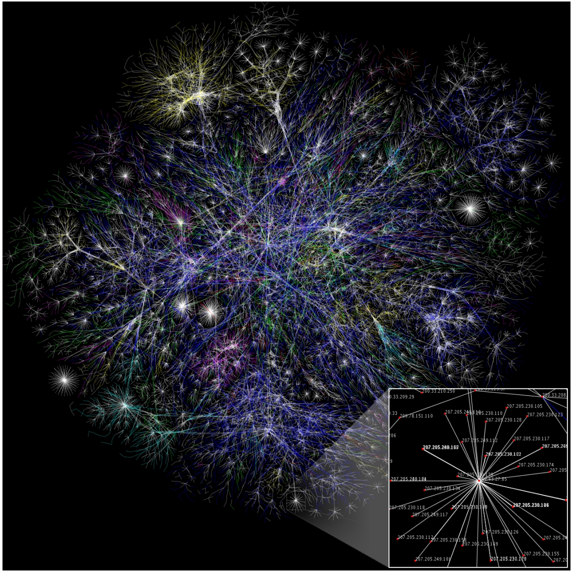
What types of networks can we consider?

Networks are:

- a collection of $\mathcal{D} = \{x_1, \ldots, x_n\}$ samples of p features;
- a binary matrix $Z \in \{0, 1\}^{n \times n}$ representing edges between each sample (nodes).

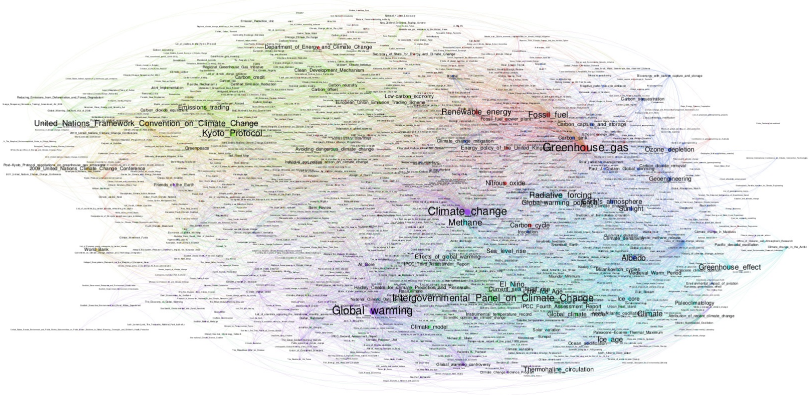
- Current networks may be directed or undirected;
- often $n = O(millions)$;
- in some networks, edges are given (e.g., "friends" in a social network);
- in other networks, edges are inferred (e.g., protein-protein interactions);
- no edge may mean edge does not exist between nodes, or edge is missing.

The internet



- relationships between IP addresses
- undirected network; edges are given; no-edge may be missing
- built for redundancy

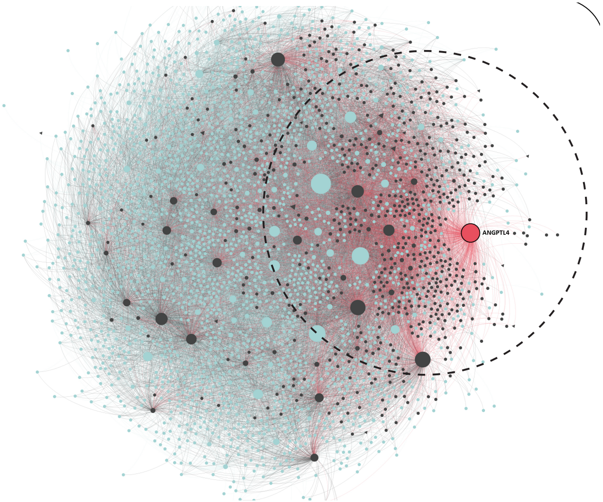
Wikipedia hyperlinks



emapsproject.com: Hyperlinks b/t wikipedia articles on climate change

- edges are hyperlinks between articles on climate change
- directed network; edges are given; no-edge may be missing

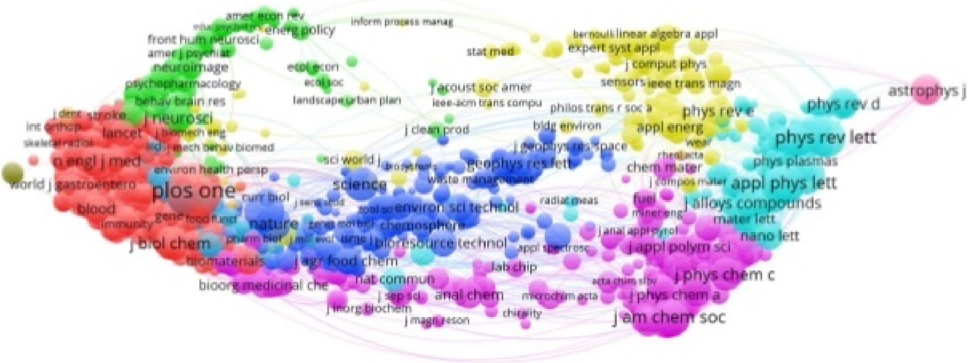
Gene interaction networks



Bianca Dumitrascu and Jonathan Lu

- pairwise interactions between genes
- directed network; edges are inferred; no-edge may be missing
- inferred from observational or time series data

Journal citation network



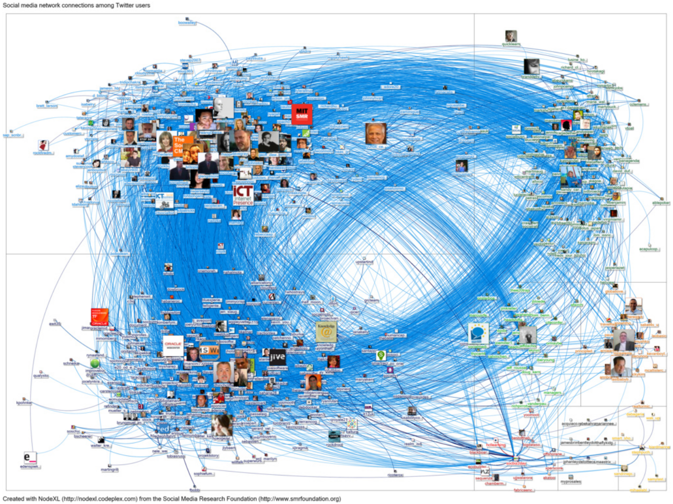
Nees Jan van Eck (Leiden University)

- edges are references between articles in scientific journals
- directed network; edges are given; no-edge is not missing

Social networks

- edges are *friend* relationships on `facebook.com`
- undirected network; edges are given; no-edge may be missing

Twitter networks



`forbes.com` article on Twitter influence networks

- pairwise interactions between users
- directed network; edges are given; no-edge may be missing
- edges represent *following*, *retweeting*, *liking*, *mentioning*

Network representation: Random graphs

First, let's describe a specific model of networks, the *random graph (Erdös-Renyi-Gilbert)*

- Represent relational data as $G = (X, Y)$
- $X \in \Re^{n \times p}$ represents n observations of p features
- $Y \in \Re^{n \times n}$ represents edge weights between n nodes
- Here, consider only binary graphs: $Y \in \{0, 1\}^{n \times n}$

Example: consider Facebook, where edges represent "friends" relationship

Network representation: ERG networks

Erdös-Renyi-Gilbert (ERG networks)

- Let $Y_{i,j}$ represent the edge between nodes i, j
- Binary edges are all sampled independently: $Y_{i,j} \sim Bernoulli(\theta)$
- Then, likelihood is written as:

$$p(G|\theta) = \prod_{i,j \in n \times n; i \neq j} \theta_{i,j}^{Y_{i,j}} (1 - \theta_{i,j})^{1 - Y_{i,j}}.$$

Much theoretical work on the value of θ and subsequent network properties

ERG networks: limitations

- Probability of edge is uniform across pairs of nodes
- cannot encode known pairwise relationships

Extend these models by adding node specific parameters

Mixed membership stochastic block model

Generative model for the mixed membership stochastic block model

- For every network node $i \in n$:
  - Draw a K dimensional vector on the simplex, given α:

$$\pi_i \sim Dirichlet(\alpha)$$

- For every pair of network nodes, $i, j \in N$:
  - Draw a membership indicator for initiator, $z_{i,j} \sim Multinomial(\pi_i)$
  - Draw a membership indicator for receiver, $z_{j,i} \sim Multinomial(\pi_j)$
  - Draw an indicator of their edge, $Y_{i,j} \sim Bernoulli(z_{i,j}^T B z_{j,i})$

Here, B is a $K \times K$ matrix capturing the probability that pairs of nodes from each of the K communities have an edge

Properties of MM stochastic block model

- Edges are *context dependent*: node may play different roles with different neighbors
- *Admixture*: each node is a combination of different communities
- $z_{i,j}$ latent variables label each possible interaction for every node with one community
- *Asymmetry*: an edge may have different underlying community labels from the two nodes
- When π, B given, can draw networks trivially from generative model
- B, and, correspondingly, Y may be free to take on arbitrary values and distributions

Mixed membership stochastic block model

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What are the latent variables? What are the parameters?

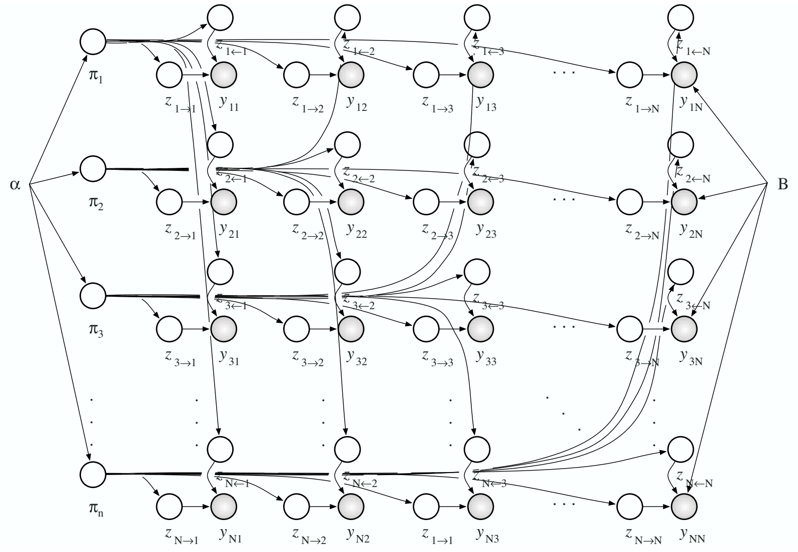
MMB: Likelihood

Mixed membership stochastic block model likelihood

$$p(Y, \pi, Z \mid \alpha, B) = \prod_{i,j \in n \times n; i \neq j} p(Y_{i,j} \mid z_{i,j}, z_{j,i}, B) p(z_{i,j} \mid \pi_j) \prod_{i \in n} p(\pi_i \mid \alpha)$$

Exercise: write out the expected complete log likelihood

Graphical model of MMB with *Z [Airoldi et al. 2008]*



Marginal mixed membership stochastic block model

Marginalize out latent variables Z:

Generative model for the marginal mixed membership SBM

- For every network node $i \in n$:
  - Draw a K dimensional vector on the simplex, given α:
$$\pi_i \sim Dirichlet(\alpha)$$
- For every pair of network nodes, $i, j \in n$:
  - Draw an indicator of their edge, $Y_{i,j} \sim Bernoulli(\pi_i^T B \pi_j)$

What are the latent variables? What are the parameters?

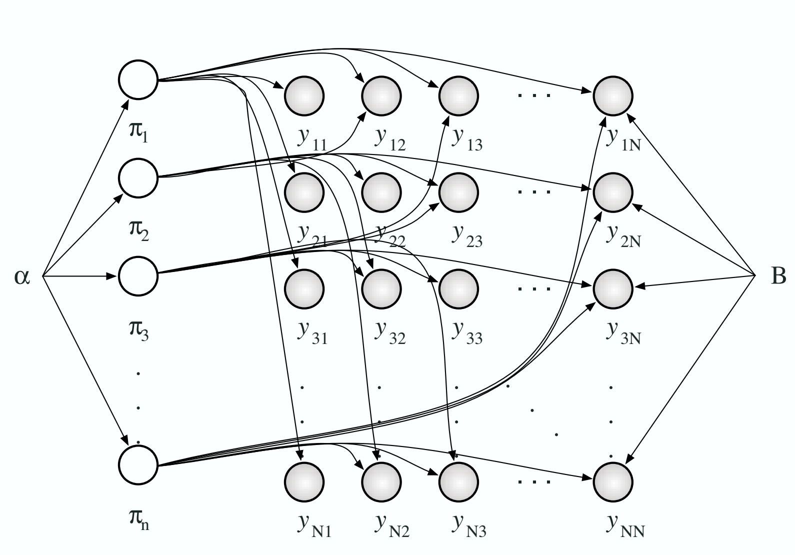
MMB: Likelihood, marginalizing out Z

Marginal mixed membership stochastic block model likelihood

$$p(Y, \pi \mid \alpha, B) \quad = \quad \prod_{i,j \in n \times n; i \neq j} p(Y_{i,j} \mid \pi_i, \pi_j, B) \prod_{i \in n} p(\pi_i \mid \alpha)$$

Exercise: write out the expected complete log likelihood

Graphical model of MMB *[Airoldi et al. 2008]*



Additional problem: sparsity

Often communities are large, graph structures are sparse.

No-edge may indicate community structure or missing data.

Extend to mixture model: data may be missing or observed; if not missing, what is the probability of an edge?

Sparse MMB likelihood

$$p(Y_{i,j} \mid Z, B, \rho) \quad = \quad \rho\delta(0) + (1-\rho) z_{i,j}^T B z_{j,i}$$

where ρ indicates probability of missing data between any pair of nodes.

Includes indicator for each pair of nodes with posterior prob of missingness

How can variables be interpreted for a MMB?

- Estimated values of π: node-specific community membership
- Estimated value of B: connectivity within and across communities
- Estimated value of $z_{i,j}$: which community gave rise to (possible) connection

What if I do not have an explicit graph to begin with?

Gaussian field or Markov random field or Gaussian graphical model

A Markov random field (MRF) is an undirected graphical model with marginal distributions on single nodes and pairs of nodes:

$$
\begin{aligned}
p(x \mid \theta) &\propto \prod_{i \sim j} \psi_{i,j}(x_i, x_j) \prod_i \psi_i(x_i) \\
\psi_i(x_i) &= \exp\left(-\frac{1}{2}\Lambda_{i,i} x_i^2 + \eta_i x_i\right) \\
\psi_{i,j}(x_i, x_j) &= \exp\left(-\frac{1}{2} x_i \Lambda_{i,j} x_j\right).
\end{aligned}
$$

Gaussian graphical models

Gaussian field or Markov random field or Gaussian graphical model

The joint distribution of a GGM then can be written as:

$$p(x|\theta) \propto \exp\left(\eta^T x - \frac{1}{2} x^T \Lambda x\right),$$

which is an unnormalized multivariate Gaussian, with $\Lambda = \Sigma^{-1}$ and $\eta = \Lambda\mu$.

Σ^{-1} is the precision matrix, and an entry is the covariance of $x_i, x_j | x_{\neg i,j}$.

When $\Lambda_{i,j} = 0$, then x_i, x_j are conditionally independent and have no edge.

Gaussian graphical models in practice

From data $X \in \Re^{n \times p}$, estimate a sparse, $n \times n$ precision matrix

This can practically be done in three ways (probably more)

- When $n \gg p$ this can be done analytically
- Graphical lasso: $x_i = \beta^T x_{\neg i} + \epsilon$
- Sparse, low dimensional precision matrix estimates.

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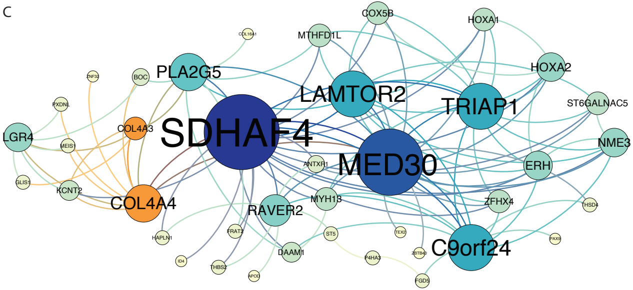


What analyses can I perform with MMB?

Dimension reduction

From network Y $(n \times n)$ to node-specific communities π $(n \times K)$ & interactions B $(K \times K)$

Prediction

Expected posterior of $Y(i, j)$ (marginalize over Z):

$$E[Y(i, j) = 1 | \hat{\pi}_i, \hat{B}, \hat{\pi}_j] \approx \hat{\pi}_i^T \hat{B} \hat{\pi}_j.$$

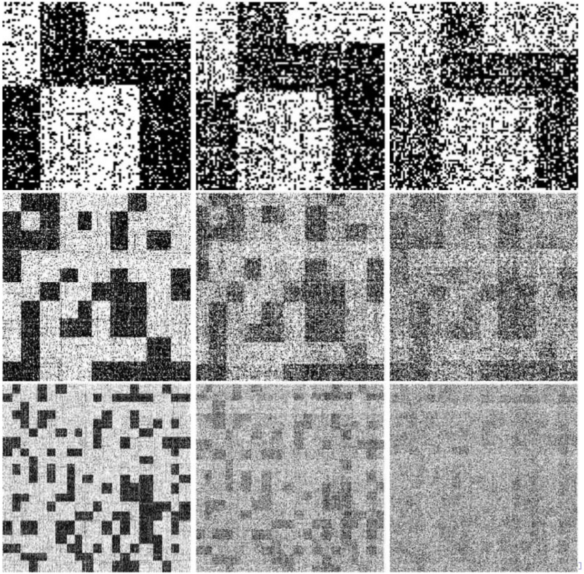
Network painting

Color each node in a network with $\hat{\pi}$; color each edge with $\hat{Z}$

Network generation

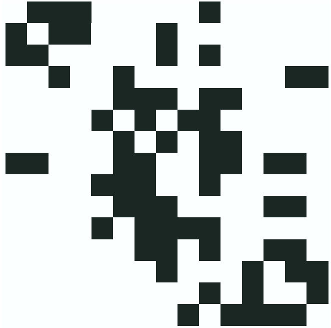
Generate networks with same latent structure as data using $\hat{\pi}, \hat{B}, \hat{Z}$

Simulated networks *[Airoldi et al. 2008]*



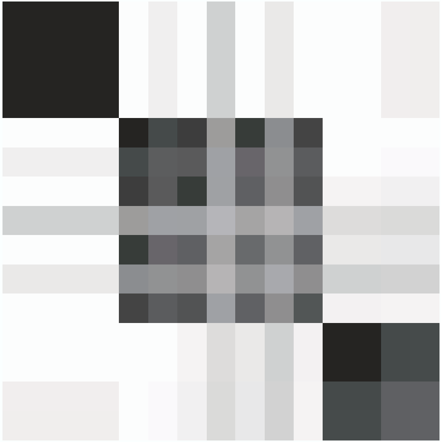
Example: Analysis of a monastery *[Airoldi et al. 2008]*

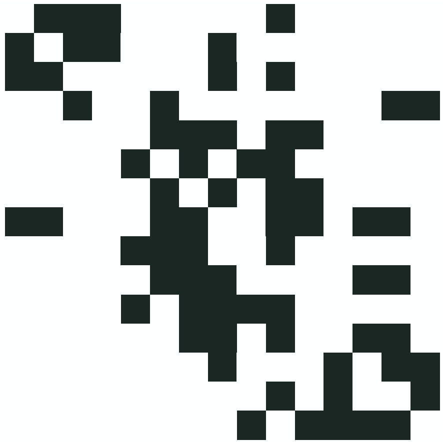
"Sampson (1968) surveyed 18 novice monks in a monastery [over two years] and asked them to rank the other novices in terms of four sociometric relations: like/dislike, esteem, personal influence, and alignment with the monastic credo."



- pairwise interactions between monks
- Sampson suggested four communities:
  - *loyal opposition*: members joined monastery first
  - *young turks*: members joined later
  - *outcasts*: members were not accepted into either main faction
  - *waverers*: members did not take sides

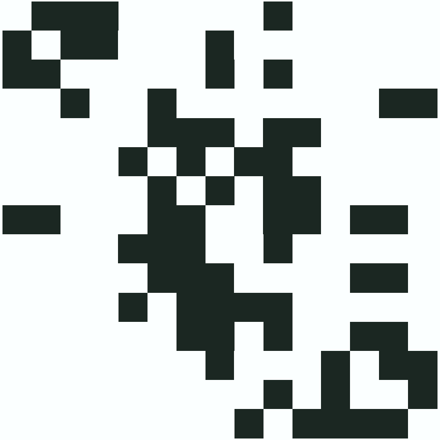
Example: Predicted interactions *[Airoldi et al. 2008]*





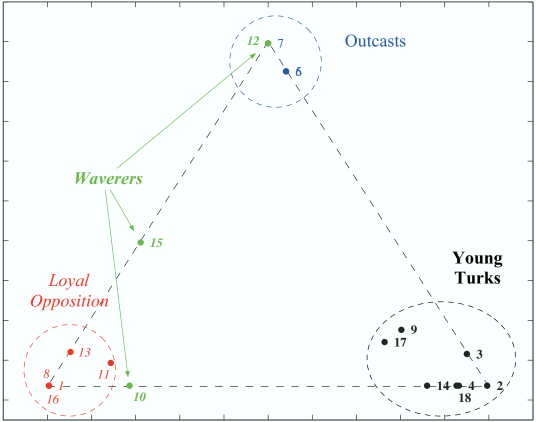
- predicted interactions between monks
- predictions are symmetric, *soft*: not binary

Example: Reconstructed interactions *[Airoldi et al. 2008]*



- reconstructed interactions between monks
- reconstruction is asymmetric, *soft*: not binary

Selecting K [Airoldi et al. 2008]

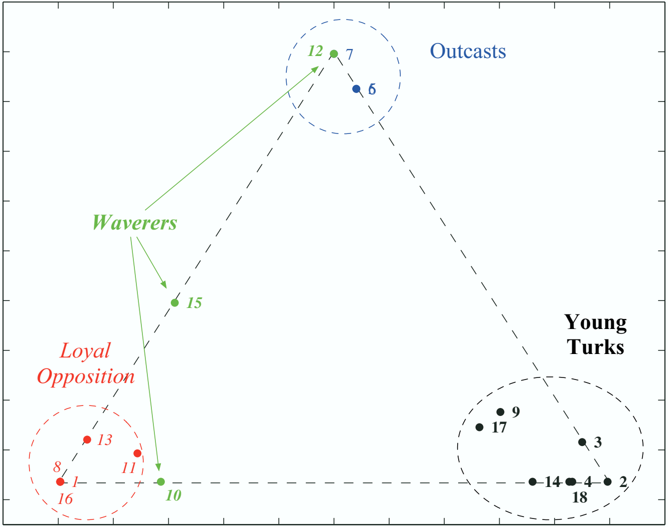


1 Ambrose
2 Boniface
3 Mark
4 Winfrid
5 Elias
6 Basil
7 Simplicius
8 Berthold
9 John Bosco
10 Victor
11 Bonaventure
12 Amand
13 Louis
14 Albert
15Ramuald
16 Peter
17 Gregory
18 Hugh

Used Bayesian information criterion (BIC) to select $K = 3$:

$$BIC = -2\log(p(Y|B,\alpha)) + \log(n)K$$

Node-specific community proportions *[Airoldi et al. 2008]*



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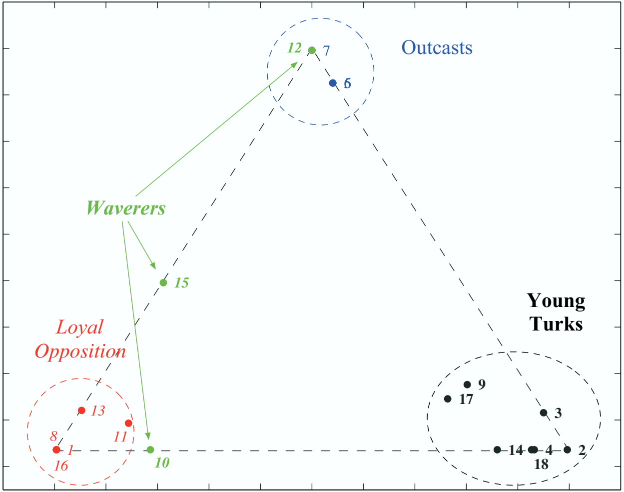
Is this type of community structure what you expected ($K = 3$)?

Node-specific community proportions *[Airoldi et al. 2008]*



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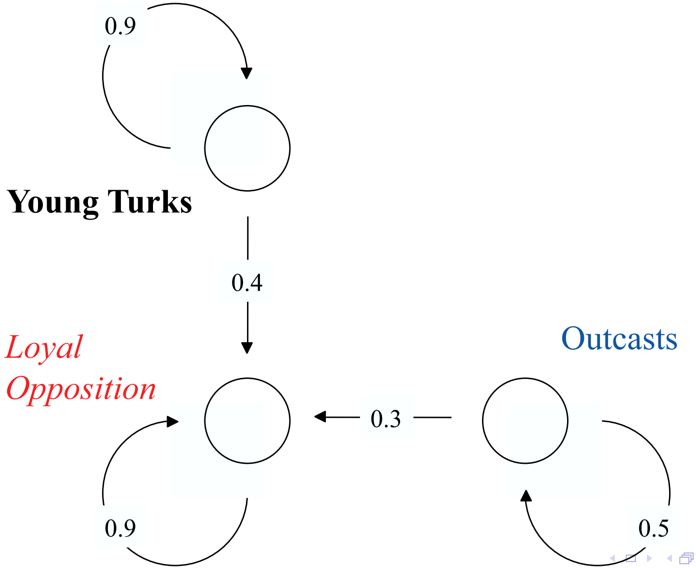
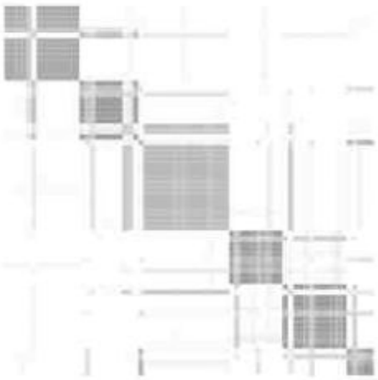
"After two years, John Bosco and Gregory get expelled, most young turks leave, and the order dissolves"

Monk block structure *[Airoldi et al. 2008]*

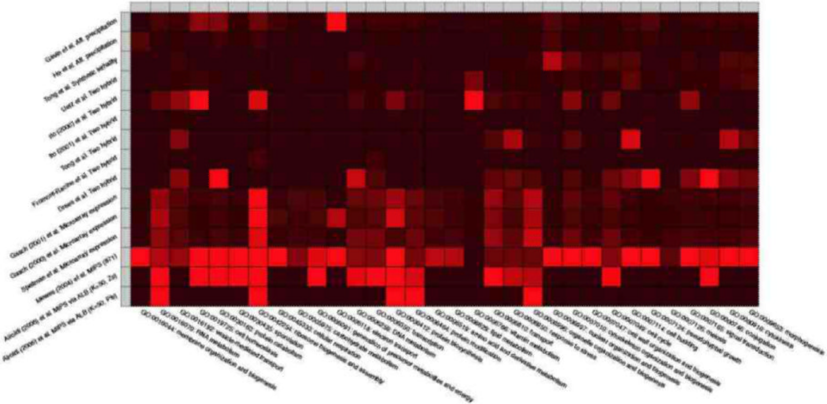


Middle+high school friendship network *[Airoldi et al. 2008]*



True (L) and estimated (R) friendships among people at a school.

Yeast protein interaction network *[Airoldi et al. 2008]*



Red is area under precision-recall curve for functions (columns) of genes in a group (rows)

Relationship with LDA model

- Both are mixed membership models, where each node/sample has proportions of membership
- Both are linear models, with expected complete log likelihood being a linear function of variables
- Latent representation, however, differs: LDA has no $K \times K$ matrix;

A note on inference

- As with LDA models, posterior inference is often intractable
- Variational expectation-maximization (and variants) has been used
- Sampling approaches have also been suggested, with various success

MMB: Extensions

- Sparsity
- Informative priors
- Fully Bayesian treatment
- Node attributes included
- Dynamic MMB
- Hierarchical relational models
- Non-parametric MMBs

MMBs: Summary & Assumptions

- Mixed membership stochastic block models assign each edge in a network to a pair of communities according to the node-specific community proportions, and draw from the blockmodel of that pair of communities
- MMBs do not use observations; instead, only use network connectivity.
- Samples and populations have alternative meaning from standard statistical framework: only one observation of network edges; samples not IID

Network analysis has a rich history

- Sociology and friendship networks (1930s)
- Sociology, mathematics, and psychology (1950s)
- Psychology (1960s)
- Statistics (1970s)
- Computer science and physics (1990s)

Additional Resources

- Many details of this lecture were borrowed from Airoldi's slides on MMBs (Guest lecture for EE380L)
- MMB paper: *[Airoldi, Blei, Fienberg, Xing 2008]*
- R code: iGraph, LDA