# Transitional Neoclassical Growth Dynamics with One-Sided Commitment

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# Motivation

- **Idiosyncratic income risk** in macro:
  - Aiyagari (1994): self-insurance. Many applications: HANK ...
  - But theoretically: **Pareto-improving trades!** What are the underlying frictions?
  - But empirically: **More consumption smoothing than "self-insurance"**. Blundell, Pistaferri and Preston (2008), ...
- Empirically: Lots of long-term **contracts with one-sided commitment**:
  - Firms and Workers.
  - Insurers and insured.
  - Financial intermediaries and depositors/borrowers.
- Theoretically: Significant literature on **limited commitment** with **exogenous outside option**: Kehoe and Levine (1993, 2001), Kocherlakota (1996), Alvarez and Jermann (2000).
- **Krueger-Uhlig, JME 2006**: Fix discount rate $r$ of intermediaries **(partial equilibrium)**.
  - Competition between intermediaries ("firms").
  - **Endogenous outside option** for agents ("workers").

# The Paper(s) Today

- **Krueger-Uhlig, 2022**: embed into neoclassical growth model.
  - Characterize **steady state**, including consumption distribution in closed form.
  - Alternative to Aiyagari (1994): heterogeneity with endogenous incomplete markets.
- **This paper**: **transition in neoclassical growth model**
  - One time permanent "MIT" shock in productivity at $t = 0$.
  - Characterize transition dynamics, including distributions analytically.
- This is a **theoretical exploration**: Take simplest version of the model, understand as much as possible.
- **Eventual Goal**, down the road: an attractive and improved quantitative **alternative to Aiyagari (1994), Krusell & Smith (1998)** workhorse model.

# Environment: Household Preferences and Endowments

- Time $t \in (-\infty, \infty)$ is continuous.
- Mass one of agents $j \in [0, 1]$.
- **Idiosyncratic labor productivity** $z_{j,t}$: iid across $j$. Earn $z_{j,t}\mathbf{w_t}$.
  - $z_{j,t} \in Z = \{\mathbf{0}, \zeta\}$ with $\zeta > 0$.
  - Poisson transition rates: $\nu\, dt = P(0 \to \zeta)$, $\xi\, dt = P(\zeta \to 0)$.
  - Stationary labor productivity distribution: $(\psi_l, \psi_h) = \left(\frac{\xi}{\xi+\nu}, \frac{\nu}{\xi+\nu}\right)$.
  - Normalize average labor productivity to one: $\frac{\nu}{\xi+\nu}\zeta = \mathbf{L} = \mathbf{1}$
- Preferences are CRRA. Lifetime utility:

$$U_0 = E_0\left[\int_0^\infty e^{-\rho t} u(c_t)dt\right], \text{ where } u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}$$

# Environment: Technology

- Neoclassical production function operated by representative firm renting capital and labor:

$$Y_t = \mathbf{A_t} K_t^{\theta} L_t^{1-\theta}$$

  where $0 < \theta < 1$.
- Capital depreciates at rate $\delta$.
- Wage, interest rate in equilibrium (with $L_t = 1$):

$$\begin{aligned} w_t &= (1-\theta) A_t K_t^{\theta} \\ r_t &= \theta A_t \left(K_t\right)^{\theta-1} - \delta \end{aligned}$$

- Aggregate resource constraint

$$C_t + \dot{K}_t = Y_t - \delta K_t.$$

- Productivity process:
  - $t < 0$: $A_t \equiv \mathbf{A}^*$, “assumed” forever.
  - At $t = 0$: “MIT” shock to path $A_t$. Perfect foresight from $t = 0$ on.
- **Closed form solution for transition** if $\sigma = 1$, $A_t \equiv \tilde{\mathbf{A}} \neq A^*$ for $t \geq 0$, as long as permanent shock $\tilde{\mathbf{A}}$ not too large.

# Financial Market Structure

- Aiyagari (1994): self-insure by precautionary capital accumulation.
- **Here instead:** (risk-neutral) competitive financial intermediaries offer consumption insurance contracts against productivity risk.
- Every $t$ households can **save** through capital with intermediary and **buy insurance** against $z$ transitions ($\approx$ Arrow securities).
  - Intermediaries honor capital and insurance contracts.
  - Key friction: **agents cannot commit**, can change intermediary at any point, without punishment. Thus capital can't become negative.
- Perfect competition: intermediaries make zero profits, offer actuarially fair contracts.
- Limited commitment & no punishment: individuals **cannot borrow**. See Krueger & Uhlig (2006), Alvarez & Jermann (2000).
- Assumptions on parameters will insure that individuals with high labor productivity ($z = \zeta$) **will not save**.
- t = 0: After "MIT shock", **capital account of agents unchanged on impact**, but future consumption allocation altered.

# The Optimal Contract: HJB Equation for Agents

## Definition

For $z \in Z$, wages $w_t$ and interest rates $r_t$, let $\tilde{z}$ be the "other" $z$ and let $p_z$ be the transition rate $z \to \tilde{z}$. An optimal consumption insurance contract

$$\mathcal{C}_t = \Big(U_t(k;z), c_t(k;z), x_t(k;z), \tilde{k}_t(k;z)\Big)_{k \geq 0, z \in Z}$$

solves

$$\begin{aligned}
\rho U_t(k;z) \quad &= \quad \max_{c, \tilde{\mathbf{k}} \geq \mathbf{0}, x} \Big\{ u(c) + U_t'(k;z)x + p_z(U_t(\tilde{k}, \tilde{z}) - U_t(k;z)) + \dot{U}_t(k;z) \Big\} \\
\text{s.t.} \quad & \quad c + x + p_z(\tilde{k} - k) = r_t k + w_t z \\
& \quad \mathbf{x \geq 0 \text{ if } k = 0}
\end{aligned}$$

# The Optimal Contract: Heuristic Derivation from Neoclassical Growth Model

- Competitive equilibrium (with constant factor prices) of standard Neoclassical growth model:

$$\begin{aligned} \rho U(k) &= \max_{c,x} \left\{ u(c) + U'(k)x \right\} \\ \text{s.t.} \quad & c + x = rk + w \end{aligned}$$

- ... or plugging in the budget constraint to eliminate $x$

$$\rho U(k) = \max_{c} \left\{ u(c) + U'(k)(rk + w - c) \right\}$$

- Often $\dot{k}$ is used to denote $x$

$$\begin{aligned} \rho U(k) &= \max_{c,\dot{k}} \left\{ u(c) + U'(k)\dot{k} \right\} \\ \text{s.t.} \quad & c + \dot{k} = rk + w \end{aligned}$$

- Denote co-state variable $\lambda = U'(k)$ associated with $k$ as

$$\rho U(k) = \max_{c,\lambda} \{ u(c) + \lambda(rk + w - c) \} = \max_{c,\lambda} \{ \mathcal{H}(k, c, \lambda) \}$$

- $\mathcal{H}(k, c, \lambda) = u(c) + \lambda(rk + w - c)$ is the current-value Hamiltonian

# The Optimal Contract: Discrete vs. Continuous Time

- Period length $\Delta$, discrete time DP, discount fac. $\beta(\Delta) = e^{-\rho\Delta} \approx 1 - \Delta\rho$

$$\begin{aligned} U(k) &= \max_{c, k_\Delta - k} \left\{ \Delta u(c) + e^{-\rho\Delta} U(k_\Delta) \right\} \\ \text{s.t.} \quad & k_\Delta - k = \Delta(rk + w - c) \end{aligned}$$

- Use $e^{-\rho\Delta} \approx 1 - \Delta\rho$, subtract $(1 - \Delta\rho)U(k)$ from both sides:

$$\begin{aligned} \rho\Delta U(k) &= \max_{c, k_\Delta - k} \left\{ \Delta u(c) + (1 - \Delta\rho) \frac{U(k_\Delta) - U(k)}{k_\Delta - k} (k_\Delta - k) \right\} \\ \text{s.t.} \quad & k_\Delta - k = \Delta(rk + w - c) \end{aligned}$$

- Now divide both sides by $\Delta$

$$\begin{aligned} \rho U(k) &= \max_{c, \frac{k_\Delta - k}{\Delta}} \left\{ u(c) + (1 - \Delta\rho) \frac{U(k_\Delta) - U(k)}{k_\Delta - k} \frac{(k_\Delta - k)}{\Delta} \right\} \\ \text{s.t.} \quad & \frac{k_\Delta - k}{\Delta} = rk + w - c \end{aligned}$$

- Now take $\Delta \to 0$

$$\rho U(k) = \max_{c, \dot{k}} \left\{ u(c) + U_t'(k)\dot{k} \right\} \text{ s.t. } \dot{k} = rk + w - c$$

# Optimal Consumption-Savings Choice in the Standard Neoclassical Growth Model

$$\begin{aligned} \rho U(k) &= \max_{c,x} \{u(c) + U_t'(k)x\} \\ \text{s.t.} & \quad c + x = rk + w \end{aligned}$$

- Determine FOCs, take derivative wrt to time, calculate:

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{u'(c)}{cu''(c)}(\rho - r) \\ \text{(for CRRA:)} &= \frac{r-\rho}{\sigma} \\ \text{(for log:)} &= r - \rho \end{aligned}$$

- For CRRA, when $w = 0$: "cake eating" problem:

$$\begin{aligned} c &= \alpha k \text{ for some } \alpha \\ \frac{\dot{k}}{k} = \frac{x}{k} &= \frac{\dot{c}}{c} \end{aligned}$$

# Heuristic Derivation from Neoclassical Growth Model

$$\begin{aligned}\rho U(k) &= \max_{c,x} \{u(c) + U_t'(k)x\} \\ \text{s.t.} \quad & c + x = rk + w\end{aligned}$$

- With idiosyncratic risk and incomplete markets (Achdou et al, 2021):

$$\begin{aligned}\rho U(k, \mathbf{z}) &= \max_{c,x} \{u(c) + U'(k)x + \mathbf{p_z(U(k,\tilde{z}) - U(k,z))}\} \\ \text{s.t.} \quad & c + x = rk + wz\end{aligned}$$

- With actuarially fair insurance contracts:

$$\begin{aligned}\rho U(k, z) &= \max_{c,\tilde{\mathbf{k}},x} \left\{u(c) + U'(k)x + p_z(U(\tilde{\mathbf{k}}, \tilde{z}) - U(k,z))\right\} \\ \text{s.t.} \quad & c + x + \mathbf{p_z(\tilde{k} - k)} = rk + wz\end{aligned}$$

- Limited Commitment

$$\begin{aligned}\rho U(k, z) &= \max_{c,x,\tilde{k}} \left\{u(c) + U'(k)x + p_z(U(\tilde{k}, \tilde{z}) - U(k,z))\right\} \\ \text{s.t.} \quad & c + x + p_z(\tilde{k} - k) = rk + wz \\ & \mathbf{\tilde{k} \geq 0, x \geq 0 \text{ if } k = 0}\end{aligned}$$

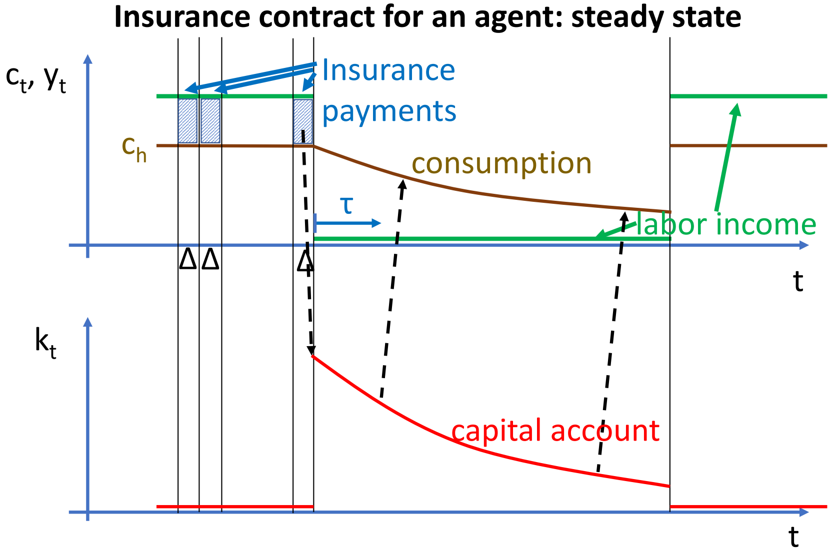
# The Optimal Contract: HJB Equation for Agents

- Transition adds time-varying $w_t, r_t$, makes $U_t(k;z)$ time-dependent and thus adds the term $\dot{U}_t(k;z)$ in the HJB equation.

- Thus contract solves:

$$\begin{aligned} \rho U_t(k;z) &= \max_{c,\tilde{\mathbf{k}}\geq\mathbf{0},x} \left\{ u(c) + U_t'(k;z)x + p_z(U_t(\tilde{k},\tilde{z}) - U_t(k;z)) + \dot{U}_t(k;z) \right\} \\ \text{s.t.} \quad & c + x + p_z(\tilde{k} - k) = r_t k + w_t z \\ & \mathbf{x \geq 0} \text{ if } \mathbf{k = 0} \end{aligned}$$

# Optimal Contract in Steady State (Assume $r < \rho$)



# Optimal Contract in Steady State with $r < \rho, \sigma = 1$

- Key 1: if $r < \rho$, individuals do not save for the $z = \zeta$ state
- Key 2: Limited commitment: poor individuals ($z = 0$) cannot borrow against the $z = \zeta$ state
- Budget constraints under these "conjectures"

$$
\begin{aligned}
c + \xi \tilde{k} &= \zeta w \\
c + x - \nu k &= rk
\end{aligned}
$$

- Optimal allocations
  - High productivity, $z = \zeta$: $\frac{c_h}{\zeta w} = \frac{\nu + \rho}{\xi + \nu + \rho}$ and $\frac{\tilde{k}}{\zeta w} = \frac{1}{\xi + \nu + \rho}$. A share of income $\frac{\xi}{\xi + \nu + \rho}$ is used to buy insurance for loss of productivity. Choice of $\tilde{k}$ guarantees continuity of consumption upon negative productivity shock.
  - Low productivity, $z = 0$: $c = (\nu + \rho)k$ and $x \equiv \dot{k} = (r - \rho)k$. Consumption and capital account drifts down at rate $\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = r - \rho < 0$ as in standard neoclassical growth model.

# Asset Distribution in Steady State and Transition

## Assumption (A1)

*For all $t \geq 0$, assume that $\sigma = 1$ and*

$$\frac{\dot{w}_t}{w_t} + \rho - r_t > 0$$

- Assumption on endogenous variables! Later replaced by assumptions on parameters only.
- In steady state, A1 requires $r < \rho$ since $\frac{\dot{w}_t}{w_t} = 0$ in the steady state.
- Assumption A1 insures that all high productivity $(z = \zeta)$-agents do not hold capital: $x_t(0, \zeta) = \tilde{k}_t(k; 0) = 0$ and are all identical.
- Low productivity $(z = 0)$ agents are only distinguished by the time $\tau$ elapsed since having had high productivity $z = \zeta$. Density of waiting times $\tau \geq 0$

  $$\psi_l(\tau) = \frac{\xi\nu}{\xi + \nu} e^{-\nu\tau}$$

  which integrates to the total mass $\xi/(\xi + \nu)$ of $z = 0$ agents.

# From Recursive to Sequential Allocations: Needed for Transition Path

- Low productivity agents hold capital $k_{s,t}$ depending on the date $t$ and the time $s = t - \tau$ of last transition to $z = 0$.
- Likewise, let $c_{s,t} = c_t(k_{s,t}, 0)$ be consumption of $z = 0$ agent at $t$, who lost productivity last at date $s \leq t$.
- Finally, (abusing notation), let $c_{h,t} = c_t(0, \zeta)$ denote consumption of individuals with currently high productivity $z = \zeta$.
- Time derivatives are always with respect to calendar time $t$.

# Definition of Dynamic Equilibrium

## Definition

Given an initial capital distribution $(k_{-\tau,0})$ for $z = 0$-agents, a dynamic equilibrium are contracts $\mathcal{C}_t$, wages $w_t$, interest rates $r_t$, aggregate capital $K_t$ and capital of $z = 0$ agents $(k_{s,t})_{s \leq t}$, for all $t \geq 0$, such that

1. Given the sequence of $w_t, r_t$, the contracts $\mathcal{C}_t$ are optimal.
2. The contracts $\mathcal{C}_t$ have the "only $z = 0$ agents hold capital" property: $\tilde{k}_t(k; 0) = 0$ for all $k = k_{t,\tau}$, $\tau \geq 0$ as well as $x_t(0; \zeta) = 0$.
3. Capital held by $z = 0$ agents are consistent with the contracts $\mathcal{C}_t$ , i.e. $k_{t,t} = \tilde{k}_t(0; \zeta)$ and $\dot{k}_{s,t} = x_t(k_{s,t}; 0)$, where $\dot{k}_{s,t} = \partial k_{s,t}/\partial t$.
4. Factor prices satisfy $r_t = \theta A_t \left(K_t\right)^{\theta-1} - \delta$ and $w_t = (1-\theta) A_t \left(K_t\right)^{\theta}$.
5. The goods markets and the capital markets clear:

$$
\begin{aligned}
\int_0^\infty c_{t-\tau,t} \psi_l(\tau) d\tau + \frac{\nu}{\xi + \nu} c_{h,t} &= A_t \left(K_t\right)^{\theta} - \delta K_t \\
\int_0^\infty k_{t-\tau,t} \psi_l(\tau) d\tau &= K_t
\end{aligned}
$$

# Partial Insurance Steady State

- Capital demand in steady state solves

$$r = \theta A \left(K^d(r)\right)^{\theta-1} - \delta$$

- Steady state allocations $(c_{-\tau}, k_{-\tau})$. Capital supply in steady state

$$K^s(r) = \int_0^\infty k_{-\tau}(r)\psi_l(\tau)d\tau$$

- Evidently, $K^d(r = -\delta) = \infty > K^s(r = -\delta)$.
- The following assumption guarantees that $K^d(r = \rho) < K^s(r = \rho)$

## Assumption (A2)

*Let the exogenous parameters of the model satisfy $\theta, \nu, \xi, \rho > 0$ and*

$$\frac{\theta}{(1-\theta)(\rho+\delta)} < \frac{\xi}{\nu(\rho+\nu+\xi)}$$

*and $\sigma = 1$ (log-utility).*

# Partial Insurance Steady State

### Proposition (Krueger-Uhlig, 2022)

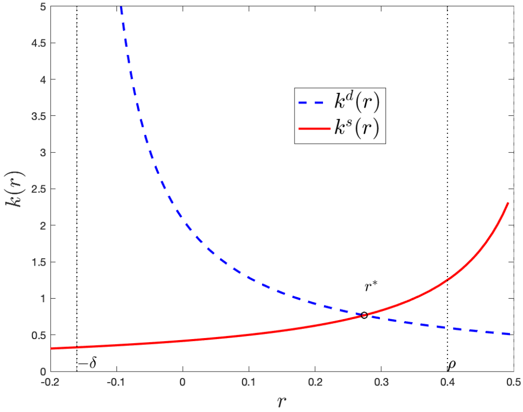
*There is a unique stationary equilibrium $r^*$ satisfying $K^d(r^*) = K^s(r^*)$ with*

$$r^* = \frac{\theta(\xi + \nu + \rho)(\nu + \rho) - \xi\delta(1-\theta)}{\xi + \theta(\nu + \rho)} \in (-\delta, \rho)$$

*Equilibrium consumption (deflated by wage $w$) distribution is truncated Pareto below mass point $c_h/w$:*

$$\phi_{r^*}(c) = \begin{cases} \frac{\xi\nu (c_h/w)^{-\frac{\nu}{\rho - r^*}}}{(\rho - r^*)(\nu+\xi)} (c/w)^{\frac{\nu}{\rho - r^*} - 1} & \text{if} \quad c/w \in (0, c_h/w) \\ \frac{\nu}{\nu + \xi} & \text{if} \quad c/w = c_h/w = \frac{\nu+\rho}{\xi+\nu+\rho}\zeta \end{cases}$$

# Capital Market: Steady State Partial Insurance




- Assumption guarantees $\frac{K^d(r=\rho)}{w} = \kappa^d(r=\rho) < \kappa^s(r=\rho) = \frac{K^s(r=\rho)}{w}$
- Since $r^* < \rho$, then $z = \zeta$-individuals don't want to save.
- Full comparative statics with respect to $(\theta, \delta, \rho, \xi, \nu)$.
- If $\sigma > 2$, two steady states with $r_1^* < r_2^* < \rho$ possible as $\kappa^s(r)$ slopes down

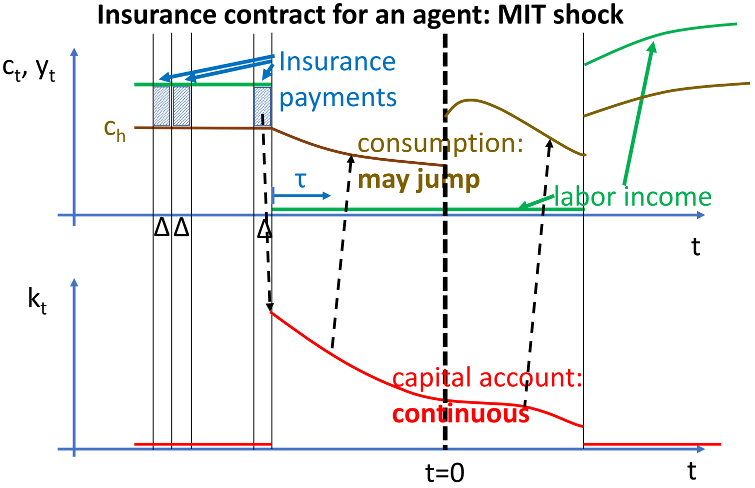
# Thought Experiment and Construction of Equilibrium

- For all $t < 0$ economy is in stationary equilibrium associated with productivity $A^*$.
- At $t = 0$ productivity changes unexpectedly to new time path $A_t$, for $t \geq 0$. Perfect foresight from $t = 0$ on.
- Transition path is solution to the following fixed point problem (computational algorithm):
    1. Conjecture a path for capital $K_t$, $t \geq 0$.
    2. Compute $r_t = \theta A_t \left(K_t\right)^{\theta-1} - \delta$ and $w_t = (1-\theta)A_t \left(K_t\right)^{\theta}$.
    3. Compute the paths of individual household consumption and capital, given the path for $r_t$ and $w_t$.
    4. Compute the path of aggregate capital supply $K_t^S$ by aggregating across individual households using density $\psi_l$.
    5. Check whether this is the conjectured path, $K_t = K_t^S$.

# Characterization of Optimal Contract under A1

- Consumption of high-income people: $c_{h,t} = c_t(0, \zeta) = \frac{\nu+\rho}{\nu+\rho+\xi}\zeta w_t$
- Capital upon receiving bad shock: $k_{t,t} = \tilde{k}(0, \zeta) = \frac{1}{\nu+\rho+\xi}\zeta w_t$
- Evolution of individual capital stock: $x_t(0, \zeta) = 0$ and $x_t(k, 0) = (r_t - \rho)k < 0$. Thus, $\frac{\dot{k}_{s,t}}{k_{s,t}} = r_t - \rho$.
- Consumption of the income poor $c_t(k, 0) = (\rho + \nu)k$
- Note (1): current allocation does not depend on future interest rates or wages. Only true with log-utility. This implies that it is irrelevant if the MIT-shock is anticipated or unanticipated.
- Note (2): All capital owners $(z = 0)$ consume and save the same share of their capital (income). High-productivity individuals $(z = \zeta)$ don't save (but buy insurance).

# Optimal Contract: Response to MIT Shock




- Consumption of $s < 0$-agents does not jump at $t = 0$: $c_{s,0} = (\rho + \nu)k^*_{-s}$.
- Consumption of $z = \zeta$-individuals remains $\frac{c_{h,t}}{w_t} = \frac{\nu+\rho}{\nu+\rho+\xi}\zeta$ for all $t \geq 0$.
- Log-utility key for both results.

# Aggregation

- Since all individuals with capital have the same propensity to save out of capital, the model aggregates.
- Law of motion for aggregate capital is given by

$$\begin{aligned}\dot{K}_t &= \left(\frac{\xi}{\rho+\nu+\xi}(1-\theta)+\theta\right)A_t K_t^{\theta} - (\delta+\rho+\nu)K_t \\ &= sA_t K_t^{\theta} - \hat{\delta}K_t\end{aligned}$$

- This differential equation is a Bernoulli equation that has a closed form solution (Jones, 2000) for arbitrary path of $\{A_t\}$.

$$K_t = \left(e^{-(1-\theta)(\delta+\rho+\nu)t}\left(K^*\right)^{1-\theta} + (1-\theta)\int_0^t e^{-(1-\theta)(\delta+\rho+\nu)(t-s)}a_s ds\right)^{\frac{1}{1-\theta}}$$

where

$$a_s = \left(\frac{(1-\theta)\xi}{\rho+\nu+\xi}+\theta\right)A_s.$$

# Summary for log utility, perm. prod. change

- Consumption of high-income people: $c_{h,t} = c_t(0, \zeta) = \frac{\rho+\nu}{\rho+\nu+\xi} \zeta w_t$
- Capital upon receiving bad shock: $k_{t,t} = \tilde{k}(0, \zeta) = \frac{1}{\rho+\nu+\xi} \zeta w_t$
- Evolution of individual capital stock: $x_t(0, \zeta) = 0$ and $x_t(k, 0) = (r_t - \rho)k < 0$. Thus, $\frac{\dot{k}_{s,t}}{k_{s,t}} = r_t - \rho$.
- Consumption of the income poor $c_t(k, 0) = (\rho + \nu)k$
- Since all individuals with capital have the same saving rate, the model aggregates. Law of motion for aggregate capital is given by

$$\begin{aligned} \dot{K}_t &= \left( \frac{\xi}{\rho + \nu + \xi} (1 - \theta) + \theta \right) A_t K_t^\theta - (\delta + \rho + \nu) K_t \\ &= s A_t K_t^\theta - \hat{\delta} K_t \end{aligned}$$

- This differential equation is a Bernoulli equation that has a closed form solution (Jones, 2000) for arbitrary path of $\{A_t\}$.
- True, even if shocks are anticipated. Could do bus. cycle analysis!

# Intuition for the Closed-Form Solution

- **No closed-form solution** in the neoclassical growth model.
- This environment has idiosyncratic risk that is not fully insured. Non-degenerate consumption distribution that changes over time. far richer model!
- So: why a closed form solution here?
- Log-utility: low-productivity agents consume according to a constant savings rate.
- High-productivity agents only insure against switch to low productivity, but they do not accumulate new capital.
- Together, the model **aggregates** since all agents with positive wealth have the same constant savings rate. See also Moll (2014).
- The Solow model (which **assumes** a constant aggregate saving rate) has a closed form solution, see Jones (2000).
- **Now: Numerical Illustration**
  - Aggregate dynamics.
  - Consumption distribution dynamics.

# Permanent Change in Productivity

- Suppose productivity changes permanently from $A^*$ to $\tilde{A}$.
- Then the aggregate capital stock is given in closed form by

$$K_t = \left( \frac{a}{b} + \left( (K^*)^{1-\theta} - \frac{a}{b} \right) e^{-(1-\theta)bt} \right)^{\frac{1}{1-\theta}}$$

where

$$\begin{aligned} a &= \left( \frac{\xi}{\rho + \nu + \xi} (1-\theta) + \theta \right) \tilde{A} \\ b &= \delta + \rho + \nu \\ K^* &= \text{Old Steady State Capital Stock} \end{aligned}$$

- If $\tilde{A} > A^*$, then $(K^*)^{1-\theta} < \frac{a}{b}$. Capital monotonically increasing from old to new steady state.
- If $\tilde{A} < A^*$, then $(K^*)^{1-\theta} > \frac{a}{b}$. Capital monotonically declining from old to new steady state.

# A Loose End

- Thus far have assumed that $\frac{\dot{w}_t}{w_t} + \rho - r_t > 0$ for all $t$.
- Now can replace this assumption with assumption on exogenous parameters: permanent increase in $A$ cannot be too large:

## Assumption (A3)

*Let $\tilde{A} < \bar{A}$, where $\bar{A} < A^*$ is defined as*

$$\bar{A} = A^* \left(1 + \frac{\nu(\rho+\delta)}{\theta(\rho+\nu+\delta)}\left(1+\frac{\xi}{\rho+\nu}\right)\left(\frac{\xi}{\nu(\rho+\nu+\xi)} - \frac{\theta}{(1-\theta)(\rho+\delta)}\right)\right)$$

## Proposition

*Assume A2 and A3. Then Assumption A1*

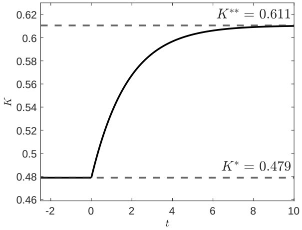
$$\frac{\dot{w}_t}{w_t} + \rho - r_t > 0$$

*is satisfied for all $t \geq 0$.*

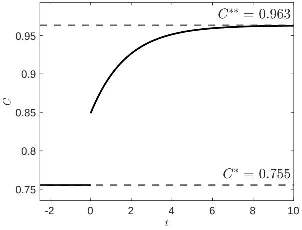
# Transitional Dynamics: Productivity Increase

- Parameters $\theta = 0.25, \delta = 0.16, \nu = \xi = 0.2, \rho = 0.4, A^* = 1, \tilde{A} = 1.2$

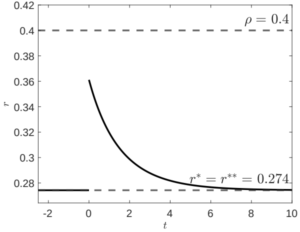
Capital:



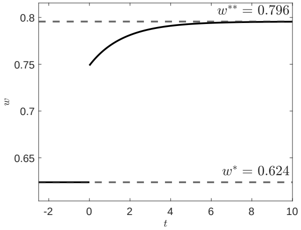

Consumption:



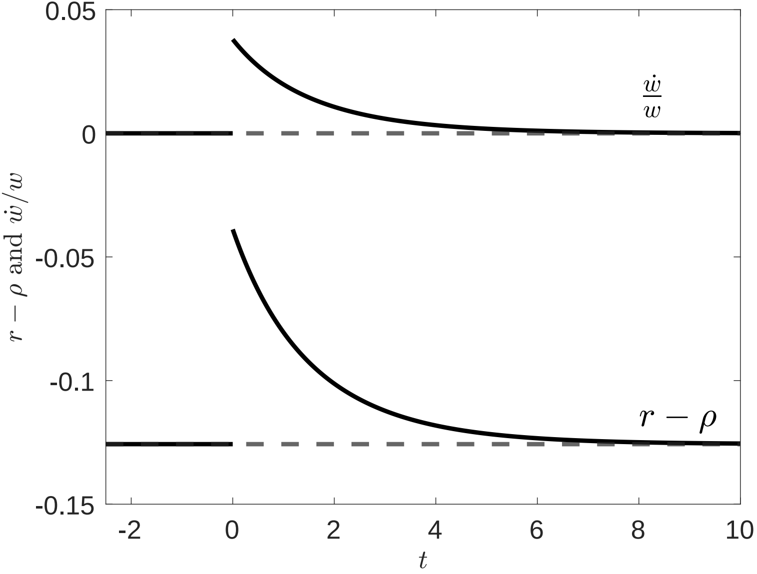

Interest Rate:




Wage:

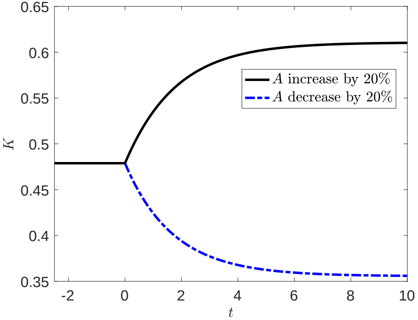


# The no-savings condition: $\frac{\dot{w}}{w} > r - \rho$

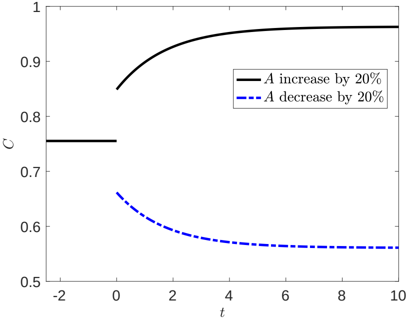


# Productivity Increase vs Decrease

Capital:

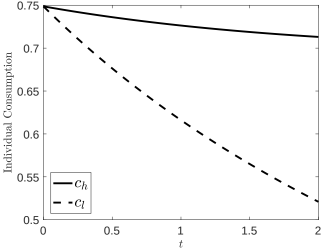



Consumption:

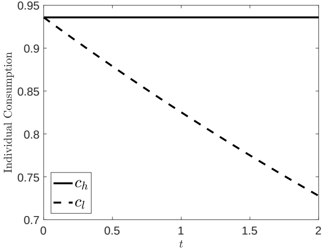


# Consumption Paths: $c_h$ vs. $c_l$ ($z = 0$ for long time)

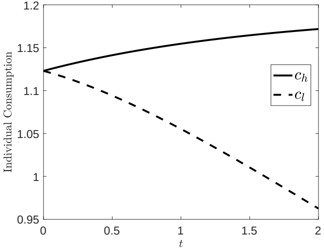
$\tilde{A} = 0.8 < A^* = 1$:



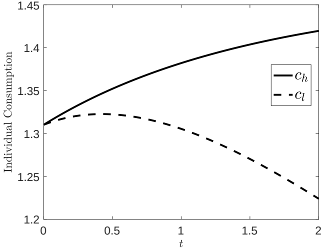

$\tilde{A} = A^* = 1$:




$\tilde{A} = 1.2 > A^* = 1$:

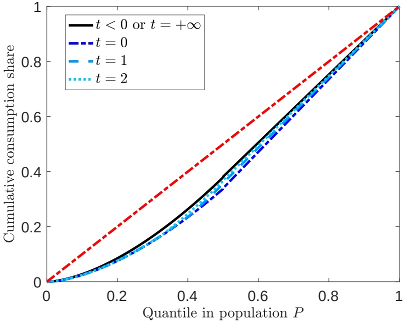



$\tilde{A} = 1.4 > A^* = 1$:

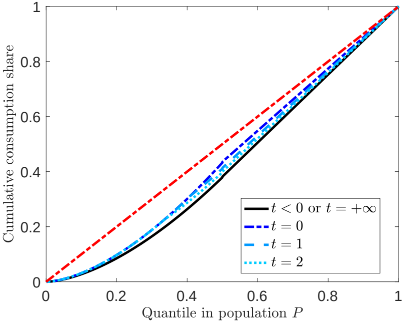


# Lorenz Curve for Consumption

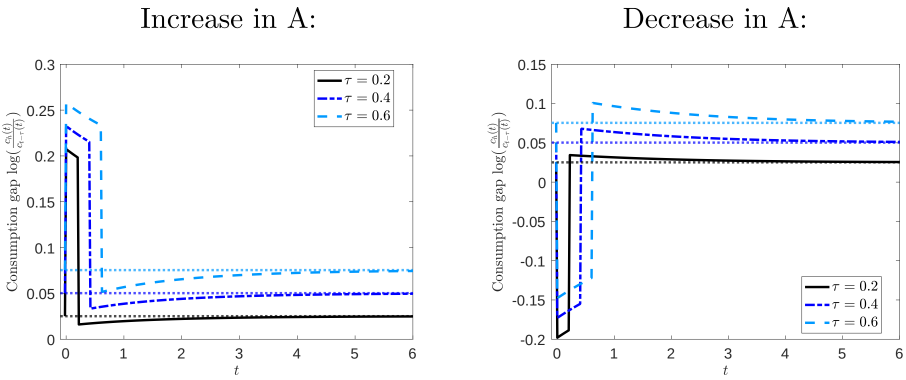
Increase in A:




Decrease in A:



# Consumption Inequality



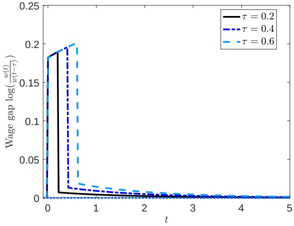

A useful decomposition:

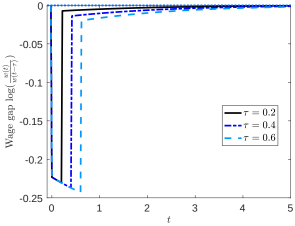
$$\underbrace{\log\left(\frac{c_t^h}{c_{t-\tau,t}}\right)}_{\text{consumption gap}} = \underbrace{\log\left(\frac{w_t}{w_{t-\tau}}\right)}_{\text{wage gap}} \underbrace{-\int_{t-\tau}^{t} g_u du}_{\text{discounting gap}}$$

# Wage Gap and Discounting Gap

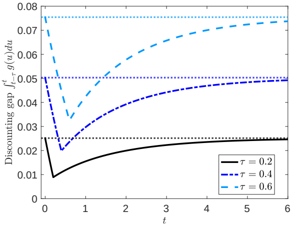
Increase in A: Decrease in A:

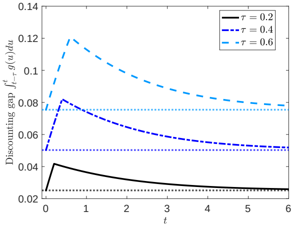
Wage Gap:




Discounting Gap:



# Conclusion

- **Model:**
  - Two-state idiosyncratic income risk, $z \in \{0, \zeta > 0\}$.
  - Households insured by intermediaries: one-sided commitment.
  - Embed in neoclassical growth model with CRRA utility, Cobb-Douglas production.
  - Characterize transition after “MIT” shock to productivity.
- **Results:**
  - **Closed-form** solution for log utility.
  - **Rich set of analytical implications** for the dynamics of the consumption and wealth distribution.
- **Why is this interesting?** Because (we think):
  - It provides a theory of imperfect consumption insurance based on **micro-founded friction**: one-side limited commitment.
  - **No “missing markets”**. Scope for meaningful policy experiments.
  - Attractive and **analytically tractable alternative to Aiyagari**-style workhorse model.
  - **Wide open questions**: quantitative implications, confront empirical facts, aggregate shocks, other frictions, nominal rigidities.