Note on the Heterogeneous Agent Model: Aiyagari (1994)

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June, 2018

Overview

The purpose of this note is to explain the details of the algorithm of the standard heterogeneous agents model developed by Aiyagari (1994, JPE).
The main features of the basic Aiyagari model are

- There are mass of households.
- Households are ex-ante homogeneous bu ex-post heterogeneous, depending on the history of realization of idiosyncratic shocks. In Aiyagari (1994), labor income shock is the only shock.
- There is only one asset (risk-free asset or capital) that are traded.
- Hence, households cannot fully insure away their idiosyncratic risks. They can only self-insure by saving.
- Only the stationary equilibrium is studied: all aggregates don't vary.
- Prices (wage and interest rate) are determined competitively $\rightarrow$ Use prices to clear markets.

1 Model

1.1 Household Problem

- Time is discrete.
- There are a continuum of households. Total measure of households is normalized to one. Each household has preference

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

  where $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$

- Households are endowed with capital k_0 initially and one unit of time in each period. Households spend all of their time in working, since leisure is not valued.
- Households can hold capital $k \geq \phi$ which yields return r_t in period t. ϕ is the borrowing constraint.
- A household's labor income in period t is $w_t s_t$ where w_t is the wage for efficiency unit of labor. s_t is idiosyncratic labor productivity shock following a Markov chain (S, Π). $S = \{s_1 < s_2 < ... < s_n\}$, each one is a realization of labor productivity. An element of Π, $\pi_{ii'}$ represents the transition probability $\pi_{ii'} = prob(s_{t+1} = s_{i'} | s_t = s_i)$. s_t of each household is independent of others' s_t.
- Recursive formulation of household problem

$$V(k, s) = max_{c,k'} \ u(c) + \beta \sum_{s'} \pi(s'|s) V(k', s')$$

subject to

$$c + k' = ws + (1 + r)k$$

$$c \geq 0$$

$$k' \geq \phi$$

1.2 Firm

There is a representative firm which has access to a CRS technology

$$Y_t = Z K_t^{\alpha} N_t^{1-\alpha}$$

where K_t is capital input and N_t is labor input. Firm rents inputs in competitive markets. Capital depreciates at a constant rate δ. Hence, the firm FOCs are

$$w_t = (1-\alpha)ZK_t^{\alpha}N_t^{-\alpha}$$
$$r_t = \alpha ZK_t^{\alpha-1}N_t^{1-\alpha} - \delta$$

1.3 Define Recursive Competitive Equilibrium

A recursive competitive equilibrium consists of prices $r(\mu)$, $w(\mu)$, value function $V(k,s,\mu)$, policy functions $g_k(k,s,\mu)$ and $g_c(k,s,\mu)$, type distribution of households $\mu(k,s)$, and aggregate capital $K(\mu)$ and aggregate labor supply $L(\mu)$, such that

- Household optimization:
  Given prices, the value function $V(k,s,\mu)$ is a solution to the household's optimization problem, and $g_k(k,s,\mu)$, $g_c(k,s,\mu)$ are associated policy functions.

- Firm optimization:

$$w = (1-\alpha)ZK^{\alpha}N^{-\alpha}$$
$$r = \alpha ZK^{\alpha-1}N^{1-\alpha} - \delta$$

- Law of motion for μ

$$\mu'(k',s') = \sum_k \sum_s \mu(k,s)\pi(s'|s)\mathbf{1}_{k'=g_k(k,s)}$$

- Markets clear

$$\text{Goods: } Y = \int c d\mu + K' - (1-\delta)K$$
$$\text{Labor: } N = \int s d\mu$$
$$\text{Capital: } K = \int k d\mu$$

2 Algorithm

Only the stationary equilibrium is studied.

1. Guess aggregate K^d
2. Given K^d, solve for r and w using firm FOCs.

   Notice that the aggregate labor supply can be computed exogenously in the stationary equilibrium. If we store the stationary distribution regarding the idiosyncratic shocks as $\{P_i\}_{i=1}^n$, the aggregate labor supply in the stationary distribution can be computed as

$$N = \sum_{i=1}^{n} s_i P_i$$

   Hence r and w are functions of K.

3. Given prices, solve household problem using value function iteration to get policy function for k', $g_k(k, s)$
4. Compute the stationary distribution μ
   (a) Make an initial guess at the stationary distribution μ^0. A convenient one is uniform distribution. Hence $\mu^0(k, s) = \frac{1}{n_k \times n_s}$ for all (k, s) on the grid.

   (b) Given initial guess μ^0, compute $\mu^1(k, s) = T^* \mu^0(k, s)$ for all (k, s) on the grid. Given the decision rule obtained in the last step, $g_k(k, s)$, and setting $\mu^1(k, s) = 0$ at the start of each new iteration before looping through all k, s, s', μ^1 can be computed by accumulation. That is, for each triple (k, s, s')

$$\mu^1(g_k(k, s), s') = \mu^1(g_k(k, s), s') + \mu^0(k, s)\pi(s'|s)$$

   Note that the expression above is pseudo-code. The μ^1 term on the RHS represents the value up until the last point in the loop through all (k, s, s'), while μ^1 term on the LHS is the new value which includes this (k, s, s').

   (c) Compute the sup-norm metric $sup_{(k,s)} |\mu^0(k, s) - \mu^1(k, s)|$

   (d) If the convergence metric from Step-c is within tolerance, exit the loop and set $\mu^1 = \mu^*$. Otherwise, set $\mu^0 = \mu^1$ and repeat Step-b and c.

5. With the policy function obtained in Step-3 and the stationary distribution obtained in Step-4, compute the aggregate capital supply

$$K^s = \int k d\mu$$

6. Define excess capital demand as $\Phi = K^d - K^s$

7. K^s is computed using policy function $g_k(k,s)$ and stationary distribution $\mu(k,s)$, which are computed given prices. And since the prices are functions of our guess for aggregate capital K^d, hence K^s depends on K^d. Therefore, $\Phi = K^d - K^s$ is a function of K^d. Then use 'fsolve' to get the the solution for K^d

A summary of the functions I used to compute the stationary equilibrium of Aiyagari model