

# Online supplement of the paper: An efficient branch-and-cut approach for large-scale competitive facility location problems with limited choice rule

Wei-Kun Chen<sup>a</sup>, Wei-Yang Zhang<sup>a</sup>, Yan-Ru Wang<sup>a</sup>, Shahin Gelareh<sup>d</sup>, Yu-Hong Dai<sup>b,c</sup>

<sup>a</sup>School of Mathematics and Statistics/Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, Beijing 100081, China

<sup>b</sup>Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

<sup>c</sup>School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

<sup>d</sup>Département R&T, IUT de Béthune, Université d'Artois, Béthune F-62000, France

In this supplement, we first present computational results to demonstrate the performance of using the greedy algorithm for solving the competitive facility location problem with limited choice rule (CFLPLCR) in Appendix A. Then, we provide the proofs of Proposition 3.6, Theorems 3.8 and 4.7 of Chen et al. (2025) in Appendices B to D, respectively. Subsequently, we present computational results to demonstrate the computational efficiency of the proposed dynamic programming algorithm over the mixed integer linear programming (MILP) based approach for computing the lifted submodular inequalities in Appendix E, compare the performance of the branch-and-cut (B&C) and cutting plane (CP) approaches for solving the CFLPLCR based on the Benders inequalities or lifted submodular inequalities in Appendix F, and perform a sensitivity analysis of the CFLPLCR under different competitive environments and customers' patronizing behaviors to derive some managerial insights in Appendix G. Finally, we consider the competitive facility location problem (CFLP) with a joint variant of limited choice rule (CFLPJLCR), and propose an iterative heuristic algorithm, based on solving CFLPLCRs with different parameters, for finding a high-quality solution for CFLPJLCRs in Appendix H.

## A. Quality of solutions returned by the greedy algorithm

As shown in Section 3.1 of Chen et al. (2025), the objective function of problem

$$\max_{S \subseteq [n]} \Psi(S) = \sum_{i \in [m]} b_i \Phi_i(S) - \sum_{j \in S} f_j, \quad (\text{CFLPLCR})$$

is submodular. Thus we can apply the greedy algorithm to solve (CFLPLCR), which usually identifies a high-quality solution for the submodular maximization problem (Nemhauser & Wolsey, 1988). The greedy algorithm starts with no facilities open. In each iteration, the greedy algorithm adds to the incumbent

solution  $\mathcal{S}^*$  the facility  $j \notin \mathcal{S}^*$  whose incremental value  $\rho_j(\mathcal{S}^*) = \Psi(\mathcal{S}^* \cup j) - \Psi(\mathcal{S}^*)$  is as large as possible, and is positive. If no such facility exists, the algorithm stops with the set  $\mathcal{S}^*$  of open facilities.

To evaluate the performance of the greedy algorithm, we compare the solution returned by the greedy algorithm (denoted as Greedy) with that returned by B&C+LSI (which is an exact approach). Tables 1 and 2 summarize the performance results of B&C+LSI and Greedy on instances in testsets T1 and T2. For each setting, we report the total CPU time in seconds (T) and the objective value of the returned solution ( $v$ ). Similar to other tables in Chen et al. (2025), for instances that cannot be solved to optimality by B&C+LSI within the given time limit, we report under column T (G%) the percentage optimality gap (G%) computed as  $\frac{\text{UB} - \text{LB}}{\text{UB}} \times 100\%$ , where UB and LB denote the upper bound and the lower bound obtained at the end of the time limit. To intuitively evaluate the quality of the solution returned by the greedy algorithm, we also report the optimality relative gap ( $G_{\text{greedy}}\%$ ) computed as  $\frac{v_{\text{LSI}} - v_{\text{Greedy}}}{v_{\text{LSI}}} \times 100\%$ , where  $v_{\text{LSI}}$  and  $v_{\text{Greedy}}$  are the objective values of the solutions returned by B&C+LSI and Greedy, respectively. The smaller the  $G_{\text{greedy}}\%$ , the higher the quality of the solution returned by the greedy algorithm.

From Tables 1 and 2, we observe that the gap between the feasible solution returned by the greedy algorithm and the (optimal) solution returned by B&C+LSI is not large, generally smaller than 2.2%. This shows that thanks to the submodularity of the objective function established in the paper, the greedy algorithm is indeed able to find a high-quality solution for the CFLPLCR.

## B. Proof of Proposition 3.6

*Proof.* It suffices to consider the case  $|\mathcal{S}| \geq \gamma + 1$ . In this case, it is easy to see

$$\begin{aligned}\Phi(\mathcal{S}) &= \Phi(\mathcal{S}^\gamma), \\ \rho_j(\mathcal{S}) &= \rho_j(\mathcal{S}^\gamma), \quad \forall j \in [n] \setminus \mathcal{S}, \\ \rho_j([n] \setminus j) &= 0, \quad \rho_j(\mathcal{S}^\gamma) = 0, \quad \forall j \in \mathcal{S} \setminus \mathcal{S}^\gamma.\end{aligned}$$

Table 1: Performance comparison of B&amp;C+LSI and Greedy on the instances in testset T1.

$m$	$n$	$\gamma$	B&C+LSI		Greedy		$G_{\text{greedy}}\%$
			T (G%)	$\nu$	T	$\nu$	
800	100	1	0.5	264362	0.1	260384	1.5
		2	9.7	264939	0.2	263906	0.4
		3	182.1	263928	0.3	263078	0.3
		NH	26.5	244915	0.2	242384	1.0
	200	1	0.7	227512	0.5	225316	1.0
		2	37.1	223636	0.6	222172	0.7
		3	2792.8	222457	0.9	221170	0.6
		NH	1212.3	204104	0.8	202501	0.8
	300	1	1.0	198515	1.2	197373	0.6
		2	89.6	193557	1.5	192409	0.6
		3	(0.4)	191211	1.9	189721	0.8
		NH	4410.7	175997	1.8	172326	2.1
400	100	1	1.4	188005	2.2	186076	1.0
		2	77.7	181431	3.0	179179	1.2
		3	(0.3)	177827	4.1	175332	1.4
		NH	726.3	158857	3.4	157263	1.0
	200	1	1.5	279108	0.7	276236	1.0
		2	97.1	277575	1.0	274374	1.2
		3	(0.2)	276919	1.6	274617	0.8
		NH	31.9	269590	1.1	267325	0.8
	300	1	1.5	250267	1.6	246929	1.3
		2	66.1	246941	2.1	243555	1.4
		3	(0.3)	245607	4.0	242939	1.1
		NH	1304.8	242150	2.5	239607	1.1
1000	200	1	2.1	227145	3.4	222778	1.9
		2	503.6	223587	4.1	221154	1.1
		3	(0.9)	220298	5.5	218542	0.8
		NH	448.8	222849	5.3	221051	0.8
	Sol.		27		32		
	Ave.		1504.5		1.8		1.0

Table 2: Performance comparison of B&amp;C+LSI and Greedy on the instances in testset T2.

$m$	$n$	$\gamma$	B&C+LSI		Greedy		$G_{\text{greedy}}\%$
			T (G%)	$\nu$	T	$\nu$	
1500	100	1	1.5	522641	0.3	520811	0.4
	200	1	2.3	459323	1.4	456504	0.6
	300	1	2.8	417261	3.8	409915	1.8
	500	1	4.7	382618	11.3	374388	2.2
	1000	1	8.0	302819	45.6	296146	2.2
	2000	1	10.1	215965	175.2	211614	2.0
2000	100	1	1.9	714512	0.4	710440	0.6
	200	1	3.8	645031	1.8	638181	1.1
	300	1	5.8	593787	4.8	586321	1.3
	500	1	9.3	532907	13.6	525691	1.4
	1000	1	14.8	433099	54.3	427195	1.4
	2000	1	20.3	321601	212.1	316829	1.5
3000	100	1	2.7	1102447	0.9	1100397	0.2
	200	1	7.7	1014199	3.6	1006286	0.8
	300	1	11.0	953657	7.8	943360	1.1
	500	1	21.8	877514	20.1	866912	1.2
	1000	1	38.3	737911	82.1	728236	1.3
	2000	1	58.2	578773	328.1	568129	1.8
5000	100	1	5.5	1904721	1.4	1903646	0.1
	200	1	11.3	1806572	6.2	1796071	0.6
	300	1	20.1	1727602	14.0	1720855	0.4
	500	1	38.2	1609712	34.7	1600347	0.6
	1000	1	157.8	1406478	145.5	1393331	0.9
	2000	1	171.8	1152049	576.3	1132166	1.7
10000	100	1	13.3	3987629	3.3	3985998	0.0
	200	1	32.6	3845691	16.2	3839134	0.2
	300	1	48.4	3749289	28.9	3739645	0.3
	500	1	124.2	3574934	109.5	3559426	0.4
	1000	1	518.0	3242958	368.4	3215581	0.8
	2000	1	2071.5	2819264	1331.1	2780255	1.4
Sol.			30		30		
Ave.			114.6		120.1		1.0

Thus,

$$\begin{aligned}
& \Phi(\mathcal{S}) + \sum_{j \in [n] \setminus \mathcal{S}} \rho_j(\mathcal{S})x_j - \sum_{j \in \mathcal{S}} \rho_j([n] \setminus j)(1 - x_j) \\
&= \Phi(\mathcal{S}) + \sum_{j \in [n] \setminus \mathcal{S}} \rho_j(\mathcal{S})x_j - \sum_{j \in \mathcal{S}^\gamma} \rho_j([n] \setminus j)(1 - x_j) - \underbrace{\sum_{j \in \mathcal{S} \setminus \mathcal{S}^\gamma} \rho_j([n] \setminus j)(1 - x_j)}_{=0} \\
&= \Phi(\mathcal{S}) + \sum_{j \in [n] \setminus \mathcal{S}^\gamma} \rho_j(\mathcal{S}^\gamma)x_j - \underbrace{\sum_{j \in \mathcal{S} \setminus \mathcal{S}^\gamma} \rho_j(\mathcal{S}^\gamma)x_j}_{=0} - \sum_{j \in \mathcal{S}^\gamma} \rho_j([n] \setminus j)(1 - x_j) \\
&= \Phi(\mathcal{S}^\gamma) + \sum_{j \in [n] \setminus \mathcal{S}^\gamma} \rho_j(\mathcal{S}^\gamma)x_j - \sum_{j \in \mathcal{S}^\gamma} \rho_j([n] \setminus j)(1 - x_j).
\end{aligned}$$

This shows statement (i).

Similarly, it is simple to check

$$\begin{aligned}
\Phi(\mathcal{S}) &= \Phi(\bar{\mathcal{S}}^\gamma), \\
\rho_j(\mathcal{S} \setminus j) &= \rho_j(\bar{\mathcal{S}}^\gamma \setminus j), \quad \forall j \in \mathcal{S}, \\
\rho_j(\bar{\mathcal{S}}^\gamma \setminus j) &= 0, \quad \forall j \in \bar{\mathcal{S}}^\gamma \setminus \mathcal{S}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \Phi(\bar{\mathcal{S}}^\gamma) - \sum_{j \in \bar{\mathcal{S}}^\gamma} \rho_j(\bar{\mathcal{S}}^\gamma \setminus j)(1 - x_j) + \sum_{j \in [n] \setminus \bar{\mathcal{S}}^\gamma} \rho_j(\emptyset)x_j \\
&= \Phi(\bar{\mathcal{S}}^\gamma) - \sum_{j \in \mathcal{S}} \rho_j(\bar{\mathcal{S}}^\gamma \setminus j)(1 - x_j) - \underbrace{\sum_{j \in \bar{\mathcal{S}}^\gamma \setminus \mathcal{S}} \rho_j(\bar{\mathcal{S}}^\gamma \setminus j)(1 - x_j)}_{=0} + \sum_{j \in [n] \setminus \bar{\mathcal{S}}^\gamma} \rho_j(\emptyset)x_j \\
&= \Phi(\mathcal{S}) - \sum_{j \in \mathcal{S}} \rho_j(\mathcal{S} \setminus j)(1 - x_j) + \sum_{j \in [n] \setminus \bar{\mathcal{S}}^\gamma} \rho_j(\emptyset)x_j \\
&\leq \Phi(\mathcal{S}) - \sum_{j \in \mathcal{S}} \rho_j(\mathcal{S} \setminus j)(1 - x_j) + \sum_{j \in [n] \setminus \mathcal{S}} \rho_j(\emptyset)x_j,
\end{aligned}$$

□

## C. Proof of Theorem 3.8

To prove Theorem 3.8, we need the following lemma.

**Lemma C.1.** Let  $f_\ell(x)$  be the right-hand sides of inequalities (24), i.e.,

$$f_\ell(x) = \frac{u_{\ell+1}}{u_{\ell+1} + u_0} + \sum_{j=1}^n \left( \frac{u_j}{u_j + u_0} - \frac{u_{\ell+1}}{u_{\ell+1} + u_0} \right)^+ x_j = \frac{u_{\ell+1}}{u_{\ell+1} + u_0} + \sum_{j=1}^\ell \left( \frac{u_j}{u_j + u_0} - \frac{u_{\ell+1}}{u_{\ell+1} + u_0} \right) x_j, \quad \forall \ell \in [n]. \quad (\text{C.1})$$

For any  $x^* \in [0, 1]^n$ , it follows

$$f_{k(x^*)}(x^*) = \min \{f_\ell(x^*) : \ell \in [n]\}, \quad (\text{C.2})$$

where

$$k(x^*) = \begin{cases} 1, & \text{if } x_1^* = 1; \\ \max \{\ell \in [n] : \sum_{j=1}^\ell x_j^* < 1\}, & \text{otherwise.} \end{cases} \quad (\text{C.3})$$

*Proof.* Given any  $\ell \in \{2, 3, \dots, n\}$ , it follows

$$\begin{aligned} f_\ell(x^*) - f_{\ell-1}(x^*) &= \frac{u_{\ell+1}}{u_{\ell+1} + u_0} + \sum_{j=1}^\ell \left( \frac{u_j}{u_j + u_0} - \frac{u_{\ell+1}}{u_{\ell+1} + u_0} \right) x_j^* - \frac{u_\ell}{u_\ell + u_0} - \sum_{j=1}^{\ell-1} \left( \frac{u_j}{u_j + u_0} - \frac{u_\ell}{u_\ell + u_0} \right) x_j^* \\ &= \left( \frac{u_\ell}{u_\ell + u_0} - \frac{u_{\ell+1}}{u_{\ell+1} + u_0} \right) \cdot \left( \sum_{j=1}^\ell x_j^* - 1 \right). \end{aligned}$$

Together with the definition of  $k(x^*)$  in (C.3), we have  $f_1(x^*) \geq f_2(x^*) \geq \dots \geq f_{k(x^*)}(x^*)$  and  $f_{k(x^*)}(x^*) \leq f_{k(x^*)+1}(x^*) \leq \dots \leq f_n(x^*)$ . Thus equation (C.2) holds true.  $\square$

*Proof of Theorem 3.8.* It suffices to show that every extreme point  $(w^*, x^*)$  of  $\mathcal{P} := \{(w, x) \in \mathbb{R} \times [0, 1]^n : w \leq f_\ell(x), \forall \ell \in [n]\}$  is integral. Suppose, otherwise, that  $(w^*, x^*)$  is a fractional extreme point of  $\mathcal{P}$ , i.e.,  $x_j^* \in (0, 1)$  for some  $j \in [n]$ . We have the following two cases.

(i)  $x_t^* \in (0, 1)$  holds for some  $t \in \{k(x^*) + 1, k(x^*) + 2, \dots, n\}$ . Let  $(w^1, x^1) = (w^*, x^* + \delta e_t)$  and  $(w^2, x^2) = (w^*, x^* - \delta e_t)$ , where  $\delta > 0$  is a sufficiently small value. By the definition of  $f$  in (C.1), we have  $f_{k(x^*)}(x^*) = f_{k(x^*)}(x^1)$ . From  $t \geq k(x^*) + 1$  and the definitions of  $x^1$  and  $k(x^*)$  in (C.3), it follows  $k(x^*) = k(x^1)$ . Thus,  $w^1 = w^* \leq f_{k(x^*)}(x^*) = f_{k(x^*)}(x^1) = f_{k(x^1)}(x^1)$ . From Lemma C.1, this implies  $w^1 \leq f_\ell(x^1)$  for all  $\ell \in [n]$ , and therefore  $(w^1, x^1) \in \mathcal{P}$ . The proof of  $(w^2, x^2) \in \mathcal{P}$  is similar. From  $(w^*, x^*) = \frac{1}{2}(w^1, x^1) + \frac{1}{2}(w^2, x^2)$  and  $\delta > 0$ ,  $(w^*, x^*)$  cannot be an extreme point of  $\mathcal{P}$ , a contradiction.

(ii)  $x_t^* \in (0, 1)$  holds for some  $t \in [k(x^*)]$  and  $x_\ell^* \in \{0, 1\}$  holds for all  $\ell \in \{k(x^*) + 1, k(x^*) + 2, \dots, n\}$ .

Then, from the definition of  $k(x^*)$  in (C.3), this implies

$$x_{k(x^*)+1}^* = 1, \quad x_1^* \neq 1, \quad \text{and} \quad \sum_{j=1}^{k(x^*)} x_j^* < 1. \quad (\text{C.4})$$

Let  $(w^3, x^3) = (w^* + \zeta, x^* + \delta e_t)$  and  $(w^4, x^4) = (w^* - \zeta, x^* - \delta e_t)$ , where  $\delta > 0$  is a sufficiently small value and  $\zeta = (\frac{u_t}{u_t + u_0} - \frac{u_{k(x^*)+1}}{u_{k(x^*)+1} + u_0}) \cdot \delta$ . As  $\delta$  is sufficiently small, by (C.4), we have  $k(x^3) = k(x^4) = k(x^*)$ , and thus

$$f_{k(x^3)}(x^3) = f_{k(x^*)}(x^3) = f_{k(x^*)}(x^*) + \left( \frac{u_t}{u_t + u_0} - \frac{u_{k(x^*)+1}}{u_{k(x^*)+1} + u_0} \right) \cdot \delta = f_{k(x^*)}(x^*) + \zeta, \quad (\text{C.5})$$

$$f_{k(x^4)}(x^4) = f_{k(x^*)}(x^4) = f_{k(x^*)}(x^*) - \left( \frac{u_t}{u_t + u_0} - \frac{u_{k(x^*)+1}}{u_{k(x^*)+1} + u_0} \right) \cdot \delta = f_{k(x^*)}(x^*) - \zeta. \quad (\text{C.6})$$

From  $w^3 = w^* + \zeta$ ,  $w^4 = w^* - \zeta$ ,  $w^* \leq f_{k(x^*)}(x^*)$ , (C.5), and (C.6), it follows  $w^3 \leq f_{k(x^3)}(x^3)$  and  $w^4 \leq f_{k(x^4)}(x^4)$ . Therefore, using Lemma C.1, this implies  $w^3 \leq f_\ell(x^3)$  and  $w^4 \leq f_\ell(x^4)$  for all  $\ell \in [n]$ , and thus  $(w^3, x^3), (w^4, x^4) \in \mathcal{P}$ . From  $(w^*, x^*) = \frac{1}{2}(w^3, x^3) + \frac{1}{2}(w^4, x^4)$  and  $\delta > 0$ ,  $(w^*, x^*)$  cannot be an extreme point of  $\mathcal{P}$ , a contradiction.  $\square$

## D. Proof of Theorem 4.7

We prove that problem (43) is as hard as the partition problem, which is NP-complete (Garey & Johnson, 1978). First, we introduce the partition problem: given a finite set  $\mathcal{N} = \{1, 2, \dots, p\}$  of  $p$  elements and a size  $a_i \in \mathbb{Z}_+$  for the  $i$ -th element with  $\sum_{i \in \mathcal{N}} a_i = 2b$ , does there exist a partition  $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$  with  $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$  such that  $\sum_{i \in \mathcal{N}_1} a_i = \sum_{i \in \mathcal{N}_2} a_i = b$ ? Without loss of generality, we assume  $b > 0$ .

Given any instance of a partition problem, we construct an instance of problem (43) by setting  $q = \gamma = p$ ,  $u_0 = b$ ,  $u_j = a_j$ , and  $\alpha_j = a_j/4b$  for  $j \in [p]$ . As  $q = \gamma = p$ , the cardinality constraint  $\sum_{j=1}^q x_j + \sum_{j=q+1}^p y_j \leq \gamma$  in problem (43) reduces to  $\sum_{j=1}^p x_j \leq p$  and thus is redundant. Therefore, problem (43) reduces to

$$\nu := \max \left\{ \frac{\sum_{j=1}^p a_j x_j}{\sum_{j=1}^p a_j x_j + b} - \frac{1}{4b} \sum_{j=1}^p a_j x_j : x \in \{0, 1\}^p \right\}. \quad (\text{D.1})$$

Letting  $f(z) = \frac{z}{z+b} - \frac{z}{4b}$  where  $z \in [0, 2b]$ , it is simple to check that  $f(z) \leq 1/4$  where the only maximum point is arrived at  $z = b$ . Using this fact,  $\nu = 1/4$  holds if and only if  $\sum_{j=1}^p a_j x_j = b$  holds for some  $x \in \{0, 1\}^p$ , which is further equivalent to the existence of a partition  $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$  with  $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$  and  $\sum_{i \in \mathcal{N}_1} a_i = \sum_{i \in \mathcal{N}_2} a_i = b$ , that is, the answer to the partition problem is yes. Since the above transformation can be done in polynomial time and the partition problem is NP-complete, we conclude that problem (D.1) is NP-hard.

## E. Computational efficiency of the proposed the dynamic programming algorithm over the MILP based approach for computing lifted submodular inequalities

In this section, we first present the MILP reformulation for the lifting problem (43) and then demonstrate the computational efficiency of the proposed dynamic programming algorithm in Section 4.3 of Chen et al. (2025) over the MILP based approach for computing the lifted submodular inequalities in the B&C context.

### E.1. An MILP reformulation for the lifting problem (43)

Let us first present an MILP reformulation, adapted from Haase (2009), for the lifting problem (43). To proceed, we note that variables  $\{y_j\}_{j \in [p] \setminus [q]}$  in problem (43) can be equivalently stated as binary variables which enables us to rewrite problem (43) as

$$\max_x \left\{ \frac{\sum_{j=1}^p u_j x_j}{\sum_{j=1}^p u_j x_j + u_0} - \sum_{j=1}^q \alpha_j x_j : \sum_{j=1}^p x_j \leq \gamma, x \in \{0, 1\}^p \right\}, \quad (\text{E.1})$$

where  $u_j \geq 0$  for  $j \in [p]$  and  $\alpha_j > 0$  for  $j \in [q]$ . Without loss of generality, we assume that  $u_j > 0$  for  $j \in [p]$ , since otherwise we can set  $x_j = 0$  for  $j \in [p]$  with  $u_j = 0$  without changing its optimal value. Note that problem (E.1) is a variant of the maximum capture facility location problem with random utilities (MCFLRU) where only a single customer exists and the company can locate at most  $\gamma$  new facilities. In the literature, there are various MILP reformulations for the MCFLRU; see (Benati & Hansen, 2002; Haase, 2009; Aros-Vera et al., 2013; Zhang et al., 2012; Freire et al., 2016). Here, we derive an MILP reformulation for the lifting problem (E.1) based on the one proposed by Haase (2009), which usually runs efficiently in practice (Haase & Müller, 2014).

Let  $\{\omega_j\}_{j \in \{0\} \cup [p]}$  denote the probabilities of the customer being attracted by the outside options and new open facilities, i.e.,  $\omega_0 := \frac{u_0}{\sum_{j=1}^p u_j x_j + u_0}$  and  $\omega_j := \frac{u_j x_j}{\sum_{j'=1}^p u_{j'} x_{j'} + u_0}$  for  $j \in [p]$ . Then problem (E.1) can be reformulated as an MILP problem:

$$\max_{x, \omega} \left\{ \sum_{j=1}^p \omega_j - \sum_{j=1}^q \alpha_j x_j : \frac{\omega_j}{u_j} \leq \frac{\omega_0}{u_0}, 0 \leq \omega_j \leq \frac{u_j}{u_j + u_0} x_j, \forall j \in [p], \sum_{j=0}^p \omega_j = 1, \sum_{j=1}^p x_j \leq \gamma, x \in \{0, 1\}^p \right\}. \quad (\text{E.2})$$

The following proposition demonstrates the equivalence between problems (E.1) and (E.2).

**Proposition E.1.** *Problems (E.1) and (E.2) are equivalent in terms of sharing the same optimal value.*

*Proof.* Let  $v_1$  and  $v_2$  be the optimal values of problems (E.1) and (E.2), respectively. We first show that  $v_1 \leq v_2$ . Letting  $\hat{x}$  be an optimal solution of problem (E.1), we define  $\hat{\omega} \in \mathbb{R}^{p+1}$  as follows:  $\hat{\omega}_0 = \frac{u_0}{\sum_{j=1}^p u_j \hat{x}_j + u_0}$  and  $\hat{\omega}_j = \frac{u_j \hat{x}_j}{\sum_{j'=1}^p u_{j'} \hat{x}_{j'} + u_0}$  for  $j \in [p]$ . From the definition, it is simple to check that  $(\hat{x}, \hat{\omega})$  is a feasible solution of problem (E.2) and has the same objective value as that of problem (E.1) at  $\hat{x}$ . Thus, we have

$$v_1 = \sum_{j=1}^p \hat{\omega}_j - \sum_{j=1}^q \alpha_j \hat{x}_j \leq v_2.$$

In order to show that  $v_2 \leq v_1$ , we first prove that, given an optimal solution  $(x^*, \omega^*)$  of problem (E.2),  $\frac{\omega_j^*}{u_j} = \frac{\omega_0^*}{u_0}$  must hold for all  $j \in [p]$  with  $x_j^* = 1$ . Suppose, otherwise, there exists  $t \in [p]$  such that  $x_t^* = 1$  and  $\frac{\omega_t^*}{u_t} < \frac{\omega_0^*}{u_0}$ . Then it must follow that  $\omega_0^* > 0$ . We define a solution  $(\bar{x}, \bar{\omega})$  as  $\bar{x} = x^*$  and

$$\bar{\omega}_j = \begin{cases} \omega_j^* + \epsilon u_j \sum_{j' \in \{0\} \cup J} u_{j'}, & \text{if } j = t; \\ \omega_j^* - \epsilon u_j u_t, & \text{if } j \in \{0\} \cup J; \quad \forall j \in \{0\} \cup [p], \\ \omega_j^*, & \text{otherwise,} \end{cases}$$

where  $\epsilon > 0$  is a sufficiently small value and  $J = \left\{ j \in [p] : \frac{\omega_j^*}{u_j} = \frac{\omega_0^*}{u_0} \text{ and } x_j^* = 1 \right\}$ . From the definition of  $(\bar{x}, \bar{\omega})$  and the fact that  $\epsilon > 0$  is sufficiently small, it is simple to check (i)  $\bar{\omega}_j \geq 0$  for  $j \in \{0\} \cup [p]$ , (ii)  $\frac{\bar{\omega}_j}{u_j} \leq \frac{\omega_0^*}{u_0}$  for  $j \in [p]$ , and (iii)  $\sum_{j=0}^p \bar{\omega}_j = \sum_{j=0}^p \omega_j^* - \sum_{j \in \{0\} \cup J} \epsilon u_j u_t + \epsilon u_t \sum_{j \in \{0\} \cup J} u_j = \sum_{j=0}^p \omega_j^* = 1$ . Moreover, it follows that

$$\frac{u_t}{u_t + u_0} \bar{x}_t \stackrel{(a)}{=} \frac{u_t}{u_t + u_0} \stackrel{(b)}{\geq} \frac{u_t}{u_t + u_0} \cdot (\bar{\omega}_t + \bar{\omega}_0) \stackrel{(c)}{>} \frac{u_t}{u_t + u_0} \cdot \left( \frac{\bar{\omega}_t}{u_t} u_t + \frac{\bar{\omega}_0}{u_t} u_0 \right) = \bar{\omega}_t,$$

where (a) follows from  $\bar{x}_t = x_t^* = 1$ , (b) follows from  $\bar{\omega}_t + \bar{\omega}_0 \leq 1$  (since  $\sum_{j=0}^p \bar{\omega}_j = 1$  and  $\bar{\omega}_j \geq 0$  for  $j \in \{0\} \cup [p]$ ), and (c) holds as  $\frac{\bar{\omega}_t}{u_t} < \frac{\omega_0^*}{u_0}$ . Hence,  $(\bar{x}, \bar{\omega})$  is a feasible solution of problem (E.2). However, since  $\sum_{j=1}^p \bar{\omega}_j - \sum_{j=1}^q \alpha_j \bar{x}_j = 1 - \bar{\omega}_0 - \sum_{j=1}^q \alpha_j \bar{x}_j = 1 - \omega_0^* + \epsilon u_0 u_t - \sum_{j=1}^q \alpha_j x_j^* > 1 - \omega_0^* - \sum_{j=1}^q \alpha_j x_j^*$ , the objective value of point  $(\bar{x}, \bar{\omega})$  is strictly greater than that of point  $(x^*, \omega^*)$ . Thus,  $(x^*, \omega^*)$  cannot be an optimal solution of problem (E.2), a contradiction.

Note that  $x^*$  is a feasible solution of problem (E.1) with an objective value of  $\frac{\sum_{j=1}^p u_j x_j^*}{\sum_{j=1}^p u_j x_j^* + u_0} - \sum_{j=1}^q \alpha_j x_j^*$ . To complete the proof, it suffices to show that the objective value of the optimal solution  $(x^*, \omega^*)$  of problem (E.2) is also equal to  $\frac{\sum_{j=1}^p u_j x_j^*}{\sum_{j=1}^p u_j x_j^* + u_0} - \sum_{j=1}^q \alpha_j x_j^*$ . Indeed, for each  $j \in [p]$ , if  $x_j^* = 1$ , then  $\frac{\omega_j^*}{u_j} = \frac{\omega_0^*}{u_0}$  follows from the above discussion; if  $x_j^* = 0$ , then  $\omega_j^* = 0$  holds as  $0 \leq \omega_j^* \leq \frac{u_j}{u_j + u_0} x_j^*$ . In both cases, we have  $\omega_j^* = \frac{u_j}{u_0} \omega_0^* x_j^*$ . This, together with  $\sum_{j=0}^p \omega_j^* = 1$ , implies that  $\omega_0^* = \frac{u_0}{\sum_{j=1}^p u_j x_j^* + u_0}$  and  $\omega_j^* = \frac{u_j x_j^*}{\sum_{j'=1}^p u_{j'} x_{j'}^* + u_0}$  for  $j \in [p]$ . Thus, the objective value of problem (E.2) at point  $(x^*, \omega^*)$  is  $\sum_{j=1}^p \omega_j^* - \sum_{j=1}^q \alpha_j x_j^* = \frac{\sum_{j=1}^p u_j x_j^*}{\sum_{j=1}^p u_j x_j^* + u_0} - \sum_{j=1}^q \alpha_j x_j^*$ , yielding

the desired result.  $\square$

### *E.2. Comparison of the proposed dynamic programming algorithm with the MILP approach for computing the lifted submodular inequalities*

Next, we compare the performance of the proposed dynamic programming algorithm and the MILP approach for computing the lifted submodular inequalities in the B&C algorithm. To do this, we compare the proposed B&C algorithm in which the lifted submodular inequalities (40a)–(40b) are computed by using the proposed dynamic programming algorithm (denoted by B&C+LSI), with the one in which the lifted submodular inequalities (40a)–(40b) are computed by using the MILP based approach that solves the MILP reformulations (E.2) of the lifting problems (43) (denoted by B&C+LSI').

Table 3 summarizes the performance results of B&C+LSI and B&C+LSI' on instances in testset T1. For each setting, we report the total CPU time in seconds (T), the CPU time in seconds spent in separating the cuts (CT), the number of explored nodes (N), and the objective of the optimal solution or the best incumbent of the instance ( $v$ ). For instances that cannot be solved to optimality within the given time limit, we report under column T (G%) the percentage optimality gap (G%) computed as  $\frac{UB-LB}{UB} \times 100\%$ , where UB and LB denote the upper bound and the lower bound obtained at the end of the time limit. For setting B&C+LSI', we also report the average CPU time in seconds spent in solving a single MILP problem ( $T^{MILP}$ ).

From Table 3, we observe that although it does not spend too much computational effort in solving a single MILP problem (see column  $T^{MILP}$  under setting B&C+LSI'), the overall computational effort spent in using the MILP approach to compute the lifted submodular inequalities is significantly huge (see column CT under setting B&C+LSI'), making it unrealistic to embed the lifted submodular inequalities into the B&C algorithm to solve the CFLPLCR. This is due to the reason that B&C+LSI' requires to solve a huge number of MILP reformulations of lifting problems to compute the lifted submodular inequalities; indeed, to compute a single lifted submodular inequality (40a) (respectively, (40b)), it requires to solve  $|\mathcal{S}|$  (respectively,  $n - |\mathcal{S}|$ ) MILP reformulations of lifting problems (43). In sharp contrast, the overall computational effort spent in using the proposed dynamic programming algorithm to compute the lifted submodular inequalities is not large (see column CT under setting B&C+LSI), making it realistic to embed the lifted submodular inequalities into the B&C algorithm to improve the computational performance of solving the CFLPLCR.

Table 3: Performance comparison of B&amp;C+LSI and B&amp;C+LSI' on the instances in testset T1.

m	n	$\gamma$	B&C+LSI'					B&C+LSI				
			T (G%)	CT	N	$\nu$	T <sup>MILP</sup>	T (G%)	CT	N	$\nu$	
800	100	1	0.5	0.0	1	264362	<0.01	0.5	0.1	1	264362	
		2	6471.1	6466.2	477	264939	0.13	9.7	5.4	477	264939	
		3	(2.5)	7198.9	1	259446	0.26	182.1	11.1	5250	263928	
		NH	(2.4)	7199.1	1	239917	0.32	26.5	8.0	1636	244915	
		200	1	0.7	0.1	1	227512	<0.01	0.7	0.1	1	227512
	200	2	(1.9)	7198.3	1	220495	0.22	37.1	11.3	959	223636	
		3	(52.5)	7200.0	1	149683	0.54	2792.8	22.3	53408	222457	
		NH	(51.0)	7199.8	1	147649	0.67	1212.3	20.1	43382	204104	
		300	1	1.0	0.1	1	198515	<0.01	1.0	0.1	1	198515
		2	(19.2)	7199.7	1	179887	0.38	89.6	30.1	3193	193557	
400	300	3	(53.5)	7200.0	1	138708	0.83	(0.4)	45.9	61587	191211	
		NH	(52.5)	7200.0	1	134333	1.47	4410.7	44.0	124297	175997	
		200	1	1.6	0.3	1	188005	<0.01	1.4	0.2	1	188005
		2	(54.6)	7200.0	1	133552	0.62	77.7	29.6	5152	181431	
		3	(52.5)	7200.0	1	138353	1.21	(0.3)	55.9	101424	177827	
		NH	(55.7)	7200.0	1	120465	2.00	726.3	71.4	24259	158857	
	400	100	1	0.7	0.1	1	330340	<0.01	0.8	0.1	1	330340
		2	(0.7)	7200.0	1	329157	0.15	12.6	5.6	295	331100	
		3	(38.6)	7199.7	1	249831	0.38	90.7	10.3	1474	330659	
		NH	(1.5)	7198.8	1	311122	0.26	14.0	7.1	428	315119	
		200	1	1.5	0.1	1	279108	<0.01	1.5	0.1	1	279108
1000	200	2	(3.5)	7198.9	1	269450	0.23	97.1	15.4	2559	277575	
		3	(43.5)	7199.8	1	217846	0.51	(0.2)	36.0	58692	276919	
		NH	(47.3)	7199.9	1	198722	0.68	31.9	15.3	1678	269590	
		300	1	1.3	0.1	1	250267	<0.01	1.5	0.2	1	250267
		2	(17.6)	7199.9	1	232979	0.40	66.1	32.3	1222	246941	
	300	3	(51.2)	7200.0	1	182315	0.80	(0.3)	49.6	90387	245607	
		NH	(51.4)	7200.0	1	177808	1.23	1304.8	32.8	28759	242150	
		200	1	1.9	0.2	1	227145	<0.01	2.1	0.2	1	227145
		2	(49.5)	7200.0	1	183490	0.60	503.6	38.0	12529	223587	
		3	(49.5)	7200.0	1	182764	1.22	(0.9)	93.7	37044	220298	
Sol.	Ave.	NH	(50.2)	7200.0	1	177324	1.48	448.8	63.9	12479	222849	
		9						27				
		720.0			54			2.1		54		

## F. Comparison of the branch-and-cut and cutting plane approaches for solving the CFLPLCR

In this section, we compare the performance of the B&C and CP approaches for solving the CFLPLCR based on the lifted submodular inequalities (40a)–(40b) and the Benders inequalities of Lin & Tian (2021). To do this, we compare the following four settings:

- CP+BI: solving the CFLPLCR by the CP approach of the Benders reformulation of Lin & Tian (2021), where the Benders inequalities are generated in each iteration and the master problem is solved again as an MILP problem.
- CP+LSI: solving the CFLPLCR by the CP approach of formulation (40), where the lifted submodular inequalities (40a)–(40b) are generated in each iteration and the master problem is solved again as an MILP problem.
- B&C+BI: solving the CFLPLCR by the B&C approach of the Benders reformulation of Lin & Tian (2021), where a single enumeration tree is created and the exponential number of Benders inequalities are separated at the nodes of the search tree (which is equivalent to setting GBD).
- B&C+LSI: solving the CFLPLCR by the B&C approach of the proposed formulation (40), where a single enumeration tree is created and the exponential number of lifted submodular inequalities (40a)–(40b) are separated at the nodes of the search tree.

When implementing CP+BI and CP+LSI, we follow Mai & Lodi (2020) to use a multi-cut implementation where (at most) one inequality per customer will be added in each iteration<sup>1</sup>. Note that since the Benders inequalities are linear outer approximations of the nonlinear constraints, they can also be called outer approximation inequalities, and thus CP+BI can be treated as an extension of the multi-cut outer approximation algorithm of Mai & Lodi (2020) for solving the CFLPLCR.

Table 4 summarizes the computational results of CP+BI, CP+LSI, B&C+BI, and B&C+LSI on instances in testset T1. For each setting, we report the total CPU time in seconds (T) and the objective value of the optimal solution or the best incumbent of the instance ( $v$ ). For settings B&C+BI and B&C+LSI, we report under column T (G%) the percentage optimality gap (G%) for instances that cannot be solved to optimality within the given time limit, computed as  $\frac{UB-LB}{UB} \times 100\%$ , where UB and LB denote the upper bound and

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<sup>1</sup>If the number of customers is too large, Mai & Lodi (2020) also suggested to aggregate the cuts among several customers to avoid adding too many cuts to the master problem. However, we found that this approach is outperformed by the disaggregated version, where at most one cut for a customer is added in each iteration.

the lower bound obtained at the end of the time limit. For settings CP+BI and CP+LSI, we also report the number of iterations ( $It$ ) in the CP approaches (which is equal to the number of solved MILP problems) and use “–” under columns  $T$  and  $v$  to denote that the instance cannot be solved to optimality within the given time limit.

From Table 4, we observe that in the CP approaches, CP+LSI outperforms CP+BI, which is due to the strength of the proposed lifted submodular inequalities over the Benders inequalities. In addition, both CP approaches, CP+BI and CP+LSI, are outperformed by the B&C approaches B&C+BI and B&C+LSI. Indeed, among the 32 instances in testset T1, B&C+LSI and B&C+BI can solve 27 and 14 instances to optimality, whereas CP+LSI and CP+BI are only able to solve 11 and 8 of them; for instances that can be solved to optimality, the CPU times under settings B&C+LSI and B&C+BI are much smaller than those under settings CP+LSI and CP+BI, respectively. This can be explained by the reason that in each iteration of the CP approach, we need to explore a search tree to find an optimal solution for an MILP relaxation of the CFLPLCR (and considerable efforts are likely spent in revisiting candidate solutions that have been eliminated in previous search trees), while the B&C approach only requires to explore a single search tree (thereby avoiding duplicate efforts in exploring the same part of the search trees). As the number of iterations required to solve a CFLPLCR instance to optimality in the CP approach is relatively large (see columns  $It$  under settings CP+BI and CP+LSI), the CP approaches CP+LSI and CP+BI are thus outperformed by the B&C approaches B&C+LSI and B&C+BI, respectively.

## G. Sensitivity analysis

In this section, we perform a sensitivity analysis of the CFLPLCR to draw some managerial insights. In particular, we analyze the results of CFLPLCRs under two key parameters: the number of existing facilities of the competitors and the number of facilities within each company that will be considered by the customers. These two parameters reflect the different competitive environments and customers’ patronizing behaviors, respectively. The results will provide insights on the relations between the market competition, the customers’ patronizing behavior under the limited choice rule, the newcomer’s strategy for opening facilities in the market, and the net profit.

We conduct experiments on instances with 100 customers and facilities. We assume that there is a single competitor in the market, and the number of the competitor’s facilities, denoted as  $n_c$ , is taken from  $\{1, 2, \dots, 10\}$ . The maximum numbers of facilities within each company, that will be considered by the

Table 4: Performance comparison of CP+BI, CP+LSI, B&amp;C+BI, and B&amp;C+LSI on the instances in testset T1.

$m$	$n$	$\gamma$	CP+BI			CP+LSI			B&C+BI			B&C+LSI	
			T	$\nu$	It	T	$\nu$	It	T (%)	$\nu$	T (%)	$\nu$	
800	100	1	318.1	264362	10	97.8	264362	6	18.9	264362	0.5	264362	
		2	2816.6	264939	11	813.0	264939	7	291.4	264939	9.7	264939	
		3	—	—	9	6491.4	263928	10	329.6	263928	182.1	263928	
		NH	4080.9	244915	11	1354.1	244915	9	159.1	244915	26.5	244915	
200	200	1	3102.3	227512	12	400.9	227512	7	32.2	227512	0.7	227512	
		2	—	—	3	—	—	3	5577.4	223636	37.1	223636	
		3	—	—	3	—	—	3	(0.2)	222457	2792.8	222457	
		NH	—	—	3	—	—	2	(0.1)	204104	1212.3	204104	
300	300	1	—	—	2	—	—	2	193.6	198515	1.0	198515	
		2	—	—	2	—	—	2	(0.3)	193557	89.6	193557	
		3	—	—	2	—	—	2	(0.8)	191211	(0.4)	191211	
		NH	—	—	2	—	—	2	(0.6)	175995	4410.7	175997	
400	400	1	—	—	2	6451.3	188005	7	630.5	188005	1.4	188005	
		2	—	—	2	—	—	2	(0.5)	181431	77.7	181431	
		3	—	—	2	—	—	2	(0.9)	177827	(0.3)	177827	
		NH	—	—	2	—	—	2	(0.5)	158857	726.3	158857	
1000	100	1	576.6	330340	13	66.5	330340	7	22.1	330340	0.8	330340	
		2	2675.2	331100	11	1789.4	331100	8	126.7	331100	12.6	331100	
		3	—	—	3	5205.8	330659	9	1169.6	330659	90.7	330659	
		NH	2148.4	315119	11	526.4	315119	8	109.7	315119	14.0	315119	
200	200	1	3179.2	279108	15	319.4	279108	6	207.3	279108	1.5	279108	
		2	—	—	3	—	—	3	(0.3)	277575	97.1	277575	
		3	—	—	3	—	—	3	(0.5)	276788	(0.2)	276919	
		NH	—	—	3	—	—	3	(<0.1)	269590	31.9	269590	
300	300	1	—	—	3	—	—	3	606.2	250267	1.5	250267	
		2	—	—	2	—	—	2	(0.3)	246941	66.1	246941	
		3	—	—	2	—	—	2	(0.7)	245607	(0.3)	245607	
		NH	—	—	2	—	—	2	(0.3)	242150	1304.8	242150	
400	400	1	—	—	2	—	—	2	(0.2)	227145	2.1	227145	
		2	—	—	2	—	—	2	(1.1)	223582	503.6	223587	
		3	—	—	2	—	—	2	(1.5)	220121	(0.9)	220298	
		NH	—	—	2	—	—	2	(0.5)	222849	448.8	222849	
Sol.			8			11			14		27		
Ave.			2362.2		12	670.9		7	120.9		8.3		

customers, are set to the same value  $\gamma$ , taken from  $\{1, 2, 3, +\infty\}$ . Other parameters of the CFLPLCR are constructed using a similar procedure as in Section 6 of Chen et al. (2025). For each fixed  $\gamma$  and  $n_c$ , 100 instances are randomly constructed and the results reported below are averaged over these instances.

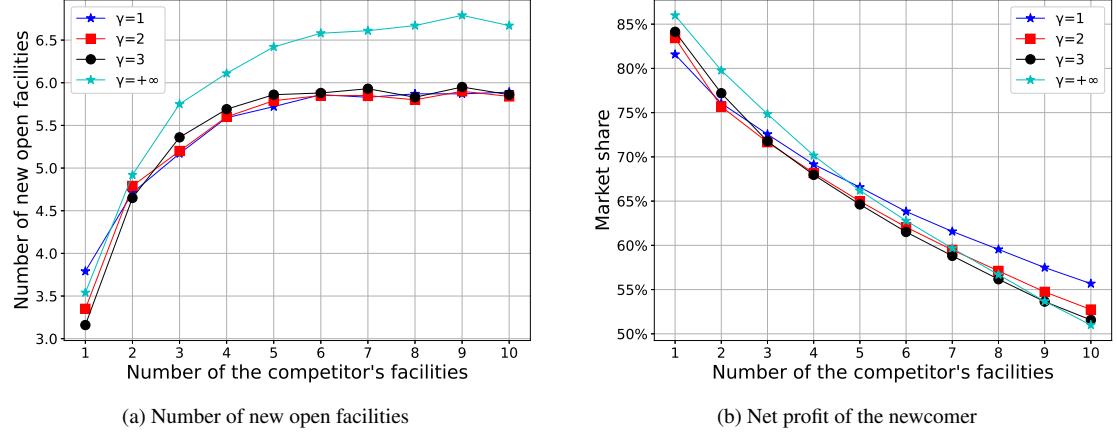


Figure 1: Computational results of the CFLPLCR under different numbers of the competitor's facilities and different  $\gamma$ .

Figures 1a and 1b report the numbers of new open facilities and the newcomer's net profit, respectively. First, we observe from Figures 1a and 1b that for all customers' patronizing behaviors (i.e., the limited choice rules with different  $\gamma$ ), when the competition is not intense (i.e.,  $n_c \leq 5$ ), the newcomer generally opens more facilities to capture the customers' buying power. Nevertheless, the net profit decreases with the increasing number of the competitor's facilities, which is due to the additional fixed costs of opening new facilities and the loss of market share captured by the competitor. When the competition is intense (i.e.,  $n_c \geq 6$ ), however, the newcomer generally does not open additional facilities with the increase in  $n_c$ , as opening more facilities to capture the market cannot compensate for the incurred fixed costs. With the increasing number of the competitor's facilities, although the net profit continues to decline, the marginal loss shows a tendency to diminish.

Second, we can observe from Figures 1a and 1b that when the competition is not intense (i.e.,  $n_c \leq 5$ ), the net profits of the newcomer under different customer patronizing behaviors are very similar. However, when the competition is intense (i.e.,  $n_c \geq 6$ ), the net profit of the newcomer tends to grow with the decrease in  $\gamma$ . This can be explained as follows. For a small  $\gamma$ , the number of facilities within a company considered by each customer is small; and if the competition is intense, it is easier for the newcomer to open some facilities (where the number is not large) to capture the demand of some "key" customers (i.e., those with a large weight), and as a result, the net profit is likely to be large. This result sheds a useful insight that in the

highly competitive environment, if the number of facilities within a company considered by the customers is small, the newcomer will achieve a better net profit.

## H. CFLPs with the joint limited choice rule

The considered (individual) limited choice rule of Lin & Tian (2021) assumes that customers first form their consideration set by choosing, for each individual company, no more than a predetermined number of open facilities in descending order of utilities, and then split their buying power among all chosen facilities in the consideration set proportional to their utilities. This rule is applicable in contexts where the services provided by facilities of the same company are “similar” while those provided by facilities of different companies are “different” (though their functionalities are identical); and related CFLPs have been widely investigated in the literature (Suárez-Vega et al., 2004; Biesinger et al., 2016; Fernández et al., 2017; Méndez-Vogel et al., 2023; Lin & Tian, 2021; García-Galán et al., 2024, 2025). When the services provided by facilities of different companies are “similar”, it would also be reasonable to consider a joint version of limited choice rule, in which customers form their consideration set by choosing no more than a predetermined number of open facilities from *those opened by all companies*, in descending order of utilities, and then split their buying power among all chosen facilities in the consideration set in proportion to their utilities.

In this section, we will consider the CFLP with the joint limited choice rule (CFLPJLCR) and investigate the solution approaches for it. To do this, we first present the mathematical formulations for the CFLPJLCR. Then, we develop a heuristic algorithm that finds a high-quality solution for the CFLPJLCR by solving a sequence of CFLPLCRs (that can be tackled by the proposed B&C algorithm in the paper). Finally, we conduct computational experiments to demonstrate the effectiveness and efficiency of the proposed heuristic algorithm for the CFLPJLCR.

### H.1. Problem formulations

#### H.1.1. Set optimization formulation and its property

Let  $\mathcal{S} \subseteq [n]$  be the subset of open facilities chosen by the newcomer and  $C := C_1 \cup \dots \cup C_c$  be the set of existing open facilities of all competitors (where  $C_k$  is the set of open facilities of competitor  $k \in [c]$ ). Let  $\tilde{\gamma}_i$  be the predetermined maximum number of open facilities of all companies that will be considered by customer  $i \in [m]$ . Then, the consideration set  $\mathcal{Y}_i(\mathcal{S})$  of customer  $i$  under the joint limited choice rule can be

written as:

$$\mathcal{Y}_i(\mathcal{S}) \in \operatorname{argmax} \left\{ \sum_{j \in \mathcal{Y}'} u_{ij} : \mathcal{Y}' \subseteq \mathcal{S} \cup C, |\mathcal{Y}'| \leq \tilde{\gamma}_i \right\}, \quad (\text{H.1})$$

and the probability of customer  $i$  patronizing an open facility  $j \in \mathcal{S} \cup C$  can be computed as

$$\tilde{p}_{ij} = \begin{cases} \frac{u_{ij}}{\sum_{j' \in \mathcal{S} \cap \mathcal{Y}_i(\mathcal{S})} u_{ij'} + \sum_{j' \in C \cap \mathcal{Y}_i(\mathcal{S})} u_{ij'}}, & \text{if } j \in \mathcal{Y}_i(\mathcal{S}); \\ 0, & \text{otherwise,} \end{cases} \quad \forall j \in \mathcal{S} \cup C. \quad (\text{H.2})$$

Thus, the probability of customer  $i$  (or the proportion of its buying power) being attracted by the open facilities of the newcomer company can be written as

$$\tilde{\Phi}_i(\mathcal{S}) := \sum_{j \in \mathcal{S} \cap \mathcal{Y}_i(\mathcal{S})} \tilde{p}_{ij} = \frac{\sum_{j \in \mathcal{S} \cap \mathcal{Y}_i(\mathcal{S})} u_{ij}}{\sum_{j \in \mathcal{S} \cap \mathcal{Y}_i(\mathcal{S})} u_{ij} + \sum_{j \in C \cap \mathcal{Y}_i(\mathcal{S})} u_{ij}}. \quad (\text{H.3})$$

Similar to the CFLPLCR, the CFLPJLCR attempts to locate new facilities in the competitive environment while maximizing the net profit of the newcomer and can be formally stated as

$$\max_{\mathcal{S} \subseteq [n]} \sum_{i \in [m]} b_i \tilde{\Phi}_i(\mathcal{S}) - \sum_{j \in \mathcal{S}} f_j. \quad (\text{CFLPJLCR})$$

In contrast to the probability functions  $\{\Phi_i\}_{i \in [m]}$  in the CFLPLCR which are shown to be submodular in Theorem 3.2, the probability functions  $\{\tilde{\Phi}_i\}_{i \in [m]}$  in the CFLPJLCR are *not* submodular, as demonstrated in the following example.

**Example H.1.** Consider an example of  $\tilde{\Phi}_i(\cdot)$  where  $n = 3$ ,  $C = \{4, 5\}$ ,  $(u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5}) = (1, 3, 4, 2, 2)$ , and  $\tilde{\gamma}_i = 3$ . The marginal gains of adding element 2 into subsets  $\mathcal{S} = \{1\}$  and  $\mathcal{T} = \{1, 3\}$  are

$$\begin{aligned} \tilde{\Phi}_i(\mathcal{S} \cup \{2\}) - \tilde{\Phi}_i(\mathcal{S}) &= \frac{u_{i2}}{u_{i2} + u_{i4} + u_{i5}} - \frac{u_{i1}}{u_{i1} + u_{i4} + u_{i5}} = \frac{3}{7} - \frac{1}{5} = \frac{8}{35}, \\ \tilde{\Phi}_i(\mathcal{T} \cup \{2\}) - \tilde{\Phi}_i(\mathcal{T}) &= \frac{u_{i2} + u_{i3}}{u_{i2} + u_{i3} + u_{i4}} - \frac{u_{i3}}{u_{i3} + u_{i4} + u_{i5}} = \frac{7}{9} - \frac{4}{8} = \frac{5}{18}. \end{aligned}$$

As  $\tilde{\Phi}_i(\mathcal{S} \cup \{2\}) - \tilde{\Phi}_i(\mathcal{S}) < \tilde{\Phi}_i(\mathcal{T} \cup \{2\}) - \tilde{\Phi}_i(\mathcal{T})$ , function  $\tilde{\Phi}_i$  is not submodular.

Example H.1 demonstrates that different from the CFLPLCR, for the CFLPJLCR, we cannot develop MILP formulations based on submodular or lifted submodular inequalities. Thus, the proposed B&C algorithm based on submodular and lifted submodular inequalities for the CFLPLCR cannot be directly applied

to solving CFLPJLCRs. In Appendix H.2, we will develop an iterative heuristic algorithm, based on solving a sequence of CFLPLCRs, to find a high-quality solution for CFLPJLCRs.

### H.1.2. Mixed integer linear programming reformulation for (CFLPJLCR)

Let  $x \in \{0, 1\}^n$  be the location variables such that  $x_j = 1$  if facility  $j$  is open and  $x_j = 0$  otherwise. Let  $y \in \{0, 1\}^{m \times n}$  and  $z \in \{0, 1\}^{m \times |C|}$  be the allocation variables such that  $y_{ij} = 1$  and  $z_{ij} = 1$  if customer  $i$  considers to patronize the facility  $j \in [n]$  of the newcomer and existing facility  $j \in C$  of the competitors, respectively. Let  $w \in \{0, 1\}^m$  be the indicator variables such that  $w_i = 1$  if the size of customer  $i$ 's consideration set is equal to  $\tilde{\gamma}_i$ , i.e.,  $\sum_{j \in [n]} y_{ij} + \sum_{j \in C} z_{ij} = \tilde{\gamma}_i$ , and  $w_i = 0$  otherwise. Then (CFLPJLCR) can be formulated as the following mixed integer nonlinear programming (MINLP) formulation:

$$\max_{x, y, z, w} \sum_{i \in [m]} b_i \frac{\sum_{j \in [n]} u_{ij} y_{ij}}{\sum_{j \in [n]} u_{ij} y_{ij} + \sum_{j \in C} u_{ij} z_{ij}} - \sum_{j \in [n]} f_j x_j \quad (\text{H.4a})$$

$$\text{s.t. } y_{ij} \leq x_j, \forall i \in [m], j \in [n], \quad (\text{H.4b})$$

$$y_{ij_1} \leq z_{ij_2}, \forall i \in [m], j_1 \in [n], j_2 \in C \text{ with } u_{ij_1} < u_{ij_2}, \quad (\text{H.4c})$$

$$z_{ij_1} \leq z_{ij_2}, \forall i \in [m], j_1, j_2 \in C \text{ with } u_{ij_1} < u_{ij_2}, \quad (\text{H.4d})$$

$$\sum_{j \in [n]} y_{ij} + \sum_{j \in C} z_{ij} \leq \tilde{\gamma}_i, \forall i \in [m], \quad (\text{H.4e})$$

$$\sum_{j \in [n]} y_{ij} + \sum_{j \in C} z_{ij} \geq \tilde{\gamma}_i w_i, \forall i \in [m], \quad (\text{H.4f})$$

$$\sum_{j \in [n]} x_j \leq n - (n + |C| - \tilde{\gamma}_i + 1)(1 - w_i), \forall i \in [m], \quad (\text{H.4g})$$

$$1 - w_i \leq z_{ij}, \forall i \in [m], j \in C, \quad (\text{H.4h})$$

$$x_j \in \{0, 1\}, \forall j \in [n], \quad (\text{H.4i})$$

$$y_{ij} \in \{0, 1\}, \forall i \in [m], j \in [n], \quad (\text{H.4j})$$

$$z_{ij} \in \{0, 1\}, \forall i \in [m], j \in C, \quad (\text{H.4k})$$

$$w_i \in \{0, 1\}, \forall i \in [m]. \quad (\text{H.4l})$$

The objective function (H.4a) maximizes the net profit, where the first term is the revenue collected by the new open facilities and the second term is the fixed cost of opening the facilities. Constraints (H.4b) indicate that facility  $j$  is considered by customer  $i$  only if it is open. Constraints (H.4c) and (H.4d) ensure that customer  $i$  considers to patronize a new open facility  $j \in [n]$  of the newcomer or an existing facility

$j \in C$  of the competitors only if the customer also considers to patronize all existing facilities in  $C$  with higher utilities than that of facility  $j$ . Constraints (H.4e) guarantee that the number of facilities considered by customer  $i$  is upper bounded by  $\tilde{\gamma}_i$  for each  $i \in [m]$ ; constraints (H.4f)–(H.4h), together with (H.4e), ensure that variables  $\{w_i\}_{i \in [m]}$  are properly defined:

- If  $w_i = 1$ , then from (H.4e) and (H.4f), equation  $\sum_{j \in [n]} y_{ij} + \sum_{j \in C} z_{ij} = \tilde{\gamma}_i$  holds, and the size of customer  $i$ 's consideration set is equal to  $\tilde{\gamma}_i$ ;
- Otherwise, constraint (H.4g) reduces to  $\sum_{j \in [n]} x_j \leq \tilde{\gamma}_i - |C| - 1$ , which together with (H.4b) and (H.4k) implies that

$$\sum_{j \in [n]} y_{ij} + \sum_{j \in C} z_{ij} \leq \sum_{j \in [n]} x_j + \sum_{j \in C} z_{ij} \leq \tilde{\gamma}_i - |C| - 1 + |C| = \tilde{\gamma}_i - 1 < \tilde{\gamma}_i,$$

and thus the size of customer  $i$ 's consideration set is smaller than  $\tilde{\gamma}_i$ .

Next, we present an equivalent MILP reformulation for (H.4). Specifically, let  $t \in [0, 1]^m$  be the probabilities of customers being attracted by the new open facilities, i.e.,

$$t_i = \frac{\sum_{j \in [n]} u_{ij} y_{ij}}{\sum_{j \in [n]} u_{ij} y_{ij} + \sum_{j \in C} u_{ij} z_{ij}}, \quad \forall i \in [m], \quad (\text{H.5})$$

let  $\omega^0 \in [0, 1]^{m \times n}$  and  $\omega^1 \in [0, 1]^{m \times |C|}$  be such that  $\omega_{ij}^0 = t_i y_{ij}$  for  $i \in [m]$  and  $j \in [n]$  and  $\omega_{ij}^1 = t_i z_{ij}$  for  $i \in [m]$  and  $j \in C$ . Then, using the linearization technique, problem (H.4) can be reformulated as the following MILP problem:

$$\max_{x, y, z, w, t, \omega^0, \omega^1} \sum_{i \in [m]} b_i t_i - \sum_{j \in [n]} f_j x_j \quad (\text{H.6a})$$

s.t. (H.4b)–(H.4l),

$$\sum_{j \in [n]} u_{ij} \omega_{ij}^0 + \sum_{j \in C} u_{ij} \omega_{ij}^1 = \sum_{j \in [n]} u_{ij} y_{ij}, \quad \forall i \in [m], \quad (\text{H.6b})$$

$$\omega_{ij}^0 \geq 0, \quad \omega_{ij}^0 \leq t_i, \quad \omega_{ij}^0 \leq y_{ij}, \quad \omega_{ij}^0 \geq t_i + y_{ij} - 1, \quad \forall i \in [m], j \in [n], \quad (\text{H.6c})$$

$$\omega_{ij}^1 \geq 0, \quad \omega_{ij}^1 \leq t_i, \quad \omega_{ij}^1 \leq z_{ij}, \quad \omega_{ij}^1 \geq t_i + z_{ij} - 1, \quad \forall i \in [m], j \in C. \quad (\text{H.6d})$$

Although (H.6) is an MILP problem, it involves  $O(m(n + |C|))$  binary variables and constraints. The large problem size, together with the extremely poor LP relaxation, makes (H.6) impractical to be directly solved

by modern MILP solvers, even for problems with moderate sizes of  $n$  and  $m$ ; see Appendix H.3 further ahead.

## H.2. An iterative heuristic procedure to find a high-quality solution for the CFLPJLCR

To find a computationally effective solution for the CFLPJLCR, here we present an iterative heuristic algorithm, which is based on the close relations between the CFLPJLCR and CFLPLCR (considered in the paper):

1. For any feasible solution  $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$  of the MINLP formulation (H.4) of the CFLPJLCR,  $(x^*, y^*) = (\bar{x}, \bar{y})$  is also a feasible solution of the MINLP formulation (2) (in the paper) of the CFLPLCR with some parameters  $\{\gamma_i\}_{i \in [m]}$  and  $\{u_{i0}\}_{i \in [m]}$ . More specifically, letting  $\lambda_i := \sum_{j \in C} \bar{z}_{ij}$  be the number of facilities of the competitors that are considered to be patronized by customer  $i$ , and

$$\gamma_i := \tilde{\gamma}_i - \lambda_i \text{ and } u_{i0} := \sum_{\tau=1}^{\lambda_i} u_{i\sigma_i(\tau)}, \quad \forall i \in [m], \quad (\text{H.7})$$

then  $(x^*, y^*)$  is a feasible solution the MINLP formulation of the CFLPLCR with parameters  $\{\gamma_i\}_{i \in [m]}$  and  $\{u_{i0}\}_{i \in [m]}$ . Here,  $\sigma_i(1), \dots, \sigma_i(|C|)$  is a permutation of  $C$  such that  $u_{i\sigma_i(1)} \geq \dots \geq u_{i\sigma_i(|C|)}$ .

2. For any feasible solution  $(x^*, y^*)$  of the MINLP formulation (2) with  $\gamma_i \leq \tilde{\gamma}_i$  and  $u_{i0} = \sum_{\tau=1}^{\tilde{\gamma}_i - \gamma_i} u_{i\sigma_i(\tau)}$  for  $i \in [m]$ , we can recover a feasible solution  $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$  for the MINLP formulation (H.4) of the CFLPJLCR as follows. First, let  $\mathcal{S} = \{i \in [n] : x_i^* = 1\}$ , and  $\pi_i(1), \dots, \pi_i(|\mathcal{S}| + |C|)$  be a permutation of  $\mathcal{S} \cup C$  such that  $u_{i\pi_i(1)} \geq \dots \geq u_{i\pi_i(|\mathcal{S}| + |C|)}$ , and  $\mathcal{P}_i = \{\pi_i(1), \dots, \pi_i(\tilde{\gamma}_i)\}$  if  $|\mathcal{S}| + |C| \geq \tilde{\gamma}_i$  and  $\mathcal{P}_i = \mathcal{S} \cup C$  otherwise. Then, one feasible solution  $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$  for the MINLP formulation (H.4) of the CFLPJLCR is:

$$\begin{aligned} \bar{x} &= x^*, \quad \bar{y}_{ij} = \begin{cases} 1, & \text{if } i \in \mathcal{P}_i \cap \mathcal{S}, \\ 0, & \text{otherwise,} \end{cases} \quad \forall i \in [m], j \in [n], \\ \bar{z}_{ij} &= \begin{cases} 1, & \text{if } i \in \mathcal{P}_i \cap C, \\ 0, & \text{otherwise,} \end{cases} \quad \forall i \in [m], j \in C, \quad \bar{w}_i = \begin{cases} 1, & \text{if } |\mathcal{S} \cup C| \geq \tilde{\gamma}_i, \\ 0, & \text{otherwise,} \end{cases} \quad \forall i \in [m]. \end{aligned} \quad (\text{H.8})$$

Based on the above relations between the CFLPJLCR and CFLPLCR, we propose to find a feasible solution for the CFLPJLCR by iteratively solving CFLPLCRs (with different parameters  $\{\gamma_i\}_{i \in [m]}$  and  $\{u_{i0}\}_{i \in [m]}$ ). Specifically, we first provide an initial estimate  $\{\lambda_i\}_{i \in [m]}$  for the numbers of facilities of the competitors that are considered to be patronized by customers, compute  $\{\gamma_i\}_{i \in [m]}$  and  $\{u_{i0}\}_{i \in [m]}$  as in (H.7), and solve the

CFLPLCR with parameters  $\{\gamma_i\}_{i \in [m]}$  and  $\{u_{i0}\}_{i \in [m]}$  to obtain a solution  $(x^*, y^*)$ . Then, we use the  $(x^*, y^*)$  to recover a feasible solution  $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$  as in (H.8), which enables to compute  $\lambda_i := \sum_{j \in C} \bar{z}_{ij}$  for  $i \in [n]$ , obtaining new parameters  $\{\gamma_i\}_{i \in [m]}$  and  $\{u_{i0}\}_{i \in [m]}$  (using (H.7)). The procedure is repeated until the parameters  $\{\lambda_i\}_{i \in [m]}$  (and thus  $\{\gamma_i\}_{i \in [m]}$  and  $\{u_{i0}\}_{i \in [m]}$ ) remain unchanged or some prespecified maximum number of iterations is reached.

The overall procedure is summarized in Algorithm 1. In each iteration of Algorithm 1, we use the proposed B&C algorithm based on lifted submodular inequalities to solve the CFLPLCR in step 4. Due to the efficiency of the proposed B&C algorithm for the CFLPLCR (see Section 6 in the paper), the above proposed iterative heuristic algorithm is usually much more efficient than directly solving the MILP reformulation (H.6) by off-the-shelf solvers, as will be shown in the next subsection.

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**Algorithm 1:** An iterative procedure to find a feasible solution for the CFLPJLCR

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**Input:** An initial estimate  $\{\lambda_i\}_{i \in [m]}$  for the numbers of facilities of the competitors that are considered to be patronized by customers, and the maximum number of iterations  $\text{MaxIter}$ .

- 1 Set  $\text{Iter} \leftarrow 0$  and  $\text{Obj} \leftarrow 0$ ;
- 2 **repeat**
- 3     Compute  $\{\gamma_i\}_{i \in [m]}$  and  $\{u_{i0}\}_{i \in [m]}$  as in (H.7);
- 4     Solve the CFLPLCR with parameters  $\{\gamma_i\}_{i \in [m]}$  and  $\{u_{i0}\}_{i \in [m]}$  to obtain an optimal solution  $(x^*, y^*)$ ;
- 5     Use (H.8) to compute a feasible solution  $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$  for the CFLPJLCR and the corresponding objective value  $\bar{v}$ ;
- 6     **if**  $\bar{v} > \text{Obj}$  **then**
- 7         Update the best found solution and set  $\text{Obj} \leftarrow \bar{v}$ ;
- 8     **end**
- 9     **if**  $\lambda_{i_0} \neq \sum_{j \in C} \bar{z}_{i_0 j}$  holds for some  $i_0 \in [m]$  **then**
- 10         Update  $\lambda_i \leftarrow \sum_{j \in C} \bar{z}_{ij}$  for all  $i \in [m]$  and mark that  $\{\lambda_i\}_{i \in [m]}$  are changed;
- 11     **end**
- 12     Set  $\text{Iter} \leftarrow \text{Iter} + 1$ ;
- 13 **until**  $\{\lambda_i\}_{i \in [m]}$  are not changed or  $\text{Iter} \geq \text{MaxIter}$ ;

---

### H.3. Effectiveness and efficiency of the proposed iterative heuristic algorithm for solving the CFLPJLCR

In this subsection, we present the computational results to demonstrate the effectiveness and efficiency of the proposed iterative heuristic algorithm in Appendix H.2 for solving the CFLPJLCR. To do this, we compare the performance of the following two settings:

- Gurobi: solving MILP formulation (H.6) of the CFLPJLCR by the B&C algorithm of Gurobi.

- **iB&C+LSI**: solving the CFLPJLCR by the iterative approach in Algorithm 1, in which the CFLPLCR subproblems are solved by the proposed B&C algorithm based on the lifted submodular inequalities.

In Algorithm 1, we set  $\lambda_i := \lceil \frac{\tilde{\gamma}_i}{2} \rceil$  for all  $i \in [m]$  and IterMax = 50.

We conduct experiments on a testset of 30 CFLPJLCR instances. The numbers of customers and facility locations  $(m, n)$  are taken from  $\{50, 100, 200\}$  and  $\{25, 50\}$ , respectively, and the values of  $\tilde{\gamma}_i$  for  $i \in [m]$  are set to the same  $\tilde{\gamma}$ , chosen from  $\{3, 4, 5, 6, 7\}$ . Other parameters of the CFLPJLCR are constructed using the same procedure as in Section 6 of Chen et al. (2025).

Table 5 summarizes the computational results of **Gurobi** and **iB&C+LSI**. For each setting, we report the total CPU time in seconds (T) and the objective value of the optimal solution or the best incumbent of the instance ( $v$ ). For setting **Gurobi**, we report under column T (G%) the percentage optimality gap (G%) for instances that cannot be solved to optimality within the given time limit, computed as  $\frac{\text{UB} - \text{LB}}{\text{UB}} \times 100\%$ , where UB and LB denote the upper bound and the lower bound obtained at the end of the time limit, and the LP relaxation bound obtained at the root node ( $v_{\text{LP}}$ ); for setting **iB&C+LSI**, we report the number of iterations (It), required by Algorithm 1. To evaluate the quality of feasible solutions returned by **iB&C+LSI**, we also report the objective ratio (OR(%)), defined by  $\frac{v_{\text{iB\&C+LSI}}}{v_{\text{Gurobi}}}$ , where  $v_{\text{Gurobi}}$  and  $v_{\text{iB\&C+LSI}}$  denote the best objective values of incumbents under settings **Gurobi** and **iB&C+LSI**, respectively. The larger the OR(%), the higher the solution quality returned by **iB&C+LSI**.

From Table 5, we first observe that it is very inefficient to use **Gurobi** to solve the MILP reformulation of the CFLPJLCR; indeed, **Gurobi** can only solve 2 small-scale instances (with  $m = 50$  and  $n = 25$ ) to optimality within the time limit of 7200 seconds and the CPU times are all larger than 700 seconds. This is due to the large problem size (i.e., the number of variables and constraints are all  $O(m(n + |C|))$ ), and particularly the extremely poor LP relaxation. The latter is evidenced by the results under columns  $v$  and  $v_{\text{LP}}$  for which we observe that the difference between the optimal value (or the best incumbent) of the MILP reformulation of the CFLPJLCR and that of its LP relaxation is very large. In sharp contrast, the proposed **iB&C+LSI** for the CFLPJLCR is very efficient; the CPU time ranges from 0.1 seconds to 23.6 seconds. This is attributed to the small number of the iterations (as shown in column It) and the efficiency of the proposed B&C algorithm for solving the CFLPLCR subproblems in Algorithm 1. Moreover, **iB&C+LSI** is competitive to **Gurobi** in finding a high-quality solution; for some cases, the solution returned by **iB&C+LSI** is even better than that returned by **Gurobi** (with the time limit of 7200 seconds). From these results, we can conclude that **iB&C+LSI** is indeed capable of finding high-quality feasible solutions for the CFLPJLCR

while enjoying high computational efficiency.

Table 5: Performance comparison of settings *Gurobi* and *iB&C+LSI* for the CFLPJLCR.

$m$	$n$	$\hat{\gamma}$	<i>Gurobi</i>				<i>iB&amp;C+LSI</i>		OR (%)		
			T (G%)	N	v	UB <sup>1</sup>	T (G%)	v			
50	25	3	623.9	48200	<b>9226</b>	18227	0.1	<b>9226</b>	2	1.00	
		4	(<0.1)	86526	<b>8920</b>	18438	0.2	8513	2	0.95	
		5	766.9	30847	<b>8648</b>	18119	0.8	8624	2	1.00	
		6	(<0.1)	47768	<b>8513</b>	17939	0.4	<b>8513</b>	1	1.00	
		7	(<0.1)	39177	<b>8513</b>	17802	2.5	<b>8513</b>	1	1.00	
		50	(<0.1)	222526	<b>7341</b>	16991	0.4	7006	2	0.95	
		4	(27.4)	209440	<b>6771</b>	17364	0.3	6661	2	0.98	
	100	5	(32.8)	138685	<b>6480</b>	17374	1.2	6320	2	0.98	
		6	(37.7)	127969	<b>6339</b>	18184	2.1	<b>6339</b>	2	1.00	
		7	(28.6)	101758	<b>6244</b>	17345	12.4	6231	2	1.00	
		100	25	3	(1.2)	283457	<b>25201</b>	37740	0.4	24473	
		4	(<0.1)	353281	<b>24921</b>	38945	0.9	24345	4	0.98	
		5	(5.3)	106213	<b>24580</b>	37779	2.6	24329	3	0.99	
		6	(7.5)	81556	<b>24562</b>	37507	4.5	24329	3	0.99	
200	25	7	(7.8)	95177	<b>24547</b>	37264	12.7	24424	2	0.99	
		50	3	(25.6)	58506	<b>20540</b>	36292	0.7	19391	3	0.94
		4	(33.7)	49660	19033	36534	0.7	<b>19059</b>	2	1.00	
		5	(35.0)	17328	<b>18925</b>	36465	4.4	18667	3	0.99	
		6	(31.4)	25297	18565	38597	6.1	<b>18632</b>	3	1.00	
		7	(30.6)	15473	<b>18537</b>	36959	10.7	18519	2	1.00	
		200	25	3	(7.8)	166055	<b>63535</b>	82611	11.4	62100	
	50	4	(15.0)	35350	61565	85141	0.8	<b>62662</b>	2	1.02	
		5	(6.2)	43981	<b>62970</b>	83445	2.6	62428	2	0.99	
		6	(8.3)	29843	<b>62516</b>	82688	3.9	61639	2	0.99	
		7	(10.0)	32066	<b>61823</b>	82126	7.6	61803	2	1.00	
		50	3	(17.5)	22998	<b>57594</b>	81524	1.3	57117	4	0.99
		4	(19.2)	13698	<b>56886</b>	81520	1.5	56237	2	0.99	
		5	(23.1)	6871	55557	81685	4.2	<b>55669</b>	2	1.00	
	7	6	(21.2)	5002	55095	85015	8.1	<b>55417</b>	2	1.01	
		7	(21.5)	5557	54885	82482	23.6	<b>55755</b>	2	1.02	

## References

- Aros-Vera, F., Marianov, V., & Mitchell, J. E. (2013). P-Hub approach for the optimal park-and-ride facility location problem. *Eur. J. Oper. Res.*, 226, 277–285.
- Benati, S., & Hansen, P. (2002). The maximum capture problem with random utilities: Problem formulation and algorithms. *Eur. J. Oper. Res.*, 143, 518–530.
- Biesinger, B., Hu, B., & Raidl, G. (2016). Models and algorithms for competitive facility location problems with different customer behavior. *Ann. Math. Artif. Intell.*, 76, 93–119.

- Chen, W.-K., Zhang, W.-Y., Wang, Y.-R., Gelareh, S., & Dai, Y.-H. (2025). An efficient branch-and-cut approach for large-scale competitive facility location problems with limited choice rule. <https://doi.org/10.48550/arXiv.2406.05775>.
- Fernández, P., Pelegrín, B., Lančinskas, A., & Žilinskas, J. (2017). New heuristic algorithms for discrete competitive location problems with binary and partially binary customer behavior. *Comput. Oper. Res.*, 79, 12–18.
- Freire, A. S., Moreno, E., & Yushimoto, W. F. (2016). A branch-and-bound algorithm for the maximum capture problem with random utilities. *Eur. J. Oper. Res.*, 252, 204–212.
- García-Galán, E., Herrán, A., & Colmenar, J. M. (2024). Iterated local search for the facility location problem with limited choice rule. In A. Alonso-Betanzos, B. Guijarro-Berdiñas, V. Bolón-Canedo, E. Hernández-Pereira, O. Fontenla-Romero, D. Camacho, J. R. Rabuñal, M. Ojeda-Aciego, J. Medina, J. C. Riquelme, & A. Troncoso (Eds.), *Advances in Artificial Intelligence* (pp. 142–151). volume 14640.
- García-Galán, E., Herrán, A., & Colmenar, J. M. (2025). An efficient variable neighborhood search approach for the facility location problem with the limited choice rule. *Int. Trans. Oper. Res.*, online.
- Garey, M. R., & Johnson, D. S. (1978). “Strong” NP-completeness results: Motivation, examples, and implications. *J. ACM*, 25, 499–508.
- Haase, K. (2009). *Discrete Location Planning*. Technical Report WP-09-07 Institute of Transport and Logistics Studies University of Syndney.
- Haase, K., & Müller, S. (2014). A comparison of linear reformulations for multinomial logit choice probabilities in facility location models. *Eur. J. Oper. Res.*, 232, 689–691.
- Lin, Y. H., & Tian, Q. (2021). Branch-and-cut approach based on generalized Benders decomposition for facility location with limited choice rule. *Eur. J. Oper. Res.*, 293, 109–119.
- Mai, T., & Lodi, A. (2020). A multicut outer-approximation approach for competitive facility location under random utilities. *Eur. J. Oper. Res.*, 284, 874–881.
- Méndez-Vogel, G., Marianov, V., Lüer-Villagra, A., & Eiselt, H. (2023). Store location with multipurpose shopping trips and a new random utility customers’ choice model. *Eur. J. Oper. Res.*, 305, 708–721.
- Nemhauser, G., & Wolsey, L. (1988). *Integer and Combinatorial Optimization*. New York: Wiley.
- Suárez-Vega, R., Santos-Peña, D. R., & Dorta-González, P. (2004). Competitive multifacility location on networks: The  $(r|X_p)$ -medianoid problem. *J. Reg. Sci.*, 44, 569–588.
- Zhang, Y., Berman, O., & Verter, V. (2012). The impact of client choice on preventive healthcare facility network design. *OR Spectrum*, 34, 349–370.