# STA 4210 HW1

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1.(R) A substance used in biological and medical research is shipped by airfreight to users in cartons of 1,000 ampules. The data below, involving 10 shipments, were collected on the number of times the carton was transferred from one aircraft to another over the shipment route ($X$) and the number of ampules found to be broken upon arrival ($Y$). Assume that the simple linear regression model is appropriate.

```
shipment <- c(1,0,2,0,3,1,0,1,2,0)
ampules <- c(16,9,17,12,22,13,8,15,19,11)
```

a. Obtain the estimated regression function and MSE "by hand" (i.e. without using the `lm` function).

## Solution

The estimated regression model we learned is

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, \qquad \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2).$$

```
# load sample size
n <- length(shipment)

# compute sample means for xbar and ybar
xbar <- mean(shipment)
ybar <- mean(ampules)

#print
xbar
```

```
## [1] 1
```

```
ybar
```

```
## [1] 14.2
```

From the formula sheet,

$$\hat{\beta}_1 = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2}, \qquad \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}.$$

```
# slope and intercept (by hand formulas)
numerator   <- sum((shipment - xbar) * (ampules - ybar))
denominator <- sum((shipment - xbar)^2)

b1 <- numerator / denominator
b0 <- ybar - b1 * xbar

b1
```

```
## [1] 4
```

```
b0
```

```
## [1] 10.2
```

Thus, the estimated regression function is

$$\hat{y} = 10.2 + 4x.$$

From the formula sheet,

$$MSE = \frac{SSE}{n-2} \qquad \text{and} \qquad SSE = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2.$$

```
# fitted values
yhat <- b0 + b1 * shipment

# SSE and MSE
SSE <- sum((ampules - yhat)^2)
MSE <- SSE / (n - 2)

SSE
```

```
## [1] 17.6
```

```
MSE
```

```
## [1] 2.2
```

$$MSE = 2.2.$$

b. Obtain a point estimate of the expected number of broken ampules when $X = 1$ transfer is made using the regression function you calculated in part (a).

## solution:

From part (a), the estimated regression function is

$$\hat{y} = 10.2 + 4x.$$

A point estimate of the expected number of broken ampules when $X = 1$ transfer is made is obtained by evaluating the regression function at $x = 1$:

$$\hat{y}(1) = 10.2 + 4(1) = 14.2.$$

Thus, the point estimate of the expected number of broken ampules when one transfer is made is 14.2.

c. Fit the simple linear regression model by the `lm` function. Plot the data and the estimated regression line. Based on your plot, do you think the linear regression model is appropriate for this data?

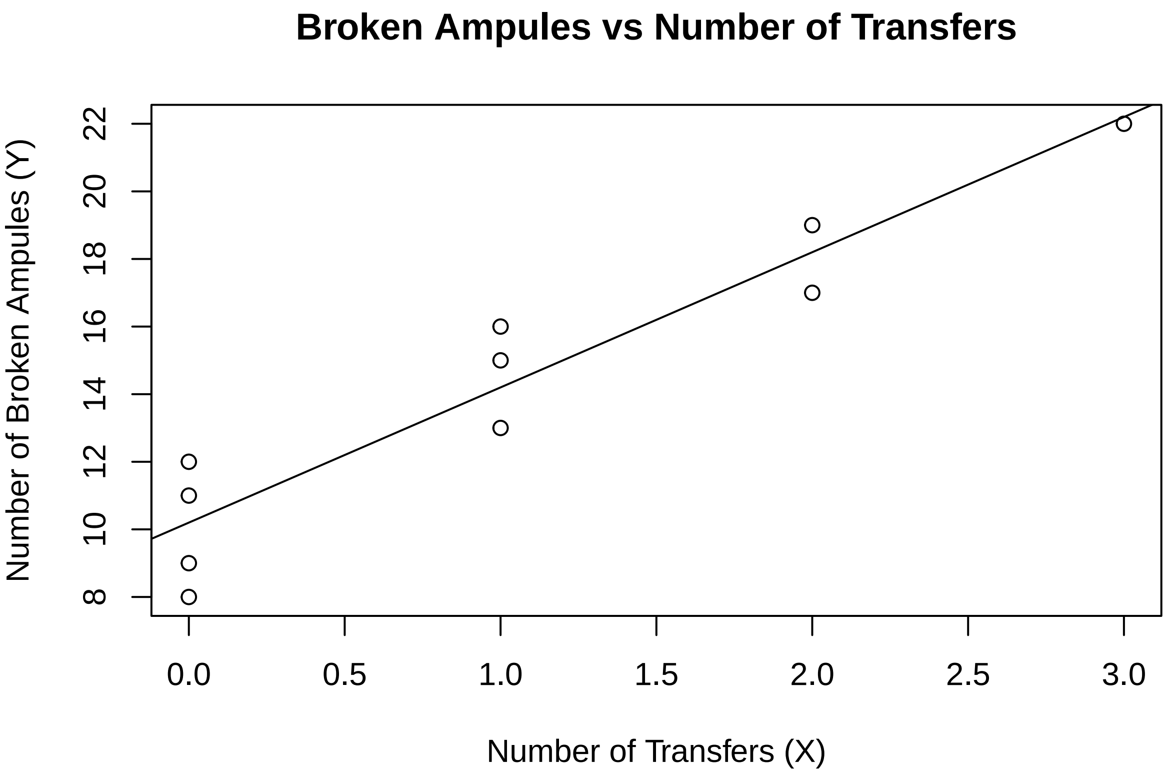
### Solution:

Fit the simple linear regression model using the `lm` function and plot the data with the estimated regression line.

```r
# fit simple linear regression model using the given data
fit <- lm(ampules ~ shipment)

# plot data
plot(shipment, ampules,
     xlab = "Number of Transfers (X)",
     ylab = "Number of Broken Ampules (Y)",
     main = "Broken Ampules vs Number of Transfers")

# add regression line
abline(fit)
```



Based on the plot, the relationship between the number of transfers and the number of broken ampules appears approximately linear, with an increasing trend and no strong curvature. Therefore, the simple linear regression model appears appropriate for this data.

d. Verify that your fitted regression line goes through the point $(\bar{X}, \bar{Y})$.

### Solution:

From part (a), the fitted regression line is

$$\hat{y} = 10.2 + 4x.$$

From the data that we computed in part a,

$$\bar{X} = 1, \qquad \bar{Y} = 14.2.$$

Plug in 1 to valuate the fitted regression function at $\bar{X} = 1$:

$$\hat{y}(\bar{X}) = 10.2 + 4(1) = 14.2.$$

Since

$$\hat{y}(\bar{X}) = \bar{Y},$$

the fitted regression line does passes through the point $(\bar{X}, \bar{Y})$. :)

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2.(R) Consider the simple linear regression model through the origin: $Y_i = \beta_1 X_i + \epsilon_i$, $\epsilon_i \sim N(0, \sigma^2)$. Consider two estimators for $\beta_1$:

- $\hat{\beta}_1 = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2}$
- $\tilde{\beta}_1 = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}$.

  We can show that $E[\hat{\beta}_1] = E[\tilde{\beta}_1] = \beta_1$. So these two estimators are both unbiased estimators for $\beta_1$. In general, an estimator with smaller variance is more efficient. For this problem, we will conduct simulation studies to analyze the Monte Carlo variance of the two estimators.

a. Run 1000 simulations. In each simulation, generate samples $(X_i, Y_i, \epsilon_i)$ with sample size $n = 30$ where $X_i \sim N(1, 9)$ (note that 9 is the variance), $\epsilon_i \sim N(0, 1)$ and $Y_i = 2X_i + \epsilon_i$ for $i = 1, .., n$ , and obtain the $\hat{\beta}_1, \tilde{\beta}_1$ based on the sample. Use `set.seed(123)` at the beginning of the code to ensure that results are reproducible.

```r
set.seed(123)

# number of simulations and sample size
num_sim <- 1000
n <- 30
beta1 <- 2

# storage for estimators
beta1_hat_values <- numeric(num_sim)      # sum(X_i Y_i) / sum(X_i^2)
beta1_tilde_values <- numeric(num_sim)    # sum(Y_i) / sum(X_i)

for (m in 1:num_sim) {

  # generate data
  X <- rnorm(n, mean = 1, sd = 3)         # X_i ~ N(1, 9)
  epsilon <- rnorm(n, mean = 0, sd = 1)   # epsilon_i ~ N(0, 1)
  Y <- beta1 * X + epsilon                # Y_i = 2 X_i + epsilon_i
  # estimators
  beta1_hat_values[m] <- sum(X * Y) / sum(X^2)
  beta1_tilde_values[m] <- sum(Y) / sum(X)
}

# empirical means across simulations
mean_hat <- mean(beta1_hat_values)
mean_tilde <- mean(beta1_tilde_values)

# sample variances across simulations
var_hat <- var(beta1_hat_values)
var_tilde <- var(beta1_tilde_values)

# output
mean_hat
```

```
## [1] 2.002106
```

```
mean_tilde
```

```
## [1] 2.001178
```

```
var_hat
```

```
## [1] 0.00374595
```

```
var_tilde
```

```
## [1] 0.4084336
```

Based on the 1000 simulated samples, the empirical mean of

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2}$$

is

$$2.002106,$$

and the empirical mean of

$$\tilde{\beta}_1 = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}$$

is

$$2.001178.$$

The sample variance of

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2}$$

across the 1000 simulations is

$$0.00374595,$$

while the sample variance of

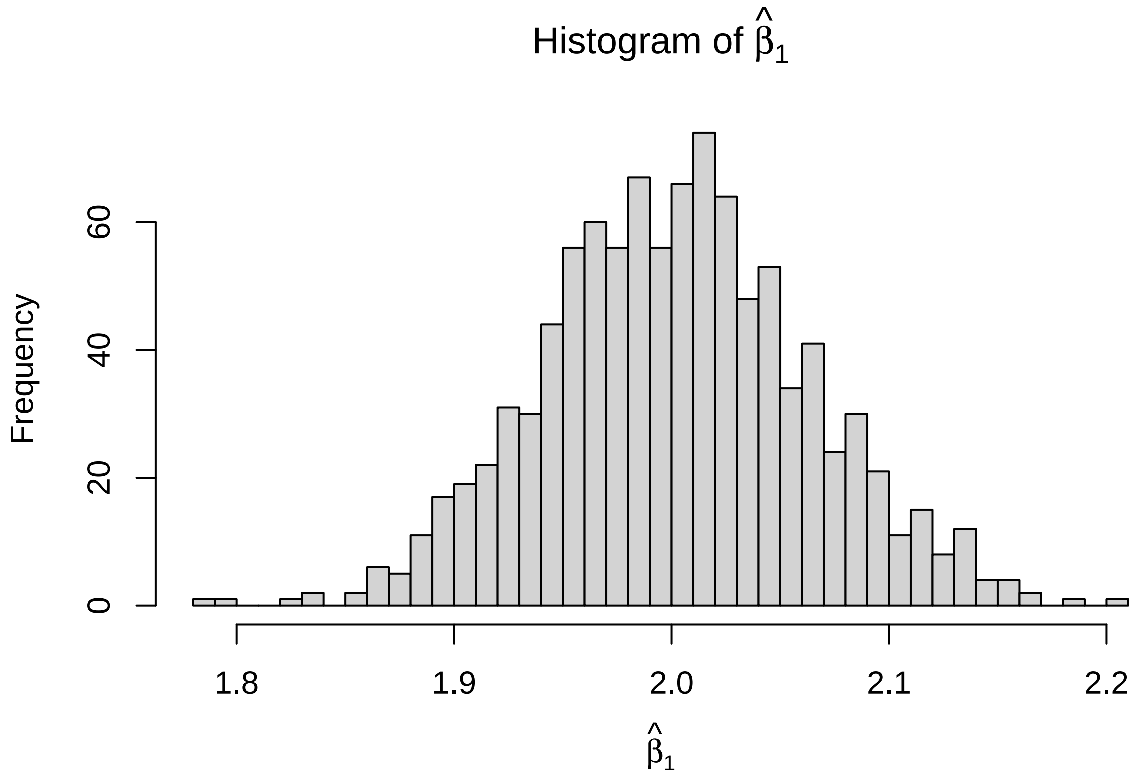
$$\tilde{\beta}_1 = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}$$

across the 1000 simulations is

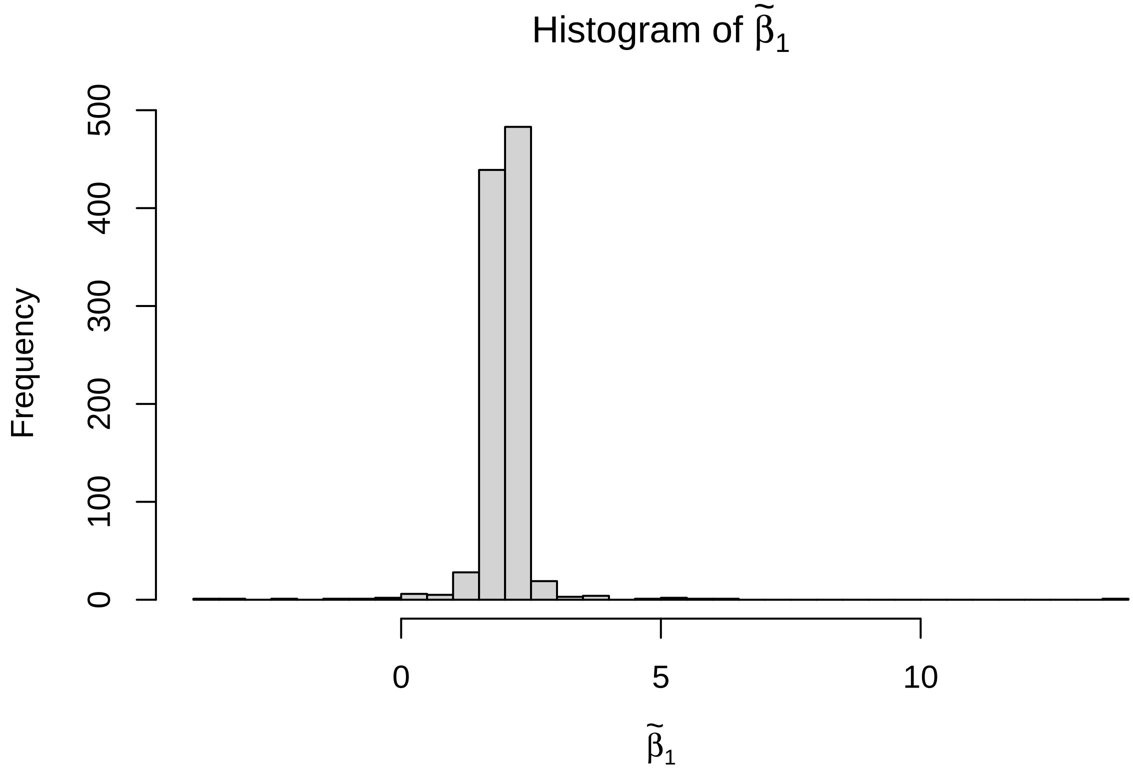
$$0.4084336.$$

b. Plot the histogram of $\hat{\beta}_1$ and $\tilde{\beta}$ across all simulations.

```r
# histogram of hat{beta}_1
hist(beta1_hat_values,
     breaks = 30,
     main = expression(paste("Histogram of ", hat(beta)[1])),
     xlab = expression(hat(beta)[1]))
```



```r
# histogram of tilde{beta}_1
hist(beta1_tilde_values,
     breaks = 30,
     main = expression(paste("Histogram of ", tilde(beta)[1])),
     xlab = expression(tilde(beta)[1]))
```



c. Compute the sample variance for $\hat{\beta}_1$ and $\tilde{\beta}_1$. Which estimator has smaller sample variance?

```
var_hat <- var(beta1_hat_values)
var_tilde <- var(beta1_tilde_values)

var_hat
```

```
## [1] 0.00374595
```

```
var_tilde
```

```
## [1] 0.4084336
```

Since both $\hat{\beta}_1$ and $\tilde{\beta}_1$ are unbiased estimators of $\beta_1$, efficiency is determined by their variances. The estimator with smaller variance is more efficient because its values are more tightly concentrated around the true parameter. Since

$$\text{Var}(\hat{\beta}_1) = 0.00374595 < \text{Var}(\tilde{\beta}_1) = 0.4084336,$$

$\hat{\beta}_1$ is the more efficient estimator of $\beta_1$.

---

3. Consider the simple linear regression model through the origin and two estimators $\hat{\beta}_1$ and $\tilde{\beta}_1$ defined in problem 2. In the problem 2, we investigate the sampling distribution of these two estimators by simulations. For this problem, we will derive the **true** sampling distribution of the these two estimators.

a. Derive the $E(\hat{\beta}_1)$ and $V(\hat{\beta}_1)$. Write down the sampling distribution of $\hat{\beta}_1$. (Hint: $\hat{\beta}_1$ is a linear combination of $Y_i$'s)

## Solution:

Through origin model, the simple linear regression model is:

$$Y_i = \beta_1 X_i + \varepsilon_i, \qquad \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2).$$

The estimator is

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2}$$

By formula from the formula sheet for linear combinations,

$$E\left(\sum_{i=1}^n a_i Y_i\right) = \sum_{i=1}^n a_i E(Y_i).$$

we subsititute the some fraction of the function to with $a_i$ so that the equation matches:

$$\hat{\beta}_1 = \sum_{i=1}^n a_i Y_i, \qquad a_i = \frac{X_i}{\sum_{j=1}^n X_j^2}.$$

Since $E(Y_i) = \beta_1 X_i$ because:

if we Taking expectations on both sides of the model,

$$E(Y_i) = E(\beta_1 X_i + \varepsilon_i).$$

By linearity of expectation and since $\beta_1$ and $X_i$ are constants,

$$E(Y_i) = \beta_1 X_i + E(\varepsilon_i).$$

Because $E(\varepsilon_i) = 0$, we obtain

$$\boxed{E(Y_i) = \beta_1 X_i.}$$

We will then have:

$$E(\hat{\beta}_1) = E\left(\sum_{i=1}^n a_i Y_i\right) = \sum_{i=1}^n a_i \beta_1 X_i = \beta_1 \frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i^2} = \beta_1.$$

**So:**

$$E(\hat{\beta}_1) = \beta_1$$

Variance of $\hat{\beta}_1$: Again, The estimator of $\beta_1$ is

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2} = \sum_{i=1}^n a_i Y_i, \qquad a_i = \frac{X_i}{\sum_{j=1}^n X_j^2}.$$

Thus, $\hat{\beta}_1$ is a linear combination of the $Y_i$'s.

By the formula:

$$\mathrm{Var}\left(\sum_{i=1}^n a_i Y_i\right) = \sum_{i=1}^n a_i^2 \,\mathrm{Var}(Y_i) + 2 \sum_{i<j} a_i a_j \,\mathrm{Cov}(Y_i, Y_j).$$

Under the model's assumptions, the error terms $\varepsilon_i$ are independent. Since

$$Y_i = \beta_1 X_i + \varepsilon_i,$$

and constants do not affect covariance, it follows that

$$\mathrm{Cov}(Y_i, Y_j) = \mathrm{Cov}(\varepsilon_i, \varepsilon_j) = 0 \quad \text{for } i \neq j.$$

Therefore, all covariance terms vanish.

Moreover,

$$\mathrm{Var}(Y_i) = \mathrm{Var}(\varepsilon_i) = \sigma^2.$$

Hence,

$$\mathrm{Var}(\hat{\beta}_1) = \sigma^2 \sum_{i=1}^{n} a_i^2 = \sigma^2 \frac{\sum_{i=1}^{n} X_i^2}{\left(\sum_{j=1}^{n} X_j^2\right)^2} = \boxed{\frac{\sigma^2}{\sum_{i=1}^{n} X_i^2}.}$$

In summary: Since $\hat{\beta}_1$ is a linear combination of normally distributed $Y_i$'s,

$$\hat{\beta}_1 \sim N\left(\beta_1,\ \frac{\sigma^2}{\sum_{i=1}^{n} X_i^2}\right).$$

---

b. Derive the $E(\tilde{\beta}_1)$ and $V(\tilde{\beta}_1)$. Write down the sampling distribution of $\tilde{\beta}_1$.

We use the same regression through the origin model:

$$Y_i = \beta_1 X_i + \varepsilon_i, \qquad \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2),$$

and we treat the regressors $X_i$ as fixed.

The estimator is

$$\tilde{\beta}_1 = \frac{\sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} X_i}.$$

Rewrite $\tilde{\beta}_1$ as a linear combination of the $Y_i$'s. We rewrite $\tilde{\beta}_1$ in the form $\sum_{i=1}^{n} b_i Y_i$:

$$\tilde{\beta}_1 = \frac{\sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} X_i} = \sum_{i=1}^{n} b_i Y_i, \qquad b_i = \frac{1}{\sum_{j=1}^{n} X_j}.$$

Expectation of $\tilde{\beta}_1$. By the formula sheet for linear combinations,

$$E\left(\sum_{i=1}^{n} b_i Y_i\right) = \sum_{i=1}^{n} b_i E(Y_i).$$

From part (a), we already proved that

$$E(Y_i) = \beta_1 X_i.$$

Therefore,

$$E(\tilde{\beta}_1) = E\left(\sum_{i=1}^{n} b_i Y_i\right) = \sum_{i=1}^{n} b_i\, \beta_1 X_i = \sum_{i=1}^{n} \frac{1}{\sum_{j=1}^{n} X_j}\, \beta_1 X_i = \frac{\beta_1 \sum_{i=1}^{n} X_i}{\sum_{j=1}^{n} X_j} = \beta_1.$$

$$\boxed{E(\tilde{\beta}_1) = \beta_1.}$$

Variance of $\tilde{\beta}_1$.: By the variance formula for a linear combination,

$$\operatorname{Var}\left(\sum_{i=1}^{n} b_i Y_i\right) = \sum_{i=1}^{n} b_i^2 \operatorname{Var}(Y_i) + 2\sum_{i<j} b_i b_j \operatorname{Cov}(Y_i, Y_j).$$

As shown in part (a), under the model assumptions the errors are independent, hence

$$\operatorname{Cov}(Y_i, Y_j) = 0 \quad \text{for } i \neq j,$$

so all covariance terms vanish.

Also from part (a), we have

$$\operatorname{Var}(Y_i) = \sigma^2.$$

Thus,

$$\operatorname{Var}(\tilde{\beta}_1) = \sum_{i=1}^{n} b_i^2 \sigma^2 = \sigma^2 \sum_{i=1}^{n} \left(\frac{1}{\sum_{j=1}^{n} X_j}\right)^2 = \sigma^2 \cdot n \cdot \frac{1}{\left(\sum_{j=1}^{n} X_j\right)^2}.$$

$$\boxed{\operatorname{Var}(\tilde{\beta}_1) = \frac{n\sigma^2}{\left(\sum_{i=1}^{n} X_i\right)^2}.}$$

Sampling distribution of $\tilde{\beta}_1$. Because each $Y_i$ is normal and $\tilde{\beta}_1 = \sum_{i=1}^{n} b_i Y_i$ is a linear combination of jointly normal variables, $\tilde{\beta}_1$ is normal. Using the mean and variance derived above,

$$\boxed{\tilde{\beta}_1 \sim N\left(\beta_1,\ \frac{n\sigma^2}{\left(\sum_{i=1}^{n} X_i\right)^2}\right).}$$

---

c. Prove that $V(\hat{\beta}_1) \leq V(\tilde{\beta}_1)$. (Hint: there are different ways to show this. One option is to prove that $\frac{1}{V(\hat{\beta}_1)} - \frac{1}{V(\tilde{\beta}_1)} = \sum_{i=1}^{n}(X_i - \bar{X})^2/\sigma^2 \geq 0$)

From parts (a) and (b),

$$\frac{1}{V(\hat{\beta}_1)} = \frac{\sum_{i=1}^{n} X_i^2}{\sigma^2}, \qquad \frac{1}{V(\tilde{\beta}_1)} = \frac{\left(\sum_{i=1}^{n} X_i\right)^2}{n\sigma^2}.$$

Subtract:

$$\frac{1}{V(\hat{\beta}_1)} - \frac{1}{V(\tilde{\beta}_1)} = \frac{1}{\sigma^2}\left[\sum_{i=1}^{n} X_i^2 - \frac{\left(\sum_{i=1}^{n} X_i\right)^2}{n}\right].$$

I FINALLY found that By definition of sample mean: $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

$$\frac{\left(\sum_{i=1}^{n} X_i\right)^2}{n} = n\bar{X}^2.$$

So the inside the bracket it becomes

$$\sum_{i=1}^{n} X_i^2 - n\bar{X}^2.$$

Now expand the sum of squared deviations:

$$\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}\left(X_i^2 - 2\bar{X}X_i + \bar{X}^2\right) = \sum_{i=1}^{n} X_i^2 - 2\bar{X}\sum_{i=1}^{n} X_i + n\bar{X}^2.$$

Since $\sum_{i=1}^n X_i = n\bar{X}$ by mutiply the sample mean equation by n, we have

$$-2\bar{X}\sum_{i=1}^n X_i = -2\bar{X}(n\bar{X}) = -2n\bar{X}^2,$$

hence

$$\sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=1}^n X_i^2 - 2n\bar{X}^2 + n\bar{X}^2 = \sum_{i=1}^n X_i^2 - n\bar{X}^2.$$

Therefore,

$$\sum_{i=1}^n X_i^2 - \frac{\left(\sum_{i=1}^n X_i\right)^2}{n} = \sum_{i=1}^n X_i^2 - n\bar{X}^2 = \sum_{i=1}^n (X_i - \bar{X})^2.$$

Substituting back,

$$\boxed{\frac{1}{V(\hat{\beta}_1)} - \frac{1}{V(\tilde{\beta}_1)} = \frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \bar{X})^2 \geq 0.}$$

Since $V(\hat{\beta}_1) > 0$ and $V(\tilde{\beta}_1) > 0$, the inequality

$$\frac{1}{V(\hat{\beta}_1)} \geq \frac{1}{V(\tilde{\beta}_1)}$$

Proves that:

$$\boxed{V(\hat{\beta}_1) \leq V(\tilde{\beta}_1).}$$