1. In the lecture notes and Homework 1, we have shown that $MA(1)$ and $MA(2)$ are stationary processes. In general, for a MA process with order q:

$$X_t = Z_t + \theta_1 Z_{t-1} + ... + \theta_q Z_{t-q}, \{Z_t\} \sim WN(0, \sigma^2)$$

(a) Prove it's stationary for all q by finding

- The mean function of $\{X_t\}$ and show it's independent of t.
- The autocovariance function of $\{X_t\}$ with lag h and show it's independent of t.

(b) Find the autocorrelation function of $\{X_t\}$ and verify that $MA(q)$ is $q-correlated$, aka the $\rho(h) = 0$ for $h > q$.

(a)

$$E(X_t) = E(Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q})$$

$$= E(Z_t) + \theta_1 E(Z_{t-1}) + \cdots + \theta_q E(Z_{t-q}) = 0$$

when $h=0$

$$Cov(X_t, X_{t+h}) = Var(X_t) = \sigma^2 + \theta_1^2 \sigma^2 + \cdots \theta_q^2 \sigma^2$$

$$= (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma^2$$

when $h=1$

$$Cov(X_t, X_{t+h}) = Cov(Z_t + \theta_1 Z_{t-1} + \cdots, Z_{t+1} + \theta_1 Z_t + \cdots)$$

$$= \theta_1 \sigma^2 + \theta_2 \theta_1 \sigma^2 + \theta_3 \theta_2 \sigma^2 + \cdots \theta_q \theta_{q-1} \sigma^2$$

$$= (\theta_1 + \theta_2 \theta_1 + \theta_3 \theta_2 \cdots + \theta_q \theta_{q-1})\sigma^2$$

when $h=2$

$$Cov(X_t, X_{t+h}) = Cov(Z_t + \theta_1 Z_{t-1} + \cdots, Z_{t+2} + \theta_1 Z_{t+1} + \theta_2 Z_t \cdots)$$

$$= (\theta_2 + \theta_3 \theta_1 + \theta_4 \theta_2 + \cdots \theta_q \theta_{q-2})\sigma^2$$

when $h \geq q$

$$Cov(X_t, X_{t+h}) = Cov(Z_t + \theta_1 Z_{t+1} + \cdots, Z_{t+q} + \theta_1 Z_{t+q-1} \cdots)$$

$$= 0$$

$$\Rightarrow \operatorname{cov}(X_t, X_{t+h}) = \begin{cases} \sum_{i=k}^{q} \theta_i \theta_{i-k} \sigma^2 & (\text{when } h = k) \text{ for } h < q \\ = 0 & \text{when } h \geq q \end{cases}$$

$\Rightarrow$ independent of t

(b) Autocorrelation

$$\operatorname{corr}(X_{t+h}, X_t) = \frac{r(h)}{r(0)}$$

when $h > q \Rightarrow r(h) = 0$ as above

$$= 0$$

$\Rightarrow$ MA(q) is q-related

2. For an AR process: $X_t = 0.8X_{t-1} - 0.5X_{t-2} + 0.25X_{t-4} + 0.4X_{t-6} + Z_t$ where $\{Z_t\} \sim WN(0, \sigma^2)$, give the generating function $\Phi^6(x)$.

$X_t = 0.8X_{t-1} - 0.5X_{t-2} + 0.25X_{t-4} + 0.4X_{t-6} + Z_t$

$X_t - 0.8X_{t-1} + 0.5X_{t-2} - 0.25X_{t-4} - 0.4X_{t-6} = Z_t$

$X_t - 0.8BX_t + 0.5B^2X_t - 0.25B^4X_t - 0.4B^6X_t = Z_t$

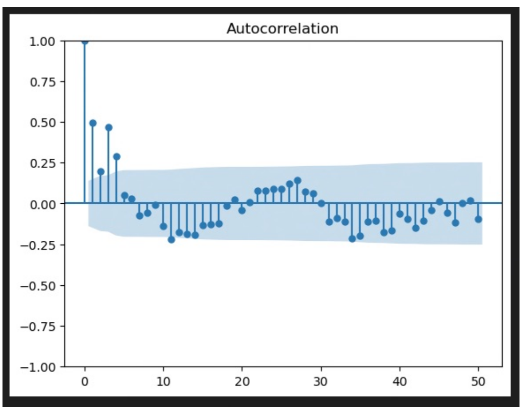
$\Phi^6(x) = 1 - 0.8x + 0.5x^2 - 0.25x^4 - 0.4x^6$

3. For a MA process: $X_t = 0.9Z_{t-1} + Z_{t-3} + 1.2Z_{t-4}$ $+Z_t$ where $\{Z_t\} \sim WN(0, \sigma^2)$, give the generating function $\Theta^4(x)$.

$X_t = 0.9BZ_t + B^3Z_t + 1.2B^4Z_t$

$\Theta^4(x) = 1 + 0.9x + x^3 + 1.2x^4$

4. Simulate the above MA process in Python and run the ACF plot. What did you observe from the ACF plot? Does it give a suggestion of stationary and/or the order of the MA process?



ACF drops quickly before 10 ⇒ suggest stationary

5. For each of the AR processes below, determine whether they are stationary by checking the stationary condition. In each case $\{Z_t\} \sim WN(0, \sigma^2)$

   (a) $X_t = -0.2X_{t-1} + 0.48X_{t-2} + Z_t$

   (b) $X_t = X_{t-1} - 0.8X_{t-2} + Z_t$

(a) $X_t + 0{,}2BX_t - 0{,}48B^2X_t = Z_t$

$\Phi^2(B) = 1 + 0{,}2B - 0{,}48B^2 = 0$

$B = \dfrac{-0{,}2 \pm \sqrt{0{,}04 + 4 \cdot 0{,}48}}{-0{,}96} = \dfrac{0{,}2 \pm 1{,}4}{0{,}96} = \begin{cases} 1{,}67 \\ -1{,}25 \end{cases} \Rightarrow |B| > 1$

all stationary

(b) $X_t - X_{t-1} + 0{,}8X_{t-2} = Z_t$

$X_t - BX_t + 0{,}8B^2X_t = Z_t$

$\Phi^2(B) = 1 - B + 0{,}8B^2 = 0$

$B = \dfrac{1 \pm \sqrt{1 - 4 \cdot 0{,}8}}{1{,}6} = \dfrac{1 \pm \sqrt{-2{,}2}}{1{,}6} = 0{,}625 \pm \sqrt{0{,}85}\, i$

$|B| = \sqrt{0.625^2 + 0.85} > 1$ $\Rightarrow$ stationary

6. Simulate the above two AR processes in Python then run the ADF test for stationary, report p values and test conclusion.

(a) p-value: $6{,}59 \times 10^{-29} < 0{,}05 \Rightarrow$ suggest stationary

(b) p-value: $0{,}000 < 0{,}05 \Rightarrow$ suggest stationary

7. For the ARMA(1,1) process $X_t - 0.8X_{t-1} = Z_t + 0.6Z_{t-1}$, $\{Z_t\} \sim WN(0, \sigma^2)$. Prove this process is stationary and invertible.

$\Phi(B) X_t = \Theta(B) Z_t$

$\Phi(B) = 1 - 0.8B = 0 \qquad B = 1.25 \Rightarrow |B| > 1$, stationary

$\Theta(B) = 1 + 0.6B = 0 \qquad B = -1.67 \Rightarrow |B| > 1$, invertible

8. The dataset *profit.csv* recorded the profits (in \$k) of an investment product in 200 days (positive number shows increased price compared to original price, negative number shows dropped price from original price).

(a) Attach the Time Series plot, and ACF plot for lags $h = 0, 1, ..., 20$ in the pdf. (Hint: when h gets larger, ACF plot doesn't provide a close-to-unbiased estimate anymore, therefore numbers become unstable and unreliable. It's reasonable just to examine the beginning part of the plots)

- In python, to choose the lags up to 20:

$$plot\_acf(data, lags = 20)$$

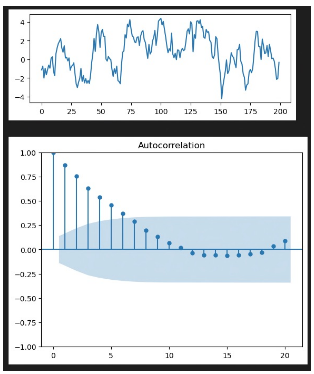
(b) Based on the plots from (a), do you think this is a stationary process? Briefly justify your answer.Perform an ADF test to verify your observation from the plots.

(c) Fit an AR(2) model to this data, and write the estimated model.

(d) Give the forecast of the next 5 days.

(e) Attach plots in the pdf showing the historical data and forecast (similar to the two plots on page 42 in L3. You can choose how close to zoom in.)

(a)



(b) I think it's stationary from the plot since it drops to 0 quickly before 10

ADF test: p-value $= 0.0033 < 0.05$

$\Rightarrow$ stationary

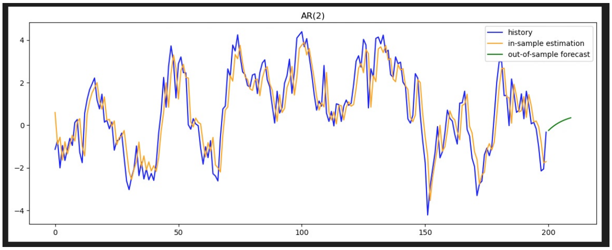
(c)

$$X_t - 0.5948 = 0.8481(X_{t-1} - 0.5948) + 0.0221(X_{t-2} - 0.5948) + Z_t$$

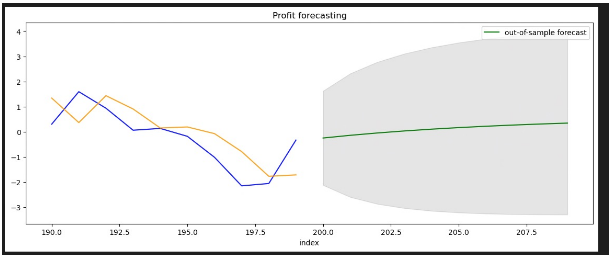
(d)

200	-0.246393
201	-0.139028
202	-0.046167
203	0.034963
204	0.105823

(e)



AR(2)
history
in-sample estimation
out-of-sample forecast
4
2
0
−2
−4
0
50
100
150
200



Profit forecasting
out-of-sample forecast
4
3
2
1
0
−1
−2
−3
190.0
192.5
195.0
197.5
200.0
202.5
205.0
207.5
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